DIFFERENTIATION OF FUNCTIONS.

NOTES

ON

DIFFERENTIATION OF FUNCTIONS

WITH EXAMPLES.

PAGE

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I. FUNCTIONS.

1. When two variable quantities are so related that the value of one depends upon that of the other, the former is said to be a *function* of the latter. Thus, the volume of a sphere is a function of its radius. Any expression containing x is a function of x, as x^2, $\log(x + x^2)$, $\sin 3x$, etc.

2. The symbols $F(x)$, $f(x)$, $\phi(x)$, $\psi(x)$, etc., are used to denote functions of x. The following examples will illustrate the notation of functions.

EXAMPLES.

1. If $\phi(x) = x^2 + 5$, show that

$$\phi(2) = 9, \quad \phi(1) = 6, \quad \phi(0) = 5, \quad \phi(a+1) = a^2 + 2a + 6,$$
$$\phi(x+h) = x^2 + 2hx + h^2 + 5.$$

2. If $f(x) = 2x^3 - x^2 - 7x + 6$, show that

$$f(3) = 30, \quad f(2) = 4, \quad f(0) = 6, \quad f(1) = 0,$$
$$f(-2) = 0, \quad f(\tfrac{3}{2}) = 0, \quad f(x-2) = 2x^3 - 13x^2 + 21x,$$
$$f(x+h) = 2x^3 + (6h-1)x^2 + (6h^2 - 2h - 7)x + 2h^3 - h^2 - 7h + 6.$$

3. Given $\phi(x) = x^2 - b^2$, $\psi(x) = x + a$; show that

$$\frac{(a)}{\psi(b)} = a - b, \quad \frac{\phi(a+b)}{\psi(2b)} = a.$$

4. Given $f_1(y) = 2y^4 - y^3 + 1$, $f_2(y) = 7y^2 - 6y + 1$; show that

$$f_1(1) = f_2(1), \quad f_1(\tfrac{3}{2}) = f_2(\tfrac{3}{2}), \quad f_1(-2) = f_2(-2),$$
$$f_1(0) = f_2(0).$$

5. If $f(x)=\frac{1-x}{1+x}$, show that

$$f^2(x)=x, \text{ where } f^2(x) \text{ denotes } f[f(x)].$$

6. If $\phi(m)=(m+1)\,m\,(m-1)\,(m-2)$, show that

$$\phi(2)=\phi(1)=\phi(0)=\phi(-1)=0, \quad \phi(3)=\phi(-2),$$

$$\frac{\phi(m+1)}{m+2}=\frac{\phi(m)}{m-2}.$$

7. If $\phi(x)=(x-a)(x-b)(x-c)$, show that

$$\phi(a)=\phi(b)=\phi(c)=0,$$

$$\frac{\phi(a+b)\cdot\phi(b+c)\cdot\phi(c+a)}{[\phi(0)]^2}=8\,\phi\left(\frac{a+b+c}{2}\right),$$

$$\frac{\phi(2a)}{a}+\frac{\phi(2b)}{b}+\frac{\phi(2c)}{c}=4(a^2+b^2+c^2)-3(ab+bc+ca),$$

$$\frac{\phi(-a)\cdot\phi(-b)\cdot\phi(-c)}{\phi(0)}=8[\phi(a+b+c)]^2.$$

II. DIFFERENTIAL COEFFICIENT.

3. In the equation $y = x^2$, if we suppose x to vary, y will vary also. To fix the attention upon a definite value of x, let us suppose $x = 10$ and therefore $y = 100$, and let us inquire what addition or increment will be produced in y by a certain increment assigned to x.

Let Δx denote the increment of x, and Δy the increment of y, and in general the symbol Δ denotes an increment in the quantity following it.

Now, x being 10, if we calculate the value of Δy resulting from different values of Δx, we find results as in the following table: —

If $\Delta x =$	then $\Delta y =$	and $\frac{\Delta y}{\Delta x} =$
3.	69.	23.
2.	44.	22.
1.	21.	21.
0.1	2.01	20.1
0.01	0.2001	20.01
0.001	0.020001	20.001

The third column gives the value of the ratio between the increments of x and of y.

It appears from the table that, as Δx diminishes and approaches zero, Δy also diminishes and approaches zero; $\frac{\Delta y}{\Delta x}$ diminishes, but instead of approaching zero, approaches 20 as its limit.

This limit of $\frac{\Delta y}{\Delta x}$ is denoted by $\frac{dy}{dx}$, and is called the *differential coefficient* of y with respect to x. So, in this case, when $x=10$, $\frac{dy}{dx}=20$.

4. Without restricting ourselves to any one numerical value, we may obtain $\frac{dy}{dx}$ from the equation $y=x^2$ thus:—

Having $y=x^2$, let $\Delta x=h$, and let the new value of y be denoted by

$$y'=(x+h)^2;$$

therefore,

$$\Delta y=y'-y=(x+h)^2-x^2=2xh+h^2.$$

Dividing by $\Delta x=h$, gives

$$\frac{\Delta y}{\Delta x}=2x+h.$$

The limit of this, when h approaches zero, is $2x$. Hence,

$$\frac{dy}{dx}=2x.$$

5. In the same way the differential coefficient of any other given function may be found.

In general, if $y=\phi(x)$,

$$y'=\phi(x+h),$$

$$\Delta y=y'-y=\phi(x+h)-\phi(x),$$

$$\frac{\Delta y}{\Delta x}=\frac{\phi(x+h)-\phi(x)}{h},$$

$$\frac{dy}{dx}=\text{limit of }\frac{\phi(x+h)-\phi(x)}{h},$$

as h approaches zero.

The *differential coefficient* of a function may then be defined as the limiting value of the ratio of the increment of the function to the increment of the variable, as these increments approach zero.

EXAMPLES.

Find the differential coefficients in the following: —

1. $y = 3x^2 - 2x.$ $\qquad \dfrac{dy}{dx} = 6x - 2.$

2. $y = x^3 + 5.$ $\qquad \dfrac{dy}{dx} = 3x^2.$

3. $y = (x - 1)(2x + 3).$ $\qquad \dfrac{dy}{dx} = 4x + 1.$

4. $y = \dfrac{1}{x}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{1}{x^2}.$

5. $y = \dfrac{a}{x^2}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{2a}{x^3}.$

6. $y = \dfrac{x - a}{x + a}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2a}{(x + a)^2}.$

7. $y = \sqrt{x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}.$

8. $y = \sqrt{x^2 - 2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2 - 2}}.$

9. $y = \dfrac{2}{\sqrt{x + 1}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{1}{(x + 1)^{\frac{3}{2}}}.$

10. $y = \sqrt[3]{x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{3x^{\frac{2}{3}}}.$

III. DIFFERENTIATION OF FUNCTIONS.

6. The process of finding the differential coefficient of a given function is called *differentiation.* The examples on page 5 are intended to explain more fully the nature of the differential coefficient. Differentiation may, however, be most readily performed by the application of certain general rules which may be expressed by formulæ.

In these formulæ the letters u, v, w, denote *variable* quantities, which may be functions of x; and c and n, *constant* quantities.

7. The following are the formulæ for

ALGEBRAIC FUNCTIONS.

$$\frac{dc}{dx} = 0 \quad \ldots \ldots \quad (1)$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx} \quad \ldots \ldots \quad (2)$$

$$\frac{d}{dx}(u+c) = \frac{du}{dx} \quad \ldots \ldots \quad (3)$$

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} \quad \ldots \ldots \quad (4)$$

$$\frac{d}{dx}(cu) = c\frac{du}{dx} \quad \ldots \ldots \quad (5)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \ldots \ldots \quad (6)$$

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx} \quad \ldots \ldots \quad (7)$$

8. The following special cases of the preceding formulæ will be useful:—

$$\frac{d}{dx}(u \pm v \pm w \pm \text{etc.}) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \frac{dw}{dx} \pm \text{etc.} \quad (2')$$

$$\frac{d}{dx}(uvw) = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx} \quad . \; . \; . \quad (4')$$

$$\frac{d}{dx}\left(\frac{u}{c}\right) = \frac{\frac{du}{dx}}{c} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (6')$$

$$\frac{d}{dx}\sqrt{u} = \frac{\frac{du}{dx}}{2\sqrt{u}} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (7')$$

We proceed to give the derivation of these formulæ.

