NAVAL POSTGRADUATE SCHOOL
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THESIS

INVESTIGATION OF PIPE FLOW INSTABILITY
AND RESULTS FOR WAVE NUMBER ZERO

by

Michael James Arnold

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Two data reduction programs were also developed to process and display data generated by the main program.

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Investigation of Pipe Flow Instability and Results for Wave Number Zero

by

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Past research by Harrison and Johnston on the stability of pipe flow yielded only tenuous results owing to errors in setup of the problem and in formulation of the complex axis boundary conditions.

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A nonuniform computational mesh was developed which provided dramatic reductions in computational time on a limited basis.

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TABLE OF SYMBOLS

C	Constant in non-uniform mesh functions given by equations (C-32) and (C-40)
$D, D^2, \ldots$	Partial derivatives with respect to r.
$D^*, D^{*2}, \ldots$	Partial derivatives with respect to η.
e	Base of natural logarithms.
$\bar{e}_x, \bar{e}_r, \bar{e}_\theta$	Unit vectors along the x, r and θ axes in cylindrical coordinates.
F, G, H	Components of the velocity vector potential defined in equation (2-6).
$f_{11}, f_{22}, \ldots$	Coefficients of D^*Q, $D^{*2}Q$, ... in equations (C-9) through (C-12) as defined in equations (C-13) through (C-22).
i	$+\sqrt{-1}$, the imaginary unit. Also used as an index in Section III and Appendix D.
N	The number of interior points in the finite difference mesh of Section III.
O	Symbol denoting the phrase "of order".
Q	The component of the velocity vector potential derived from the component H by the change of variable, $H = rQ$.
R_e	Reynolds number based on mean velocity and pipe radius.
t	Time.
U	The streamwise velocity in Pipe Poiseuille Flow as defined by equation (2-11).
u, v, w	Components of the complex perturbation velocity defined in equation (D-1).
$\bar{W}$	Complex vector potential of perturbation velocity defined in equation (D-2).
x, r, θ	Cylindrical coordinates.
α	$\alpha_R + i\alpha_I$. Complex wave number of the perturbation in the x-direction.

β	in. Complex wave number of the perturbation in the θ direction, where n = 0,1,2,3, ...
δ	1/(N+1). The r or η increment in the finite difference approximations of the derivatives of Q.
η	The independent variable replacing r in the nonuniform mesh of Appendix C.
γ	$\gamma_R + i\gamma_I$. Complex frequency of the perturbation.
$\overline{\Gamma}$	The vorticity transport equation expressed in abbreviated notation as defined in equation (2-7).
$\Gamma_x, \Gamma_r, \Gamma_\theta$	The components of $\overline{\Gamma}$ in cylindrical coordinates as defined in equation (2-7).
λ	Mesh offset parameter as defined in equations (C-32) and (C-40).
∇	Linear vector operator (nabla)
$\times$	Vector cross-product operator.
[ ]	Brackets enclosing a matrix.
{ }	Brackets enclosing a column vector.

I. INTRODUCTION

The problem of finding an analytical solution to the pipe flow stability problem has been pursued actively ever since the classical experiments of Osborne Reynolds [10] about 100 years ago. Up to now, however, no investigation has been able to satisfactorily predict flow instabilities, although many approaches have been taken.

Salwen and Grosch [11] studied pipe flow with various angular wave numbers and sinusoidal streamwise perturbations and concluded that it was stable for all axial and angular wave numbers. Perturbations with exponential growth in space but a purely sinusoidal time variation were researched by Garg and Rouleau [2] and those with both exponential growth in space and in time by Gill [3]. Both concluded that the flows were stable.

Because of this inability of linear theory to account for experimental fact, explanations by Davey and Drazin [1] involving finite disturbances and by Huang and Chen [5] and Leite [7] involving conditions at the pipe entrance have been offered. While these investigations have indeed shown instabilities to exist, a completely general solution to the linear problem has never been achieved.

Recently a more general theory was presented by Harrison [4] and further investigated by Johnston [6]. These two studies, however, failed to produce conclusive results due

to mathematical errors in the problem setup and inadequate formulation of the boundary conditions at the axis. Gawain [9] has subsequently formulated the axis boundary conditions in a new way which corrects the previous discrepancies and promises further advances.

For angular wave number, n, equal to zero, radical simplifications result in the governing equations (Section II), indicating that this case should be approached first. This investigation centers on that case.

Preliminary checks using the computer program of Ref. 6 revealed that, of the two eigenfunctions, G and H, which occur in this problem and which are uncoupled for n = 0, the latter appeared to be the more critical. Hence the present research was arbitrarily restricted to investigation of the stability of eigenfunction H. A similar study of the other eigenfunction, G, for n = 0 remains to be completed at some future time. Comparable calculations for other wave numbers (n = 1,2,3, ...) also remain to be accomplished in the future. Extensive and systematic calculations of this type will be essential to provide the factual basis for a comprehensive theory of pipe flow stability.

Reverting to the case at hand, eigenfunction H for wave number n = 0, we note that the program of Ref. 6 was rewritten for this case, incorporating the newly formulated boundary conditions of Ref. 9. In addition, a new, generalized stability criteria was adopted. Moreover, a new technique was introduced which allows the use of nonuniform meshes to reduce computational time.

Lastly, two data reduction programs were written to process data produced by the main investigative program.

II. THE VORTICITY TRANSPORT EQUATION

Although a complete treatment of this subject is contained in Appendix A of Ref. 4 and further addressed in Ref. 6 and Ref. 9, it is felt that a brief overview is still required here to maintain continuity with previously referenced works. This discussion is an abbreviated version of Section II of Ref. 6.

Laminar flow of an incompressible fluid of constant viscosity is governed by the Navier-Stokes equation and the continuity equation. Taking the curl ($\nabla\times$) of the Navier-Stokes equation and introducing a perturbation velocity ($\overline{v}$) and vorticity ($\overline{\omega}$) gives the vorticity transport equation which is equation (A-10) of Appendix A, Ref. 4.

Expressing this equation in terms of the complex velocity vector potential, $\overline{W}$, gives

$$W(x,r,\theta,t) = (\overline{e}_x F(r) + \overline{e}_r G(r) + \overline{e}_\theta H(r))e^X \qquad (2\text{-}1)$$

where

$$X = \alpha x + \beta\theta + \gamma t \qquad (2\text{-}2)$$

and

$$\overline{v} = \nabla \times \overline{W} \qquad (2\text{-}3)$$

$$\bar{\omega} = \nabla \times \bar{v} \, . \qquad (2\text{-}4)$$

It should also be noted that, as shown in part one of Appendix G in Ref. 4, α and γ are complex while β is a purely imaginary quantity defined by

$$\beta = in \qquad n = 0,1,2, \ldots \qquad (2\text{-}5)$$

When expressed in the form of equation (2-1), the vorticity transport equation becomes three simultaneous fourth-order differential equations of the form

$$[M_4] \begin{Bmatrix} D^4F \\ D^4G \\ D^4H \end{Bmatrix} + [M_3] \begin{Bmatrix} D^3F \\ D^3G \\ D^3H \end{Bmatrix} + [M_2] \begin{Bmatrix} D^2F \\ D^2G \\ D^2H \end{Bmatrix}$$

$$+ [M_1] \begin{Bmatrix} DF \\ DG \\ DH \end{Bmatrix} + [M_0] \begin{Bmatrix} F \\ G \\ H \end{Bmatrix} - \gamma ([N_2] \begin{Bmatrix} D^2F \\ D^2G \\ D^2H \end{Bmatrix}$$

$$+ [N_1] \begin{Bmatrix} DF \\ DG \\ DH \end{Bmatrix} + [N_0] \begin{Bmatrix} F \\ G \\ H \end{Bmatrix}) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}6)$$

Equations (2-5) may be further expressed in the abbreviated form

$$\overline{\Gamma} = \begin{Bmatrix} \overline{\Gamma}_x \\ \overline{\Gamma}_r \\ \overline{\Gamma}_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{2-7}$$

where $\overline{\Gamma}$ appears to be a set of three coupled equations in the components of $\overline{W}$. As given in Appendix B of Ref. 4, equations (2-7) actually represent only two independent conditions and by an appropriate linear combination of Γ_x and Γ_θ, equations (2-6) can be expressed as a set of two equations in three unknowns. The appropriate linear combination is given in Appendix B of Ref. 4 and yields the set of equations

$$\begin{aligned} \Gamma_r &= 0 \\ -\frac{in}{r}\Gamma_x + \alpha\Gamma_\theta &= 0 . \end{aligned} \tag{2-8}$$

Except for the case where n is equal to zero, equations (2-8) do not uncouple. The linear combination given by the second of equations (2-8) does, however, reduce the highest order derivative of G(r) in equations (2-6) to second order. Appendix C of Ref. 4 illustrates the redundancy of the three components of $\overline{W}$, allowing one of these components to be arbitrarily set to zero for all r. The maximum benefits of equations (2-8) are obtained if

$$F(r) = 0 \tag{2-9}$$

Incorporating equations (2-8) and (2-9) into equations (2-6) results in the form

$$[M_4'] \begin{Bmatrix} D^4G \\ D^4H \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3G \\ D^3H \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix}$$

$$+ [M_1'] \begin{Bmatrix} DG \\ DH \end{Bmatrix} + [M_o'] \begin{Bmatrix} G \\ H \end{Bmatrix} - \gamma([N_2'] \begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix}$$

$$+ [N_1'] \begin{Bmatrix} DG \\ DH \end{Bmatrix} + [N_o'] \begin{Bmatrix} G \\ H \end{Bmatrix}) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}10)$$

where the coefficient matrices are given by equations (2-10) through (2-17) of Ref. 6. It is appropriate to note that these same coefficient matrices appear in Ref. 9, equations (A1) through (A9), in a slightly different form resulting from the substitutions

$$U = 2(1 - r^2) \qquad (2\text{-}11)$$

$$t = \alpha^2 + \frac{\beta^2}{r^2} \quad \text{and} \qquad (2\text{-}12)$$

$$T = \alpha U - \frac{1}{R_e}(\alpha^2 + \frac{\beta^2}{r^2}) \ . \qquad (2\text{-}13)$$

As discussed in the previous section, the case where

$$\beta = in \ , \quad n = 0 \qquad (2\text{-}14)$$

leads to great simplifications in equations (2-10), (2-12) and (2-13). In particular, equations (2-10) uncouple and allow an independent investigation of either H or G. As a result of the findings discussed in Section I, it was decided to explore the function H only. This reduced equation (2-10) to that of equation (A-6) of Appendix A, which is a linear, homogeneous fourth order differential equation in H(r).

III. NUMERICAL METHODS

Substituting the change of variable H = rQ as given in equation (A-1) and the coefficients defined in equations (A-11) through (A-18) into the vorticity transport relation, equation (A-6), gives the expression

$$M_4 D^4 Q + M_3 D^3 Q + M_2 D^2 Q + M_1 DQ + M_0 Q - \gamma[N_2 D^2 Q + N_1 DQ + N_0 Q] = 0 , \quad (3\text{-}1)$$

which is a homogeneous fourth order differential equation in Q(r). The boundary conditions for this case are derived in detail in Ref. 9 as

$$\begin{aligned} Q(1) &= 0 \\ DQ(1) &= 0 \\ DQ(0) &= 0 \\ D^3 Q(0) &= 0 . \end{aligned} \quad (3\text{-}2)$$

The boundary finite difference equations derived in Appendix B from equations (3-2), along with the standard central difference equations given in Ref. 6, allow the function Q(r) to be approximated by a finite number of discrete unknowns. As shown by Figure 3-1 below, the non-dimensionalized radius of the pipe is divided into a one-dimensional

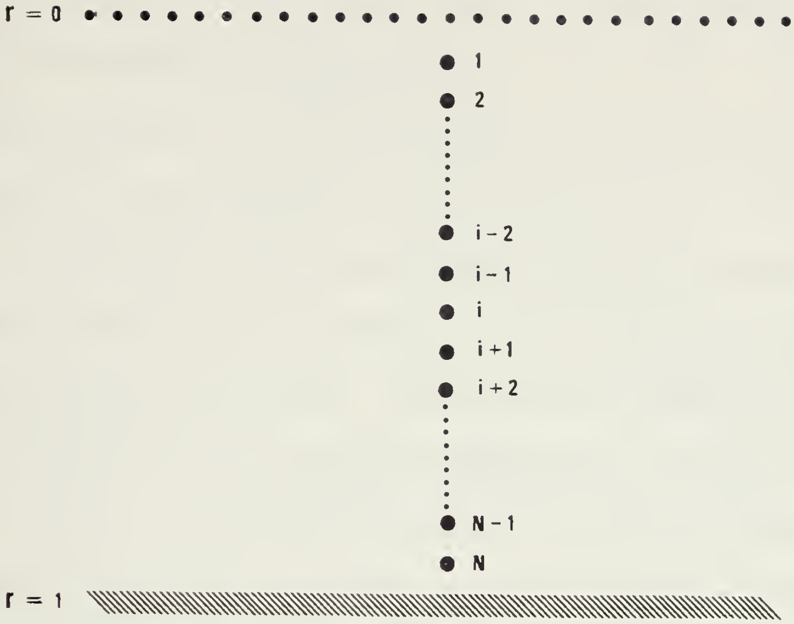

Figure 3-1 Finite Difference Mesh

computational mesh consisting of N interior points, N+1 intervals, and N+2 total points, including the boundary points at r = 1 and r = 0. As will be discussed later, the spacing between these points may or may not be uniform. For the uniform case, the spacing is defined by

$$\delta = 1/(N+1) \ . \tag{3-3}$$

For the nonuniform case, a change of independent variable is performed. The spacing of the new independent variable, η, is still given by equation (3-3).

With a nonuniform mesh, the points shown in Figure 3-1 will be concentrated near the axis or near the wall according

to the type of offset specified. These effects are discussed in detail in Section IV.

Substitution of the finite difference equations of Appendix B into equation (3-1) results in a set of N, linear, algebraic difference equations in terms of the unknown value of Q at each of the N interior points of the computational mesh. Since each of these equations is of the form of a linear combination of the ith, central, point and the two, three or four adjacent points (depending on the order of the derivative being approximated), this system of equations consists of a coefficient array multiplying a vector containing the unknown value of the function Q at each of the N interior points. This technique allows the problem to be converted into an eigenvalue problem of the form

$$[X]\{Q\} - \gamma [Y]\{Q\} = 0 \tag{3-4}$$

with the basic composition of the arrays [X] and [Y] and the vector {Q} as illustrated in Figure 3-2 below.

It should be noted at this point that Figure 3-2 differs somewhat from the normal finite difference banded matrix in the first two rows and last row because of the method of deriving the finite difference approximations at the boundaries. Additionally, the order of the N unknowns has been reversed from that of Ref. 6. This was done to conform to standard matrix notation.

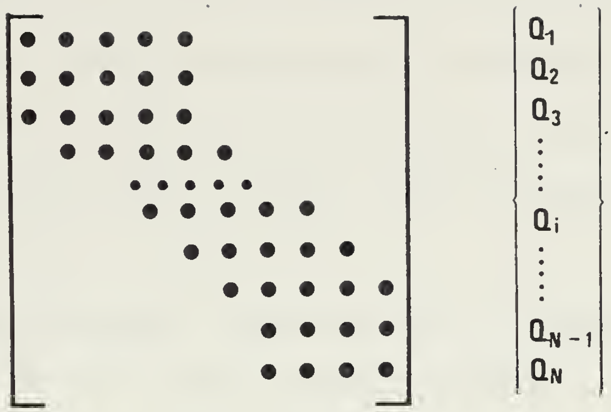

Figure 3-2 Basic Composition of Coefficient Arrays and Vector of Unknowns

This array is established by the subroutine MSET2 in conjunction with the subroutine MSET1 and function subprograms CQM1E1 and CQM2E1, which compute the numerical value for each element in the array. Subroutine MSET1 provides the coefficients given by equations (A-11) through (A-18) of Appendix A or by equations (C-24) through (C-31) if a nonuniform mesh is specified. Function CQM1E1 then computes the values for each of the elements of array [X] in equation (3-4) using the coefficients passed from subroutine MSET1 in vector CQM1. Function subprogram CQM2E1 performs the same function for matrix [Y] in equation (3-4) using the coefficients passed in vector CQM2.

The solution of the eigenvalue problem as formulated to this point is carried out by the controlling subroutine of program PIPEO, subroutine STAB, by the following steps:

1) Subroutine MSET2 is called twice to set up the coefficient matrices [X] and [Y] of equation (3-4).

2) Subroutine CDMTIN is then called to invert matrix [Y], the second coefficient array in equation (3-4). CDMTIN was obtained from the IBM Library routine CMTRIN by modifying it to accept double precision arrays.

3) Both coefficient arrays, [X] and [Y], are then pre-multiplied by $[Y]^{-1}$. Since multiplication of an array by its inverse invariably results in the identity matrix, [I], only the product $[Y]^{-1}[X]$ is computed using subroutine MULM. This converts the eigenvalue problem of equation (3-4) to the more conventional form

$$([Z] - \gamma[I])\{Q\} = 0 \tag{3-5}$$

where

$$[Z] = [Y]^{-1}[X] \tag{3-6}$$

4) Since all programs currently available for solving equations (3-5) require that the real and imaginary parts of the elements of [Z] be presented in separate arrays, subroutine DSPLIT is called to accomplish this.

5) The eigenvalues and eigenvectors of equations (3-5) are computed using subroutines EBALAC, EHESSC, ELRH2C and EBBCKC which are available through the International

Math and Statistics Library. Subroutine EBALAC balances matrix [Z] by equalizing the exponents of all terms. The details of this transformation are retained for later use. The balanced matrix is then passed to subroutine EHESSC where it is reduced into the complex upper Hessenberg form. Subroutine ELRH2C then solves for the eigenvalues and eigenvectors. To transform the eigenvectors back into the original unbalanced form, EBBCKC is finally called using information passed from subroutine EBALAC.

For each solution, subroutine STAB determines the least stable eigenvalue (largest algebraic value) and then writes the values of N, R_e, α_R, α_I, λ, γ_{RL}, γ_{IL} and KSET to file FT02F001. The eigenvector corresponding to the least stable eigenvalue is also written to FILE FT02F001 when MODENO is set equal to one.

Control of subroutine STAB is accomplished by the main program, PIPE0. This program is a time-sharing (CP/CMS) program. Modes one and three compute the stability of the flow for a given set of input conditions. Mode one writes the least stable eigenvector to FILE FT02F001 while this output is inhibited when MODENO is set equal to three. To generate data for program EIGFCN, program PIPE0 must be run with MODENO equal to one.

Mode two operation generates a grid of stability values (stability map) based on parameters read in from FILE FT01F001. Due to the long run time in this mode, only small meshes can be generated under CP/CMS. Longer runs must be accomplished under batch, with changes to the program as specified in the comments section. Data is output to file FT03F001 when MODENO is equal to two and is compatible with program STBCONT.

The plotting programs EIGFCN and STBCONT were used to process the data generated by program PIPEO in modes one and three, respectively. Program EIGFCN generates normalized plots of the perturbation velocity, u, as a function of radius, r. The perturbation velocities generated in accordance with Appendix D were normalized in two steps. First the perturbation velocity of largest magnitude was determined. Letting this velocity be termed u_C, a normalizing constant producing unit magnitude and zero phase angle in u_C was found in the following manner:

If

$$u_C = u_{RC} + iu_{iC} \, , \qquad (3\text{-}7)$$

then

$$Cu_C = 1 + i(0) \qquad (3\text{-}8)$$

where C is the normalizing constant. Thus,

$$C = \frac{1}{u_{RC} + iu_{iC}} = \frac{u_{RC} - u_{iC}}{(u_{RC}^2 - u_{iC}^2)} \quad (3\text{-}9)$$

$$= \frac{\bar{u}_C}{|u_C|^2} \quad (3\text{-}10)$$

where $\bar{u}_C$ is the complex conjugate of u_C.

The nondimensionalized radius values were taken directly from the data cards for uniform meshes or computed from equations (C-32) or (C-40) in the case of a nonuniform mesh.

Program STBCONT plots the stability contours against α_R and α_I. The stability map generated by program PIPE0 is searched columnwise and rowwise for sign changes for each of the three stability criteria discussed in Section V and Ref. 9. The points are then plotted, producing contours of incipient, critical and fully developed instability and areas that denote stable flow and subcritical, supercritical and hypercritical instability.

Both programs, EIGFCN and STBCONT, utilize the NPS VERSATEC plotter, certain built-in VERSATEC subroutines, and subroutine PLOTG. These routines are only accessable when running under FORTCLGW.

IV. RESULTS

A. STABILITY

Since an understanding of the term stability is necessary to interpret the results of this investigation, a brief discussion is presented here. A complete discussion of the generalized criteria of stability is given by Gawain [9].

The characteristics of the flow for the case n = 0 are set by the parameters R_e and α. For fixed values of these parameters, the solution of equations (3-5) is a set of N eigenvalues, γ, and their corresponding eigenvectors, Q. As can readily be seen from equation (2-1), the value of the real part of the complex eigenvalue γ will determine the growth or decay rate in time of the perturbation. Since positive values of the real part of γ represent an exponential growth rate in time, the most important γ is the one having the largest algebraic value for its real part. This root is termed the least stable root and will be represented by the symbol γ_{RL}. As the stability represented by γ_{RL} is that seen by a fixed observer, it is not the most general criterion. As derived in Ref. 9, a more appropriate stability criterion is that based on an axis system moving at the average volumetric velocity of the flow. This criteria is termed γ_{RL}^* and is defined by Ref. 9 as

$$\gamma_{RL}^{*} = \gamma_{RL} + \alpha_R \; . \tag{4-1}$$

For this and subsequent discussions, the subscript will be dropped and γ^{*} will refer to the quantity defined by equation (4-1). Three stability cases arise from this equation. The first is termed incipient instability and is defined by

$$\gamma^{*} = -|\alpha_R| \; . \tag{4-2}$$

The second case, termed critical instability, is given by

$$\gamma^{*} = 0 \tag{4-3}$$

and, lastly, the case termed fully developed instability is said to exist when

$$\gamma^{*} = +|\alpha_R| \; . \tag{4-4}$$

The transition from stable flow to fully developed instability is progressive and several distinct stages are given in Ref. 9 to describe this transition. The region from incipient to critical instability is termed subcritical instability, that from critical instability to fully developed instability is called supercritical instability while that beyond fully developed instability is termed hypercritical instability.

B. PERTURBATION VELOCITY PLOTS

Initial investigation of the function Q was centered around plotting its appearance in the region of interest. A Reynolds number of 1150 (2300 based on diameter) was chosen as this value is generally accepted as the nominal value for transition to turbulent flow. The value of α was set at -0.5 + i 10.0 for the major part of the investigation as preliminary checks revealed that supercritical instabilities were present for this value. A secondary Reynolds number of 4000 was chosen to show trends.

The quantity chosen as the most realistic and representative of the eigenfunction Q is the axial perturbation velocity, u. This quantity was derived from the elements of the least stable eigenvector as outlined in Appendix D. Initially, R_e and α_I were held fixed and α_R was varied over a range of positive and negative values. For values of α_R below about two, the normalized perturbation velocity was found to have all activity near the axis with a decay essentially to zero by r = 0.3. A typical plot of u versus r for an α_R in this range is shown in Figure 4-1. When α_R was made sufficiently positive, the plot changed significantly in both appearance and region of activity. Figure 4-2 shows a plot of u for α_R = 2.5. The activity can now be seen to be concentrated near the wall, with most of the activity occurring at r values greater than 0.7.

Although no particular relationship between the nature of u and the stability of the flow was evident or expected,

the plots were nevertheless valuable as indicators for various parameters involved in the investigation.

First, as can be seen by the differences in Figures 4-1 and 4-2, the plots were ideal indicators of changes in the nature of the function Q. Secondly, the adequacy of the mesh could be directly observed by noting the number of points defining the curves in regions of high activity. Figures 4-3, 4-4 and 4-5 show the same conditions as Figure 4-1 but with decreasing number of mesh points, N. Lastly, the effects of nonuniform meshes could be observed as will be discussed later in this section.

C. STABILITY CONTOUR PLOTS

The principal results of this investigation are shown in Figures 4-6 and 4-7. Although these two figures pertain to only a limited portion of the complex α plane, they do represent a significant advance in the investigation of pipe flow stability. As can be seen in these figures, the flow is characterized by regions of differing stability, ranging from stable through supercritical instability. Note that these two figures correspond to Reynolds numbers of 1150 and 4000, respectively. This is a result that has not, to this writer's knowledge, been heretofore achieved by a linearized analysis of fully developed pipe flow. The figures also show that, as has been born out by previous investigations, flow for purely sinusoidal oscillations ($\alpha_R = 0$) is stable. Additionally, a comparison of Figures 4-6

and 4-7 shows the effect of Reynolds number on the flow stability. It is clear from this comparison that an increase in Reynolds number reduces the size of the stable regions in the complex α plane; in other words, stability decreases with increasing Reynolds number. This trend agrees with our general experience pertaining to fluid flow. Lastly, the effect of the real and imaginary parts of the wave number α can readily be seen. For α_R, increasingly negative values produce successively greater levels of instability. While a contour plot was not produced for positive values of α_R, point checks of stability in this region suggest that somewhat similar contours exist in the right half-plane also. For α_I, increasing values produce increasing stability. This effect is also more pronounced at the lower Reynolds number.