9. *Formula* (1). This follows from the nature of a constant, which can undergo no change. Hence

$$\Delta c = 0 \quad \text{and} \quad \frac{\Delta c}{\Delta x} = 0;$$

therefore, its limit

$$\frac{dc}{dx} = 0.$$

10. *Formula* (2). Let $y = u + v$, and suppose that when x is changed into $x + h$, y, u, and v become y', u', and v'; then

$$y' = u' + v';$$

therefore

$$y' - y = u' - u + v' - v;$$

that is,

$$\Delta y = \Delta u + \Delta v.$$

Divide by Δx; then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}.$$

Now suppose Δx to diminish and approach zero, and we have, for the limits of these fractions,

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}.$$

If in this we substitute for y, $u + v$, we have formula (2).

11. *Formula* (3). This is a special case of (2), $\frac{dc}{dx}$ being zero.

12. *Formula* (4). Let $y = uv$; then, as in (2),

$$y' = u'v',$$

$$y' - y = u'v' - uv = (u' - u)v' + u(v' - v),$$

that is,

$$\Delta y = v'\Delta u + u\Delta v.$$

Divide by Δx; then

$$\frac{\Delta y}{\Delta x} = v'\frac{\Delta u}{\Delta x} + u\frac{\Delta v}{\Delta x}.$$

Now suppose Δx to approach zero, and, noticing that the limit of v' is v, we have

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}.$$

13. *Formula* (5). This is a special case of (4), $\frac{dc}{dx}$ being zero. But we may derive it independently thus:—

$$y = cu,$$

$$y' = cu',$$

$$y' - y = c(u' - u),$$

$$\Delta y = c\Delta u,$$

$$\frac{\Delta y}{\Delta x} = c\frac{\Delta u}{\Delta x},$$

$$\frac{dy}{dx} = c\frac{du}{dx}.$$

14. *Formula* (6). Let $y=\frac{u}{v}$,

then

$$y'=\frac{u'}{v'};$$

therefore

$$y'-y=\frac{u'}{v'}-\frac{u}{v}=\frac{u'v-uv'}{v'v}=\frac{(u'-u)v-u(v'-v)}{v'v},$$

that is,

$$\Delta y=\frac{v\Delta u-u\Delta v}{v'v},$$

$$\frac{\Delta y}{\Delta x}=\frac{v\frac{\Delta u}{\Delta x}-u\frac{\Delta v}{\Delta x}}{v'v}.$$

Now suppose Δx to diminish towards zero, and, noticing that the limit of v' is v, we have

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

Or we may derive (6) from (4) thus:—

Since

$$y=\frac{u}{v},$$

therefore

$$yv=u.$$

By (4),

$$v\frac{dy}{dx}+y\frac{dv}{dx}=\frac{du}{dx},$$

$$v\frac{dy}{dx}=\frac{du}{dx}-\frac{u}{v}\frac{dv}{dx};$$

therefore

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

15. *Formula* (7). *First*, suppose n to be a positive integer. Let

$$y = u^n,$$

and, as before,

$$y' = u'^n,$$

$$y' - y = u'^n - u^n$$

$$= (u' - u)(u'^{n-1} + u'^{n-2}u + u'^{n-3}u^2 \cdots u^{n-1}),$$

that is,

$$\Delta y = \Delta u\,(u'^{n-1} + u'^{n-2}u + u'^{n-3}u^2 \cdots u^{n-1}),$$

$$\frac{\Delta y}{\Delta x} = (u'^{n-1} + u'^{n-2}u + u'^{n-3}u^2 \cdots u^{n-1})\frac{\Delta u}{\Delta x}.$$

Now let Δx diminish; then, u being the limit of u', each of the n terms within the parenthesis becomes u^{n-1}; therefore

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}.$$

Second, suppose n to be a positive fraction, $\frac{p}{q}$.

Let

$$y = u^{\frac{p}{q}},$$

then

$$y^q = u^p;$$

therefore

$$\frac{d}{dx}(y^q) = \frac{d}{dx}(u^p).$$

But we have already shown (7) to be true when the exponent is a positive integer; hence we may apply it to each member of this equation. This gives

$$qy^{q-1}\frac{dy}{dx} = pu^{p-1}\frac{du}{dx};$$

therefore

$$\frac{dy}{dx} = \frac{p}{q}\,\frac{u^{p-1}}{y^{q-1}}\,\frac{du}{dx}.$$

Substituting for y, $u^{\frac{p}{q}}$, gives

$$\frac{dy}{dx}=\frac{p}{q}u^{\frac{p}{q}-1}\frac{du}{dx},$$

which shows (7) to be true in this case also. Hence that formula applies to any positive value of n, whether integral or fractional.

Third, suppose n to be negative and equal to $-m$.

Let

$$y=u^{-m}=\frac{1}{u^m};$$

by (6),

$$\frac{dy}{dx}=\frac{-\frac{d}{dx}(u^m)}{u^{2m}}=\frac{-mu^{m-1}\frac{du}{dx}}{u^{2m}}=-mu^{-m-1}\frac{du}{dx}.$$

Hence (7) is universally true.

16. *Formula* (2′) is an extension of (2), and may be derived in the same way.

Let

$$y=u\pm v\pm w+\text{etc.},$$

$$y'=u'\pm v'\pm w'+\text{etc.},$$

$$y'-y=u'-u\pm(v'-v)\pm(w'-w)+\text{etc.};$$

that is,

$$\Delta y=\Delta u\pm\Delta v\pm\Delta w+\text{etc.},$$

$$\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}\pm\frac{\Delta v}{\Delta x}\pm\frac{\Delta w}{\Delta x}+\text{etc.}$$

Hence

$$\frac{dy}{dx}=\frac{du}{dx}\pm\frac{dv}{dx}\pm\frac{dw}{dx}+\text{etc.}$$

17. *Formula* (4′) is an extension of (4), and may be derived from it thus: —

$$\frac{d}{dx}(uv\cdot w) = w\frac{d}{dx}(uv) + uv\frac{dw}{dx}$$
$$= w\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + uv\frac{dw}{dx}$$
$$= vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}.$$

Similarly, for the product of four functions, we have

$$\frac{d}{dx}(uvwz) = vwz\frac{du}{dx} + wzu\frac{dv}{dx} + zuv\frac{dw}{dx} + uvw\frac{dz}{dx}.$$

The formula may be extended to the product of any number of functions.

18. *Formula* (6′) is easily derived from (6), $\frac{dc}{dx}$ being zero.

19. *Formula* (7′) is an application of (7) to the square root of a function.

$$\frac{d}{dx}\sqrt{u} = \frac{d}{dx}(u^{\frac{1}{2}}) = \tfrac{1}{2}u^{-\frac{1}{2}}\frac{du}{dx} = \frac{\frac{du}{dx}}{2\sqrt{u}}.$$

EXAMPLES.

1. $y = x^3 + x^2.$ $\qquad \frac{dy}{dx} = 3x^2 + 2x.$

2. $y = 3x^4 - 4x^3 + 6x^2 - 12x + 5.$ $\qquad \frac{dy}{dx} = 12(x^3 - x^2 + x - 1).$

3. $y = \frac{x^5}{5} - \frac{x^4}{12}.$ $\qquad \frac{dy}{dx} = x^4 - \frac{x^3}{3}.$

4. $y = 6x^{\frac{5}{3}} + 10x^{\frac{3}{5}} - x^{-\frac{2}{3}}.$ $\qquad \frac{dy}{dx} = 10x^{\frac{2}{3}} + 6x^{-\frac{2}{5}} + \frac{2x^{-\frac{5}{3}}}{3}.$

5. $y = (x^2 + 2)^{\frac{5}{2}}$. $\qquad \dfrac{dy}{dx} = 5x(x^2 + 2)^{\frac{3}{2}}$.