D. NONUNIFORM MESH EFFECTS

One of the difficulties in this investigation was the relatively long computing time required to obtain an accurate solution, especially when operating under CP/CMS (time-sharing). The major factor controlling computing time was the number of interior mesh points, N. As an example, an increase in N of 50 percent resulted in a fourfold increase in computing time. Therefore, the desired objectives of rapidity and accuracy were in direct conflict. Additionally, follow-on investigations for values of angular wave number n other than zero involve matrices twice the order required for this case because of the coupling of equations (2-8).

For these reasons, a nonuniform mesh was developed to obtain increased accuracy at lower values of N. The nature of the velocities as seen in Figures 4-1 and 4-2 shows that a high degree of resolution in the computational mesh is only required in the vicinity of the axis (α_R less than about 2) or the wall (α_R greater than about 2). It was therefore theoretically possible to redistribute the points at moderate values of N to attain resolutions equivalent to much finer (and more time-consuming) uniform meshes.

As can be seen from Figures 4-8 and 4-9, the value of γ^* varies with the number of mesh points, N. Theoretically, each of these curves would approach some limiting value if N were increased without bound, and it is this theoretical limit that represents the required solution. In practice, it is adequate to approximate the unknown limit by a point that lies on the relatively flat portion of the curve at a value of N which is practically attainable and which does not involve a prohibitively long computing time. It has been found in this investigation that N = 79 fulfills these conditions.

The conversion to a nonuniform mesh involved a change of independent variable and the introduction of an analytical function to control the distribution of the mesh points. The details of these steps are given in Appendix C. By varying the mesh offset parameter, λ, it was possible to vary γ^* over a wide range. To determine when the high

accuracy solution (N = 79) and the nonuniform solutions were approximately equal, γ^* was plotted versus λ for fixed values of R_e, α and N with the value of γ^* for N = 79 as a reference. Figure 4-10 shows a plot of this type for N = 31. The appropriate value of λ can be seen to be approximately 1.1. Figure 4-11 is the perturbation velocity plot of the solution for N = 31 and $\lambda = 1.1$ for the same R_e and α as Figure 4-1. Note that the γ^* values are equal for these two figures. While the resolution of Figure 4-11 is not quite as fine as that of Figure 4-1, a comparison of Figure 4-11 with Figure 4-5 makes the improved resolution obvious. Figures 4-2 and 4-12 are similar to Figures 4-1 and 4-11 except that a wall offset was used. Note that for this case $\lambda = 1.2$, which points to a drawback of the nonuniform mesh, that of dependence on input conditions. While a check of λ dependence on α was not made, it most probably exists. There is also, however, the possibility that for small regions of the complex α plane, the variations in λ are small enough to allow an average value of λ to be nearly optimum for the entire region. While not used for the main results of this study, the method as developed here may well prove to be of maximum utility in follow-on investigations of higher angular wave numbers.

E. NUMERICAL ACCURACY

To ensure that the solutions presented here were of sufficient accuracy, two separate checks were made. The

first, γ^* dependence on N, is the most commonly used criterion.

For a solution to be accurate, it should be virtually independent of mesh fineness, that is, of N. The required magnitude of N for an accurate solution was found by plotting γ^* against N. Figures 4-8 and 4-9 both show that the solution is well converged for N = 79 at Reynolds numbers of 1150 and 4000, as γ^* changes by only .001 to .003 from N = 31 to N = 79 for both values of Reynolds number.

The second verification of the solution, so obvious that it is sometimes overlooked, involves simply substituting the numerical solution (least stable eigenvector) into the governing equation to ensure that it is indeed being satisfied. A short program was independently written to check the finite difference representations of equation (3-1) at the first and last interior stations and at a mid-radius station. Initial checks of numerical solutions yielded unsatisfactory results and led to the discovery of various programming errors. In particular, it was discovered that four double precision constants in the finite difference approximations were lacking the required "D0" exponent. Elimination of these seemingly trivial errors resulted in a surprising four order-of-magnitude improvement in the accuracy of the solution, with the left side of equation (3-1) improving from order 10^{-4} to order 10^{-8}.

It is instructive to note at this point that the order of magnitude of the left side of equation (3-1) is not the

true measure of its satisfaction. A more correct procedure is to compare this value with the largest term in the equation. When examined from this viewpoint, the relative error for solutions at $R_e = 1150$ and $R_e = 4000$ are found to be of order 10^{-11} to 10^{-12}, a very satisfactory result.

Therefore, by these results, the solutions presented here are both virtually independent of N and satisfy the governing differential equation to a high degree. The efforts expended to reach these conclusions were well worth the result and also point out that attention to detail is fundamental to accurate numerical results.

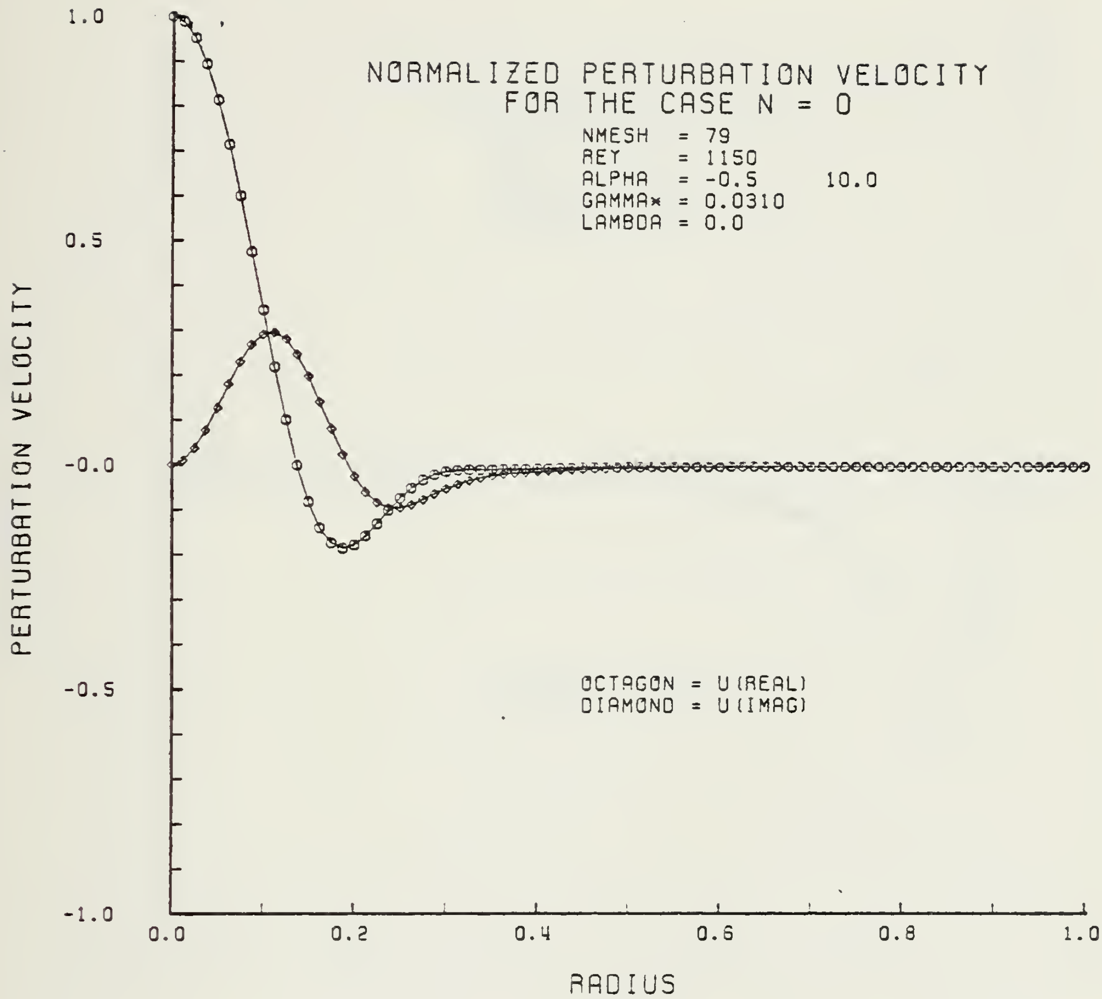
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 79
REY = 1150
ALPHA = -0.5 10.0
GAMMA* = 0.0310
LAMBDA = 0.0
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-1

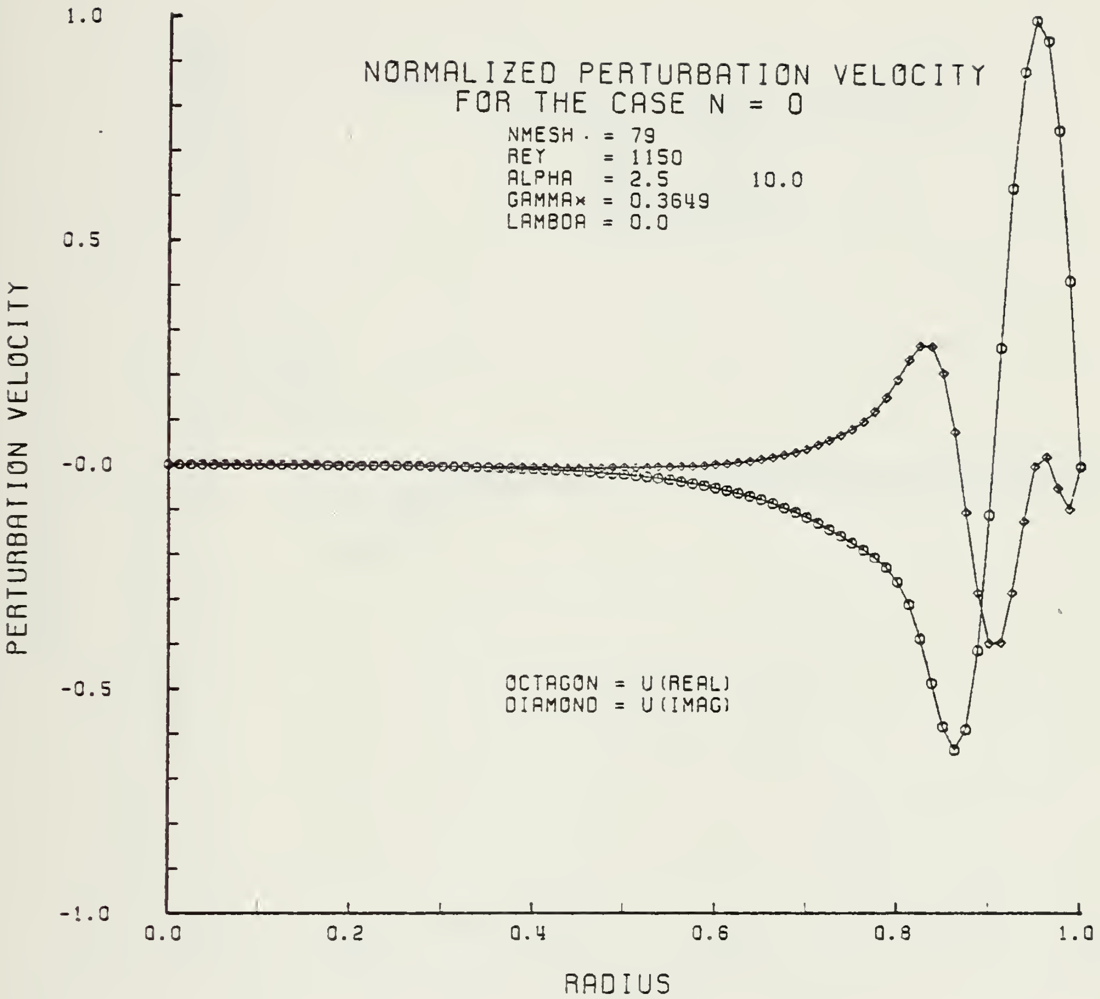
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 79
REY = 1150
ALPHA = 2.5 10.0
GAMMA* = 0.3649
LAMBDA = 0.0
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-2

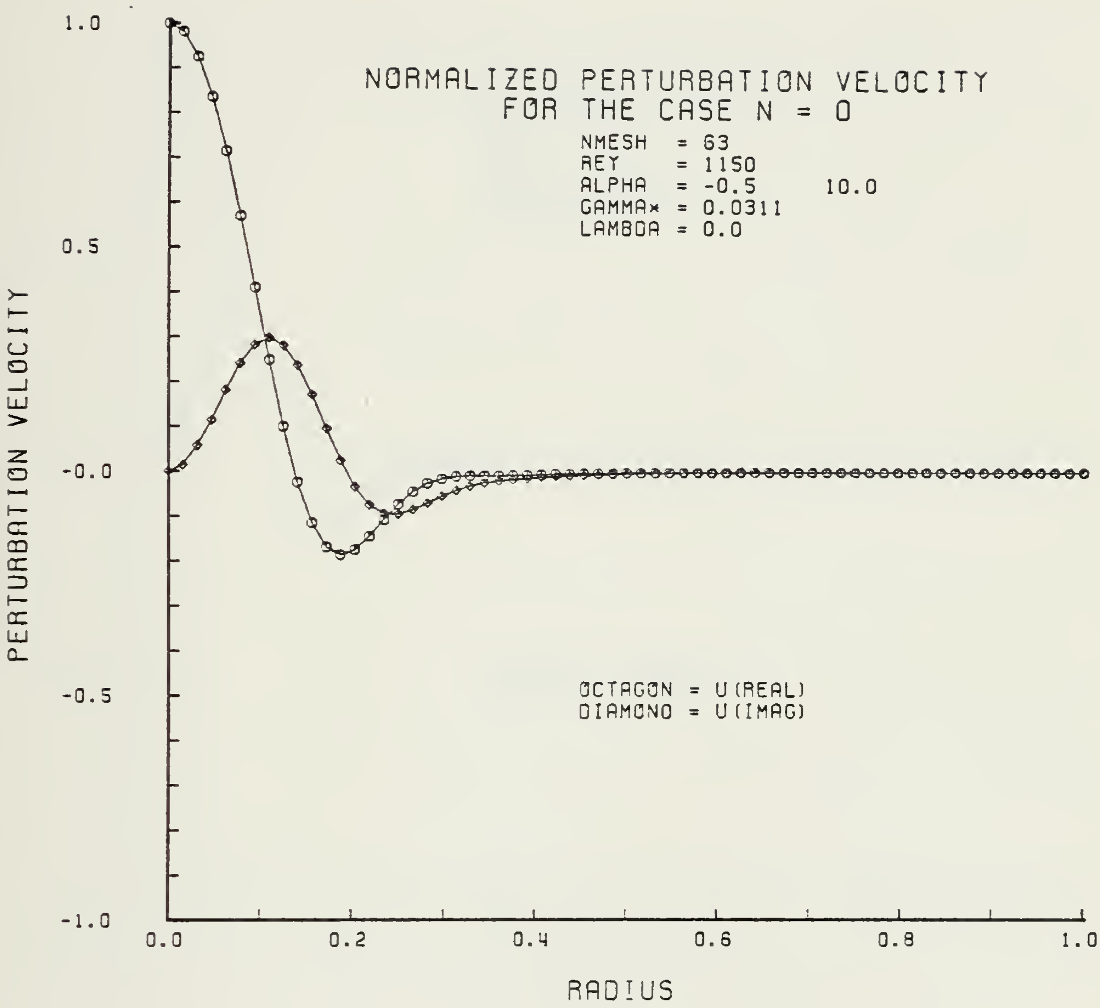
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 63
REY = 1150
ALPHA = -0.5 10.0
GAMMA* = 0.0311
LAMBDA = 0.0
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-3

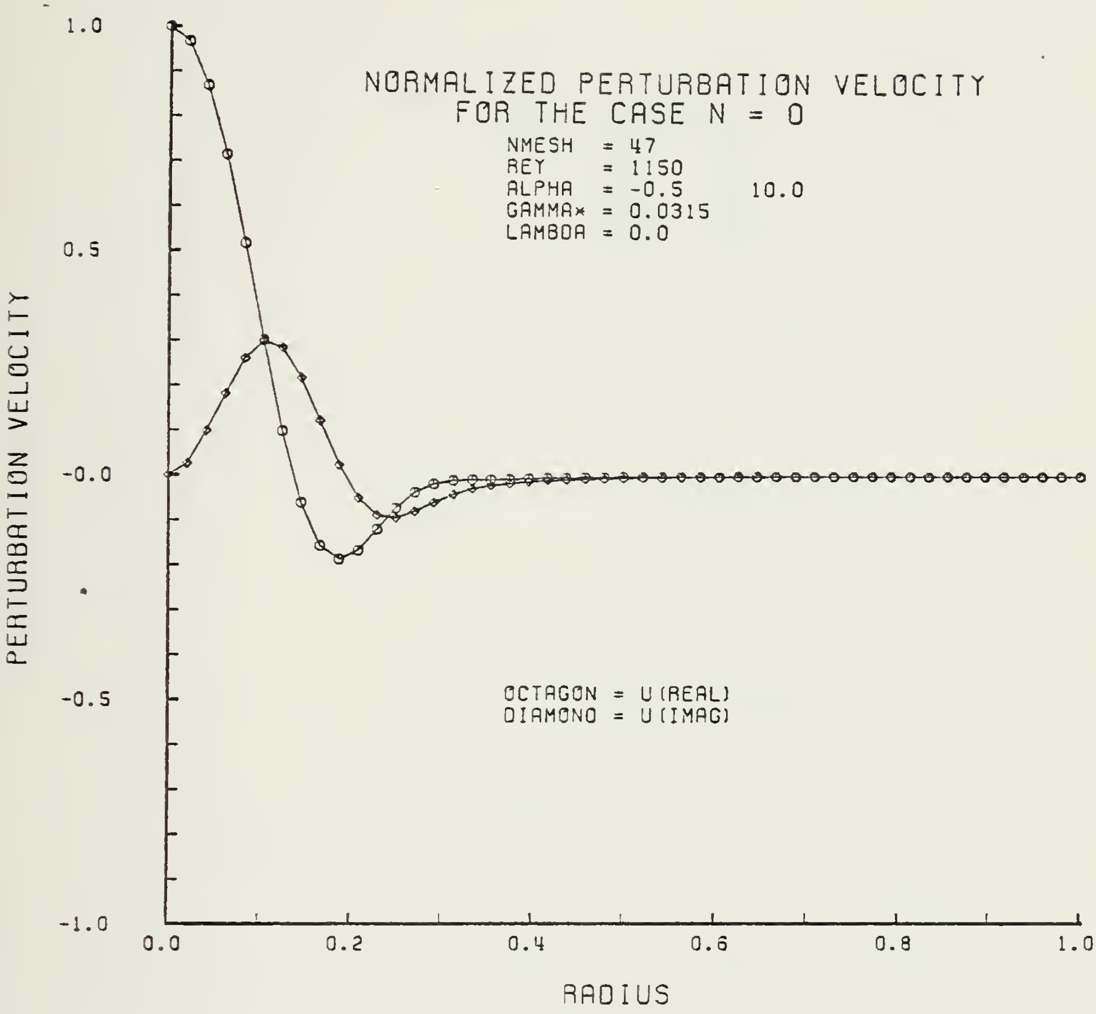
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 47
REY = 1150
ALPHA = -0.5 10.0
GAMMA* = 0.0315
LAMBDA = 0.0
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-4

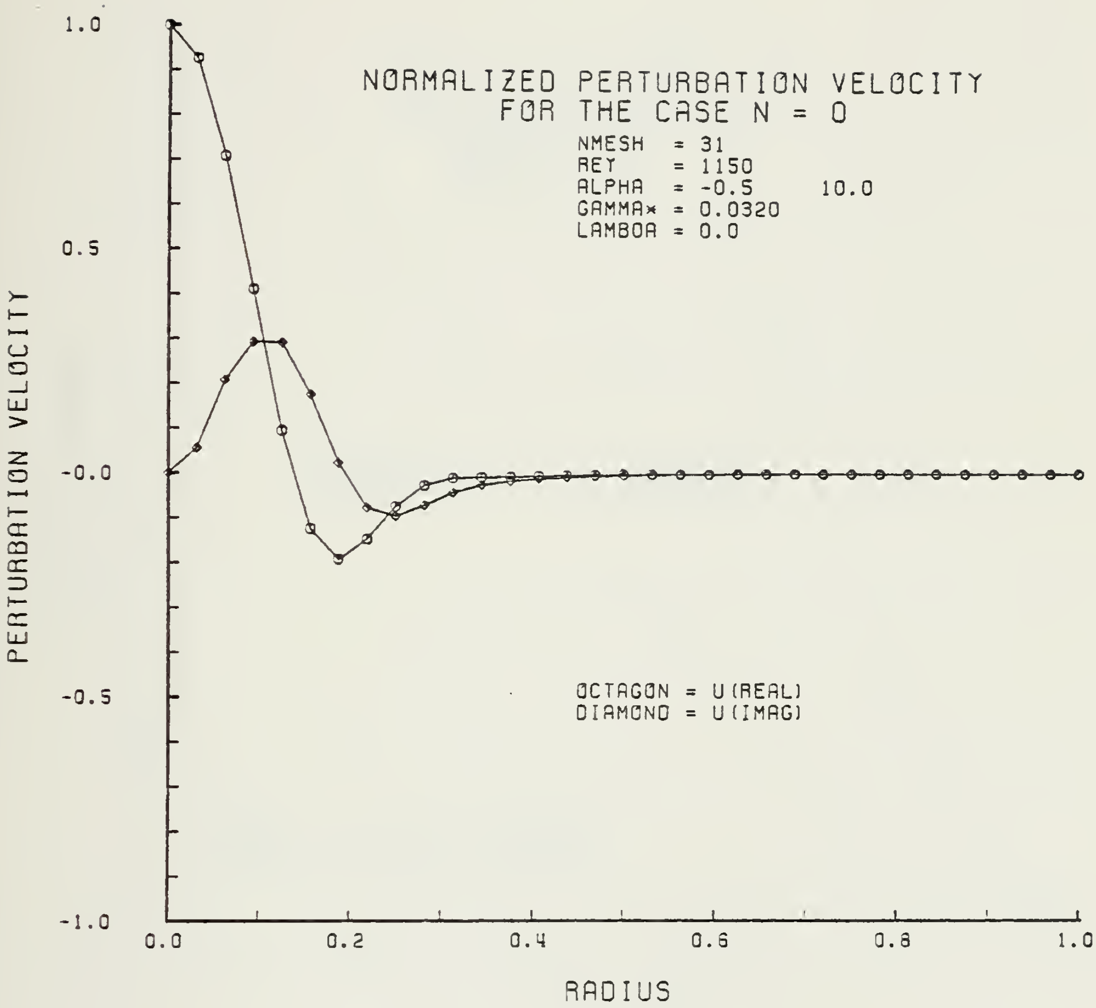
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 31
REY = 1150
ALPHA = -0.5 10.0
GAMMA* = 0.0320
LAMBDA = 0.0
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
RADIUS

FIGURE 4-5

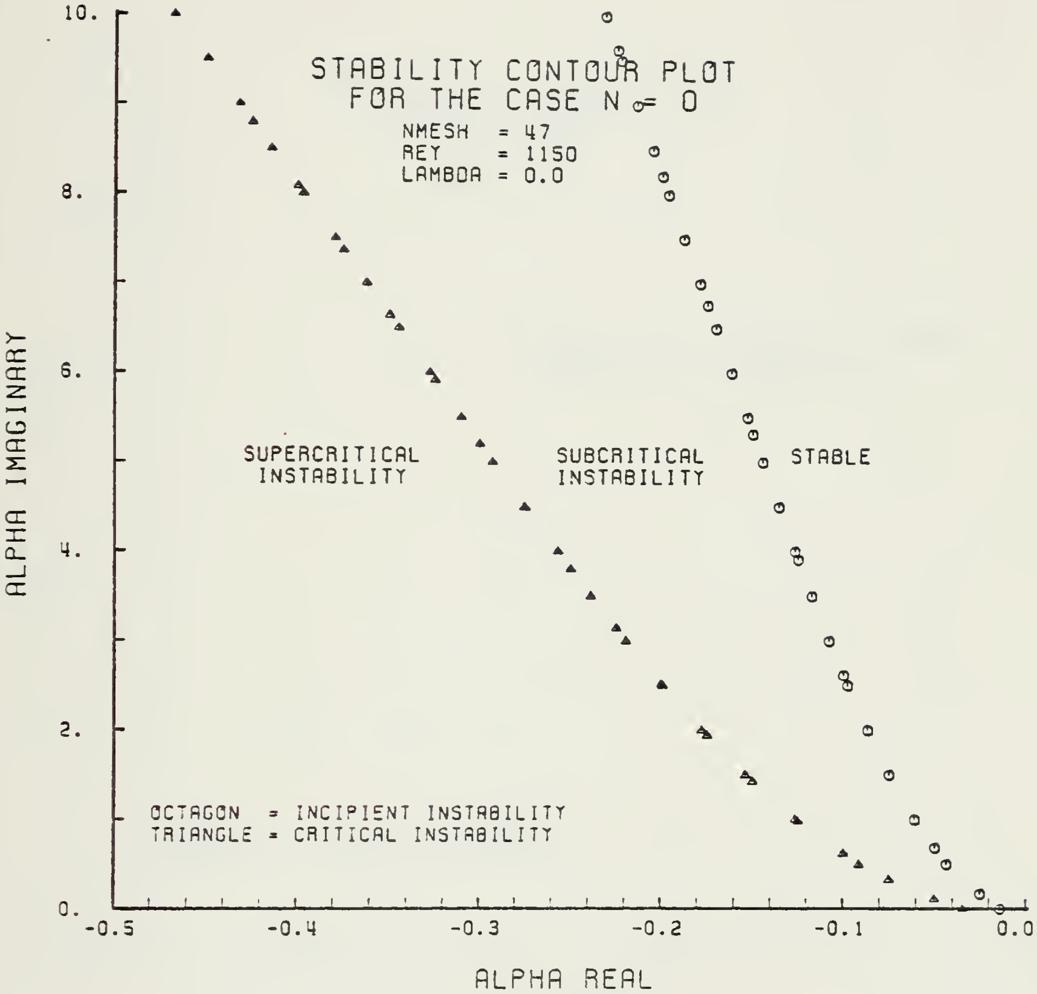
STABILITY CONTOUR PLOT
FOR THE CASE N = 0
NMESH = 47
REY = 1150
LAMBDA = 0.0
SUPERCRITICAL INSTABILITY
SUBCRITICAL INSTABILITY
STABLE
OCTAGON = INCIPIENT INSTABILITY
TRIANGLE = CRITICAL INSTABILITY
ALPHA IMAGINARY
ALPHA REAL
10.
8.
6.
4.
2.
0.
-0.5
-0.4
-0.3
-0.2
-0.1
0.0