6. $y = \sqrt{5x^4 - 4}$. $\qquad \dfrac{dy}{dx} = \dfrac{10x^3}{\sqrt{5x^4 - 4}}$.

7. $y = (x + 1)^5 (2x - 1)^3$. $\qquad \dfrac{dy}{dx} = (16x + 1)(x+1)^4 (2x-1)^2$.

8. $y = \dfrac{a + bx + cx^2}{x}$. $\qquad \dfrac{dy}{dx} = c - \dfrac{a}{x^2}$.

9. $y = \dfrac{(x - 1)^3}{x^{\frac{1}{3}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{8}{3}x^{\frac{5}{3}} - 5x^{\frac{2}{3}} + 2x^{-\frac{1}{3}} + \dfrac{1}{3}x^{-\frac{4}{3}}$.

10. $y = \dfrac{x + 3}{x^2 + 3}$. $\qquad \dfrac{dy}{dx} = \dfrac{3 - 6x - x^2}{(x^2 + 3)^2}$.

11. $y = \dfrac{x^{\frac{5}{2}} + x - x^{\frac{1}{2}} + a}{x^{\frac{3}{2}}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x^{\frac{5}{2}} - x + 2x^{\frac{1}{2}} - 3a}{2x^{\frac{5}{2}}}$.

12. Given

$$(a + x)^5 = a^5 + 5a^4x + 10a^3x^2 + 10a^2x^3 + 5ax^4 + x^5;$$

derive by differentiation the expansion of $(a + x)^4$.

13. Given $1 + x + x^2 \cdots + x^n = \dfrac{x^{n+1} - 1}{x - 1}$;

derive the sum of the series $1 + 2x + 3x^2 \cdots + nx^{n-1}$.

Ans. $\dfrac{nx^{n+1} - (n + 1)x^n + 1}{(x - 1)^2}$.

14. $y = \sqrt{\dfrac{1 + x}{1 - x}}$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1 - x)\sqrt{1 - x^2}}$.

15. $y = \dfrac{x^n}{(1 + x)^n}$. $\qquad \dfrac{dy}{dx} = \dfrac{nx^{n-1}}{(1 + x)^{n+1}}$.

16. $y = (1 - 2x + 3x^2 - 4x^3)(1 + x)^2$. $\qquad \dfrac{dy}{dx} = -20x^3(1 + x)$.

17. $y = (1 - 3x^2 + 6x^4)(1 + x^2)^3$. $\qquad \dfrac{dy}{dx} = 60x^5(1 + x^2)^2$.

18. $y = x^5 (a + 3x)^3 (a - 2x)^2.$

$$\frac{dy}{dx} = 5x^4 (a + 3x)^2 (a - 2x)(a^2 + 2ax - 12x^2).$$

19. $y = x^{15} (a - 3x)^5 (a + 5x)^3.$

$$\frac{dy}{dx} = 15x^{14} (a - 3x)^4 (a + 5x)^2 (a^2 + 2ax - 23x^2).$$

20. $y = (a + x)^m (b + x)^n.$

$$\frac{dy}{dx} = [m(b + x) + n(a + x)] (a + x)^{m-1} (b + x)^{n-1}.$$

21. $y = \dfrac{1}{(a + x)^m (b + x)^n}.$

$$\frac{dy}{dx} = -\frac{m(b + x) + n(a + x)}{(a + x)^{m+1} (b + x)^{n+1}}.$$

22. $y = \dfrac{x}{\sqrt{1 - x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{(1 - x^2)^{\frac{3}{2}}}.$

23. $y = \dfrac{1 - x}{\sqrt{1 + x^2}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{1 + x}{(1 + x^2)^{\frac{3}{2}}}.$

24. $y = \dfrac{2\sqrt{x}}{3 + x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3(1 - x^2)}{(3 + x^2)^2 \sqrt{x}}.$

25. $y = \dfrac{1}{x + \sqrt{1 + x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 + x^2}} - 1.$

26. $y = \dfrac{x}{x + \sqrt{1 - x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2x(1 - x^2) + \sqrt{1 - x^2}}.$

27. $y = \dfrac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{a^2 + a\sqrt{a^2 - x^2}}{x^2 \sqrt{a^2 - x^2}}.$

28. $y = \dfrac{3x^3 + 2}{x(x^3 + 1)^{\frac{2}{3}}}.$ $\qquad \dfrac{dy}{dx} = -\dfrac{2}{x^2 (x^3 + 1)^{\frac{5}{3}}}.$

29. $y = 3(x^2 + 1)^{\frac{4}{3}} (4x^2 - 3).$ $\qquad \dfrac{dy}{dx} = 56x^3 (x^2 + 1)^{\frac{1}{3}}.$

30. $y=\frac{\sqrt{(x+a)^3}}{\sqrt{x-a}}$. $\frac{dy}{dx}=\frac{(x-2a)\sqrt{x+a}}{(x-a)^{\frac{3}{2}}}$.

31. $y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$. $\frac{dy}{dx}=-\frac{2}{x^3}\left(1+\frac{1}{\sqrt{1-x^4}}\right)$.

32. $y=\left(\frac{x}{1+\sqrt{1-x^2}}\right)^n$. $\frac{dy}{dx}=\frac{ny}{x\sqrt{1-x^2}}$.

LOGARITHMIC AND EXPONENTIAL FUNCTIONS.

20. The formulæ for differentiation are as follows: —

$$\frac{d}{dx}\log_a u=\frac{M}{u}\frac{du}{dx} \quad \ldots \ldots \quad (8)$$

where $M=\log_a e=\frac{1}{\log_e a}$.

$$\frac{d}{dx}\log u=\frac{1}{u}\frac{du}{dx}\text{*} \quad \ldots \ldots \quad (9)$$

$$\frac{d}{dx}a^u=\log a\cdot a^u\frac{du}{dx} \quad \ldots \ldots \quad (10)$$

$$\frac{d}{dx}e^u=e^u\frac{du}{dx} \quad \ldots \ldots \quad (11)$$

$$\frac{d}{dx}u^v=vu^{v-1}\frac{du}{dx}+\log u\cdot u^v\frac{dv}{dx} \quad \ldots \ldots \quad (12)$$

21. Before deriving these formulæ it is necessary to find the limit of the expression

$\left(1+\frac{1}{z}\right)^z$, as z approaches infinity.

By the Binomial Theorem

$$\left(1+\frac{1}{z}\right)^z=1+z\frac{1}{z}+\frac{z(z-1)}{\lfloor 2}\left(\frac{1}{z}\right)^2+\frac{z(z-1)(z-2)}{\lfloor 3}\left(\frac{1}{z}\right)^3+\text{etc.},$$

which may be written

$$\left(1+\frac{1}{z}\right)^z=1+1+\frac{1-\frac{1}{z}}{\lfloor 2}+\frac{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)}{\lfloor 3}+\text{etc.}$$

---

* The base of the logarithm is e, unless otherwise expressed.

Now when z increases indefinitely, we have

$$\text{limit of } \left(1+\frac{1}{z}\right)^z = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{etc.}$$

This quantity is usually denoted by e, so that

$$e = 1 + \frac{1}{1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \text{etc.}$$

The value of e can be easily calculated to any desired number of decimals by computing the values of the successive terms of this series. For seven decimal places the calculation is as follows, —

$$\begin{array}{r} 1. \\ 1. \\ .5 \\ .166666667 \\ .041666667 \\ .008333333 \\ .001388889 \\ .000198413 \\ .000024802 \\ .000002756 \\ .000000276 \\ .000000025 \\ .000000002 \\ \hline e = 2.7182818\cdots \end{array}$$

By calculating the value of $\left(1+\frac{1}{z}\right)^z$ for different values of z, we may verify its limit. Thus

$$\begin{aligned} (1+\tfrac{1}{2})^2 &= 2.25 \\ (1+\tfrac{1}{5})^5 &= 2.48832 \\ (1+\tfrac{1}{10})^{10} &= 2.59374 \\ (1.01)^{100} &= 2.70481 \\ (1.001)^{1000} &= 2.71692 \\ (1.0001)^{10000} &= 2.71815 \\ (1.00001)^{100000} &= 2.71827 \\ (1.000001)^{1000000} &= 2.71828 \end{aligned}$$

22. *Formula* (8). Let $y = \log_a u$,

then

$$y' = \log_a (u + \Delta u),$$

$$\Delta y = \log_a (u + \Delta u) - \log_a u = \log_a \frac{u + \Delta u}{u}$$

$$= \log_a \left(1 + \frac{\Delta u}{u}\right) = \frac{\Delta u}{u} \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}.$$

Dividing by Δx,

$$\frac{\Delta y}{\Delta x} = \log_a \left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}} \cdot \frac{1}{u} \frac{\Delta u}{\Delta x}.$$

Now if Δx approach zero, Δu at the same time approaches zero; then the limit of $\left(1 + \frac{\Delta u}{u}\right)^{\frac{u}{\Delta u}}$ is the same as the limit of $\left(1 + \frac{1}{z}\right)^z$ as z increases indefinitely. But we have already found the latter limit to be e. Hence we have

$$\frac{dy}{dx} = \frac{\log_a e}{u} \frac{du}{dx}.$$

23. *Formula* (9) is a special case of (8) when $a = e$. In this case

$$M = \log_e e = 1.$$

24. *Formula* (10) may be derived from (9) as follows, —

Let $y = a^u$;

taking the logarithm of each member, we have

$$\log y = u \log a;$$

therefore by (9)

$$\frac{1}{y} \frac{dy}{dx} = \log a \frac{du}{dx};$$

multiplying by $y = a^u$, we have

$$\frac{dy}{dx} = \log a \cdot a^u \frac{du}{dx}.$$

25. *Formula* (11) is a special case of (10) when $a = e$.

26. *Formula* (12). Let $y = u^v$; taking the logarithm of each member, we have

$$\log y = v \log u;$$

therefore by (9)

$$\frac{1}{y}\frac{dy}{dx} = \frac{v}{u}\frac{du}{dx} + \log u \frac{dv}{dx};$$

multiplying by $y = u^v$, we have

$$\frac{dy}{dx} = vu^{v-1}\frac{du}{dx} + \log u \cdot u^v \frac{dv}{dx}.$$

EXAMPLES.

1. $y = \log(3x^2 + x)$. $\qquad \dfrac{dy}{dx} = \dfrac{6x+1}{3x^2+x}$.

2. $y = x \log x$. $\qquad \dfrac{dy}{dx} = 1 + \log x$.

3. $y = x^n \log x$. $\qquad \dfrac{dy}{dx} = x^{n-1}(1 + n \log x)$.

4. $y = \log \sqrt{1 - x^2}$. $\qquad \dfrac{dy}{dx} = -\dfrac{x}{1-x^2}$.

5. $y = e^x(1 - x^3)$. $\qquad \dfrac{dy}{dx} = e^x(1 - 3x^2 - x^3)$.

6. $y = \sqrt{x} - \log(\sqrt{x} + 1)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{2(\sqrt{x}+1)}$.

7. $y = \log(\log x)$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{x \log x}$.

8. $y = \log(e^x + e^{-x})$. $\qquad \dfrac{dy}{dx} = \dfrac{e^{2x}-1}{e^{2x}+1}$.

9. $y=(x-3)e^{2x}+4xe^{x}+x.$ $\quad\frac{dy}{dx}=(2x-5)e^{2x}+4(x+1)e^{x}+1.$

10. $y=\log_{10}(5x+x^3).$ $\quad\frac{dy}{dx}=M\frac{5+3x^2}{5x+x^3},$

where $M=\frac{1}{\log_e 10}=\log_{10}e=.434294$

11. $y=5^{x^2+2x}.$ $\quad\frac{dy}{dx}=2(x+1)\,5^{x^2+2x}\log 5,$

$\log 5=1.609440$

12. $y=\frac{e^x-e^{-x}}{e^x+e^{-x}}.$ $\quad\frac{dy}{dx}=\frac{4}{(e^x+e^{-x})^2}.$

What is the result of differentiating both members of each of the three following equations?

13. $\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

Ans. $\frac{1}{1+x}=1-x+x^2-x^3+\cdots$

14. $\log\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots\right).$

Ans. $\frac{1}{1-x^2}=1+x^2+x^4+x^6+\cdots$

15. $e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots$

Ans. $e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots$

16. $y=x^n a^x.$ $\quad\frac{dy}{dx}=x^{n-1}a^x(n+x\log a).$

17. $y=\log(x-2)-\frac{4(x-1)}{(x-2)^2}.$ $\quad\frac{dy}{dx}=\frac{x^2+4}{(x-2)^3}.$

18. $y=\log\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}.$ $\quad\frac{dy}{dx}=\frac{\sqrt{a}}{(a-x)\sqrt{x}}.$

19. $y = \dfrac{x \log x}{1-x} + \log(1-x)$. $\qquad \dfrac{dy}{dx} = \dfrac{\log x}{(1-x)^2}$.

20. $y = e^{\sqrt{x}}(x^{\frac{3}{2}} - 3x + 6x^{\frac{1}{2}} - 6)$. $\qquad \dfrac{dy}{dx} = \frac{1}{2} x e^{\sqrt{x}}$.

21. $y = \dfrac{x^4}{4}[(\log x)^2 - \log\sqrt{x} + \frac{1}{8}]$. $\qquad \dfrac{dy}{dx} = x^3(\log x)^2$.

22. $y = e^{ax}\left(x^3 - \dfrac{3x^2}{a} + \dfrac{6x}{a^2} - \dfrac{6}{a^3}\right)$. $\qquad \dfrac{dy}{dx} = ax^3 e^{ax}$.

23. $y = \log x \cdot \log(\log x) - \log x$. $\qquad \dfrac{dy}{dx} = \dfrac{\log(\log x)}{x}$.

24. $y = \log(x - 3 + \sqrt{x^2 - 6x + 13})$. $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2 - 6x + 13}}$.

25. $y = m\log(\sqrt{x} + \sqrt{x+m}) + \sqrt{mx + x^2}$.

$$\frac{dy}{dx} = \sqrt{\frac{m+x}{x}}.$$

26. $y = \log\dfrac{x}{a - \sqrt{a^2 - x^2}}$. $\qquad \dfrac{dy}{dx} = -\dfrac{a}{x\sqrt{a^2 - x^2}}$.

27. $y = \log\dfrac{x\sqrt{2} + \sqrt{1+x^2}}{\sqrt{1-x^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{2}}{(1-x^2)\sqrt{1+x^2}}$.

28. $y = \log\dfrac{\sqrt{x^2+a^2} + \sqrt{x^2+b^2}}{\sqrt{x^2+a^2} - \sqrt{x^2+b^2}}$. $\qquad \dfrac{dy}{dx} = \dfrac{2x}{\sqrt{x^2+a^2}\sqrt{x^2+b^2}}$.

29. $y = \log\sqrt{\dfrac{x-1}{x+1}} + \log\sqrt{\dfrac{x^3+1}{x^3-1}}$. $\qquad \dfrac{dy}{dx} = \dfrac{x^2-1}{x^4+x^2+1}$.