FIGURE 4-6

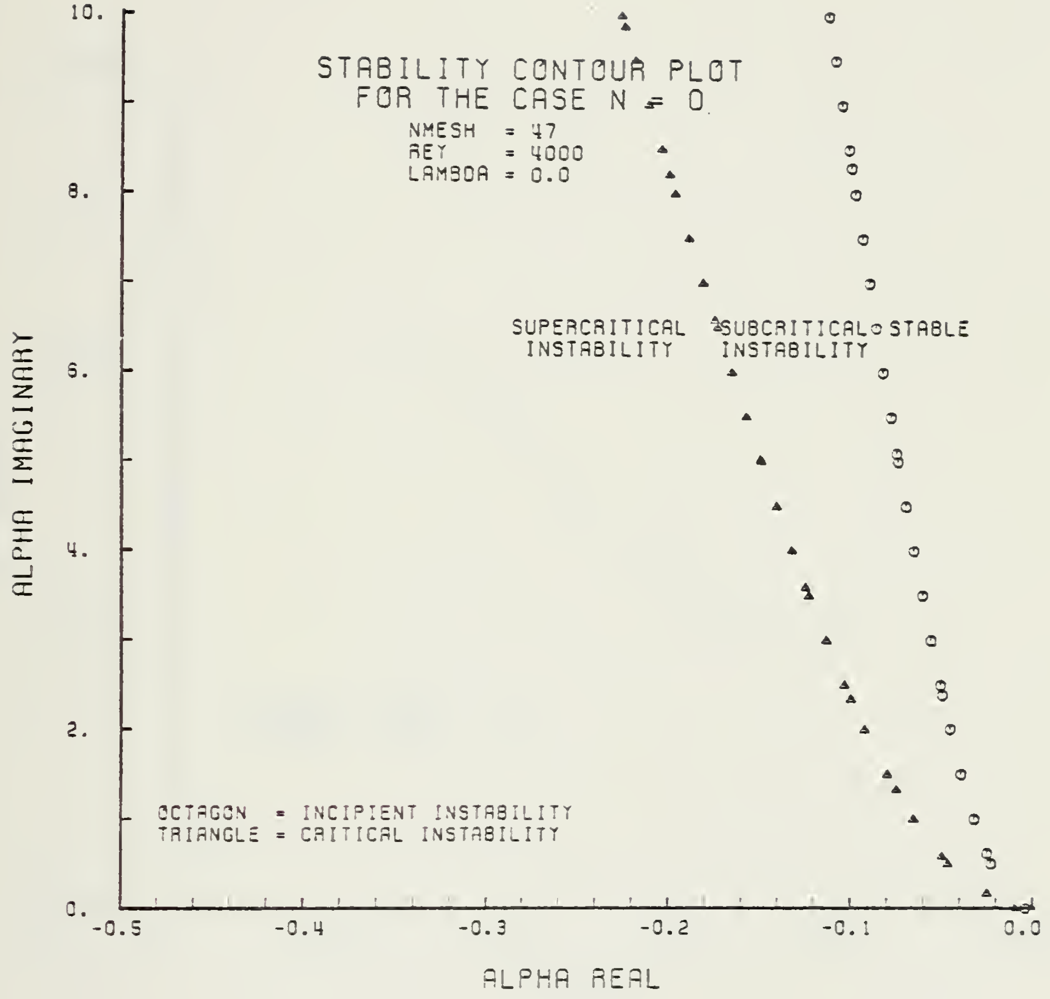
STABILITY CONTOUR PLOT
FOR THE CASE N = 0
NMESH = 47
REY = 4000
LAMBDA = 0.0
SUPERCRITICAL INSTABILITY
SUBCRITICAL INSTABILITY
STABLE
OCTAGON = INCIPIENT INSTABILITY
TRIANGLE = CRITICAL INSTABILITY
ALPHA IMAGINARY
ALPHA REAL
10.
8.
6.
4.
2.
0.
-0.5
-0.4
-0.3
-0.2
-0.1
0.0

FIGURE 4-7

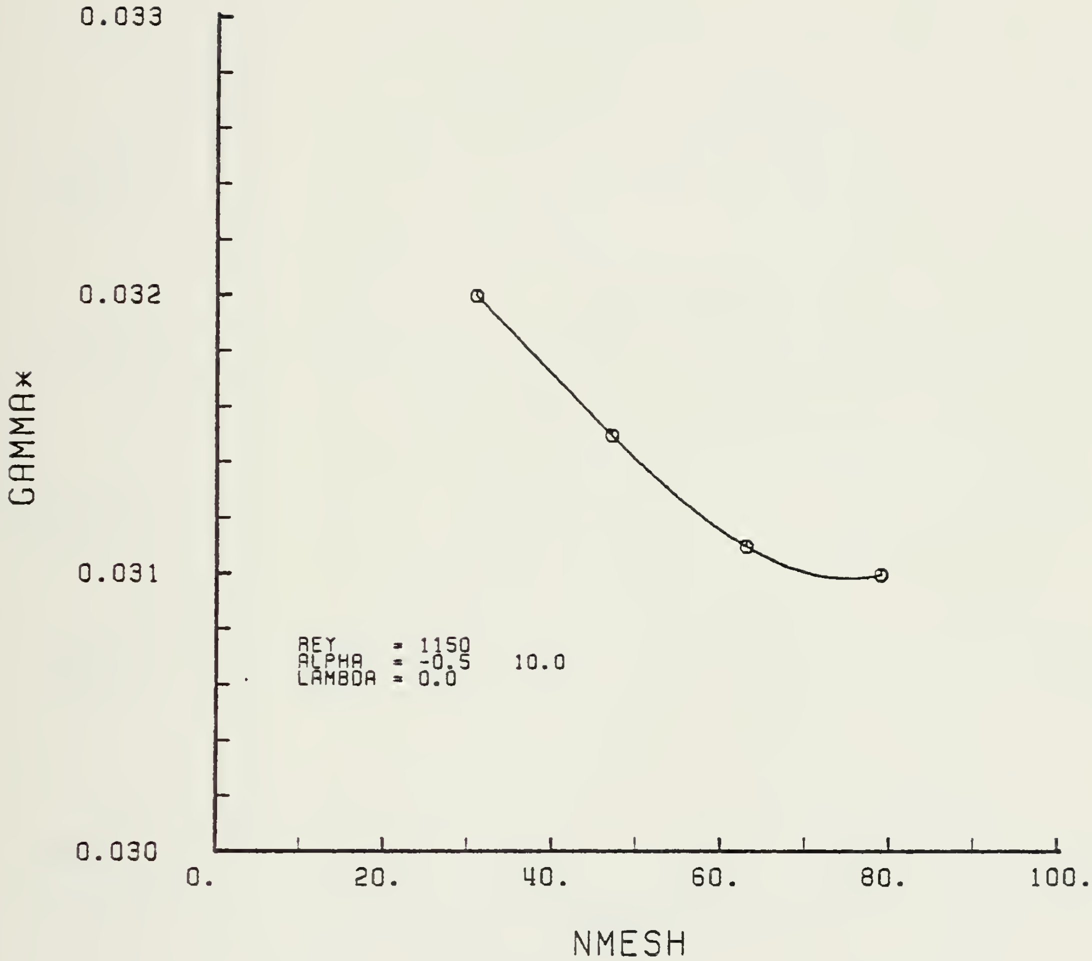

FIGURE 4-8. γ* Versus Number of Mesh Points, N.

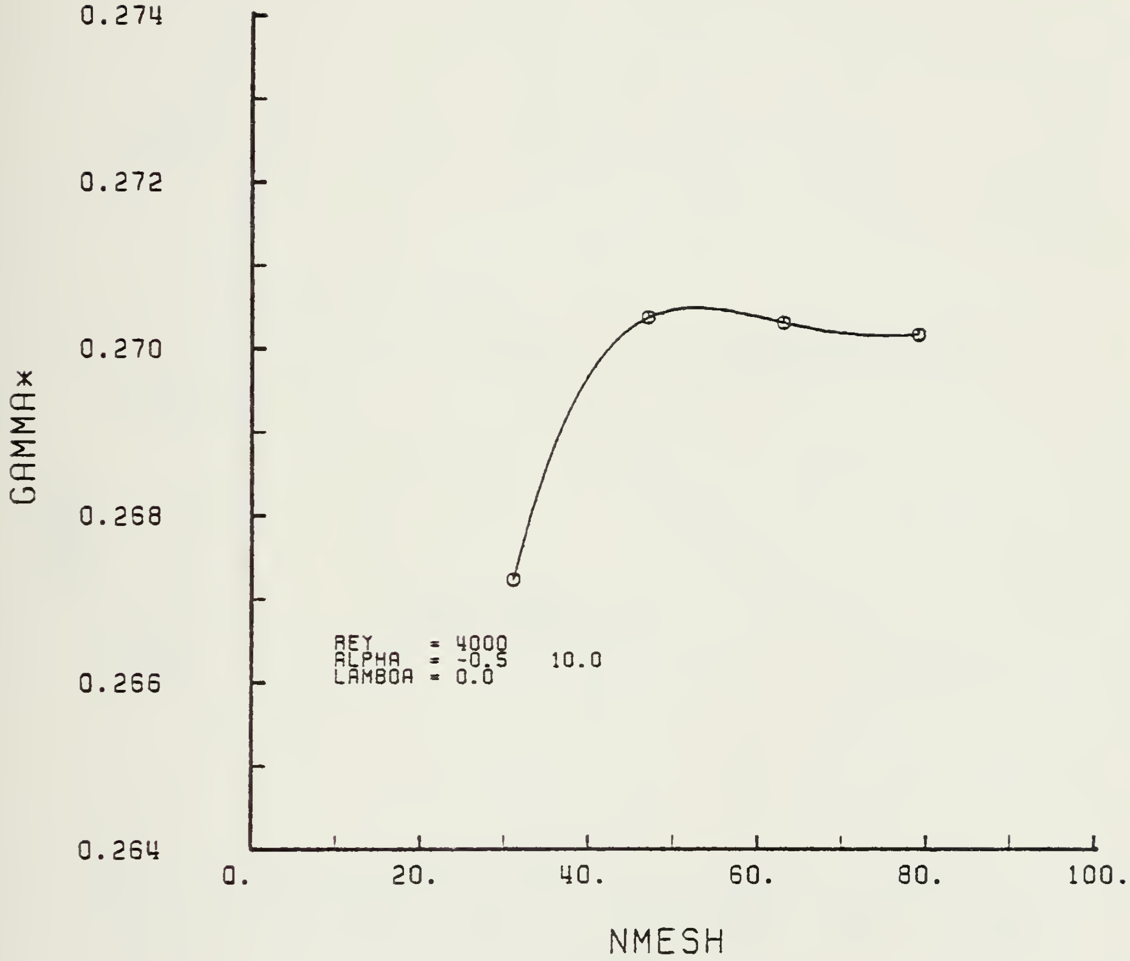

FIGURE 4-9. γ^* Versus Number of Mesh Points, N.

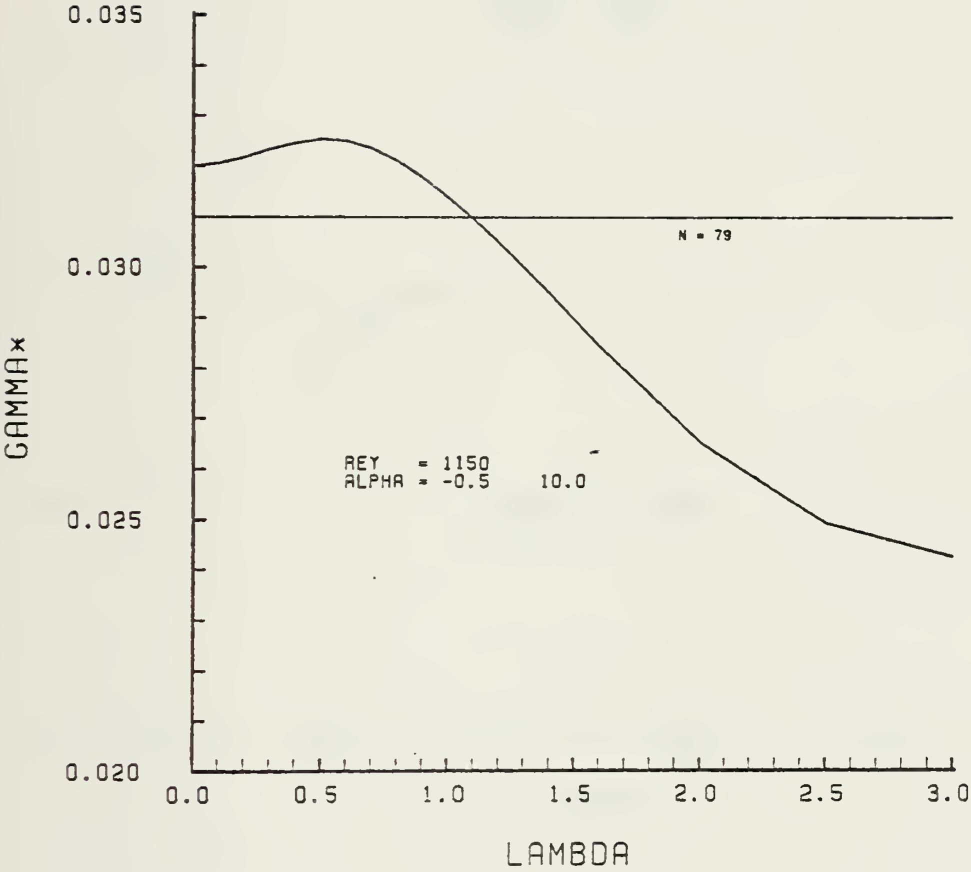

FIGURE 4-10. γ^* Versus Mesh Parameter, Lambda

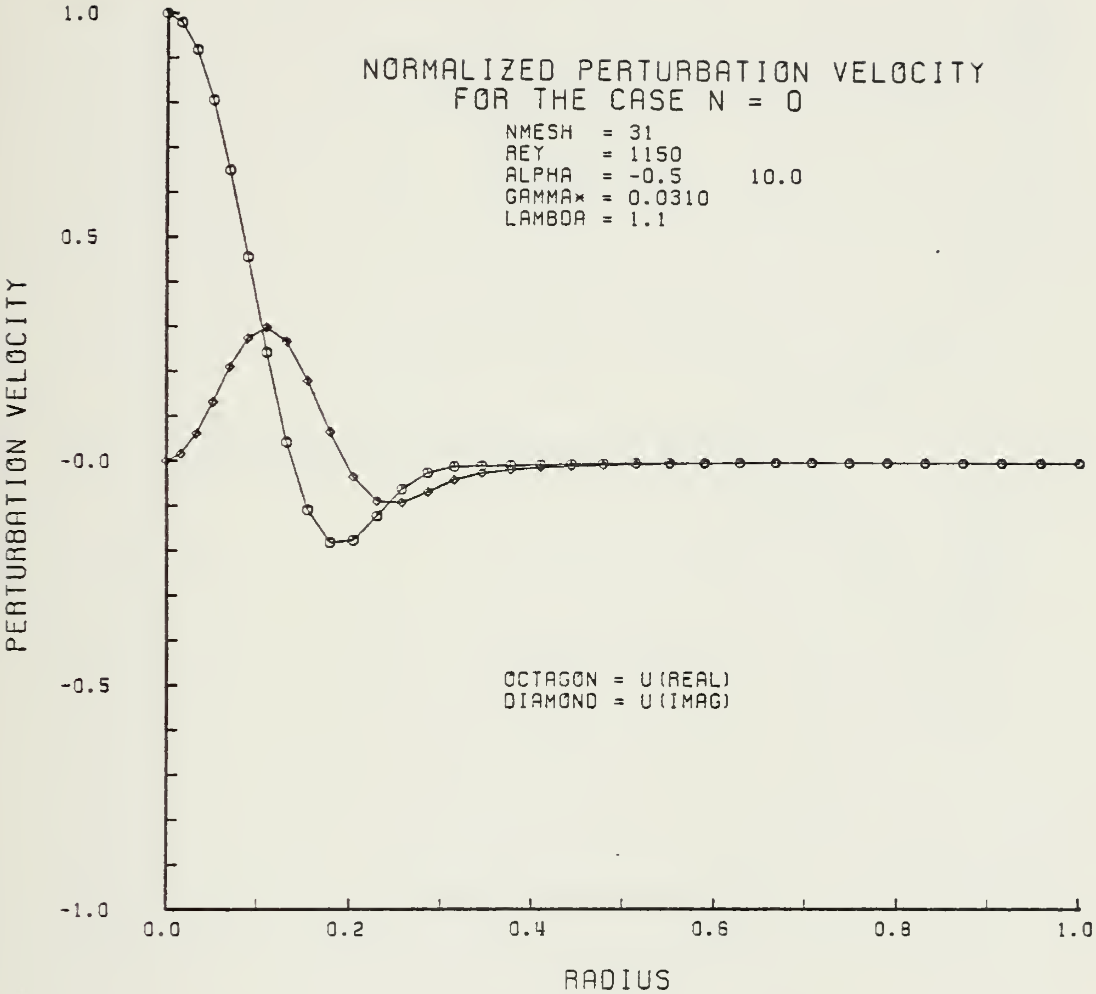
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 31
REY = 1150
ALPHA = -0.5 10.0
GAMMA* = 0.0310
LAMBDA = 1.1
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-11

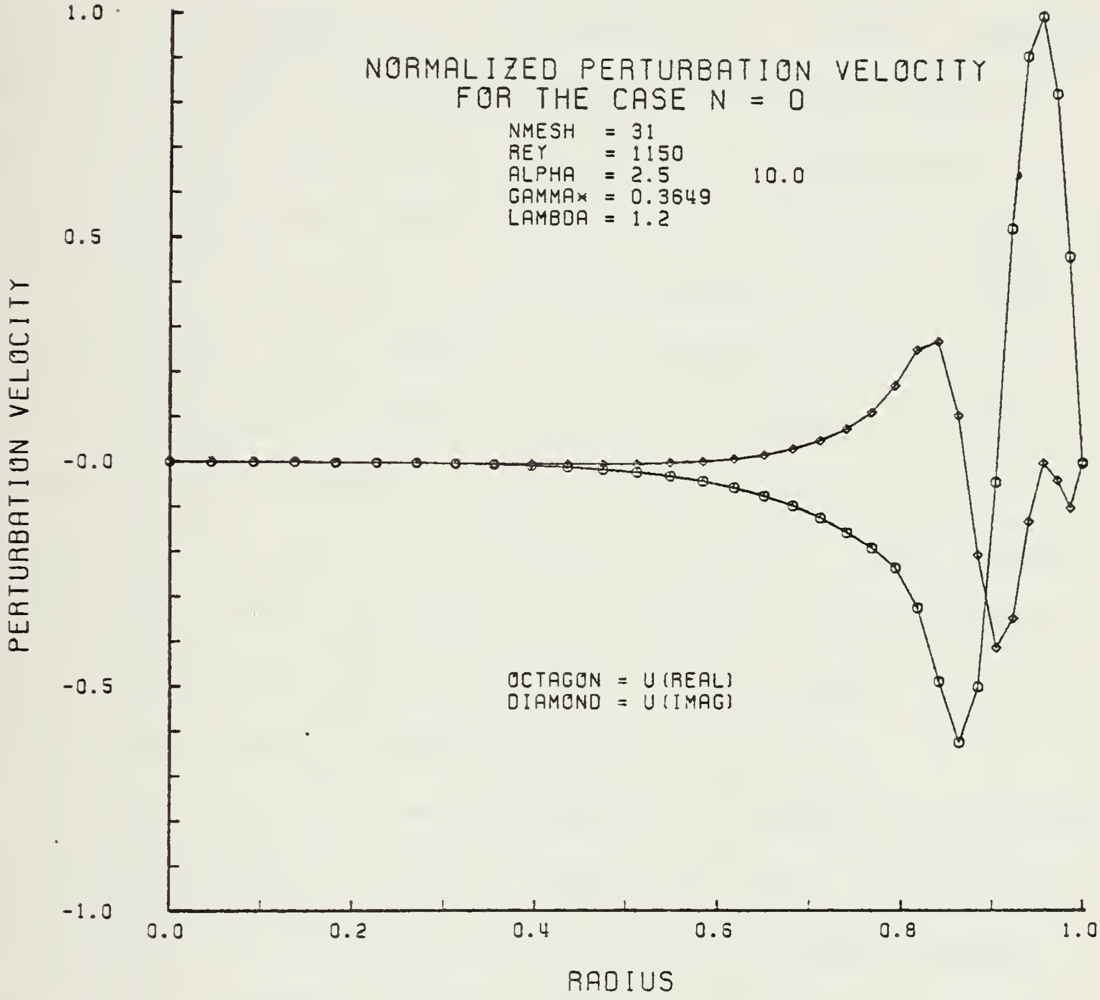
NORMALIZED PERTURBATION VELOCITY
FOR THE CASE N = 0
NMESH = 31
REY = 1150
ALPHA = 2.5 10.0
GAMMA* = 0.3649
LAMBDA = 1.2
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
PERTURBATION VELOCITY
RADIUS
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0

FIGURE 4-12

V. CONCLUSIONS AND RECOMMENDATIONS

The implementation of the newly developed boundary conditions of Gawain [9] has permitted a stable, numerical solution to the linearized vorticity transport equation. The results of the numerical solution are presented in Section IV and show that the stability of pipe Poiseuille flow is governed by the three parameters, α_R, α_I and R_e. In particular, both positive and negative values of α_R, that is, streamwise growth and decay in space, if sufficiently large, produce unstable growth rates in time. This result is new and it is consistent with the known experimental fact that transition to turbulent flow depends not only on Reynolds number but also on the general character of the perturbations which exist in the flow.

The perturbation velocity plots of Section IV represent the first practical look at the function Q. These plots were valuable indicators for adequacy of mesh fineness, that is, N, changes in the nature of the function Q and effects of a nonuniform mesh.

No instabilities were discovered for purely sinusoidal perturbations ($\alpha_R = 0$). This is consistent with the previous investigation of Ref. 11, but should not be assumed for investigations of other angular wave numbers, ($n = 1,2,3, \ldots$).

Adequate numerical accuracy was proven by demonstrating that the solution was virtually independent of the number of mesh points, N, and that it satisfied to a high degree an independent check of the governing differential equation. This procedure should also be carried out in future investigations prior to conducting full scale data runs.

This study suggests that similar, and perhaps even more rewarding results will be obtained for the higher angular wave numbers. Although lengthy, programming is straightforward if approached systematically. The general organization of the programs of Ref. 4 or Ref. 6 should be helpful in this task. It is recommended that the case for $n = 1$ be undertaken as a follow-on to this study.

The nonuniform computational mesh was shown to be a powerful tool in the reduction of computational time. At the same time, however, the dependence of the mesh offset parameter, λ, on input conditions needs to be investigated further to realize the full potential of this technique.