30. $y = (e^x - e^{-x})^2(e^{2x} + 2e^{4x} + 3e^{6x})$. $\qquad \dfrac{dy}{dx} = 24e^{6x}(e^{2x} - 1)$.

31. $y = x^{\frac{1}{x}}$. $\qquad \dfrac{dy}{dx} = x^{\frac{1-2x}{x}}(1 - \log x)$.

32. $y = \left(\dfrac{x}{n}\right)^{nx}$. $\qquad \dfrac{dy}{dx} = n\left(\dfrac{x}{n}\right)^{nx}\left(1 + \log\dfrac{x}{n}\right)$.

33. $y = (ex)^x$. $\qquad \frac{dy}{dx} = (ex)^x(2 + \log x)$.

34. $y = \left(\frac{x}{e}\right)^{\frac{x}{e}}$. $\qquad \frac{dy}{dx} = \frac{1}{e}\left(\frac{x}{e}\right)^{\frac{x}{e}} \log x$.

35. $y = x^{\log x}$. $\qquad \frac{dy}{dx} = \log x^2 \cdot x^{\log x - 1}$.

36. $y = x^{\frac{1}{\log x}}$. $\qquad \frac{dy}{dx} = 0$.

37. $y = e^{e^x}$. $\qquad \frac{dy}{dx} = e^{e^x} e^x$.

38. $y = e^{x^x}$. $\qquad \frac{dy}{dx} = e^{x^x} x^x (1 + \log x)$.

39. $y = x^{x^x}$. $\qquad \frac{dy}{dx} = yx^x\left[\frac{1}{x} + \log x + (\log x)^2\right]$.

TRIGONOMETRIC FUNCTIONS.

27. The formulæ for differentiation are as follows: —

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx} \qquad (13)$$

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx} \qquad (14)$$

$$\frac{d}{dx}\tan u = \sec^2 u \frac{du}{dx} \qquad (15)$$

$$\frac{d}{dx}\cot u = -\operatorname{cosec}^2 u \frac{du}{dx} \qquad (16)$$

$$\frac{d}{dx}\sec u = \sec u \tan u \frac{du}{dx} \qquad (17)$$

$$\frac{d}{dx}\operatorname{cosec} u = -\operatorname{cosec} u \cot u \frac{du}{dx} \qquad (18)$$

$$\frac{d}{dx}\operatorname{vers} u = \sin u \frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$\frac{d}{dx}\operatorname{covers} u = -\cos u \frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

28. *Formula* (13). Let $y = \sin u$,

then

$$y' = \sin(u + \Delta u),$$

therefore

$$\Delta y = \sin(u + \Delta u) - \sin u$$

$$= 2\cos\left(u + \frac{\Delta u}{2}\right)\sin\frac{\Delta u}{2}.$$

Hence

$$\frac{\Delta y}{\Delta x} = \cos\left(u + \frac{\Delta u}{2}\right)\frac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}.$$

Now when Δx approaches zero, Δu likewise approaches zero, and the limit of $\dfrac{\sin\frac{\Delta u}{2}}{\frac{\Delta u}{2}}$ is unity;

therefore

$$\frac{dy}{dx} = \cos u \frac{du}{dx}.$$

29. *Formula* (14) may be derived by substituting in (13) for u, $\frac{\pi}{2} - u$.

Then

$$\frac{d}{dx}\sin\left(\frac{\pi}{2} - u\right) = \cos\left(\frac{\pi}{2} - u\right)\frac{d}{dx}\left(\frac{\pi}{2} - u\right),$$

or

$$\frac{d}{dx}\cos u = \sin u\left(-\frac{du}{dx}\right) = -\sin u\frac{du}{dx}.$$

30. *Formula* (15). Since $\tan u = \frac{\sin u}{\cos u}$,

by (6)
$$\frac{d}{dx}\tan u = \frac{\cos u \frac{d}{dx}\sin u - \sin u \frac{d}{dx}\cos u}{\cos^2 u}$$
$$= \frac{\cos^2 u \frac{du}{dx} + \sin^2 u \frac{du}{dx}}{\cos^2 u} = \frac{\frac{du}{dx}}{\cos^2 u}$$
$$= \sec^2 u \frac{du}{dx}.$$

31. *Formula* (16) may be derived from (15) by substituting $\frac{\pi}{2} - u$ for u in the same way that (14) was derived from (13).

32. *Formula* (17). Since $\sec u = \frac{1}{\cos u}$,

by (6)
$$\frac{d}{dx}\sec u = \frac{-\frac{d}{dx}\cos u}{\cos^2 u} = \frac{\sin u \frac{du}{dx}}{\cos^2 u}$$
$$= \sec u \tan u \frac{du}{dx}.$$

33. *Formula* (18) may be derived from (17) by substituting $\frac{\pi}{2} - u$ for u.

34. *Formulæ* (19) and (20) are readily obtained from (13) and (14) by the relations

$$\text{vers}\, u = 1 - \cos u,$$
$$\text{covers}\, u = 1 - \sin u.$$

EXAMPLES.

1. $y = \sin 2x \cos x.$ $\qquad \dfrac{dy}{dx} = 2\cos 2x \cos x - \sin 2x \sin x.$

2. $y = \tan^2 5x.$ $\qquad \dfrac{dy}{dx} = 10 \tan 5x \sec^2 5x.$

3. $y = \tan x - x.$ $\qquad \dfrac{dy}{dx} = \tan^2 x.$

4. $y = \sin(nx + m).$ $\qquad \dfrac{dy}{dx} = n \cos(nx + m).$

5. $y = \dfrac{\tan x - 1}{\sec x}.$ $\qquad \dfrac{dy}{dx} = \sin x + \cos x.$

6. $y = \sin^3 x \cos x.$ $\qquad \dfrac{dy}{dx} = \sin^2 x(3\cos^2 x - \sin^2 x).$

7. $y = \sin(x + a)\cos(x - a).$ $\qquad \dfrac{dy}{dx} = \cos 2x.$

8. $y = \dfrac{\sin(a - x)}{\sin(a + x)}.$ $\qquad \dfrac{dy}{dx} = -\sin 2a \operatorname{cosec}^2(a + x).$

9. $y = \tan^2 x - \log(\sec^2 x).$ $\qquad \dfrac{dy}{dx} = 2\tan^3 x.$

10. $y = \tan^4 x - 2\tan^2 x + \log(\sec^4 x).$

$$\frac{dy}{dx} = 4\tan^5 x.$$

11. $y = (a \sin^2 x + b \cos^2 x)^n.$

$$\frac{dy}{dx} = n(a - b)\sin 2x(a \sin^2 x + b \cos^2 x)^{n-1}.$$

12. $y = \log \sin x.$ $\qquad \dfrac{dy}{dx} = \cot x.$

13. $y = \log \tan x.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{\sin 2x}.$

14. $y = \log \sec x.$ $\qquad \dfrac{dy}{dx} = \tan x.$

15. $y = \text{vers}\left(\frac{\pi}{2} + x\right)\text{vers}\left(\frac{\pi}{2} - x\right).$

$$\frac{dy}{dx} = -\sin 2x.$$

16. $y = \frac{e^{ax}(a \sin x - \cos x)}{a^2 + 1}.$ $\qquad \frac{dy}{dx} = e^{ax}\sin x.$

17. $y = x^{\sin x}.$ $\qquad \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x \log x\right).$

18. $y = \sin nx \sin^n x.$ $\qquad \frac{dy}{dx} = n \sin^{n-1} x \sin(n+1)x.$

19. $y = \frac{\sin^m nx}{\cos^n mx}.$ $\qquad \frac{dy}{dx} = \frac{mn \sin^{m-1} nx \cos(m-n)x}{\cos^{n+1} mx}.$

20. $y = x + \log \cos\left(x - \frac{\pi}{4}\right).$ $\qquad \frac{dy}{dx} = \frac{2}{1 + \tan x}.$

21. $y = \log \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$ $\qquad \frac{dy}{dx} = \sec x.$

22. $y = \log \sqrt{\frac{1 - \cos x}{1 + \cos x}}.$ $\qquad \frac{dy}{dx} = \text{cosec}\, x.$

23. $y = \log \sqrt{\frac{a \cos x - b \sin x}{a \cos x + b \sin x}}.$ $\qquad \frac{dy}{dx} = \frac{-ab}{a^2 \cos^2 x - b^2 \sin^2 x}.$

24. $y = \frac{\tan x - \tan^3 x}{\sec^4 x}.$ $\qquad \frac{dy}{dx} = \cos 4x.$

In each of the following pairs of equations derive by differentiation each of the two equations from the other: —

25. $\sin 2x = 2 \sin x \cos x,$

$\cos 2x = \cos^2 x - \sin^2 x.$

26. $\sin 2x = \frac{2 \tan x}{1 + \tan^2 x},$

$\cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x}.$

27. $\sin 3x = 3\sin x - 4\sin^3 x,$

$\cos 3x = 4\cos^3 x - 3\cos x.$

28. $\sin 4x = 4\sin x\cos^3 x - 4\cos x\sin^3 x,$

$\cos 4x = 1 - 8\sin^2 x\cos^2 x.$

29. $\sin(m+n)x = \sin mx\cos nx + \cos mx\sin nx,$

$\cos(m+n)x = \cos mx\cos nx - \sin mx\sin nx.$

30. $\sin x = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots$

$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots$

31. $\sin x = \frac{e^{x\sqrt{-1}} - e^{-x\sqrt{-1}}}{2\sqrt{-1}},$

$\cos x = \frac{e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}}{2}.$

INVERSE TRIGONOMETRIC FUNCTIONS.

35. The formulæ for differentiation are as follows:—

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

$$\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$\frac{d}{dx}\operatorname{cosec}^{-1}u = -\frac{1}{u\sqrt{u^2-1}}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$\frac{d}{dx}\operatorname{vers}^{-1}u = \frac{1}{\sqrt{2u-u^2}}\frac{du}{dx} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

36. *Formula* (21). Let $y = \sin^{-1}u$,

therefore

$$\sin y = u;$$

by (13)

$$\cos y\frac{dy}{dx} = \frac{du}{dx},$$

therefore

$$\frac{dy}{dx} = \frac{1}{\cos y}\frac{du}{dx};$$

but

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-u^2},$$

therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}.$$

37. *Formula* (22) may be derived by a similar method to that used for (21).

38. *Formula* (23). Let $y = \tan^{-1}u$,

therefore

$$\tan y = u;$$

by (15)

$$\sec^2 y\frac{dy}{dx} = \frac{du}{dx},$$

therefore

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}\frac{du}{dx};$$

but

$$\sec^2 y = 1 + \tan^2 y = 1 + u^2,$$

therefore

$$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}.$$

39. *Formula* (24) may be derived by a similar method to that used for (23).

40. *Formula* (25). Let $y = \sec^{-1} u$,

therefore

$$\sec y = u;$$

by (17)

$$\sec y \tan y \frac{dy}{dx} = \frac{du}{dx}$$

therefore

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y} \frac{du}{dx};$$

but

$$\sec y \tan y = \sec y \sqrt{\sec^2 y - 1} = u \sqrt{u^2 - 1},$$

therefore

$$\frac{dy}{dx} = \frac{1}{u\sqrt{u^2-1}} \frac{du}{dx}.$$

41. *Formula* (26) may be derived in a similar manner.

42. *Formula* (27). Let $y = \text{vers}^{-1} u$,

therefore

$$u = \text{vers}\, y = 1 - \cos y;$$

by (14) or (19)

$$\frac{du}{dx} = \sin y \frac{dy}{dx},$$

therefore

$$\frac{dy}{dx} = \frac{1}{\sin y} \frac{du}{dx};$$

but

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - (1-u)^2} = \sqrt{2u - u^2},$$

therefore

$$\frac{dy}{dx} = \frac{1}{\sqrt{2u - u^2}} \frac{du}{dx}.$$

EXAMPLES.

1. $y = \tan^{-1} mx.$ $\qquad \frac{dy}{dx} = \frac{m}{1 + m^2x^2}.$

2. $y = \sin^{-1}(3x - 1).$ $\qquad \frac{dy}{dx} = \frac{3}{\sqrt{6x - 9x^2}}.$

3. $y = \text{vers}^{-1}\frac{8x}{9}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{9x - 4x^2}}.$

4. $y = \sin^{-1}(3x - 4x^3).$ $\qquad \frac{dy}{dx} = \frac{3}{\sqrt{1 - x^2}}.$

5. $y = \tan^{-1}\frac{2x}{1 - x^2}.$ $\qquad \frac{dy}{dx} = \frac{2}{1 + x^2}.$

6. $y = \tan^{-1} e^x.$ $\qquad \frac{dy}{dx} = \frac{1}{e^x + e^{-x}}.$

7. $y = \tan^{-1}(n \tan x).$ $\qquad \frac{dy}{dx} = \frac{n}{\cos^2 x + n^2 \sin^2 x}.$

8. $y = \text{cosec}^{-1}\frac{3}{2x}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{9 - 4x^2}}.$

9. $y = \text{vers}^{-1} 2x^2.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{1 - x^2}}.$

10. $y = \text{vers}^{-1}\frac{2x^2}{1 + x^2}.$ $\qquad \frac{dy}{dx} = \frac{2}{1 + x^2}.$

11. $y = \tan^{-1}\frac{e^x - e^{-x}}{2}.$ $\qquad \frac{dy}{dx} = \frac{2}{e^x + e^{-x}}.$

12. $y = \text{cosec}^{-1}\frac{1}{2x^2 - 1}.$ $\qquad \frac{dy}{dx} = \frac{2}{\sqrt{1 - x^2}}.$

13. $y = \sec^{-1}\frac{x^2 + 1}{x^2 - 1}.$ $\qquad \frac{dy}{dx} = \frac{-2}{x^2 + 1}.$

14. $y = \sin^{-1}\dfrac{x+1}{\sqrt{2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{\sqrt{1-2x-x^2}}.$

15. $y = \tan^{-1}\dfrac{4\sin x}{3+5\cos x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{4}{5+3\cos x}.$

16. $y = \cos^{-1}\dfrac{3+5\cos x}{5+3\cos x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{4}{5+3\cos x}.$

17. $y = \sin^{-1}\dfrac{1-x^2}{1+x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{-2}{1+x^2}.$

18. $y = \operatorname{cosec}^{-1}\dfrac{1+x^2}{2x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}.$

19. $y = \tan^{-1}\dfrac{x+a}{1-ax}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{1+x^2}.$

20. $y = \sin^{-1}\sqrt{\sin x}.$ $\qquad \dfrac{dy}{dx} = \frac{1}{2}\sqrt{1+\operatorname{cosec} x}.$

21. $y = \tan^{-1}\sqrt{\dfrac{1-\cos x}{1+\cos x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2}.$

22. $y = \tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}(1+x)}.$

23. $y = \cot^{-1}\dfrac{a}{x} + \log\sqrt{\dfrac{x-a}{x+a}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2ax^2}{x^4-a^4}.$

24. $y = \tan^{-1}(x+\sqrt{1-x^2}).$ $\qquad \dfrac{dy}{dx} = \dfrac{\sqrt{1-x^2}-x}{2\sqrt{1-x^2}(1+x\sqrt{1-x^2})}.$

25. $y = \cos^{-1}\dfrac{e^x-e^{-x}}{e^x+e^{-x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{2}{e^x+e^{-x}}.$

26. $y = \sec^{-1}\sqrt{\dfrac{2}{1+x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{-1}{2\sqrt{1-x^2}}.$