APPENDIX A

DERIVATION OF VORTICITY TRANSPORT EQUATION COEFFICIENTS

From the change of variable introduced in Ref. 9, the function H for the case n = 0 is expressed by

$$H = rQ \tag{A-1}$$

Taking derivatives

$$DH = rDQ + Q \tag{A-2}$$

$$D^2H = rD^2Q + 2DQ \tag{A-3}$$

$$D^3H = rD^3Q + 3D^2Q \tag{A-4}$$

$$D^4H = rD^4Q + 4D^3Q \tag{A-5}$$

Let the '*' superscript denote element (2,2) of matrices (A1) through (A9) of Ref. 9. Since for n = 0, only the function H was investigated, equations (2-10) become

$$M_4^*D^4H + M_3^*D^3H + M_2^*D^2H + M_1^*DH + M_0^*H - \gamma[N_2^*D^2H + N_1^*DH + N_0^*H] = 0 \tag{A-6}$$

Substituting for H, equation (A-6) becomes

$$M_4^*\{rD^4Q+4D^3Q\} + M_3^*\{rD^3Q+3D^2Q\} + M_2^*\{rD^2Q+2DQ\}$$

$$+ M_1^*\{rDQ+Q\} + M_0^*\{rQ\} - \gamma[N_2^*\{rD^2Q+2DQ\}$$

$$+ N_1^*\{rDQ+Q\} + N_0^*\{rQ\}] = 0 \qquad \text{(A-7)}$$

Before proceeding further, it should be noted that the Ref. 9 matrices from which the coefficients for equation (A-7) were taken were obtained from matrices (2-10) through (2-17) of Ref. 6 by means of the following substitutions:

$$U = 2(1 - r^2) \qquad \text{(A-8)}$$

$$t = \alpha^2 \frac{n^2}{r^2} \qquad \text{(A-9)}$$

$$T = \alpha U - \frac{1}{R_e}(\alpha^2 - \frac{n^2}{r^2}) \qquad \text{(A-10)}$$

Defining the new coefficients for equation (A-7) as M_0 through M_4 and N_0 through N_2

$$M_4 = rM_4^* = -\frac{r}{R_e} \qquad \text{(A-11)}$$

$$M_3 = 4M_4^* + rM_3^* = -\frac{6}{R_e} \qquad \text{(A-12)}$$

$$M_2 = 3M_3^* + rM_2^* = r\alpha U - \frac{1}{R_e}\{\frac{3}{r}+2\alpha^2 r\} \qquad \text{(A-13)}$$

$$M_1 = 2M_2^* + rM_1^* = 3\alpha U + \frac{3}{R_e}\{\frac{1}{r^2}-2\alpha^2\} \qquad \text{(A-14)}$$

$$M_0 = M_1^* + rM_0^* = r\alpha^3 U - \frac{\alpha^4 r}{R_e} \qquad \text{(A-15)}$$

$$N_2 = rN_2^* = -r \tag{A-16}$$

$$N_1 = 2N_2^* + rN_1^* = -3 \tag{A-17}$$

$$N_0 = N_1^* + rN_0^* = -\alpha^2 r \tag{A-18}$$

Upon making use of the foregoing substitutions, the governing relation can finally be reduced to the form previously shown in equation (3-1).

APPENDIX B

FINITE DIFFERENCE EQUATIONS

Improved finite difference equations for the boundaries were obtained by not using the virtual point method of Ref. 4 and Ref. 6 and deriving the forms directly from the boundary conditions of Appendix A. The equations thus formed are also of consistent order truncation error, significantly improving the accuracy of the solution [Ref. 8].

Because of a peculiarity in the form of the consistent second order truncation error equations at the axis, a singularity resulted for α equal to zero. Consistent third order truncation error equations eliminated this problem.

From Appendix A, the axis boundary conditions are

$$DQ(0) = 0 \quad \text{and} \quad D^3Q(0) = 0 \qquad \text{(B-1)}$$

Representing Q by a power series and applying equations (B-1) yields

$$Q(r) = Q(0) + D^2Q(0)\frac{r^2}{2!} + D^4Q(0)\frac{r^4}{4!} + D^5Q(0)\frac{r^5}{5!} + D^6Q(0)\frac{r^6}{6!} + \dots \qquad \text{(B-2)}$$

Using five mesh points at $r = \delta$, 2δ, 3δ, 4δ and 5δ results in the matrix

$$
\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{24} & \frac{1}{120} & \frac{1}{720} \\ 1 & 2 & \frac{16}{24} & \frac{32}{120} & \frac{64}{720} \\ 1 & \frac{9}{2} & \frac{81}{24} & \frac{243}{120} & \frac{729}{720} \\ 1 & 8 & \frac{256}{24} & \frac{1024}{120} & \frac{4096}{720} \\ 1 & \frac{25}{2} & \frac{625}{24} & \frac{3125}{120} & \frac{15625}{720} \end{bmatrix} \begin{Bmatrix} Q(0) \\ \delta^2 D^2 Q(0) \\ \delta^4 D^4 Q(0) \\ \delta^5 D^5 Q(0) \\ \delta^6 D^6 Q(0) \end{Bmatrix} + O\delta^7 \qquad \text{(B-3)}
$$

Differentiating equation (B-2) and substituting $r = \delta$ gives (in matrix form)

$$
\begin{Bmatrix} Q(\delta) \\ \delta DQ(\delta) \\ \delta^2 D^2 Q(\delta) \\ \delta^3 D^3 Q(\delta) \\ \delta^4 D^4 Q(\delta) \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{2!} & \frac{1}{4!} & \frac{1}{5!} & \frac{1}{6!} \\ 0 & 1 & \frac{1}{3!} & \frac{1}{4!} & \frac{1}{5!} \\ 0 & 1 & \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} \\ 0 & 0 & 1 & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 1 & 1 & \frac{1}{2!} \end{bmatrix} \begin{Bmatrix} Q(0) \\ \delta^2 D^2 Q(0) \\ \delta^4 D^4 Q(0) \\ \delta^5 D^5 Q(0) \\ \delta^6 D^6 Q(0) \end{Bmatrix} + O\delta^7 \qquad \text{(B-4)}
$$

Let [A] and [B] denote the coefficient matrices of equations (B-3) and (B-4) respectively. The values of $Q(0)$, $\delta^2 D^2 Q(0)$, $\delta^4 D^4 Q(0)$, $\delta^5 D^5 Q(0)$ and $\delta^6 D^6 Q(0)$ may be solved for by

$$\begin{Bmatrix} Q(0) \\ \delta^2 D^2 Q(0) \\ \delta^4 D^4 Q(0) \\ \delta^5 D^5 Q(0) \\ \delta^6 D^6 Q(0) \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} + O\delta^7 \qquad \text{(B-5)}$$

Putting equation (B-5) into equation (B-4),

$$\begin{Bmatrix} Q(\delta) \\ \delta D Q(\delta) \\ \delta^2 D^2 Q(\delta) \\ \delta^3 D^3 Q(\delta) \\ \delta^4 D^4 Q(\delta) \end{Bmatrix} = [B][A]^{-1} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{Bmatrix} + O\delta^7 \qquad \text{(B-6)}$$

The last line of this set of equations gives

$$D^4Q(\delta) = \frac{1}{\delta^4}(-.9115646260Q_1 + 2.750242955Q_2 - 3.043731779Q_3$$

$$+ 1.424684160Q_4 - .219630709Q_5) + O\delta^3 \qquad \text{(B-7)}$$

To solve for $D^3Q(\delta)$, the rightmost column and bottom row are eliminated from matrices [A] and [B] then these new matrices are inserted into equations (B-5) and (B-6).

The bottom line of equation (B-6) will now give the expression for $D^3Q(\delta)$ with a consistent third order truncation error. $D^2Q(\delta)$ and $DQ(\delta)$ were solved for in a similar manner.

$$D^3Q(\delta) = \frac{1}{\delta^3}(1.825165563Q_1 - 3.250331126Q_2 + 1.660927152Q_3 - .235761589Q_4) + O\delta^3 \qquad \text{(B-8)}$$

$$D^2Q = \frac{1}{\delta^2}(- \frac{35}{60} Q_1 + \frac{8}{15} Q_2 + \frac{1}{20} Q_3) + O\delta^3 \qquad \text{(B-9)}$$

$$DQ = \frac{1}{\delta}(- \frac{2}{3} Q_1 + \frac{2}{3} Q_2) + O\delta^3 \qquad \text{(B-10)}$$

Due to the complexity of the boundary conditions, it was decided that consistent third order truncation error equations should also be used at $r = 2\delta$. For this the [B] matrix only need be changed as equation (B-2) is unchanged at this station. The new matrix [B] is formed by differentiating equation (B-2) and making the substitution $r = 2\delta$. Proceeding as for $r = \delta$ gives the following finite difference approximations

$$D^4Q(2\delta) = \frac{1}{\delta^4}(-3.10340136Q_1 + 6.903012634Q_2 - 5.342274053Q_3 + 1.66083577Q_4 - 0.123420797Q_5) + O\delta^3 \qquad \text{(B-11)}$$

$$D^3Q(2\delta) = \frac{1}{\delta^3}(.868874172Q_1 - .937748345Q_2 - .254304636Q_3 + .323178808Q_4) + O\delta^3 \qquad \text{(B-12)}$$

$$D^2Q(2\delta) = \frac{1}{\delta^2}\left(\frac{11}{12}Q_1 - \frac{28}{15}Q_2 + \frac{19}{20}Q_3\right) + O\delta^3 \tag{B-13}$$

$$DQ(2\delta) = \frac{1}{\delta}\left(-\frac{4}{3}Q_1 + \frac{4}{3}Q_2\right) + O\delta^3 \tag{B-14}$$

It should also be noted that the value of Q at r = 0 may be solved for from the top line of equations (B-5)

$$Q(0) = (1.7959183670Q_1 - 1.24781341Q_2 + .606413994Q_3 - .177842566Q_4 + .023323615Q_5) + O\delta^3 \tag{B-15}$$

The central difference equations given by Ref. 6 were already consistent second order truncation error equations as confirmed by Ref. 8 and were retained.

For the wall, the clamped end, consistent second order equations (5) through (8) of Table II, Ref. 8 were modified for the "right boundary" using the procedure given in Section 5 of that reference.

$$D^4Q(1-\delta) = \frac{1}{\delta^4}\left(-\frac{1}{4}Q_{N-3} + \frac{8}{3}Q_{N-2} - 9Q_{N-1} + 16Q_N\right) + O\delta^2 \tag{B-16}$$

$$D^3Q(1-\delta) = \frac{1}{\delta^3}\left(-\frac{1}{3}Q_{N-2} + 3Q_N\right) + O\delta^2 \tag{B-17}$$

$$D^2Q(1-\delta) = \frac{1}{\delta^2}\left(Q_{N-1} - 2Q_N\right) + O\delta^2 \tag{B-18}$$

$$DQ(1-\delta) = \frac{1}{\delta}\left(-\frac{1}{2}Q_{N-1}\right) + O\delta^2 \tag{B-19}$$

Since the wall finite difference approximations were of only second order truncation error, the approximations for DQ through D^4Q at $r = 1-2\delta$ were obtained directly from the central difference equations with $Q(1) = 0$.

$$D^4Q(1-2\delta) = \frac{1}{\delta^4}(Q_{N-3} - 4Q_{N-2} + 6Q_{N-1} - 4Q_N) + O\delta^2 \quad \text{(B-20)}$$

$$D^3Q(1-2\delta) = \frac{1}{\delta^3}(- \frac{1}{2}Q_{N-3} + Q_{N-2} - Q_N) + O\delta^2 \quad \text{(B-21)}$$

$$D^2Q(1-2\delta) = \frac{1}{\delta^2}(Q_{N-2} - 2Q_{N-1} + Q_N) + O\delta^2 \quad \text{(B-22)}$$

$$DQ(1-2\delta) = \frac{1}{\delta}(- \frac{1}{2}Q_{N-2} + \frac{1}{2}Q_N) + O\delta^2 \quad \text{(B-23)}$$

APPENDIX C

NONUNIFORM MESH

To control the distribution of a fixed number of mesh points, a change of the independent variable from r to η was performed.

$$Q = Q(\eta) \tag{C-1}$$

$$r = r(\eta) \tag{C-2}$$

The derivative with respect to r becomes

$$D = (D^{*}r)^{-1}D^{*} \tag{C-3}$$

where

$$D^{*} = \frac{d}{d\eta} \quad \text{and} \quad D = \frac{d}{dr} \tag{C-4}$$

DQ, D^2Q ... can now be expressed in terms of the new independent variable, η.

$$DQ = (D^{*}r)^{-1}D^{*}Q \tag{C-5}$$

$$D^2Q = D(DQ) = (D^{*}r)^{-1}D^{*}(DQ)$$

$$= (D^{*}r)^{-2}D^{*2}Q - (D^{*}r)^{-3}(D^{*2}r)D^{*}Q \tag{C-6}$$

$$D^3Q = D(D^2Q) = (D^*R)^{-1}D^*(D^2Q)$$

$$= (D^*r)^{-3}D^{*3}Q - 3(D^*r)^{-4}(D^{*2}r)D^{*2}Q$$

$$- [(D^*r)^{-4}(D^{*3}r) - 3(D^*r)^{-5}(D^{*2}r)^2]DQ \qquad (C-7)$$

$$D^4Q = D(D^3Q) = (D^*r)^{-1}D^*(D^3Q)$$

$$= (D^*r)^{-4}D^{*4}Q - 6(D^*r)^{-5}(D^{*2}r)D^{*3}Q$$

$$+ [15(D^*r)^{-6}(D^{*2}r) - 4(D^*r)^{-5}(D^{*3}r)]D^{*2}Q$$

$$- [15(D^*r)^{-7}(D^{*2}r)^3 - 10(D^*r)^{-6}(D^{*2}r)(D^{*3}r)$$

$$+ (D^*r)^{-5}(D^{*4}r)]DQ \qquad (C-8)$$

The derivatives of Q with respect to r can now be written

$$DQ = f_{11}D^*Q \qquad (C-9)$$

$$D^2Q = f_{22}D^*2Q + f_{21}D^*Q \qquad (C-10)$$

$$D^3Q = f_{33}D^{*3}Q + f_{32}D^{*2}Q + f_{31}D^*Q \qquad (C-11)$$

$$D^4Q = f_{44}D^{*4}Q + f_{43}D^{*3}Q + f_{42}D^{*2}Q + f_{41}D^*Q \qquad (C-12)$$

where

$$f_{11} = (D^*r)^{-1} \qquad (C-13)$$

$$f_{22} = (D^* r)^{-2} \quad \text{(C-14)}$$

$$f_{21} = -(D^* r)^{-3}(D^{*2} r) \quad \text{(C-15)}$$

$$f_{33} = (D^* r)^{-3} \quad \text{(C-16)}$$

$$f_{32} = -3(D^* r)^{-4}(D^{*2} r) \quad \text{(C-17)}$$

$$f_{31} = 3(D^* r)^{-5}(D^{*2} r)^2 - (D^* r)^{-4}(D^{*3} r) \quad \text{(C-18)}$$

$$f_{44} = (D^* r)^{-4} \quad \text{(C-19)}$$

$$f_{43} = -6(D^* r)^{-5}(D^{*2} r) \quad \text{(C-20)}$$

$$f_{42} = 15(D^* r)^{-6}(D^{*2} r)^2 - 4(D^* r)^{-5}(D^{*3} r) \quad \text{(C-21)}$$

$$f_{41} = -15(D^* r)^{-7}(D^{*2} r)^3 + 10(D^* r)^{-6}(D^{*2} r)(D^{*3} r) - (D^* r)^{-5}(D^{*4} r) \quad \text{(C-22)}$$

Substituting equations (C-9) through (C-12) into the vorticity transport equation (A-6) yields

$$M_4^* D^{*4} Q + M_3^* D^{*3} Q + M_2^* D^{*2} Q + M_1^* D^* Q + M_0^* Q - \gamma [N_2^* D^{*2} Q + N_1^* D^* Q + N_0^* Q] = 0 \quad \text{(C-23)}$$

where

$$M_4^* = M_4 f_{44} \qquad (C-24)$$

$$M_3^* = M_4 f_{43} + M_3 f_{33} \qquad (C-25)$$

$$M_2^* = M_4 f_{42} + M_3 f_{32} + M_2 f_{22} \qquad (C-26)$$

$$M_1^* = M_4 f_{41} + M_3 f_{31} + M_2 f_{21} \qquad (C-27)$$

$$M_0^* = M_0 \qquad (C-28)$$

$$N_2^* = N_2 f_{22} \qquad (C-29)$$

$$N_1^* = N_2 f_{21} + N_1 f_{11} \qquad (C-30)$$

$$N_0^* = N \qquad (C-31)$$

In order to concentrate the mesh points at the axis, the function

$$r = 1 - C \tanh \lambda(1-\eta) \qquad (C-32)$$

was chosen where λ is a parameter controlling the degree of concentration of mesh points near the axis. Equation (C-32) must satisfy the two conditions

$$r = 0 \quad \text{at} \quad \eta = 0 \tag{C-33}$$

and

$$r = 1 \quad \text{at} \quad \eta = 1 .$$

Substituting equation (C-33) into (C-32) gives

$$C = 1/\tanh \lambda . \tag{C-35}$$

Computing derivatives

$$D^{*}r = C\lambda/\cosh^{2}\lambda(1-\eta) \tag{C-36}$$

$$D^{*2}r = 2C\lambda^{2}[\tanh\lambda(1-\eta)/\cosh^{2}\lambda(1-\eta)] \tag{C-37}$$

$$D^{*3}r = -2C\lambda^{3}\{[1-2\sinh^{2}\lambda(1-\eta)]/\cosh^{4}\lambda(1-\eta)\} \tag{C-38}$$

$$D^{*4}r = 8C\lambda^{4}[\tanh^{3}\lambda(1-\eta)/\cosh^{2}\lambda(1-\eta)] \tag{C-39}$$

To shift the mesh point concentration to the wall, the function

$$r = C \tanh \lambda\eta \tag{C-40}$$

was selected. Satisfying equations (C-33) and (C-34) for this equation also gives equation (C-35). The derivatives

of (C-40) are given by equations (C-36) through (C-39) if η is substituted for all occurrences of $(1-\eta)$ and the signs of equations (C-37) and (C-39) are reversed. Figures C-1 and C-2 show equations (C-32) and (C-40) for four selected values of the parameter λ.

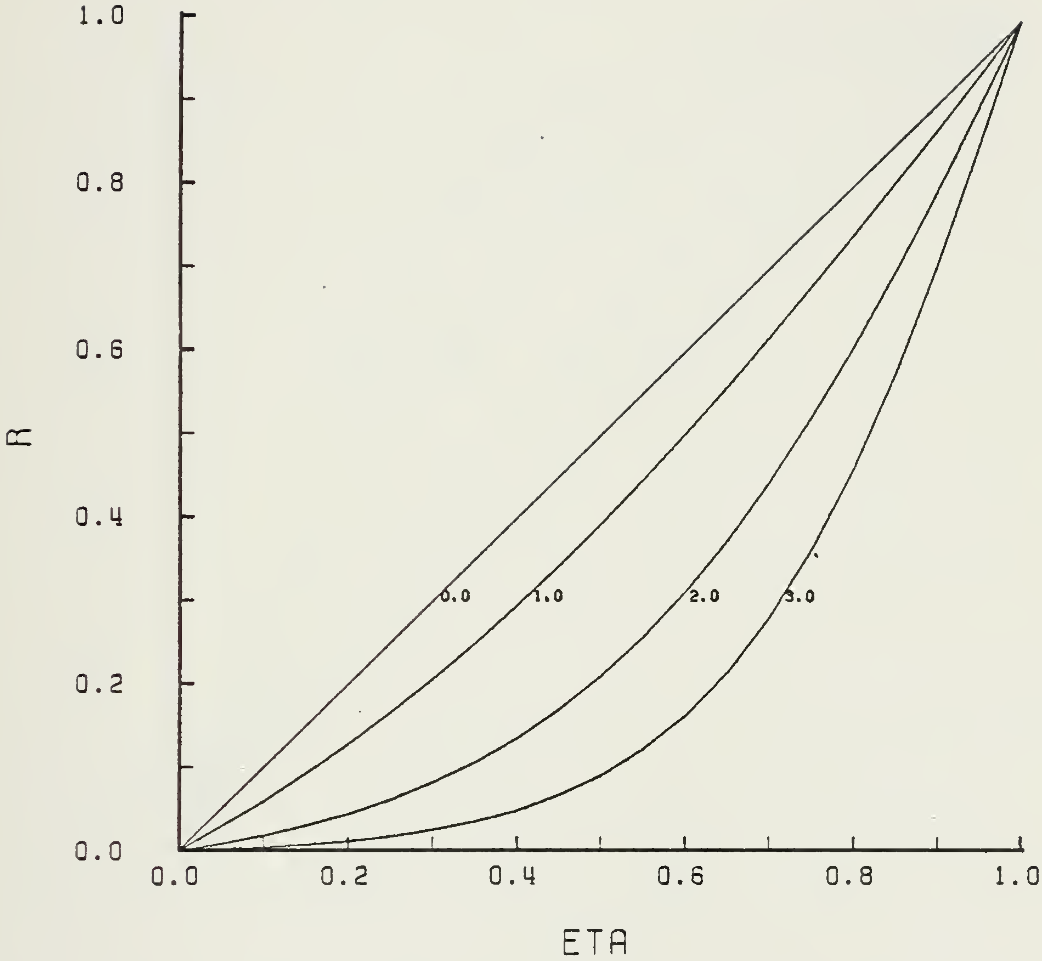

FIGURE C-1. R Versus η for Four Selected Values of Lambda - Axis Offset

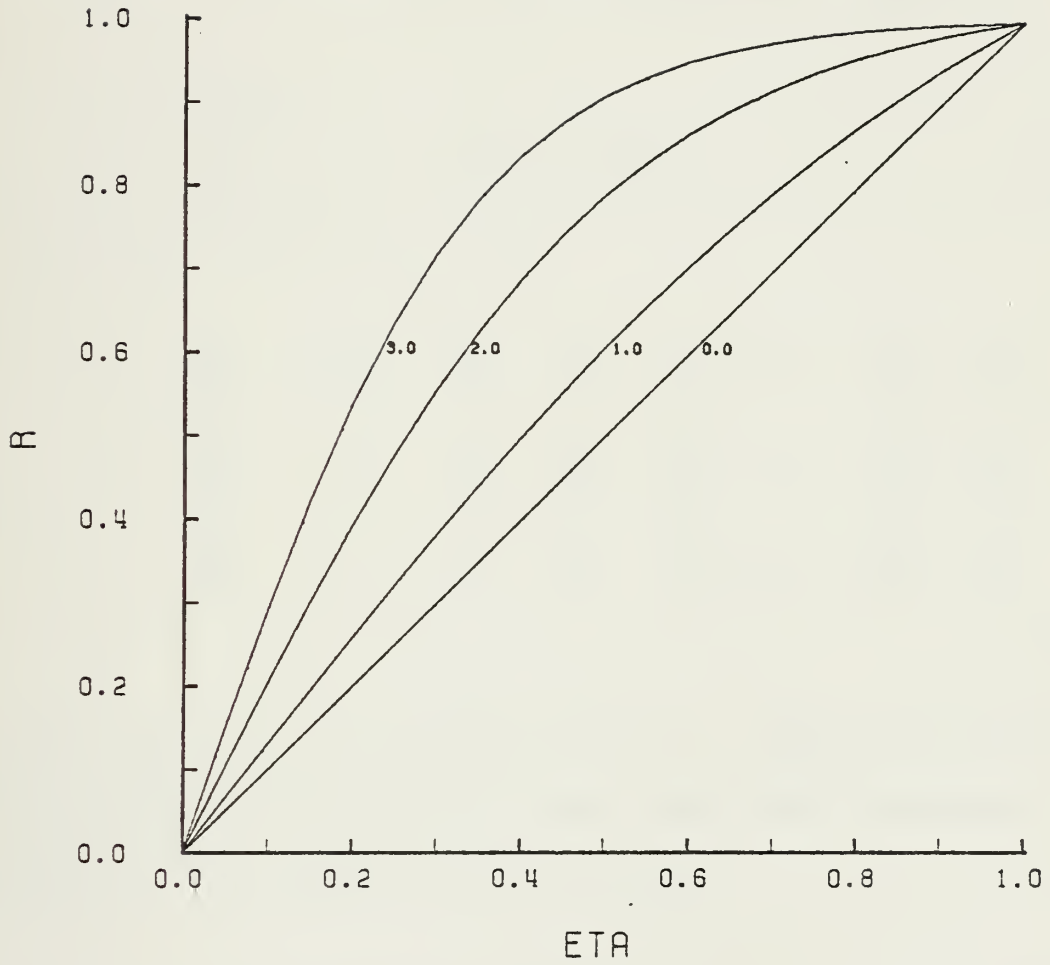

FIGURE C-2. R Versus η for Four Selected Values of Lambda - Wall Offset

APPENDIX D

DERIVATION OF PERTURBATION VELOCITIES

From Ref. 4, Appendix E, equations E-6 through E-8:

$$\begin{Bmatrix} u(r) \\ v(r) \\ w(r) \end{Bmatrix} = [A]\overline{W} + [B]D\overline{W} \qquad \text{(D-1)}$$

$$= \begin{bmatrix} 0 & -\frac{\beta}{r} & \frac{1}{r} \\ \frac{\beta}{r} & 0 & -\alpha \\ 0 & \alpha & 0 \end{bmatrix} \begin{Bmatrix} F \\ G \\ H \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} DF \\ DG \\ DG \end{Bmatrix} \qquad \text{(D-2)}$$

For this case $\beta = ni = 0$ and $F = DF = 0$. Restricting the investigation to the function H for the reason expressed in Section I and solving for u(r) gives

$$u(r) = \frac{H}{r} + DH \qquad \text{(D-3)}$$

Performing the change of variable

$$H = rQ \qquad \text{(D-4)}$$

$$DH = Q + rDQ \qquad \text{(D-5)}$$

$$u(r) = \frac{rQ}{r} + (Q + rDQ) = 2Q + rDQ \qquad (D-6)$$

In order to implement this derivation in a numerical analysis, equation (D-6) was rewritten as

$$u_i = 2Q_i + r_i DQ_i \qquad (D-7)$$

Performing the change of independent variable (Appendix C) to accommodate a nonuniform mesh

$$Q_i = Q(\eta_i) \qquad (D-8)$$

$$r_i = r(\eta_i) \qquad (D-9)$$

$$DQ_i = (D^* r_i)^{-1} D^* Q(\eta_i) \qquad (D-10)$$

Substituting equations (D-8), (D-9) and (D-10) into equation (D-7) gives

$$u_i = 2Q(\eta_i) + r(\eta_i) D^* r_i)^{-1} D^* Q(\eta_i) \qquad (D-11)$$

For the axis offset nonuniform mesh, $r(\eta)$ is given by equation (C-32) and $(D^* r)$ by equation (C-36). Substituting into equation (D-11) using equation (C-35) results in

$$u_i = 2Q(\eta_i) + \{1 - \frac{\tanh[\lambda(1-\eta_i)]}{\tanh\lambda}\}\{\frac{\cosh^2[\lambda(1-\eta_i)]}{C\lambda}\}D^*Q(\eta_i)$$

$$= 2Q(\eta_i) + \{1 - \frac{\tanh[\lambda(1-\eta_i)]}{\tanh\lambda} \quad \frac{\tanh\lambda\cosh^2[\lambda(1-\eta_i)]}{\lambda}\}D^*Q(\eta_i) \tag{D-12}$$

For the wall offset mesh, equation (C-40) is substituted for equation (C-32) and all occurrences of the term $1-\eta_i$ are replaced by the term η_i.

The value of u at the axis (u_o) and at the wall (u_{N+1}) were solved for by using the boundary conditions specified in Ref. 9, namely

$$Q(1) = 0 \tag{D-13}$$

$$DQ(1) = 0 \tag{D-14}$$

$$DQ(0) = 0 \tag{D-15}$$

$$D^3Q(0) = 0 \tag{D-16}$$

From equations (D-13) and (D-14), using equation (D-7) it is obvious that

$$u_{N+1} = 0 \tag{D-17}$$

and from equations (D-15) and (D-7), it is similarly found that

$$u_o = 2Q(0) , \qquad (D-18)$$

where the finite difference approximation for Q(0) is given by equation (B-15).

```
C.......................................................................
C
C     PROGRAM PIPEO  (CP/CMS VERSION)
C
C        PROGRAM TO INVESTIGATE FLOW STABILITY AND CHARACTERISTICS
C               FOR THE 3-D CYLINDRICAL FLOW PROBLEM
C                   NI=0;  FOR THE FUNCTION Q
C
C        TO OBTAIN A FLOW CHART OF THIS PROGRAM, CONSULT NAVAL
C        POSTGRADUATE SCHOOL TECHNICAL NOTE TN 0141-25, "USER'S
C        GUIDE TO THE PROGRAMMING AIDS LIBRARY",UNDER PROGRAM
C        FLOWCH.
C
C.......................................................................
C
C-----------------------------------------------------------------------
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AIPLT(20), REYPLT(20)
      COMPLEX *16A,G
      INTEGER *4CLOCK(6)
      COMMON /COEFNT/ A,G,REY,DELR,AMDA
C-----------------------------------------------------------------------
C
C
C     INPUT DESIRED MODE NUMBER
C
C
      WRITE (6,6)
      READ (5,7) MODENO
      CALL IXCLOK (CLOCK)
      WRITE (6,8) CLOCK(3),CLOCK(4)
      IF (MODENO.EQ.2) GO TO 2
C
C
C***********************************************************************
C
C     CALCULATE STABILITY AT A POINT (MODENO = 1 & 3)
C
C        OUTPUTS DATA TO FILE FT02F001 COMPATIBLE WITH PROGRAM EIGFCN &
C        PRINTS THE VALUE OF THE LEAST STABLE EIGENVALUE(GAMMA*) AT THE
C        CONSOLE FOR EACH SET OF INPUT CONDITIONS.  DATA IS ENTERED IN
C        'D' FORMAT WITH A ZERO EXPONENT (I.E.,1.D0,1150.D0,-.5D0,ETC.).
C        RUN IS TERMINATED WHEN A NEGATIVE VALUE FOR REYNOLDS NUMBER IS
C        ENTERED.  THE TYPE OF MESH OFFSET IS DETERMINED BY THE VALUE
C        OF LAMBDA.  IF LAMBDA < -1.D-10, KSET IS SET EQUAL TO -1; IF
C        LAMDA > 1.D-10, KSET IS SET EQUAL TO 1; OTHERWISE KSET = 0.
C        IF MODENO = 1, THE EIGENVECTOR CORRESPONDING TO THE LEAST
C        STABLE EIGENVALUE (GAMMA*) WILL BE WRITTEN TO FILE FT02F001.
```

```
C         IF MODENO = 3, THIS OUTPUT IS INHIBITED.  MODENO MUST BE
C         SET EQUAL TO ONE TO GENERATE CORRECT DATA FOR PROGRAM EIGFCN.
C
C***********************************************************************
C
    1 WRITE (6,9)
      READ (5,11) AMDA
      KSET = 0
      IF (AMDA.LT.-1.D-10) KSET=-1
      IF (AMDA.GT.1.D-10) KSET=1
      WRITE (6,10)
      READ (5,11) AR
      WRITE (6,12)
      READ (5,11) AI
      WRITE (6,13)
      READ (5,11) REY
      IF (REY.LE.0.D0) GO TO 5
      CALL STAB (AR,AI,GRMAX,KSET,MODENO)
      WRITE (6,14) GRMAX
      GO TO 1
C
C
C***********************************************************************
C
C     COMPUTE STABILITY MAP (MODENO = 2)
C
C         COMPUTES A STABILITY MAP AT MESH POINTS ESTABLISHED BY
C         THE FOLLOWING PARAMTERS READ FROM FILE FT01F001:
C
C              NXSTP - NO OF MESH PTS IN X-DIRECTION
C              NYSTP - NO OF MESH PTS IN Y-DIRECTION
C              N     - DIMENSION OF MATRICES X & Y IN SUBROUTINE STAB
C              DELAR - MAGNITUDE OF THE X-DIRECTION STEP
C              DELAI - MAGNITUDE OF THE Y-DIRECTION STEP
C              REY   - REYNOLDS NUMBER
C
C         **NOTE - RUN TIME IS LONG IN THIS MODE,SO CP/CMS
C         RUNS SHOULD BE LIMITED TO 10X10 MESHES OR LESS WITH
C         N = 31.  LARGER RUNS SHOULD BE MADE UNDER BATCH.
C         LITTLE OS MAY ALSO BE USED AS LESS THAN 18CK OF
C         CORE IS REQUIRED IN THIS MODE FOR N <= 47.
C
C         **NOTE - TO RUN THE MAPPING PORTION UNDER OS OR LITTLE OS,
C         PERFORM THE FOLLOWING:
C
C              1)RETAIN ALL MAIN PROGRAM SECTIONS BRACKETED BY '----'.
C              2)CHANGE ALL READ DEVICES TO '5' VICE '1' IN MAIN PROGRAM.
C              3)CHANGE THE DEVICE CODE OF THE LAST WRITE STATEMENT
```

```
C              PRIOR TO THE 'STOP' IN THE MAIN PROGRAM TO '7' VICE '3'.
C              CHANGE ALL OTHER MAIN PROGRAM WRITE STATEMENTS TO DEVICE
C              CODE '6'.
C           4)IMMEDIATELY AFTER STATEMENT 116 IN MAIN PROGRAM, INSERT
C              THE FOLLOWING: 'MODENO = 2'
C           5)DELETE PORTION BETWEEN '---' MARKINGS IN
C              SUBROUTINE STAB.
C
C        APPROXIMATE RUNNING TIMES ARE 490 MINUTES(CPU) FOR N=47 AND A
C        21 X 21 MESH AND 430 MINUTES(CPU) FOR N=31 AND A 41 X 41 MESH.
C
C        NXSTP MUST EQUAL NYSTP IF PLOT DESIRED BY PROGRAM STBCONT.
C
C*****************************************************************
C
C-----------------------------------------------------------------
    2 READ (1,15) NXSTP,NYSTP,N,DELAR,DELAI,REY
      READ (1,16) ARSTRT,AISTRT
      READ (1,16) AMDA
      KSET = 0
      IF (AMDA.LT.-1.D-10) KSET=-1
      IF (AMDA.GT.1.D-10) KSET=1
      AR = ARSTRT
      WRITE (6,17) REY
      WRITE (3,18) NXSTP
C
      DO 4 I=1,NXSTP
      AI = AISTRT
C
      DO 3 J=1,NYSTP
      CALL STAB (AR,AI,GRMAX,KSET,MODENO)
      WRITE (6,19) AR,AI,GRMAX
      WRITE (3,20) AR,AI,GRMAX
    3 AI = AI+DELAI
C
    4 AR = AR+DELAR
C
C
C-----------------------------------------------------------------
    5 CALL IXCLOK (CLOCK)
      WRITE (6,21) CLOCK(3),CLOCK(4)
C-----------------------------------------------------------------
      STOP
C
    6 FORMAT ('0',5X,'INPUT MODENO')
    7 FORMAT (I1)
    8 FORMAT ('0',5X,'START TIME =',2A4)
    9 FORMAT ('0',5X,'INPUT LAMBDA')
```

```
   10 FORMAT (' ',5X,'INPUT ALPHA REAL')
   11 FORMAT (D20.10)
   12 FORMAT (' ',5X,'INPUT ALPHA IMAG')
   13 FORMAT (' ',5X,'INPUT REYNOLDS NO')
   14 FORMAT (' ',5X,'STAB=',D20.10)
C-----------------------------------------------------------------------
   15 FORMAT (3I2,3D20.10)
   16 FORMAT (2D20.10)
   17 FORMAT (' ','REYNOLDS NUMBER = ',F8.1,/,10X,'AR',18X,
     1    'AI',16X,'STAB',//)
   18 FORMAT (I2)
   19 FORMAT (' ',3D20.10)
   20 FORMAT (3E20.10)
C-----------------------------------------------------------------------
   21 FORMAT ('0','STOP TIME=',2A4)
C-----------------------------------------------------------------------
      END
C-----------------------------------------------------------------------
C
C
C.................SUBROUTINE STAB(AR,AI,GRMAX,KSET,MODENO)..............
C
C       PURPOSE
C
C          RETURNS THE REAL PORTION OF THE LEAST STABLE EIGENVALUE
C          FOR THE GIVEN INPUT CONDITIONS. THIS VALUE DETERMINES THE
C          STABILITY OF THE FLOW.
C
C       USAGE
C
C          CALL STAB(AR,AI,GRMAX,KSET,MODENO)
C
C       DESCRIPTION OF PARAMETERS
C
C          AR - THE REAL PART OF ALPHA
C
C          AI - THE IMAGINARY PART OF ALPHA.
C
C          GRMAX - THE REAL PCRTION OF THE LEAST STABLE EIGENVALUE.
C                  THIS VALUE IS RETURNED TO THE CALLING PROGRAM.
C
C          N - THE NUMBER OF INTERIOR MESH POINTS
C
C          KSET - AN INTEGER DENOTING THE TYPE OF MESH OFFSET
C                 USED.
C                    KSET = -1 FOR WALL OFFSET
C                    KSET =  0 FOR UNIFORM MESH
C                    KSET =  1 FOR AXIS OFFSET
```

```
C
C        MODENO - AN INTEGER CONTROLLING THE OUTPUT OF
C                 OF EIGENVECTORS TO FILE FT02F001.  IF MODENO
C                 IS EQUAL TO +1, EIGENVECTORS ARE OUTPUT;
C                 OTHERWISE OUTPUT IS INHIBITED.
C
C     OTHER ROUTINES NEEDED
C
C        MSET2,CDMTIN,MULM,DSPLIT,EHESSC,ELRH2C,EBALAC, EBBCKC
C
C.......................................................................
C
      SUBROUTINE STAB (AR,AI,GRMAX,KSET,MODENO)
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX *16A,G
      COMPLEX *16CQM1E1,CQM2E1
C
C        NOTE--CHANGE DIMENSIONS FROM HERE THROUGH 'N = ' FOR
C        NEW NMESH. CP/CMS MAX NMESH IS 79. LOGIN WITH 520K OF CORE.
C
      REAL *8GR(79),GI(79),ZR(79,79),ZI(79,79),RADIUS(79),CVEC(79)
      COMPLEX *16XMAT(79,79),YMAT(79,79),WV(79)
      DIMENSION IVEC(79)
      COMMON /COEFNT/ A,G,REY,DELR,AMDA
      EXTERNAL CQM1E1,CQM2E1
C
C
      A = DCMPLX(AR,AI)
      MDIM = 79
      N = 79
C
C
C        SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C        THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION
C        WHICH DO NOT CONTAIN GAMMA AS A FACTOR.
C
C
      CALL MSET2 (XMAT,N,MDIM,CQM1E1,KSET)
C
C
C        SET UP THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C        THE MESH FOR THOSE TERMS IN THE VORTICITY TRANSPORT EQUATION
C        WHICH CONTAIN GAMMA AS A FACTOR.
C
C
      CALL MSET2 (YMAT,N,MDIM,CQM2E1,KSET)
C
C
```

```
C           INVERT THE RESULTING ARRAY
C
C
      CALL CDMTIN (MDIM,YMAT,MDIM,IERR)
C
C
C           PREMULTIPLY XMAT BY THE INVERSE OF YMAT TO CONVERT THE PROBLEM
C           TO THE STANDARD FORM:
C                              (A - GAMMA*I)*X = 0
C
C
      CALL MULM (YMAT,XMAT,MDIM,MDIM,WV)
C
C
C           SPLIT THE RESULTING ARRAY INTO REAL (XMAT) AND IMAGINARY (YMAT)
C           PARTS.
C
C
      CALL DSPLIT (MDIM,MDIM,YMAT,XMAT,YMAT)
C
C
C           CALCULATE THE EIGENVALUES (GAMMA) FOR THE EQUATION.
C
C
      CALL EBALAC (XMAT,YMAT,MDIM,MDIM,KBND,LBND,DVEC)
      CALL EHESSC (XMAT,YMAT,KBND,LBND,MDIM,MDIM,IVEC)
      CALL ELRH2C (XMAT,YMAT,KBND,LBND,MDIM,MDIM,GR,GI,ZR,ZI,IVEC,INERR,
     1IER)
      CALL EBBCKC (ZR,ZI,MDIM,MDIM,KBND,LBND,MDIM,DVEC)
      IF (INERR.NE.0) WRITE (6,4) INERR,IER
C
C     DETERMINE LARGEST GAMMA REAL
C
      NEIG = 1
      GRMAX = -1.0D02
C
      DO 1 I=1,MDIM
      IF (GR(I).GT.GRMAX) NEIG=I
      IF (GR(I).GT.GRMAX) GRMAX=GR(I)
    1 CONTINUE
C
      GRMAX = GRMAX+AR
C-----------------------------------------------------------------------
      DEL10 = 1.D0/DFLOAT(N+1)
      WRITE (2,5) MDIM,REY,AR,AI
      WRITE (2,7) AMDA,GR(NEIG),GI(NEIG)
      WRITE (2,6) KSET
      IF (MODENO.NE.1) GO TO 3
```

```
C
      DO 2 I=1,MDIM
      RADIUS(I) = DFLOAT(I)*DEL10
      WRITE (2,7) RADIUS(I),ZR(I,NEIG),ZI(I,NEIG)
C-----------------------------------------------------------------------
    2 CONTINUE
C
    3 RETURN
C
    4 FCRMAT ('0* * * ERROR NUMBER ',I7,' ON EIGENVALUE',
     1        I7,' * * *',///)
    5 FORMAT (I2,3D20.10)
    6 FORMAT (I2)
    7 FORMAT (F15.7,2(1PD20.10))
      END
C
C
C.....................SUBROUTINE MSET1(ETA,CQM1,CQM2,KSET)..............
C
C     PLRPOSE
C
C        MSET1 GENERATES THE COEFFICIENTS FOR THE FINITE DIFFERENCE
C
C        APPROXIMATION OF THE COMPONENT Q.
C     USAGE
C
C        CALL MSET1(ETA,CQM1,CQM2,KSET)
C
C     DESCRIPTION OF PARAMETERS
C
C        ETA  -  INDEPENDENT VARIABLE REPLACING R
C                IN NONUNIFORM MESH.
C
C        CQM1 -  COEFFICIENTS OF Q AND ITS DERIVATIVES
C                IN THE FINITE DIFFERENCE APPROXIMATION OF
C                THE NON-GAMMA TERMS. CQM1(1) IS THE
C                COEFFICIENT FOR Q, CQM1(2) IS THE COEF-
C                FICIENT FOR DQ, AND SO ON TO CQM1(5).
C
C        CQM2 -  SAME AS CQM1 EXCEPT GAMMA TERMS AND
C                DIMENSIONED 3 INSTEAD OF 5.
C
C        KSET -  MESH OFFSET PARAMETER AS DESCRIBED FOR
C                SUBROUTINE STAB.
C
C     OTHER ROUTINES NEEDED
C
C        NONE
```

```
C
C.............................................................................
C
      SUBROUTINE MSET1 (ETA,CQM1,CQM2,KSET)
      IMPLICIT REAL*8(A-H,O-Z)
      COMPLEX *16A,G,CQM1(5),CQM2(3),M4,M3,M2,M1,MO,N2,N1,NO
      COMMON /COEFNT/ A,G,REY,DEL,AMDA
C
      AMDA = DABS(AMDA)
      IF (KSET.EQ.0) GO TO 1
      TETA = ETA
      IF (KSET.EQ.1) TETA=1DO-ETA
      ETAP = AMDA*TETA
      CNST = 1DO/DTANH(AMDA)
      R = 1DO-CNST*DTANH(ETAP)
      IF (KSET.EQ.-1) R = CNST*DTANH(ETAP)
C
      S1 = DSINH(ETAP)
      C1 = DCOSH(ETAP)
      T1 = DTANH(ETAP)
C
      D4N = (8DO*CNST*AMDA**4)*(T1**3/C1**2)
      IF (KSET.EQ.-1) D4N=-D4N
      D3N = (-2DO*CNST*AMDA**3)*(1DO-2DO*S1**2)/C1**4
      D2N = (2DO*CNST*AMDA**2)*T1/C1**2
      IF (KSET.EQ.-1) D2N=-D2N
      D1N = (CNST*AMDA)*(1DO/C1**2)
C
      F11 = 1DO/D1N
      F22 = 1DO/D1N**2
      F21 = (-1DO/D1N**3)*D2N
      F33 = 1DO/D1N**3
      F32 = (-3DO/D1N**4)*D2N
      F31 = (3DO/D1N**5)*D2N**2-(1DO/D1N**4)*D3N
      F44 = 1DO/D1N**4
      F43 = (-6DO/D1N**5)*D2N
      F42 = (15DO/D1N**6)*D2N**2-(4DO/D1N**5)*D3N
      F41 = (-15DO/D1N**7)*D2N**3+(10DO/D1N**6)*D2N*D3N-(1DO/D1N**5)*D4N
C
    1 IF (KSET.EQ.0) R = ETA
C
C        DEFINE THE RECURRING PARAMETER U(R).
C
C
      U = 2DO*(1DO-R**2)
C
C
C        COEFFICIENTS OF THE COMPONENT C.
```

```
C                                                                         MST1 800
C                                                                         MST1 810
      M4 = -R /REY                                                        MST1 820
      M3 = -6 D0/R EY                                                     MST1 830
      M2 = R*A*U-1D0/REY*(3D0/R+2D0*A**2*R)                               MST1 840
      M1 = 3D0*A*U+1D0/REY*(3D0/R**2-6D0*A**2)                            MST1 850
      M0 = R*A**3*U-1D0/REY*(A**4*R)                                      MST1 860
C                                                                         MST1 870
      N2 = -R                                                             MST1 880
      N1 = -3D0                                                           MST1 890
      N0 = -A**2*R                                                        MST1 900
C                                                                         MST1 910
      IF (KSET.EQ.0) GO TO 2                                              MST1 920
C                                                                         MST1 930
      CQM1(5) = M4*F44                                                    MST1 940
      CQM1(4) = M4*F43+M3*F33                                             MST1 950
      CQM1(3) = M4*F42+M3*F32+M2*F22                                      MST1 960
      CQM1(2) = M4*F41+M3*F31+M2*F21+M1*F11                               MST1 970
      CQM1(1) = M0                                                        MST1 980
C                                                                         MST1 990
      CQM2(3) = N2*F22                                                    MST11000
      CQM2(2) = N2*F21+N1*F11                                             MST11010
      CQM2(1) = N0                                                        MST11020
      RETURN                                                              MST11030
C                                                                         MST11040
    2 CQM1(5) = M4                                                        MST11050
      CQM1(4) = M3                                                        MST11060
      CQM1(3) = M2                                                        MST11070
      CQM1(2) = M1                                                        MST11080
      CQM1(1) = M0                                                        MST11090
C                                                                         MST11100
      CQM2(3) = N2                                                        MST11110
      CQM2(2) = N1                                                        MST11120
      CQM2(1) = N0                                                        MST11130
C                                                                         MST11140
      RETURN                                                              MST11150
      END                                                                 MST11160
C
C
C.....................SUBROUTINE MSET2(X,N,MDIM,CFMAT,KSET).............MST2   10
C                                                                         MST2   20
C     PURPOSE                                                             MST2   30
C                                                                         MST2   40
C        MSET2 GENERATES THE ARRAYS REPRESENTING THE CENTRAL DIFFERENCE   MST2   50
C        APPROXIMATION OF THE VORTICITY TRANSPORT EQUATION IN TERMS OF    MST2   60
C        THE VELOCITY VECTOR POTENTIAL.                                   MST2   70
C                                                                         MST2   80
C     USAGE                                                               MST2   90
```

```
C
C        CALL MSET2(X,N,MDIM,CFMAT,KSET)
C
C     DESCRIPTION OF PARAMETERS
C
C        X - THE NAME OF THE ARRAY BEING GENERATED. MUST BE DIMENSIONED
C            IN THE CALLING PROGRAM
C
C        N- THE ROW DIMENSION OF THE MATRIX X.  MUST BE .GE. N.
C
C        MDIM - THE COLUMN DIMENSION OF THE MATRIX X. MUST BE .GE. N.
C
C        CFMAT - THE NAME OF A FUNCTION SUBPROGRAM WITH 4 PARAMETERS,
C                JSTA, K,CQM1 & CQM2.  CFMAT MUST BE DECLARED
C                EXTERNAL IN THE CALLING PROGRAM.
C
C     THE FOLLOWING IS OUTPUT BY MSET2
C
C        X - THE N BY N MATRIX INTO WHICH THE COEFFICIENTS OF THE CENTRAL
C            DIFFERENCING ARE PUT.
C
C     OTHER ROUTINES NEEDED
C
C        FUNCTION SUBPROGRAM NAME PASSED IN THE CALLING PARAMETER 'CFMAT'
C        AND MSET1.
C.......................................................................
C
      SUBROUTINE MSET2 (X,N,MDIM,CFMAT,KSET)
      REAL *8REY,R,DEL,DFLOAT,AMDA,ETA
      COMPLEX *16X(MDIM,MDIM),CQM1(5),CQM2(3)
      COMPLEX *16A,G
      COMPLEX *16CFMAT
      COMMON /COEFNT/  A,G,REY,DEL,AMDA
C
C
C        DEFINE THE SPACING OF THE INTERIOR MESH POINTS.
C
C
      DEL = 1D0/DFLOAT(N+1)
C
C
C        INITIALIZE ALL ELEMENTS IN THE ARRAY TO ZERO.
C
C
C
      DO 1 I=1,N
C
      DO 1 J=1,N
```

```
    1 X(I,J) = (0D0,0D0)
C
C
C
C        ESTABLISH THE CENTRAL DIFFERENCE APPROXIMATION AT EACH POINT IN
C        THE MESH.
C
C
      ETA = DEL
      CALL MSET1 (ETA,CQM1,CQM2,KSET)
      X(1,1) = CFMAT(1,1,CQM1,CQM2)
      X(1,2) = CFMAT(1,2,CQM1,CQM2)
      X(1,3) = CFMAT(1,3,CQM1,CQM2)
      X(1,4) = CFMAT(1,4,CQM1,CQM2)
      X(1,5) = CFMAT(1,5,CQM1,CQM2)
C
      ETA = 2D0*DEL
      CALL MSET1 (ETA,CQM1,CQM2,KSET)
      X(2,1) = CFMAT(2,1,CQM1,CQM2)
      X(2,2) = CFMAT(2,2,CQM1,CQM2)
      X(2,3) = CFMAT(2,3,CQM1,CQM2)
      X(2,4) = CFMAT(2,4,CQM1,CQM2)
      X(2,5) = CFMAT(2,5,CQM1,CQM2)
C
      IL = N-2
C
      DO 2 I=3,IL
      K = I-3
      ETA = DEL*DFLOAT(I)
      CALL MSET1 (ETA,CQM1,CQM2,KSET)
C
      DO 2 J=1,5
    2 X(I,K+J) = CFMAT(3,J,CQM1,CQM2)
C
C
      ETA = 1D0-2D0*DEL
      CALL MSET1 (ETA,CQM1,CQM2,KSET)
      X(N-1,N-3) = CFMAT(4,1,CQM1,CQM2)
      X(N-1,N-2) = CFMAT(4,2,CQM1,CQM2)
      X(N-1,N-1) = CFMAT(4,3,CQM1,CQM2)
      X(N-1,N) = CFMAT(4,4,CQM1,CQM2)
C
      ETA = 1D0-DEL
      CALL MSET1 (ETA,CQM1,CQM2,KSET)
      X(N,N-3) = CFMAT(5,1,CQM1,CQM2)
      X(N,N-2) = CFMAT(5,2,CQM1,CQM2)
      X(N,N-1) = CFMAT(5,3,CQM1,CQM2)
      X(N,N) = CFMAT(5,4,CQM1,CQM2)
```

```
      RETURN                                                            MST21060
      END                                                               MST21070
C
C
C.......................FUNCTION COM1E1(JSTA,K,CQM1,CQM2)..............CQM1   10
C                       (POLAR COORDINATES)                             CQM1   20
C                                                                       CQM1   30
C     PURPOSE                                                           CQM1   40
C                                                                       CQM1   50
C        RETURNS THE VALUES FOR THE COEFFICIENTS IN THE ARRAYS          CQM1   60
C        REPRESENTING THE CENTRAL DIFFERENCE APPROXIMATICN OF THE       CQM1   70
C        VORTICITY TRANSPORT EQUATION USING THE COEFFICIENTS COMPUTED   CQM1   80
C        BY SUBROUTINE MSET1.                                           CQM1   90
C                                                                       CQM1  100
C     DESCRIPTION OF PARAMETERS                                         CQM1  110
C                                                                       CQM1  120
C        JSTA - INDICATES WHICH DIFFERENCE EQUATION SET WILL BE USED.   CQM1  130
C                                                                       CQM1  140
C           JSTA=1 - CONSISTENT 3RD ORDER TRUNCATION ERROR FINITE       CQM1  150
C                    DIFFERENCE EQUATIONS FOR R=DEL WILL BE USED.       CQM1  160
C           JSTA=2 - SAME AS ABOVE BUT R=2D0*DEL                        CQM1  170
C           JSTA=3 - CENTRAL DIFFERENCE EQUATIONS WITH CONSISTENT 2ND   CQM1  180
C                    ORDER TRUNCATION ERROR WILL BE USED.               CQM1  190
C           JSTA=4 - SAME AS JSTA=3 BUT FOR R=1D0-2D0*DEL.              CQM1  200
C           JSTA=5 - SAME AS ABOVE BUT FOR R=1D0-DEL.                   CQM1  210
C                                                                       CQM1  220
C                                                                       CQM1  230
C        K - INDICATES THE ABSOLUTE POSITION OF THE POINT IN EACH ROW   CQM1  240
C            OF THE FINITE DIFFERENCE MESH.  IF THE FIRST NON-ZERO ENTRYCQM1  250
C            IN ROW J IS ELEMENT(J,3), THEN K=1 DENOTES ELEMENT(J,3),   CQM1  260
C            K=2 DENOTES ELEMENT(J,4),ETC.                              CQM1  270
C                                                                       CQM1  280
C        CQM1,CQM2 - THE COEFFICIENT ARRAYS FOR THE FINITE DIFFERENCE   CQM1  290
C           APPROXIMATION OF THE FUNCTION Q.  CQM1 CONTAINS THE         CQM1  300
C           COEFFICIENTS FOR THE NON-GAMMA TERMS, WHILE CQM2 CONTAINS   CQM1  310
C           THE COEFFICIENTS OF THE GAMMA TERMS. BOTH ARRAYS MUST BE    CQM1  320
C           DIMENSIONED COMPLEX*16.                                     CQM1  330
C                                                                       CQM1  340
C     EXAMPLE OF THE CALLING ARGUMENT:                                  CQM1  350
C                                                                       CQM1  360
C        CQ M(1,2) E1(JSTA,K,CQM1,CQM2)                                 CQM1  370
C                                                                       CQM1  380
C           CQ - Q COMPONENT OF THE VELOCITY VECTOR POTENTIAL.          CQM1  390
C                                                                       CQM1  400
C           M(1,2) - 1 REFERS TO TERMS NOT CONTAINING GAMMA AS A        CQM1  410
C                      FACTOR.                                          CQM1  420
C                    2 REFERS TO TERMS CONTAINING GAMMA AS A FACTOR.    CQM1  430
C                                                                       CQM1  440
```

```
C                E1 - REFERS TC THE LINEAR COMBINATICN OF THE FIRST AND
C                     THIRD EQUATIONS RESULTING FROM EXPRESSION OF THE
C                     VORTICITY TRANSPORT EQUATION IN TERMS OF THE VELOCITY
C                     VECTOR POTENTIAL.
C
C       USAGE
C
C         CQM1E1 MUST BE DECLARED COMPLEX*16 IN THE CALLING PROGRAM.
C
C       OTHER ROUTINES REQUIRED
C
C         NONE
C.......................................................................
C
      FUNCTION CQM1E1 (JSTA,K,CQM1,CQM2)
      IMPLICIT COMPLEX*16(A-H,O-Z)
      CCMMON /COEFNT/ A,G,REY,DEL,AMDA
      COMPLEX *16CQM1(5),CQM2(3)
      REAL *8REY,R,DEL
C
C
      GC TO (1,7,13,19,24), JSTA
C
C        FINITE DIFFERENCE EQUATIONS AT ETA=DEL (NON GAMMA).
C
C
    1 GO TO (2,3,4,5,6), K
    2 CCM1E1 = -0.911564626D0*CQM1(5)/DEL**4+1.825165563D0*CQM1(4)/DEL**
     13-35D0*CQM1(3)/(60D0*DEL**2)-2D0*CQM1(2)/(3C0*DEL)+CQM1(1)
      GO TO 29
    3 CQM1E1 = 2.750242955D0*CQM1(5)/DEL**4-3.250331126D0*CQM1(4)/DEL**3
     1+8D0*CQM1(3)/(15D0*DEL**2)+2D0*CQM1(2)/(3D0*DEL)
      GC TO 29
    4 CCM1E1 = -3.043731779D0*CQM1(5)/CEL**4+1.660927152D0*CQM1(4)/DEL**
     13+CQM1(3)/(20D0*DEL**2)
      GO TO 29
    5 CQM1E1 = 1.42468416D0*CQM1(5)/DEL**4-.235761589D0*CQM1(4)/DEL**3
      GO TO 29
    6 CCM1E1 = -0.219630709D0*CQM1(5)/CEL**4
      GO TO 29
C
C        FINITE DIFFERENCE EQUATIONS AT ETA=2D0*DEL (NON GAMMA).
C
    7 GO TO (8,9,10,11,12), K
    8 CCM1E1 = -3.103401336D0*CQM1(5)/DEL**4+.868874172D0*CQM1(4)/DEL**3+
     111D0*CQM1(3)/(12D0*DEL**2)-4D0*CQM1(2)/(3D0*DEL)
      GO TO 29
    9 CCM1E1 = 6.903012634D0*CQM1(5)/DEL**4-.937748345D0*CQM1(4)/DEL**3-
```

```
 123D0*CQM1(3)/(15D0*DEL**2)+4D0*CQM1(2)/(3D0*DEL)+CQM1(1)
   GO TO 29
10 CCMIE1 = -5.342274053D0*CQM1(5)/DEL**4-.254304636D0*CQM1(4)/DEL**3
  1+19D0*CQM1(3)/(20D0*DEL**2)
   GO TO 29
11 CQM1E1 = 1.666083577D0*CQM1(5)/DEL**4+.323178808D0*CQM1(4)/DEL**3
   GO TO 29
12 CQM1E1 = -0.123420797D0*CQM1(5)/DEL**4
   GO TO 29
C
C       CENTRAL DIFFERENCE APPROXIMATION FOR COMPONENT Q (NON GAMMA).
C
13 GO TO (14,15,16,17,18), K
14 CCM1E1 = CQM1(5)/DEL**4-CQM1(4)/(2D0*DEL**3)
   GO TO 29
15 CQM1E1 = -4D0*CQM1(5)/DEL**4+CQM1(4)/DEL**3+CQM1(3)/DEL**2-CQM1(2)
  1/(2D0*DEL)
   GO TO 29
16 CQM1E1 = 6D0*CQM1(5)/DEL**4-2D0*CQM1(3)/DEL**2+CQM1(1)
   GO TO 29
17 CQM1E1 = -4D0*CQM1(5)/DEL**4-CQM1(4)/DEL**3+CQM1(3)/DEL**2+CQM1(2)
  1/(2D0*DEL)
   GO TO 29
18 CQM1E1 = CQM1(5)/DEL**4+CQM1(4)/(2D0*DEL**3)
   GO TO 29
C
C       FINITE DIFFERENCE EQUATIONS AT ETA=1D0-2D0*DEL (NCN-GAMMA).
C
19 GO TO (20,21,22,23), K
20 CCM1E1 = CQM1(5)/DEL**4-0.5D0*CQM1(4)/DEL**3
   GO TO 29
21 CQM1E1 = -4D0*CQM1(5)/DEL**4+CQM1(4)/DEL**3+CQM1(3)/DEL**2-CQM1(2)
  1/(2D0*DEL)
   GO TO 29
22 CQM1E1 = 6D0*CQM1(5)/DEL**4-2D0*CQM1(3)/DEL**2+CQM1(1)
   GO TO 29
23 CQM1E1 = -4D0*CQM1(5)/DEL**4-CQM1(4)/DEL**3+CQM1(3)/DEL**2+CQM1(2)
  1/(2D0*DEL)
   GC TO 29
C
C       FINITE DIFFERENCE EQUATIONS AT ETA=1D0-DEL (NON GAMMA).
C
24 GO TO (25,26,27,28), K
25 CQM1E1 = -0.25D0*CQM1(5)/DEL**4
   GO TO 29
26 CQM1E1 = 8D0*CQM1(5)/(3D0*DEL**4)-CQM1(4)/(3D0*DEL**3)
   GO TO 29
27 CQM1E1 = -9D0*CQM1(5)/DEL**4+CQM1(3)/DEL**2-CQM1(2)/(2D0*DEL)
```

```
      GO TO 29                                                          CQM11410
   28 CQM1E1 = 16D0*CQM1(5)/DEL**4+3D0*CQM1(4)/DEL**3-2D0*CQM1(3)/DEL**2CQM11420
     1+CQM1(1)                                                          CQM11430
   29 RETURN                                                            CQM11440
C                                                                       CQM11450
      ENTRY CQM2E1(JSTA,K,CQM1,CQM2)                                    CQM11460
C                                                                       CQM11470
      GO TO (30,35,40,45,50), JSTA                                      CQM11480
C                                                                       CQM11490
C        FINITE DIFFERENCE EQUATIONS AT ETA=DEL (GAMMA).                CQM11500
C                                                                       CQM11510
   30 GO TO (31,32,33,34,34), K                                         CQM11520
   31 CQM2E1 = -35D0*CQM2(3)/(60D0*DEL**2)-2D0*CQM2(2)/(3D0*DEL)+CQM2(1)CQM11530
      GO TO 54                                                          CQM11540
   32 CQM2E1 = 8D0*CQM2(3)/(15D0*DEL**2)+2D0*CQM2(2)/(3D0*DEL)          CQM11550
      GO TO 54                                                          CQM11560
   33 CQM2E1 = CQM2(3)/(20D0*DEL**2)                                    CQM11570
      GO TO 54                                                          CQM11580
   34 CQM2E1 = (0D0,0D0)                                                CQM11590
      GO TO 54                                                          CQM11600
C                                                                       CQM11610
C        FINITE DIFFERENCE EQUATIONS AT ETA=2D0*DEL (GAMMA)             CQM11620
C                                                                       CQM11630
   35 GO TO (36,37,38,39,39), K                                         CQM11640
   36 CQM2E1 = 11D0*CQM2(3)/(12D0*DEL**2)-4D0*CQM2(2)/(3D0*DEL)         CQM11650
      GO TO 54                                                          CQM11660
   37 CQM2E1 = -28D0*CQM2(3)/(15D0*DEL**2)+4D0*CQM2(2)/(3D0*DEL)+CQM2(1)CQM11670
      GO TO 54                                                          CQM11680
   38 CQM2E1 = 19D0*CQM2(3)/(20D0*DEL**2)                               CQM11690
      GO TO 54                                                          CQM11700
   39 CQM2E1 = (0D0,0D0)                                                CQM11710
      GO TO 54                                                          CQM11720
C        CENTRAL DIFFERENCE EQUATIONS FOR THE COMPONENT Q (GAMMA).      CQM11730
C                                                                       CQM11740
   40 GO TO (41,42,43,44,41), K                                         CQM11750
   41 CQM2E1 = (0D0,0D0)                                                CQM11760
      GO TO 54                                                          CQM11770
   42 CQM2E1 = CQM2(3)/DEL**2-CQM2(2)/(2D0*DEL)                         CQM11780
      GO TO 54                                                          CQM11790
   43 CQM2E1 = -2D0*CQM2(3)/DEL**2+CQM2(1)                              CQM11800
      GO TO 54                                                          CQM11810
   44 CQM2E1 = CQM2(3)/DEL**2+CQM2(2)/(2D0*DEL)                         CQM11820
      GO TO 54                                                          CQM11830
C                                                                       CQM11840
C        FINITE DIFFERENCE EQUATIONS AT ETA=1D0-2D0*DEL (GAMMA).        CQM11850
C                                                                       CQM11860
   45 GO TO (49,46,47,48), K                                            CQM11870
   46 CQM2E1 = CQM2(3)/DEL**2-CQM2(2)/(2D0*DEL)                         CQM11880
```

```
      GC TO 54                                                          CQM11890
   47 CQM2E1 = -2DO*CQM2(3)/DEL**2+CQM2(1)                              CQM11900
      GC TO 54                                                          CQM11910
   48 CQM2E1 = CQM2(3)/DEL**2+CQM2(2)/(2DO*DEL)                         CQM11920
      GO TO 54                                                          CQM11930
   49 CQM2E1 = (0D0,0D0)                                                CQM11940
      GO TO 54                                                          CQM11950
C                                                                       CQM11960
C        FINITE DIFFERENCE EQUATIONS AT ETA=1DO-DEL (GAMMA).            CQM11970
C                                                                       CQM11980
   50 GO TO (53,53,51,52), K                                            CQM11990
   51 CQM2E1 = CQM2(3)/DEL**2-CQM2(2)/(2DO*DEL)                         CQM12000
      GO TO 54                                                          CQM12010
   52 CQM2E1 = -2DO*CQM2(3)/DEL**2+CQM2(1)                              CQM12020
      GO TO 54                                                          CQM12030
   53 CQM2E1 = (0D0,0D0)                                                CQM12040
   54 RETURN                                                            CQM12050
      END                                                               CQM12060
C
C
C..................SUBROUTINE CDMTIN(N,A,NDIM,IERR)...................CDMT  10
C                                                                       CDMT  20
C     PURPOSE                                                           CDMT  30
C                                                                       CDMT  40
C        INVERT A COMPLEX*16 MATRIX                                     CDMT  50
C                                                                       CDMT  60
C     USAGE                                                             CDMT  70
C                                                                       CDMT  80
C        CALL CDMTIN(N,A,NCIM,DETERM)                                   CDMT  90
C                                                                       CDMT 100
C     DESCRIPTION OF PARAMETERS                                         CDMT 110
C                                                                       CDMT 120
C        N      - ORDER OF COMPLEX*16 MATRIX TO BE INVERTED             CDMT 130
C                 (INTEGER) MAXIMUM 'N' IS 100                          CDMT 140
C                                                                       CDMT 150
C        A      - COMPLEX*16 INPUT MATRIX (DESTROYED).  THE             CDMT 160
C                 INVERSE OF 'A' IS RETURNED IN ITS PLACE               CDMT 170
C                                                                       CDMT 180
C        NDIM   - THE SIZE TO WHICH 'A' IS DIMENSIONED                  CDMT 190
C                 (ROW DIMENSION OF 'A' ACTUALLY APPEARING              CDMT 200
C                 IN THE DIMENSION STATMENT OF USER'S                   CDMT 210
C                 CALLING PROGRAM)                                      CDMT 220
C                                                                       CDMT 230
C        IERR - ERROR PARAMETER RETURNED BY CDMTIN. IERR = 0 INDICATES  CDMT 240
C               NORMAL INVERSION. IERR = 9999 INDICATES SINGULAR MATRIX.CDMT 250
C                                                                       CDMT 260
C     REMARKS                                                           CDMT 270
C                                                                       CDMT 280
```

```
C           MATRIX 'A' MUST BE A COMPLEX*16 GENERAL MATRIX
C           IF MATRIX 'A' IS SINGULAR THAT MESSAGE IS PRINTED
C           'N' MUST BE .LE. NDIM
C
C        SLBROUTINES AND FUNCTIONS REQUIREC
C
C           ONLY BUILT-IN FORTRAN FUNCTIONS
C
C        METHOD
C
C           GAUSSIAN ELIMINATION WITH COLUMN PIVOTING IS USED.
C
C.......................................................................
C
      SUBROUTINE CDMTIN (N,A,NDIM,IERR)
      IMPLICIT REAL*8(A-H,O-Z)
      INTEGER *4IPIVOT(100),INDEX(100,2),IERR
      REAL *8TEMP,ALPHA(100)
      COMPLEX *16A(NDIM,NDIM),PIVOT(100),AMAX,T,SWAP,U
C
C       INITIALIZATION
C
C
      DO 2 J=1,N
      IERR = 0
      ALPHA(J) = 0D0
C
      DO 1 I=1,N
    1 ALPHA(J) = ALPHA(J)+A(J,I)*DCONJG(A(J,I))
C
      ALPHA(J) = DSQRT(ALPHA(J))
    2 IPIVOT(J) = 0
C
C
      DO 16 I=1,N
C
C       SEARCH FOR PIVOT ELEMENT
C
      AMAX = (0D0,0D0)
C
      DO 7 J=1,N
      IF (IPIVOT(J)-1) 3,7,3
C
    3 DO 6 K=1,N
      IF (IPIVOT(K)-1) 4,6,21
    4 TEMP = AMAX*DCONJG(AMAX)-A(J,K)*DCONJG(A(J,K))
      IF (TEMP) 5,5,6
    5 IROW = J
```

```
      ICOLUM = K
      AMAX = A(J,K)
    6 CONTINUE
C
    7 CCNTINUE
C
      IPIVOT(ICOLUM) = IPIVOT(ICOLUM)+1
C
C      INTERCHANGE ROWS TO FUT PIVOT ELEMENT ON DIAGONAL
C
      IF (IROW-ICCLUM) 8,10,8
    8 CONTINUE
C
      DC 9 L=1,N
      SWAP = A(IROW,L)
      A(IROW,L) = A(ICOLUM,L)
    9 A(ICOLUM,L) = SWAP
C
      SWAP = ALPHA(IROW)
      ALPHA(IROW) = ALPHA(ICOLUM)
      ALPHA(ICOLUM) = SWAP
   10 INDEX(I,1) = IROW
      INDEX(I,2) = ICOLUM
      PIVOT(I) = A(ICOLUM,ICOLUM)
      U = PIVOT(I)
      TEMP = PIVOT(I)*DCONJG(PIVOT(I))
      IF (TEMP) 11,20,11
C
C      DIVIDE PIVOT ROW BY PIVOT ELEMENT
C
   11 A(ICOLUM,ICOLUM) = (1C0,0D0)
C
      DC 12 L=1,N
      U = PIVOT(I)
   12 A(ICOLUM,L) = A(ICOLUM,L)/U
C
C
C      REDUCE NON-PIVOT ROWS
C
C
      CO 15 L1=1,N
      IF (L1-ICOLUM) 13,15,13
   13 T = A(L1,ICOLUM)
      A(L1,ICOLUM) = (0C0,0D0)
C
      CO 14 L=1,N
      U = A(ICOLUM,L)
   14 A(L1,L) = A(L1,L)-U*T
```

```
C
   15 CONTINUE
C
   16 CONTINUE
C
C
C        INTERCHANGE COLUMNS
C
C
      DO 19 I=1,N
      L = N+1-I
      IF (INDEX(L,1)-INDEX(L,2)) 17,19,17
   17 JROW = INDEX(L,1)
      JCOLUM = INDEX(L,2)
C
      DO 18 K=1,N
      SWAP = A(K,JROW)
      A(K,JROW) = A(K,JCOLUM)
      A(K,JCOLUM) = SWAP
   18 CONTINUE
C
   19 CONTINUE
C
      RETURN
   20 WRITE (6,22)
      IERR = 9999
   21 RETURN
C
   22 FORMAT (20H  MATRIX IS SINGULAR)
      END
C
C
C................SUBROUTINE MULM(X1,X2,N,MDIM,TEMPV)...................
C
C        PURPOSE
C
C           PERFORMS THE MATRIX MULTIPLICATION OF A SQUARE MATRIX BY A
C           SQUARE MATRIX.  THE RESULT IS RETURNED IN MATRIX X1.
C
C        USAGE
C
C           CALL MULM(X1,X2,N,MDIM,TEMPV)
C
C        DESCRIPTION OF PARAMETERS
C
C           X1 - THE MULTIPLYING MATRIX ON INPUT AND THE RESULTANT PRODUCT
C                ON OUTPUT.
C
```

```
C         X2 - THE MULTIPLIED MATRIX.
C
C         N - THE ORDER OF X1 AND X2.
C
C         MDIM - THE DIMENSION OF X1 AND X2 FROM THE CALLING PRCGRAM.
C
C         TEMPV - A WORKING VECTOR. MUST BE DIMENSICNED MDIM.
C
C       OTHER ROUTINES REQUIRED
C
C         NONE
C
C.......................................................................
C
      SUBROUTINE MULM (X1,X2,N,MDIM,TEMPV)
      COMPLEX *16X1(MDIM,MDIM),X2(MDIM,MCIM),TEMPV(MCIM),TEMP
C
C
C          STORE ROW I OF X1 IN TEMPV.
C
C
C
      DO 4 I=1,N
C
      DC 1 J=1,N
    1 TEMPV(J) = X1(I,J)
C
C
C
C          MULTIPLY COLUMN J OF X2 BY ROW I OF X1 AND STORE IN X1(I,J).
C
C
C
      DO 3 J=1,N
      TEMP = (0D0,0D0)
C
C
      DO 2 K=1,N
    2 TEMP = TEMP+TEMPV(K)*X2(K,J)
C
C
    3 X1(I,J) = TEMP
C
    4 CONTINUE
C
      RETURN
      END
C
C
```

```
C.................SUBROUTINE DSPLIT(N,MDIM,A,AR,AI).......................
C
C
C     PURPOSE
C
C       DSPLIT TAKES A MATRIX OF COMPLEX*16 NUMBERS AND
C       SPLITS IT INTO TWO MATRICES, ONE CONTAINING THE REAL
C       PART OF THE ORIGINAL MATRIX, AND ONE CONTAINING THE
C       IMAGINARY PART.
C
C     USAGE
C
C       CALL DSPLIT(N,MDIM,A,AREAL,AIMAG)
C
C     DESCRIPTION OF PARAMETERS
C
C       N - THE SIZE OF THE MATRIX A, AN N BY N SQUARE
C         MATRIX.
C
C       MDIM - THE COLUMN DIMENSION OF MATRIX A
C
C       A - THE INPUT MATRIX.  MUST BE DIMENSIONEC MDIM BY
C         AT LEAST N IN THE CALLING PROGRAM  (COMPLEX*16)
C
C       AREAL,AIMAG - THE OUTPUT MATRICES CONTAINING THE
C         REAL AND IMAGINARY PARTS, RESPECTIVELY, OF
C         MATRIX A.  MUST BE DIMENSIONED (MDIM,MDIM) IN THE
C         CALLING PROGRAM.
C
C     NCTES...
C
C       MATRIX A AND MATRIX AREAL MAY OVERLAP IF THEY ARE
C         DIMENSICNED IN THE CALLING PROGRAM AS FCLLOWS...
C
C               COMPLEX*16 A(MDIM,MDIM)
C               REAL*8 AREAL(MDIM,MDIM),AIMAG(MDIM,MDIM)
C               EQUIVALENCE(A(1,1),AREAL(1,1))
C
C     OTHER ROUTINES NEEDED
C
C       NONE
C
C
C.........................................................................
C
      SUBROUTINE DSPLIT (N,MDIM,A,AR,AI)
      REAL *8A(2,MDIM,MDIM),AR(MDIM,MDIM),AI(MDIM,MDIM)
C
```

```
C                                                                       DSPL 490
      DO 1 J=1,N                                                        DSPL 500
C                                                                       DSPL 510
      DO 1 I=1,N                                                        DSPL 520
      AR(I,J) = A(1,I,J)                                                DSPL 530
    1 AI(I,J) = A(2,I,J)                                                DSPL 540
C                                                                       DSPL 550
C                                                                       DSPL 560
      RETURN                                                            DSPL 570
      END                                                               DSPL 580
C
C
C     SUBROUTINE EBALAC (AR,AI,N,IA,K,L,D)                              EBAC0010
C                                                                       EBAC0020
C-EBALAC--------D-------LIBRARY 1---------------------------------------EBAC0030
C                                                                       EBAC0040
C   FUNCTION            - BALANCES A COMPLEX GENERAL MATRIX AND ISOLATESEBAC0050
C                           EIGENVALUES WHENEVER POSSIBLE.              EBAC0060
C   USAGE               - CALL EBALAC (AR,AI,N,IA,K,L,D)                EBAC0070
C   PARAMETERS   AR     - INPUT/OUTPUT MATRICES OF DIMENSION N BY N.    EBAC0080
C                AI         ON INPUT, AR AND AND AI CONTAIN THE REAL    EBAC0090
C                           AND IMAGINARY PARTS, RESPECTIVELY, OF       EBAC0100
C                           THE COMPLEX MATRIX OF ORDER N TO BE         EBAC0110
C                           BALANCED. ON OUTPUT, AR AND AI CONTAIN THE  EBAC0120
C                           REAL AND IMAGINARY PARTS OF THE             EBAC0130
C                           TRANSFORMED MATRIX.                         EBAC0140
C                N      - INPUT VARIABLE CONTAINING THE ORDER           EBAC0150
C                           OF THE MATRIX A = (AR,AI) TO BE BALANCED.   EBAC0160
C                IA     - INPUT VARIABLE CONTAINING THE ROW DIMENSION OFEBAC0170
C                           AR AND AI IN THE CALLING PROGRAM.           EBAC0180
C                K      - OUTPUT INTEGERS CONTAINING THE BOUNDARY       EBAC0190