27. $y = (x+a)\tan^{-1}\sqrt{\dfrac{x}{a}} - \sqrt{ax}.$ $\qquad \dfrac{dy}{dx} = \tan^{-1}\sqrt{\dfrac{x}{a}}.$

28. $y = \cot^{-1}\dfrac{1+\sqrt{1+x^2}}{x}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}.$

29. $y = \sin^{-1}\dfrac{x\tan\alpha}{\sqrt{a^2-x^2}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{a^2\tan\alpha}{a^2-x^2}\cdot\dfrac{1}{\sqrt{a^2-x^2\sec^2\alpha}}.$

30. $y = \cot^{-1}\sqrt{\dfrac{1-x}{2+x}}.$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{2-x-x^2}}.$

31. $y = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{1+x^2}.$

32. $y = \tan^{-1}\dfrac{2x}{1+3x^2} + \cot^{-1}\dfrac{x}{1+2x^2} + \tan^{-1}2x.$ $\qquad \dfrac{dy}{dx} = \dfrac{3}{1+9x^2}.$

33. $y = \tan^{-1}\dfrac{2x-b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b-x}{x\sqrt{3}}.$ $\qquad \dfrac{dy}{dx} = 0.$

34. $y = \log\sqrt{\dfrac{2x^2-2x+1}{2x^2+2x+1}} + \tan^{-1}\dfrac{2x}{1-2x^2}.$ $\qquad \dfrac{dy}{dx} = \dfrac{8x^2}{4x^4+1}.$

43. *Given* $\frac{dx}{dy}$; *to find* $\frac{dy}{dx}$.

It is evident that

$$\frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}},$$

however small the values of Δx and Δy. As these quantities approach zero, we have, for the limits of the members of this equation,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \quad \text{. . . . . (28)}$$

44. *Given* $\frac{dy}{dz}$ *and* $\frac{dz}{dx}$; *to find* $\frac{dy}{dx}$.

It is evident that

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x},$$

however small Δx, Δy, and Δz. As these quantities approach zero, we have, for the limits of the members of this equation,

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \cdot \quad \text{. . . . . (29)}$$

EXAMPLES.

In the following eight examples find by differentiation $\frac{dx}{dy}$, and then $\frac{dy}{dx}$ by (28), —

1. $x = \frac{a}{y+1}$. $\qquad \frac{dy}{dx} = -\frac{(y+1)^2}{a} = -\frac{a}{x^2}$.

2. $x = \frac{2y}{y-1}$. $\qquad \frac{dy}{dx} = -\frac{(y-1)^2}{2} = -\frac{2}{(x-2)^2}$.

3. $x = \sqrt{y^2+1} - y$. $\qquad \frac{dy}{dx} = \frac{\sqrt{y^2+1}}{y - \sqrt{y^2+1}} = -\frac{x^2+1}{2x^2}$.

4. $x = \sqrt{1 + \sin y}$. $\qquad \frac{dy}{dx} = \frac{2\sqrt{1+\sin y}}{\cos y} = \frac{2}{\sqrt{2 - x^2}}$.

5. $x=\tan^{-1}(y+\sqrt{y^2-1}).$ $\dfrac{dy}{dx}=2y\sqrt{y^2-1}=\frac{1}{2}(\sec^2 x-\operatorname{cosec}^2 x).$

6. $x=\dfrac{y}{1+\log y}.$ $\dfrac{dy}{dx}=\dfrac{(1+\log y)^2}{\log y}=\dfrac{y^2}{xy-x^2}.$

7. $x=\log\dfrac{e^y+\sqrt{e^{2y}-4}}{2}.$ $\dfrac{dy}{dx}=\dfrac{\sqrt{e^{2y}-4}}{e^y}=\dfrac{e^{2x}-1}{e^{2x}+1}.$

8. $x=2\log\dfrac{\sqrt{e^y+2}+\sqrt{e^y-2}}{2}.$ $\dfrac{dy}{dx}=\dfrac{\sqrt{e^{2y}-4}}{e^y}=\dfrac{e^{2x}-1}{e^{2x}+1}.$

Compare the two preceding examples with Ex. 8, page 18.

In the following examples find by differentiation $\dfrac{dy}{dz}$ and $\dfrac{dz}{dx}$, and then $\dfrac{dy}{dx}$ by (29),—

9. $y=z^5,\quad z=a^2-x^2.$
$$\frac{dy}{dx}=5z^4(-2x)$$
$$=-10x(a^2-x^2)^4.$$

10. $y=\dfrac{2z}{3z-2},\quad z=\dfrac{x}{2x-1}.$ $\dfrac{dy}{dx}=\dfrac{4}{(x-2)^2}.$

11. $y=e^z+e^{2z},\quad z=\log(x-x^2).$ $\dfrac{dy}{dx}=4x^3-6x^2+1.$

12. $y=\log(z^{\frac{5}{3}}-z),\quad z=e^{3x}.$ $\dfrac{dy}{dx}=\dfrac{5e^{2x}-3}{e^{2x}-1}.$

13. $y=\log\dfrac{1+z^2}{z},\quad z=e^x.$ $\dfrac{dy}{dx}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}.$

14. $y=\tan 2z,\quad z=\tan^{-1}(2x-1).$ $\dfrac{dy}{dx}=\dfrac{2x^2-2x+1}{2(x-x^2)^2}.$

15. $y=\sqrt{\dfrac{2z^2-1}{z^2-1}},\quad z=\sqrt{\dfrac{x^2-2x}{2x^2-4x+1}}.$ $\dfrac{dy}{dx}=-\dfrac{1}{(x-1)^2}.$

16. $y=\frac{1}{6}\log\dfrac{(z+1)^2}{z^2-z+1}-\dfrac{1}{\sqrt{3}}\tan^{-1}\dfrac{2z-1}{\sqrt{3}},\quad z=\dfrac{\sqrt[3]{1+3x+3x^2}}{x}.$
$$\frac{dy}{dx}=\frac{1}{xz(1+x)}.$$

IV. SUCCESSIVE DIFFERENTIATION.

45. It has been shown that when y is equal to a given function of x, we may obtain by differentiation the differential coefficient, $\frac{dy}{dx}\cdot$ As $\frac{dy}{dx}$ will be, in general, also a function of x, this may likewise be differentiated, and the result is called the *second differential coefficient.* If this be again differentiated, the result is called the *third differential coefficient;* and so on.

Thus, if

$$y = x^4,$$

$$\frac{dy}{dx} = 4x^3,$$

$$\frac{d}{dx}\frac{dy}{dx} = 12x^2,$$

$$\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx} = 24x.$$

46. The second differential coefficient of y with respect to x is denoted by $\frac{d^2y}{dx^2}\cdot$ That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}.$$

Similarly

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d}{dx}\frac{dy}{dx},$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\frac{dy}{dx}.$$

Or

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\frac{d^2y}{dx^2},$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\frac{d^3y}{dx^3},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\frac{d^{n-1}y}{dx^{n-1}}.$$

Thus if

$$y = x^4,$$

$$\frac{dy}{dx} = 4x^3,$$

$$\frac{d^2y}{dx^2} = 12x^2,$$

$$\frac{d^3y}{dx^3} = 24x.$$

EXAMPLES.