C                L          INDICES FOR THE BALANCED MATRIX A = (AR,AI) EBAC0200
C                           SUCH THAT                                   EBAC0210
C                             AR(I,J) = 0. AND AI(I,J) = 0. IF          EBAC0220
C                             (1) I IS GREATER THAN J AND               EBAC0230
C                             (2) J = 1,...,K-1 OR                      EBAC0240
C                                 I = L+1,...,N                         EBAC0250
C                D      - OUTPUT VECTOR OF LENGTH N CONTAINING          EBAC0260
C                           INFORMATICN DETERMINING THE PERMUTATIONS    EBAC0270
C                           USED AND THE SCALING FACTORS.               EBAC0280
C   PRECISION           - SINGLE/DOUBLE                                 EBAC0290
C   LANGUAGE            - FORTRAN                                       EBAC0300
C-----------------------------------------------------------------------EBAC0310
C   LATEST REVISION     - MARCH 9, 1977                                 EBAC0320
C                                                                       EBAC0330
      SUBROUTINE EBALAC (AR,AI,N,IA,K,L,D)                              EBAC0340
C                                                                       EBAC0350
      DIMENSION          AR(IA,1),AI(IA,1),D(N)                         EBAC0360
```

```
      LOGICAL              NOCONV                                              EBAC0370
C                                    RADIX IS A MACHINE DEPENDENT              EBAC0380
C                                      PARAMETER SPECIFYING THE BASE OF        EBAC0390
C                                      THE MACHINE FLOATING POINT REPRE-       EBAC0400
C                                      SENTATION                               EBAC0410
      DOUBLE PRECISION     AR,AI,D,RADIX,ZERO,ONE,PT95,B2,F,C,G,R,S            EBAC0420
      DOUBLE PRECISION     RRADIX,RB2                                          EBAC0430
      DATA                 RADIX/16.0D0/                                       EBAC0440
      DATA                 ZERO,ONE,PT95/0.0D0,1.0D0,0.95D0/                   EBAC0450
      B2 = RADIX*RADIX                                                         EBAC0480
      RRADIX = ONE/RADIX                                                       EBAC0490
      RB2 = RRADIX*RRADIX                                                      EBAC0500
      K = 1                                                                    EBAC0510
      L = N                                                                    EBAC0520
      GO TO 30                                                                 EBAC0530
C                                    IN-LINE PROCEDURE FOR ROW AND COLUMN      EBAC0540
C                                      EXCHANGE                                EBAC0550
    5 D(M) = J                                                                 EBAC0560
      IF (J .EQ. M) GO TO 20                                                   EBAC0570
      DO 10 I = 1,L                                                            EBAC0580
         F = AR(I,J)                                                           EBAC0590
         AR(I,J) = AR(I,M)                                                     EBAC0600
         AR(I,M) = F                                                           EBAC0610
         F = AI(I,J)                                                           EBAC0620
         AI(I,J) = AI(I,M)                                                     EBAC0630
         AI(I,M) = F                                                           EBAC0640
   10 CONTINUE                                                                 EBAC0650
      DO 15 I=K,N                                                              EBAC0660
         F = AR(J,I)                                                           EBAC0670
         AR(J,I) = AR(M,I)                                                     EBAC0680
         AR(M,I) = F                                                           EBAC0690
         F = AI(J,I)                                                           EBAC0700
         AI(J,I) = AI(M,I)                                                     EBAC0710
         AI(M,I) = F                                                           EBAC0720
   15 CONTINUE                                                                 EBAC0730
   20 GO TO (25,45), IEXC                                                      EBAC0740
C                                    SEARCH FOR ROWS ISOLATING AN              EBAC0750
C                                      EIGENVALUE AND PUSH THEM DOWN           EBAC0760
   25 IF (L .EQ. 1) GO TO 115                                                  EBAC0770
      L = L-1                                                                  EBAC0780
C                                    DO J=L,1,-1                               EBAC0790
   30 L1 = L+1                                                                 EBAC0800
      DO 40 JJ = 1,L                                                           EBAC0810
         J = L1-JJ                                                             EBAC0820
         DO 35 I = 1,L                                                         EBAC0830
            IF (I .EQ. J) GO TO 35                                             EBAC0840
            IF (AR(J,I) .NE. ZERO .OR. AI(J,I) .NE. ZERO) GO TO 40             EBAC0850
   35    CONTINUE                                                              EBAC0860
```

```
      M = L                                                             EBAC0870
      IEXC = 1                                                          EBAC0880
      GO TO 5                                                           EBAC0890
   40 CONTINUE                                                          EBAC0900
      GO TO 50                                                          EBAC0910
C                                 SEARCH FOR COLUMNS ISOLATING AN       EBAC0920
C                                   EIGENVALUE AND PUSH THEM LEFT       EBAC0930
   45 K = K+1                                                           EBAC0940
   50 DO 60 J = K,L                                                     EBAC0950
         DO 55 I = K,L                                                  EBAC0960
            IF (I .EQ. J) GO TO 55                                      EBAC0970
            IF (AR(I,J) .NE. ZERO .OR. AI(I,J) .NE. ZERO) GO TO 60      EBAC0980
   55    CONTINUE                                                       EBAC0990
         M = K                                                          EBAC1000
         IEXC = 2                                                       EBAC1010
         GO TO 5                                                        EBAC1020
   60 CONTINUE                                                          EBAC1030
C                                 BALANCE THE SUBMATRIX IN ROWS         EBAC1040
C                                   K TO L                              EBAC1050
      DO 65 I = K,L                                                     EBAC1060
         D(I) = ONE                                                     EBAC1070
   65 CONTINUE                                                          EBAC1080
C                                 ITERATIVE LOOP FOR NORM REDUCTION     EBAC1090
   70 NOCONV = .FALSE.                                                  EBAC1100
      DO 110 I = K,L                                                    EBAC1110
         C = ZERO                                                       EBAC1120
         R = ZERO                                                       EBAC1130
         DO 75 J = K,L                                                  EBAC1140
            IF (J .EQ. I) GO TO 75                                      EBAC1150
            C = C+DABS(AR(J,I))+DABS(AI(J,I))                           EBAC1160
            R = R+DABS(AR(I,J))+DABS(AI(I,J))                           EBAC1180
   75    CONTINUE                                                       EBAC1200
         G = R*RRADIX                                                   EBAC1210
         F = ONE                                                        EBAC1220
         S = C+R                                                        EBAC1230
   80    IF (C .GE. G) GO TO 85                                         EBAC1240
         F = F*RADIX                                                    EBAC1250
         C = C*B2                                                       EBAC1260
         GO TO 80                                                       EBAC1270
   85    G = R*RADIX                                                    EBAC1280
   90    IF (C .LT. G) GO TO 95                                         EBAC1290
         F = F*RRADIX                                                   EBAC1300
         C = C*RB2                                                      EBAC1310
         GO TO 90                                                       EBAC1320
C                                 BALANCE                               EBAC1330
   95    IF ((C+R)/F .GE. PT95*S) GO TO 110                             EBAC1340
         G = ONE/F                                                      EBAC1350
         D(I) = D(I)*F                                                  EBAC1360
```

```
      NOCONV = .TRUE.                                                   EBAC1370
      DO 100 J = K,N                                                    EBAC1380
         AR(I,J) = AR(I,J)*G                                            EBAC1390
         AI(I,J) = AI(I,J)*G                                            EBAC1400
  100 CONTINUE                                                          EBAC1410
      DO 105 J = 1,L                                                    EBAC1420
         AR(J,I) = AR(J,I)*F                                            EBAC1430
         AI(J,I) = AI(J,I)*F                                            EBAC1440
  105 CONTINUE                                                          EBAC1450
  110 CONTINUE                                                          EBAC1460
      IF (NOCONV) GO TO 70                                              EBAC1470
  115 RETURN                                                            EBAC1480
      END                                                               EBAC1490
C
      SUBROUTINE EHESSC (AR,AI,K,L,N,IA,ID)                             EHEC0010
C                                                                       EHEC0020
C-EHESSC--------D-------LIBRARY 1---------------------------------------EHEC0030
C                                                                       EHEC0040
C   FUNCTION            - REDUCTION OF A COMPLEX MATRIX TO COMPLEX      EHEC0050
C                           UPPER HESSENBERG FORM.                      EHEC0060
C   USAGE               - CALL EHESSC(AR,AI,K,L,N,IA,ID)                EHEC0070
C   PARAMETERS   AR     - INPUT/OUTPUT MATRIX OF DIMENSION N BY N.      EHEC0080
C                           ON INPUT CONTAINS THE REAL COMPONENTS OF    EHEC0090
C                           THE MATRIX TO BE REDUCED.                   EHEC0100
C                           ON OUTPUT CONTAINS THE REAL COMPONENTS      EHEC0110
C                           OF THE REDUCED HESSENBERG FORM IN THE       EHEC0120
C                           UPPER TRIANGULAR PORTION (INCLUDING MAIN    EHEC0130
C                           AND SUB-DIAGONAL) AND THE DETAILS OF        EHEC0140
C                           THE REDUCTION IN THE LOWER TRIANGULAR       EHEC0150
C                           PORTION.                                    EHEC0160
C                AI     - INPUT/OUTPUT MATRIX OF DIMENSION N BY N       EHEC0170
C                           CONTAINING THE IMAGINARY COUNTERPARTS       EHEC0180
C                           TO AR, ABOVE.                               EHEC0190
C                K      - INPUT SCALAR CONTAINING THE ROW AND COLUMN    EHEC0200
C                           INDEX OF THE STARTING ELEMENT TO BE         EHEC0210
C                           REDUCED BY ROW SCALING.  FOR UNBALANCED     EHEC0220
C                           MATRICES SET K = 1.                         EHEC0230
C                L      - INPUT SCALAR CONTAINING THE ROW AND           EHEC0240
C                           COLUMN INDEX OF THE LAST ELEMENT TO BE      EHEC0250
C                           REDUCED BY ROW SCALING. FOR UNBALANCED      EHEC0260
C                           MATRICES SET L = N.                         EHEC0270
C                N      - INPUT SCALAR CONTAINING THE ORDER OF          EHEC0280
C                           THE MATRIX TO BE REDUCED.                   EHEC0290
C                IA     - INPUT SCALAR CONTAINING ROW DIMENSION         EHEC0300
C                           OF AR AND AI IN THE CALLING PROGRAM.        EHEC0310
C                ID     - OUTPUT VECTOR OF LENGTH L CONTAINING          EHEC0320
C                           DETAILS OF THE TRANSFORMATIONS.             EHEC0330
C   PRECISION           - SINGLE/DOUBLE                                 EHEC0340
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C    LANGUAGE              - FORTRAN                                        EHEC0350
C-----------------------------------------------------------------------EHEC0360
C    LATEST REVISION       - FEBRUARY 7, 1973                               EHEC0370
C                                                                          EHEC0380
      SUBROUTINE EHESSC (AR,AI,K,L,N,IA,ID)                                 EHEC0390
C                                                                          EHEC0400
      DIMENSION          AR(IA,1),AI(IA,1),ID(1),T1(2),T2(2)                EHEC0410
      DOUBLE PRECISION   AR,AI,XR,XI,YR,YI,T1,T2,ZERO                       EHEC0420
      COMPLEX*16         X,Y                                                EHEC0430
      EQUIVALENCE        (X,T1(1),XR),(T1(2),XI),(Y,T2(1),YR),              EHEC0450
     1                   (T2(2),YI)                                         EHEC0460
      DATA               ZERO/0.0D0/                                        EHEC0470
      LA=L-1                                                                EHEC0490
      KP1=K+1                                                               EHEC0500
      IF (LA .LT. KP1) GO TO 45                                             EHEC0510
      DC 40 M=KP1,LA                                                        EHEC0520
         I=M                                                                EHEC0530
         XR=ZERO                                                            EHEC0540
         XI=ZERO                                                            EHEC0550
         DO 5 J=M,L                                                         EHEC0560
            IF (DABS(AR(J,M-1))+DABS(AI(J,M-1)) .LE. DABS(XR)+DABS(XI))     EHEC0570
     1      GO TO 5                                                         EHEC0580
            XR=AR(J,M-1)                                                    EHEC0610
            XI=AI(J,M-1)                                                    EHEC0620
            I=J                                                             EHEC0630
    5    CONTINUE                                                           EHEC0640
         ID(M)=I                                                            EHEC0650
         IF (I .EQ. M) GO TO 20                                             EHEC0660
C                                  INTERCHANGE ROWS AND COLUMNS OF          EHEC0670
C                                    ARRAYS AR ANC AI                       EHEC0680
         MM1=M-1                                                            EHEC0690
         DO 10 J=MM1,N                                                      EHEC0700
            YR=AR(I,J)                                                      EHEC0710
            AR(I,J)=AR(M,J)                                                 EHEC0720
            AR(M,J)=YR                                                      EHEC0730
            YI=AI(I,J)                                                      EHEC0740
            AI(I,J)=AI(M,J)                                                 EHEC0750
            AI(M,J)=YI                                                      EHEC0760
   10    CONTINUE                                                           EHEC0770
         DO 15 J=1,L                                                        EHEC0780
            YR=AR(J,I)                                                      EHEC0790
            AR(J,I)=AR(J,M)                                                 EHEC0800
            AR(J,M)=YR                                                      EHEC0810
            YI=AI(J,I)                                                      EHEC0820
            AI(J,I)=AI(J,M)                                                 EHEC0830
            AI(J,M)=YI                                                      EHEC0840
   15    CONTINUE                                                           EHEC0850
C                                  END INTERCHANGE                          EHEC0860
```

```
20    IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 40                    EHEC0870
      MP1=M+1                                                          EHEC0880
      DO 35 I=MP1,L                                                    EHEC0890
         YR=AR(I,M-1)                                                  EHEC0900
         YI=AI(I,M-1)                                                  EHEC0910
         IF (YR .EQ. ZERO .AND. YI .EQ. ZERO) GO TO 35                 EHEC0920
         Y=Y/X                                                         EHEC0930
         AR(I,M-1)=YR                                                  EHEC0940
            AR(I,J)=AR(I,J)-YR*AR(M,J)+YI*AI(M,J)                      EHEC0970
         AI(I,M-1)=YI                                                  EHEC0950
            AI(I,J)=AI(I,J)-YR*AI(M,J)-YI*AR(M,J)                      EHEC0980
         DO 25 J=M,N                                                   EHEC0960
25       CONTINUE                                                      EHEC0990
         DO 30 J=1,L                                                   EHEC1000
            AR(J,M)=AR(J,M)+YR*AR(J,I)-YI*AI(J,I)                      EHEC1010
            AI(J,M)=AI(J,M)+YR*AI(J,I)+YI*AR(J,I)                      EHEC1020
30       CONTINUE                                                      EHEC1030
35    CONTINUE                                                         EHEC1040
40 CONTINUE                                                            EHEC1050
45 RETURN                                                              EHEC1060
   END                                                                 EHEC1070
C
C
C     SUBROUTINE ELRH2C (HR,HI,K,L,N,IH,WR,WI,ZR,ZI,ID,INFER,IER)      ELR20010
C                                                                      ELR20020
C-ELRH2C---------D--------LIBRARY 1------------------------------------ELR20030
C                                                                      ELR20040
C   FUNCTION            - COMPUTE THE EIGENVALUES AND EIGENVECTORS OF  ELR20050
C                           A COMPLEX UPPER HESSENBERG MATRIX AND      ELR20060
C                           BACK TRANSFORM THE EIGENVECTORS.           ELR20070
C   USAGE               - CALL ELRH2C (HR,HI,K,L,N,IH,WR,WI,ZR,ZI,ID,  ELR20080
C                           INFER,IER)                                 ELR20090
C   PARAMETERS   HR     - INPUT MATRIX OF DIMENSION N BY N CONTAINING  ELR20100
C                           THE REAL COMPONENTS OF THE COMPLEX         ELR20110
C                           HESSENBERG MATRIX. HR IS DESTROYED ON      ELR20120
C                           OUTPUT.                                    ELR20130
C                HI     - INPUT MATRIX OF DIMENSION N BY N CONTAINING  ELR20140
C                           THE IMAGINARY COUNTERPARTS TO HR, ABOVE.   ELR20150
C                           HI IS DESTROYED ON OUTPUT.                 ELR20160
C                K      - INPUT SCALAR CONTAINING THE LOWER BOUNDARY   ELR20170
C                           INDEX FOR THE INPUT MATRIX.                ELR20180
C                           FOR UNBALANCED MATRICES SET K = 1.         ELR20190
C                L      - INPUT SCALAR CONTAINING THE UPPER BOUNDARY   ELR20200
C                           INDEX FOR THE INPUT MATRIX.                ELR20210
C                           FOR UNBALANCED MATRICES SET L = N.         ELR20220
C                N      - INPUT SCALAR CONTAINING THE ORDER OF THE     ELR20230
C                           HESSENBERG MATRIX AND THE EIGENVECTOR      ELR20240
C                           MATRIX.                                    ELR20250
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C                IH       - INPUT SCALAR CONTAINING THE ROW DIMENSION       ELR20260
C                              OF MATRICES HR, HI, ZR AND ZI IN THE         ELR20270
C                              CALLING PROGRAM.                             ELR20280
C                WR       - OUTPUT VECTOR OF LENGTH N CONTAINING THE REAL ELR20290
C                              COMPONENTS OF THE EIGENVALUES.               ELR20300
C                WI       - OUTPUT VECTOR OF LENGTH N CONTAINING THE        ELR20310
C                              IMAGINARY COMPONENTS OF THE EIGENVALUES.     ELR20320
C                ZR       - OUTPUT MATRIX OF DIMENSION N BY N CONTAINING    ELR20330
C                              THE REAL COMPONENTS OF THE EIGENVECTORS.     ELR20340
C                              THE EIGENVECTORS ARE NOT NORMALIZED.         ELR20350
C                ZI       - OUTPUT MATRIX OF DIMENSION N BY N CONTAINING    ELR20360
C                              THE IMAGINARY COUNTERPARTS TO ZR, ABOVE.     ELR20370
C                ID       - INPUT VECTOR OF LENGTH N CONTAINING THE         ELR20380
C                              INFORMATION GENERATED BY IMSL ROUTINE        ELR20390
C                              EHESSC IDENTIFYING THE ROWS AND COLUMNS      ELR20400
C                              INTERCHANGED DURING THE REDUCTION TO         ELR20410
C                              HESSENBERG FORM. ONLY COMPONENTS K           ELR20420
C                              THROUGH L ARE USED.                          ELR20430
C                INFER    - OUTPUT SCALAR CONTAINING THE INDEX OF THE       ELR20440
C                              EIGENVALUE WHICH GENERATED THE TERMINAL      ELR20450
C                              ERROR(SEE DESCRIPTION OF IER, BELOW).        ELR20460
C                IER      - ERROR PARAMETER                                 ELR20470
C                              TERMINAL ERROR = 128 + N                     ELR20480
C                                N = 1 INDICATES THE EIGENVALUE RECORDED    ELR20490
C                                      IN THE OUTPUT PARAMETER, INFER,      ELR20500
C                                      COULD NOT BE DETERMINED AFTER 30     ELR20510
C                                      ITERATIONS. IF THE J-TH EIGENVALUE   ELR20520
C                                      COULD NOT BE SO DETERMINED,          ELR20530
C                                      THEN THE EIGENVALUES J+1,J+2,...,N   ELR20540
C                                      SHOULD BE CORRECT.                   ELR20550
C   PRECISION           - SINGLE/DOUBLE                                     ELR20560
C   REQD. IMSL ROUTINES - UERTST                                            ELR20570
C   LANGUAGE            - FORTRAN                                           ELR20580
C-----------------------------------------------------------------------ELR20590
C   LATEST REVISION     - APRIL 5, 1977                                     ELR20600
C                                                                           ELR20610
      SUBROUTINE ELRH2C (HR,HI,K,L,N,IH,WR,WI,ZR,ZI,ID,INFER,IER)           ELR20620
C                                                                           ELR20630
      DIMENSION          HR(IH,1),HI(IH,1),WR(1),WI(1),ZR(IH,1),ZI(IH,1)ELR20640
      DIMENSION          T1(2),T2(2),T3(2),ID(1)                            ELR20650
      COMPLEX*16         X,Y,Z                                              ELR20660
      DOUBLE PRECISION   HR,HI,WR,WI,ZR,ZI,EPS,FNORM                        ELR20680
      DOUBLE PRECISION   T1,T2,T3,XR,XI,YR,YI,ZZR,ZZI,SR,SI,TR,TI           ELR20690
      DOUBLE PRECISION   ZERO,ONE,TWO                                       ELR20700
      EQUIVALENCE        (X,T1(1),XR),(T1(2),XI),                           ELR20710
     1                   (Y,T2(1),YR),(T2(2),YI),                           ELR20720
     2                   (Z,T3(1),ZZR),(T3(2),ZZI)                          ELR20730
      DATA               ZERO,ONE,TWO/0.0D0,1.0D0,2.0D0/                    ELR20740
```

```
      DATA             EPS/Z3410000000000000/                           ELR20750
C                                 INITIALIZE IER                        ELR20770
      IER=0                                                             ELR20780
      INFER=0                                                           ELR20790
      TR=ZERO                                                           ELR20800
      TI=ZERO                                                           ELR20810
      DC 5 I=1,N                                                        ELR20820
         DO 3 J=1,N                                                     ELR20830
            ZR(I,J)=ZERO                                                ELR20840
            ZI(I,J)=ZERO                                                ELR20850
    3    CONTINUE                                                       ELR20860
         ZR(I,I) = ONE                                                  ELR2C870
    5 CONTINUE                                                          ELR20880
C                                 FORM THE MATRIX OF ACCUMULATED        ELR20890
C                                   TRANSFORMATICNS FROM THE INFOR-     ELR20900
C                                   MATION LEFT BY ROUTINE 'EHESSC'     ELR20910
      IEND=L-K-1                                                        ELR20920
      IF (IEND .LE. 0) GO TO 25                                         ELR20930
C                                 DO I=L-1,K+1,-1                       ELR20940
      DO 20 II=1,IEND                                                   ELR20950
         I=L-II                                                         ELR20960
         IP1=I+1                                                        ELR20970
         IM1=I-1                                                        ELR20980
         DO 10 M=IP1,L                                                  ELR20990
            ZR(M,I)=HR(M,IM1)                                           ELR21000
            ZI(M,I)=HI(M,IM1)                                           ELR21010
   10    CONTINUE                                                       ELR21020
         J=ID(I)                                                        ELR21030
         IF (I .EC. J) GO TO 20                                         ELR21040
         DO 15 M=I,L                                                    ELR21050
            ZR(I,M)=ZR(J,M)                                             ELR21060
            ZI(I,M)=ZI(J,M)                                             ELR21070
            ZR(J,M)=ZERO                                                ELR21080
            ZI(J,M)=ZERC                                                ELR2109C
   15    CONTINUE                                                       ELR21100
         ZR(J,I)=ONE                                                    ELR21110
   20 CONTINUE                                                          ELR21120
   25 DO 30 I=1,N                                                       ELR21130
         IF (I .GE. K .AND. I .LE. L) GO TO 30                          ELR21140
         WR(I)=HR(I,I)                                                  ELR21150
         WI(I)=HI(I,I)                                                  ELR21160
   30 CCNTINUE                                                          ELR21170
      NN=L                                                              ELR21180
C                                 SEARCH FOR NEXT EIGENVALUE            ELR21190
   35 IF (NN .LT. K) GO TO 150                                          ELR21200
      ITS=0                                                             ELR21210
      NNM1=NN-1                                                         ELR21220
      NNM2=NN-2                                                         ELR21230
```

```
      IF (NN .EQ. K) GO TO 50                                           ELR21240
C                                   LOOK FOR SINGLE SMALL SUB-DIAGONAL  ELR21250
C                                     ELEMENT                           ELR21260
   40 NPL=NN+K                                                          ELR21270
C                                   DO M=NN,K+1,-1                      ELR21280
      DO 45 KK=K,NNM1                                                   ELR21290
         M=NPL-KK                                                       ELR21300
         MM1=M-1                                                        ELR21310
         IF (DABS(HR(M,MM1))+DABS(HI(M,MM1)) .LE. EPS*(DABS(HR(MM1,MM1))ELR21320
     1      +DABS(HI(MM1,MM1))+DABS(HR(M,M))+DABS(HI(M,M)))) GO TO 55   ELR21330
   45 CONTINUE                                                          ELR21360
   50 M=K                                                               ELR21370
   55 IF (M .EQ. NN) GO TO 145                                          ELR21380
      IF (ITS .EQ. 30) GO TO 205                                        ELR21390
C                                   FORM SHIFT                          ELR21400
      IF (ITS .EQ. 10 .OR. ITS .EQ. 20) GO TO 60                        ELR21410
      SR=HR(NN,NN)                                                      ELR21420
      SI=HI(NN,NN)                                                      ELR21430
      XR=HR(NNM1,NN)*HR(NN,NNM1)-HI(NNM1,NN)*HI(NN,NNM1)                ELR21440
      XI=HR(NNM1,NN)*HI(NN,NNM1)+HI(NNM1,NN)*HR(NN,NNM1)                ELR21450
      IF (XR .EQ. ZERO .AND. XI .EQ. ZERO) GO TO 65                     ELR21460
      YR=(HR(NNM1,NNM1)-SR)/TWO                                         ELR21470
      YI=(HI(NNM1,NNM1)-SI)/TWO                                         ELR21480
      Z=CDSQRT(DCMPLX(YR**2-YI**2+XR,TWO*YR*YI+XI))                     ELR21490
      IF (YR*ZZR+YI*ZZI .LT. ZERO) Z=-Z                                 ELR21510
      X=X/(Y+Z)                                                         ELR21520
      SR=SR-XR                                                          ELR21530
      SI=SI-XI                                                          ELR21540
      GO TO 65                                                          ELR21550
   60 SR=DABS(HR(NN,NNM1))+DABS(HR(NNM1,NNM2))                          ELR21560
      SI=DABS(HI(NN,NNM1))+DABS(HI(NNM1,NNM2))                          ELR21570
   65 DO 70 I=K,NN                                                      ELR21600
         HR(I,I)=HR(I,I)-SR                                             ELR21610
         HI(I,I)=HI(I,I)-SI                                             ELR21620
   70 CONTINUE                                                          ELR21630
      TR=TR+SR                                                          ELR21640
      TI=TI+SI                                                          ELR21650
      ITS=ITS+1                                                         ELR21660
C                                   LOOK FOR TWO CONSECUTIVE SMALL      ELR21670
C                                     SUB-DIAGONAL ELEMENTS             ELR21680
      XR=DABS(HR(NNM1,NNM1))+DABS(HI(NNM1,NNM1))                        ELR21690
      YR=DABS(HR(NN,NNM1))+DABS(HI(NN,NNM1))                            ELR21700
      ZZR=DABS(HR(NN,NN))+DABS(HI(NN,NN))                               ELR21710
      NNMJ=NNM1-M                                                       ELR21750
      IF (NNMJ .EQ. 0) GO TO 80                                         ELR21760
C                                   DO MM=NN-1,M+1,-1                   ELR21770
      DO 75 NM=1,NNMJ                                                   ELR21780
         MM=NN-NM                                                       ELR21790
```

```
      MMM1=MM-1                                                         ELR21800
      YI=YR                                                             ELR21810
      YR=DABS(HR(MM,MMM1))+DABS(HI(MM,MMM1))                            ELR21820
      XI=ZZR                                                            ELR21840
      ZZR=XR                                                            ELR21850
      XR=DABS(HR(MMM1,MMM1))+DABS(HI(MMM1,MMM1))                        ELR21860
      IF (YR .LE. EPS*ZZR/YI*(ZZR+XR+XI)) GO TO 85                      ELR21880
   75 CONTINUE                                                          ELR21890
   80 MM=M                                                              ELR21900
C                                      TRIANGULAR DECOMPOSITION H=L*R   ELR21910
   85 MP1=MM+1                                                          ELR21920
      DO 110 I=MP1,NN                                                   ELR21930
         IM1=I-1                                                        ELR21940
         XR=HR(IM1,IM1)                                                 ELR21950
         XI=HI(IM1,IM1)                                                 ELR21960
         YR=HR(I,IM1)                                                   ELR21970
         YI=HI(I,IM1)                                                   ELR21980
         IF (DABS(XR)+DABS(XI) .GE. DABS(YR)+DABS(YI)) GO TO 95         ELR21990
C                                      INTERCHANGE ROWS OF HR AND HI    ELR22010
         DO 90 J=IM1,N                                                  ELR22020
            ZZR=HR(IM1,J)                                               ELR22030
            HR(IM1,J)=HR(I,J)                                           ELR22040
            HR(I,J)=ZZR                                                 ELR22050
            ZZI=HI(IM1,J)                                               ELR22060
            HI(IM1,J)=HI(I,J)                                           ELR22070
            HI(I,J)=ZZI                                                 ELR22080
   90    CONTINUE                                                       ELR22090
         Z=X/Y                                                          ELR22100
         WR(I)=ONE                                                      ELR22110
         GO TO 100                                                      ELR22120
   95    Z=Y/X                                                          ELR22130
         WR(I)=-ONE                                                     ELR22140
  100    HR(I,IM1)=ZZR                                                  ELR22150
         HI(I,IM1)=ZZI                                                  ELR22160
         DO 105 J=I,N                                                   ELR22170
            HR(I,J)=HR(I,J)-ZZR*HR(IM1,J)+ZZI*HI(IM1,J)                 ELR22180
            HI(I,J)=HI(I,J)-ZZR*HI(IM1,J)-ZZI*HR(IM1,J)                 ELR22190
  105    CONTINUE                                                       ELR22200
  110 CONTINUE                                                          ELR22210
C                                      COMPOSITION R*L=H                ELR22220
      DO 140 J=MP1,NN                                                   ELR22230
         JM1=J-1                                                        ELR22240
         XR=HR(J,JM1)                                                   ELR22250
         XI=HI(J,JM1)                                                   ELR22260
         HR(J,JM1)=ZERO                                                 ELR22270
         HI(J,JM1)=ZERO                                                 ELR22280
C                                      INTERCHANGE COLUMNS OF HR, HI,   ELR22290
C                                        ZR, AND ZI IF NECESSARY        ELR22300
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```
      IF (WR(J) .LE. ZERO) GO TO 125                                    ELR22310
      DO 115 I=1,J                                                      ELR22320
         ZZR=HR(I,JM1)                                                  ELR22330
         HR(I,JM1)=HR(I,J)                                              ELR22340
         HR(I,J)=ZZR                                                    ELR22350
         ZZI=HI(I,JM1)                                                  ELR22360
         HI(I,JM1)=HI(I,J)                                              ELR22370
         HI(I,J)=ZZI                                                    ELR22380
  115 CONTINUE                                                          ELR22390
      DO 120 I=K,L                                                      ELR22400
         ZZR=ZR(I,JM1)                                                  ELR22410
         ZR(I,JM1)=ZR(I,J)                                              ELR22420
         ZR(I,J)=ZZR                                                    ELR22430
         ZZI=ZI(I,JM1)                                                  ELR22440
         ZI(I,JM1)=ZI(I,J)                                              ELR22450
         ZI(I,J)=ZZI                                                    ELR22460
  120 CONTINUE                                                          ELR22470
C                                  END INTERCHANGE COLUMNS              ELR22480
  125 DO 130 I=1,J                                                      ELR22490
         HR(I,JM1)=HR(I,JM1)+XR*HR(I,J)-XI*HI(I,J)                      ELR22500
         HI(I,JM1)=HI(I,JM1)+XR*HI(I,J)+XI*HR(I,J)                      ELR22510
  130 CONTINUE                                                          ELR22520
      DO 135 I=K,L                                                      ELR22530
         ZR(I,JM1)=ZR(I,JM1)+XR*ZR(I,J)-XI*ZI(I,J)                      ELR22540
         ZI(I,JM1)=ZI(I,JM1)+XR*ZI(I,J)+XI*ZR(I,J)                      ELR22550
  135 CONTINUE                                                          ELR22560
C                                  END ACCUMULATE TRANSFORMATIONS       ELR22570
  140 CONTINUE                                                          ELR22580
      GC TO 40                                                          ELR22590
C                                  A ROOT FOJND                         ELR22600
  145 WR(NN)=HR(NN,NN)+TR                                               ELR22610
      WI(NN)=HI(NN,NN)+TI                                               ELR22620
      NN=NNM1                                                           ELR22630
      GO TO 35                                                          ELR22640
C                                  ALL ROOTS FOUND. BACKSUBSTITUTE TO   ELR22650
C                                    FIND VECTORS OF UPPER TRIANGULAR   ELR22660
C                                    FORM                               ELR22670
  150 IF (N .EQ. 1) GO TO 9005                                          ELR22680
      FNORM=ZERO                                                        ELR22690
      DC 160 I=1,N                                                      ELR22700
         FNORM=FNORM+DABS(WR(I))+DABS(WI(I))                            ELR22710
         IF (I .EQ. N) GO TO 160                                        ELR22730
         IP1=I+1                                                        ELR22740
         DO 155 J=IP1,N                                                 ELR22750
            FNORM=FNORM+DABS(HR(I,J))+DABS(HI(I,J))                     ELR22760
  155    CONTINUE                                                       ELR22780
  160 CONTINUE                                                          ELR22790
      IF (FNORM .EQ. ZERO) GO TO 9005                                   ELR22800
```