1. $y = x^4 - 4x^3 + 6x^2 - 4x + 1.$ $\quad \dfrac{d^2y}{dx^2} = 12(x^2 - 2x + 1).$

2. $y = x^5.$ $\quad \dfrac{d^5y}{dx^5} = \underline{|5}.$

3. $y = (x-3)e^{2x} + 4xe^x + x.$ $\quad \dfrac{d^2y}{dx^2} = 4e^x[(x-2)e^x + x + 2].$

4. $y = \dfrac{a}{x^m}.$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{m(m+1)a}{x^{m+2}}.$

5. $y = x\log x.$ $\quad \dfrac{d^2y}{dx^2} = \dfrac{1}{x}.$

6. $y = x^3\log x.$ $\quad \dfrac{d^4y}{dx^4} = \dfrac{6}{x}.$

7. $y = \log(e^x + e^{-x}).$ $\quad \dfrac{d^3y}{dx^3} = -\dfrac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}.$

8. $y=(x^2-6x+12)e^x.$ $\qquad \dfrac{d^3y}{dx^3}=x^2e^x.$

9. $y=\dfrac{x^3}{6}\left(\log x-\dfrac{5}{6}\right).$ $\qquad \dfrac{d^2y}{dx^2}=x\log x.$

10. $y=\log\sin x.$ $\qquad \dfrac{d^3y}{dx^3}=\dfrac{2\cos x}{\sin^3 x}.$

11. $y=(x^2+a^2)\tan^{-1}\dfrac{x}{a}.$ $\qquad \dfrac{d^3y}{dx^3}=\dfrac{4a^3}{(a^2+x^2)^2}.$

12. $y=e^{-x}\cos x.$ $\qquad \dfrac{d^4y}{dx^4}=-4e^{-x}\cos x.$

13. $y=\tan x.$ $\qquad \dfrac{d^3y}{dx^3}=6\sec^4 x-4\sec^2 x.$

14. $y=\dfrac{5x+1}{x^2-1}.$ $\qquad \dfrac{d^6y}{dx^6}=|\underline{6}\left[\dfrac{3}{(x-1)^7}+\dfrac{2}{(x+1)^7}\right].$

Decompose the fraction before differentiating.

15. $y=\sqrt{\sec 2x}.$ $\qquad \dfrac{d^2y}{dx^2}=3y^5-y.$

16. $y=(e^x+e^{-x})^n.$ $\qquad \dfrac{d^2y}{dx^2}=n^2y-4n(n-1)y^{\frac{n-2}{n}}.$

17. $y=\dfrac{7\cos x}{9}-\dfrac{\cos^3 x}{27}.$ $\qquad \dfrac{d^3y}{dx^3}=\sin^3 x.$

18. $y=\tan^2 x+8\log\cos x+3x^2.$ $\qquad \dfrac{d^2y}{dx^2}=6\tan^4 x.$

19. $y=(x^2-3x+3)e^{2x}.$ $\qquad \dfrac{d^3y}{dx^3}=8x^2e^{2x}.$

20. $y=x^3\left[3(\log x)^2-11\log x+\dfrac{85}{6}\right].$ $\qquad \dfrac{d^3y}{dx^3}=18(\log x)^2.$

21. $y=e^{ax}\sin bx.$ $\qquad \dfrac{d^2y}{dx^2}-2a\dfrac{dy}{dx}+(a^2+b^2)y=0.$

22. $y=\sin(m\sin^{-1}x).$ $\qquad (1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0.$

23. $y = a \cos(\log x) + b \sin(\log x)$. $\quad x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$

24. $y = \frac{1}{x+2}$. $\quad \frac{d^n y}{dx^n} = \frac{(-1)^n \lfloor n}{(x+2)^{n+1}}.$

25. $y = \frac{1}{3x+4}$. $\quad \frac{d^n y}{dx^n} = \frac{(-1)^n 3^n \lfloor n}{(3x+4)^{n+1}}.$

26. $y = e^{ax}$. $\quad \frac{d^n y}{dx^n} = a^n e^{ax}.$

27. $y = \log x$. $\quad \frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \lfloor n-1}{x^n}.$

28. $y = \frac{1-x}{1+x}$. $\quad \frac{d^n y}{dx^n} = \frac{2(-1)^n \lfloor n}{(1+x)^{n+1}}.$

Reduce the fraction to a mixed quantity, $-1 + \frac{2}{1+x}$, before differentiating.

29. $y = \frac{3x+2}{x^2-4}$. $\quad \frac{d^n y}{dx^n} = (-1)^n \lfloor n \left[\frac{1}{(x+2)^{n+1}} + \frac{2}{(x-2)^{n+1}}\right].$

V. DIFFERENTIALS.

47. The differential coefficient $\frac{dy}{dx}$ has been defined, not as a fraction having a numerator and denominator, but as a single symbol representing the limiting value of $\frac{\Delta y}{\Delta x}$, as Δx and Δy approach zero.

Another view of the differential coefficient regards it as an actual fraction, dx and dy being defined as *infinitely small increments* of x and y, and called *differentials* of x and y. That is, dx is an *infinitely small* Δx, and dy an *infinitely small* Δy.

For instance, if we differentiate $y = x^2$, we obtain $\frac{dy}{dx} = 2x$. Using differentials, this result might be written $dy = 2xdx$. The first form, $\frac{dy}{dx} = 2x$, asserts that the limit of the ratio of the increment of y to that of x, as these increments approach zero, is equal to $2x$. The second form, $dy = 2xdx$, states that an infinitely small increment of y is $2x$ times the corresponding infinitely small increment of x. The two express the same relation in different language, the first being, however, the more precise and less open to logical criticism.

These two modes of expression are analogous to the following two expressions in trigonometry: —

"An infinitely small arc is equal to its chord."

"The limit of the ratio, $\frac{\text{arc}}{\text{chord}}$, as these quantities approach zero, is unity."

The first statement may be regarded as an abbreviation of the fuller and more explicit language of the second. Similarly the equation $dy = 2xdx$ may be regarded as a convenient substitute for $\frac{dy}{dx} = 2x$.

48. The formulæ for differentiation may be expressed in the form of differentials by omitting the dx in each member. Thus, formula (4) becomes

$$d(uv) = vdu + udv;$$

formula (23),

$$d \tan^{-1} u = \frac{du}{1+u^2};$$

and the others may be similarly expressed.

Differentiation by the new forms of the formulæ is substantially the same as by the original forms, differing only in using the symbol d instead of $\frac{d}{dx}$.

For example, take Ex. 10, p. 13.

$$dy = d\left(\frac{x+3}{x^2+3}\right) = \frac{(x^2+3)\,d(x+3) - (x+3)\,d(x^2+3)}{(x^2+3)^2}$$

$$= \frac{(x^2+3)\,dx - (x+3)\,2xdx}{(x^2+3)^2}$$

$$= \frac{(x^2+3-2x^2-6x)\,dx}{(x^2+3)^2} = \frac{(3-6x-x^2)\,dx}{(x^2+3)^2}.$$

Dividing by dx gives

$$\frac{dy}{dx} = \frac{3-6x-x^2}{(x^2+3)^2}.$$

49. *Successive Differentials.* Successive differential coefficients, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, etc., which have been defined as single symbols, may also be interpreted as fractions, the numerators, d^2y, d^3y, etc., denoting $d(dy)$, $d[d(dy)]$, etc., and called the second, third, etc., differentials of y, while the denominators are $(dx)^2$, $(dx)^3$, etc.

This will be better understood from an example. Let $y = x^4$, then

$$dy = 4x^3 dx.$$

Now, $4x^2 dx$ being a variable, dy is a variable, and may be again differentiated. Now, x being the independent variable, its increment dx may be supposed the same infinitely small quantity for all values of x; that is, we may regard dx as a constant in the preceding equation. Thus we obtain

$$d\,(dy) = 12x^2 dx \cdot dx = 12x^2 (dx)^2.$$

Denoting $d(dy)$ by d^2y,

$$d^2y = 12x^2 (dx)^2.$$

Differentiating again, and still regarding dx as constant,

$$d(d^2y) = 24x dx (dx)^2 = 24 (dx)^3,$$

or

$$d^3y = 24\,(dx)^3.$$

From these equations, by dividing by the power of dx in the second members, we find

$$\frac{d^2y}{(dx)^2} = 12x^2,$$

$$\frac{d^3y}{(dx)^3} = 24x.$$

The variable x, whose differential is supposed constant, is called the *equicrescent* variable.