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C                                     DO NN=N,2,-1                      ELR22810
      NP2 = N+2                                                         ELR22820
      DO 180 NM=2,N                                                     ELR22830
         NN=NP2-NM                                                      ELR22840
         XR=WR(NN)                                                      ELR22850
         XI=WI(NN)                                                      ELR22860
         NNM1 =NN-1                                                     ELR22870
C                                     DO I=NN-1,1,-1                    ELR22880
         DO 175 II=1,NNM1                                               ELR22890
            I =NN-II                                                    ELR22900
            ZZR=HR(I,NN)                                                ELR22910
            ZZI=HI(I,NN)                                                ELR22920
            IF (I .EQ. NNM1) GO TO 170                                  ELR22930
            IP1=I+1                                                     ELR22940
            DO 165 J=IP1,NNM1                                           ELR22950
               ZZR=ZZR+HR(I,J)*HR(J,NN)-HI(I,J)*HI(J,NN)                ELR22960
               ZZI=ZZI+HR(I,J)*HI(J,NN)+HI(I,J)*HR(J,NN)                ELR22970
  165       CONTINUE                                                    ELR22980
  170       YR=XR-WR(I)                                                 ELR22990
            YI=XI-WI(I)                                                 ELR23000
            IF (YR .EQ. ZERO .AND. YI .EQ. ZERO) YR=EPS*FNORM           ELR23010
            Z=Z/Y                                                       ELR23020
            HR(I,NN)=T3(1)                                              ELR23030
            HI(I,NN)=T3(2)                                              ELR23040
  175    CONTINUE                                                       ELR23050
  180 CONTINUE                                                          ELR23060
C                                     END BACKSUBSTITUTION              ELR23070
      NM1=N-1                                                           ELR23080
C                                     VECTOR OF ISOLATED ROOTS          ELR23090
      DO 190 I=1,NM1                                                    ELR23100
         IF (I .GE. K .AND. I .LE. L) GO TO 190                         ELR23110
         IP1=I+1                                                        ELR23120
         DO 185 J=IP1,N                                                 ELR23130
            ZR(I,J)=HR(I,J)                                             ELR23140
            ZI(I,J)=HI(I,J)                                             ELR23150
  185    CONTINUE                                                       ELR23160
  190 CONTINUE                                                          ELR23170
      IF (L .EQ. 0) GO TO 9005                                          ELR23180
C                                     MULTIPLY BY TRANSFORMATION MATRIX ELR23190
C                                       TO GIVE VECTORS OF ORIGINAL FULL ELR23200
C                                       MATRIX                          ELR23210
      NPL=N+K                                                           ELR23220
C                                     DO J=N,K+1,-1                     ELR23230
      DO 200 JJ=K,NM1                                                   ELR23240
         J=NPL-JJ                                                       ELR23250
         JM1=J-1                                                        ELR23260
         DO 200 I=K,L                                                   ELR23270
            ZZR=ZR(I,J)                                                 ELR23280
```

```
            ZZI=ZI(I,J)                                                 ELR23290
            MM=JM1                                                      ELR23300
            IF (L .LT. J) MM=L                                          ELR23310
            DO 195 M=K,MM                                               ELR23320
               ZZR=ZZR+ZR(I,M)*HR(M,J)-ZI(I,M)*HI(M,J)                  ELR23330
               ZZI=ZZI+ZR(I,M)*HI(M,J)+ZI(I,M)*HR(M,J)                  ELR23340
  195       CONTINUE                                                    ELR23350
            ZR(I,J)=ZZR                                                 ELR23360
            ZI(I,J)=ZZI                                                 ELR23370
  200 CONTINUE                                                          ELR23380
      GO TO 9005                                                        ELR23390
C                                  SET ERROR - NO CONVERGENCE TO AN     ELR23400
C                                    EIGENVALUE AFTER 30 ITERATIONS     ELR23410
  205 IER=129                                                           ELR23420
      INFER=NN                                                          ELR23430
 9000 CONTINUE                                                          ELR23440
      CALL UERTST (IER,6HELRH2C)                                        ELR23450
 9005 RETURN                                                            ELR23460
      END                                                               ELR23470
C
C
C     SUBROUTINE EBBCKC (ZR,ZI,N,IZ,K,L,M,D)                            EBBC0010
C                                                                       EBBC0020
C-EBBCKC--------D--------LIBRARY 1--------------------------------------EBBC0030
C                                                                       EBBC0040
C   FUNCTION            - BACKTRANSFORM THE EIGENVECTORS OF A BALANCED  EBBC0050
C                           COMPLEX GENERAL MATRIX.                     EBBC0060
C   USAGE               - CALL EBBCKC (ZR,ZI,N,IZ,K,L,M,D)              EBBC0070
C   PARAMETERS   ZR     - INPUT/OUTPUT MATRICES OF DIMENSION N BY M.    EBBC0080
C                ZI         ON INPUT, THE FIRST M COLUMNS OF ZR AND     EBBC0090
C                           ZI CONTAIN THE REAL AND IMAGINARY PARTS,    EBBC0100
C                           RESPECTIVELY, OF THE EIGENVECTORS TO BE     EBBC0110
C                           BACK TRANSFORMED. ON OUTPUT, THESE M        EBBC0120
C                           COLUMNS CONTAIN THE REAL AND IMAGINARY      EBBC0130
C                           PARTS OF THE TRANSFORMED EIGENVECTORS.      EBBC0140
C                N      - INPUT SCALAR CONTAINING THE NUMBER OF         EBBC0150
C                           ROWS IN THE MATRIX Z = (ZR,ZI). N MUST      EBBC0160
C                           NOT BE GREATER THAN IZ.                     EBBC0170
C                IZ     - INPUT SCALAR CONTAINING THE ROW DIMENSION     EBBC0180
C                           OF MATRICES ZR AND ZI IN THE CALLING        EBBC0190
C                           PROGRAM.                                    EBBC0200
C                K      - INPUT SCALARS CONTAINING THE BOUNDARY         EBBC0210
C                L          INDICES FOR THE BALANCED MATRIX. K AND L    EBBC0220
C                           ARE TWO OUTPUT PARAMETERS FROM IMSL ROUTINE EBBC0230
C                           EBALAC.                                     EBBC0240
C                M      - INPUT SCALAR CONTAINING THE NUMBER OF COLUMNS EBBC0250
C                           OF Z = (ZR,ZI) TO BE BACK TRANSFORMED.      EBBC0260
C                D      - INPUT VECTOR OF LENGTH N CONTAINING THE       EBBC0270
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C                                    DETAILS OF THE TRANSFORMATIONS PRODUCED
C                                    BY IMSL ROUTINE EBALAC.
C   PRECISION           - SINGLE/DOUBLE
C   LANGUAGE            - FORTRAN
C-----------------------------------------------------------------------
C   LATEST REVISION     - MARCH 9, 1977
C
      SUBROUTINE EBBCKC (ZR,ZI,N,IZ,K,L,M,D)
C
      DIMENSION          ZR(IZ,1),ZI(IZ,1),D(1)
      DOUBLE PRECISION   ZR,ZI,D,S
      IF (L .EC. K) GO TO 15
      DO 10 I = K,L
         S = D(I)
C                                  LEFT HAND EIGENVECTORS ARE BACK
C                                    TRANSFORMED IF THE ABOVE
C                                    STATEMENT IS REPLACED BY S=1.0/D(I)
         DO 5 J = 1,M
            ZR(I,J) = ZR(I,J)*S
            ZI(I,J) = ZI(I,J)*S
    5    CONTINUE
   10 CONTINUE
C                                  DO 25 I=K-1,1,-1 AND
C                                    DO I=L+1,N,1
   15 DO 25 II = 1,N
         I = II
         IF (I .GE. K .AND. I .LE. L) GO TO 25
         IF (I .LT. K) I = K-II
         KK = D(I)
         IF (KK .EQ. I) GO TO 25
         DO 20 J = 1,M
            S = ZR(I,J)
            ZR(I,J) = ZR(KK,J)
            ZR(KK,J) = S
            S = ZI(I,J)
            ZI(I,J) = ZI(KK,J)
            ZI(KK,J) = S
   20    CONTINUE
   25 CONTINUE
      RETURN
      END
C
C
C     SUBROUTINE UERTST (IER,NAME)
C
C-UERTST----------------LIBRARY 1----------------------------------------
C
C   FUNCTION            - ERROR MESSAGE GENERATION
```

```
C    USAGE                  - CALL UERTST(IER,NAME)                            UERT0060
C    PARAMETERS   IER       - ERROR PARAMETER. TYPE + N  WHERE                 UERT0070
C                              TYPE= 128 IMPLIES TERMINAL ERROR                UERT0080
C                                     64 IMPLIES WARNING WITH FIX              UERT0090
C                                     32 IMPLIES WARNING                       UERT0100
C                              N    = ERROR CODE RELEVANT TO CALLING ROUTINEUERT0110
C                 NAME      - INPUT VECTOR CONTAINING THE NAME OF THE          UERT0120
C                              CALLING ROUTINE AS A SIX CHARACTER LITERAL      UERT0130
C                              STRING.                                         UERT0140
C    LANGUAGE               - FORTRAN                                          UERT0150
C-----------------------------------------------------------------------UERT0160
C    LATEST REVISION        - JANUARY 18, 1974                                 UERT0170
C                                                                              UERT0180
      SUBROUTINE UERTST(IER,NAME)                                              UERT0190
C                                                                              UERT0200
      DIMENSION          ITYP(5,4),IBIT(4)                                     UERT0210
      INTEGER*2          NAME(3)                                               UERT0220
      INTEGER            WARN,WARF,TERM,PRINTR                                 UERT0230
      EQUIVALENCE        (IBIT(1),WARN),(IBIT(2),WARF),(IBIT(3),TERM)          UERT0240
      DATA      ITYP     /'WARN','ING ','    ','    ','    ',                  UERT0250
     *                    'WARN','ING(','WITH',' FIX',')   ',                  UERT0260
     *                    'TERM','INAL','    ','    ','    ',                  UERT0270
     *                    'NON-','DEFI','NED ','    ','    '/,                 UERT0280
     *          IBIT     / 32,64,128,0/                                        UERT0290
      DATA      PRINTR   / 6/                                                  UERT0300
      IER2=IER                                                                 UERT0310
      IF (IER2 .GE. WARN) GO TO 5                                              UERT0320
C                                  NON-DEFINED                                 UERT0330
      IER1=4                                                                   UERT0340
      GO TO 20                                                                 UERT0350
    5 IF (IER2 .LT. TERM) GO TO 10                                             UERT0360
C                                  TERMINAL                                    UERT0370
      IER1=3                                                                   UERT0380
      GO TO 20                                                                 UERT0390
   10 IF (IER2 .LT. WARF) GO TO 15                                             UERT0400
C                                  WARNING(WITH FIX)                           UERT0410
      IER1=2                                                                   UERT0420
      GO TO 20                                                                 UERT0430
C                                  WARNING                                     UERT0440
   15 IER1=1                                                                   UERT0450
C                                  EXTRACT 'N'                                 UERT0460
   20 IER2=IER2-IBIT(IER1)                                                     UERT0470
C                                  PRINT ERROR MESSAGE                         UERT0480
      WRITE (PRINTR,25) (ITYP(I,IER1),I=1,5),NAME,IER2,IER                     UERT0490
   25 FORMAT(' *** I M S L(UERTST) ***  ',5A4,4X,3A2,4X,I2,                    UERT0500
     *   ' (IER = ',I3,')')                                                    UERT0510
      RETURN                                                                   UERT0520
      END                                                                      UERT0530
```

```
//EIG$FCN JOB (1719,0947,AX74),'SMC 1882',TIME=2
// EXEC FORTCLGW
//FORT.SYSIN DD *
C.......................................................................EIGF  10
C                                                                       EIGF  20
C      PROGRAM EIGFCN                                                   EIGF  30
C                                                                       EIGF  40
C      PERTURBATION VELOCITY PLOT PROGRAM                               EIGF  50
C           NI = 0                                                      EIGF  60
C                                                                       EIGF  70
C                                                                       EIGF  80
C      PURPOSE                                                          EIGF  90
C                                                                       EIGF 100
C         TO PLOT THE NONDIMENSIONALIZED PERTURBATION VELOCITY U AGAINST EIGF 110
C             NONDIMENSIONALIZED RADIUS UTILIZING THE DATA GENERATED    EIGF 120
C             BY PROGRAM PIPEO (MODE=1).  PLOTTING IS PERFORMED ON      EIGF 130
C             THE NPS VERSATEC PLOTTER USING SUBROUTINE PLOTG.          EIGF 140
C                                                                       EIGF 150
C                                                                       EIGF 160
C.......................................................................EIGF 170
C                                                                       EIGF 180
C                                                                       EIGF 190
      IMPLICIT REAL*8(A-H,O-Z)                                          EIGF 200
      COMPLEX *16QPRIM(85),DQPRIM(85),ALPHA,UP(85),UPPRIM(85),CONST,GAMMEIGF 210
     1A                                                                 EIGF 220
      REAL *8ETA(85),U(85),UR(85),UI(85)                                EIGF 230
      REAL *4UR1(85),UI1(85),RAD1(85),SREY,SLAMDA,SGAMMA,AR,AI          EIGF 240
C                                                                       EIGF 250
C     READ N,REY,ALPHA,LAMBDA,GAMMA & QPRIME'S                          EIGF 260
C                                                                       EIGF 270
      READ (5,6) N,REY,ALPHA                                            EIGF 280
      NO = N+1                                                          EIGF 290
      N1 = N+2                                                          EIGF 300
      READ (5,7) AMDA,GAMMA                                             EIGF 310
      READ (5,8) KSET                                                   EIGF 320
      SLAMDA = AMDA                                                     EIGF 340
      SREY = REY                                                        EIGF 330
      AR = ALPHA                                                        EIGF 350
      AI = AIMAG(ALPHA)                                                 EIGF 360
      SGAMMA = GAMMA                                                    EIGF 370
      SGAMMA = SGAMMA +AR                                               EIGF 380
C                                                                       EIGF 390
C                                                                       EIGF 400
      DO 1 I=2,NO                                                       EIGF 410
      READ (5,9) ETA(I),QPRIM(I)                                        EIGF 420
    1 CONTINUE                                                          EIGF 430
C                                                                       EIGF 440
C                                                                       EIGF 450
```

```
C     COMPUTE UPPRIME'S
C
      DEL = 1D0/DFLOAT(N+1)
      ETA(1) = 0.0D0
      QPRIM(1) = 1.79591836 7D0*QPRIM(2)-1.24781341D0*QPRIM(3)+0.60641399
     14D0*QPRIM(4)-0.177842566D0*QPRIM(5)+0.023323615D0*QPRIM(6)
      UPPRIM(1) = 2D0*QPRIM(1)
C
      CALL COEFNT (ETA(2),AMDA,COEF,KSET)
      DQPRIM(2) = 2D0*(-QPRIM(2)+QPRIM(3))/(3D0*DEL)
      DQPRIM(2) = COEF*DQPRIM(2)
      UPPRIM(2) = 2D0*QPRIM(2)+ETA(2)*DQPRIM(2)
C
      CALL COEFNT (ETA(3),AMDA,COEF,KSET)
      DQPRIM(3) = 4D0*(-QPRIM(2)+QPRIM(3))/(3D0*DEL)
      DQPRIM(3) = COEF*DQPRIM(3)
      UPPRIM(3) = 2D0*QPRIM(3)+ETA(3)*DQPRIM(3)
C
C
      DO 2 I=4,N
      CALL COEFNT (ETA(I),AMDA,COEF,KSET)
      DQPRIM(I) = (QPRIM(I+1)-QPRIM(I-1))/(2D0*DEL)
      DCPRIM(I) = COEF*DQPRIM(I)
      UPPRIM(I) = 2D0*QPRIM(I)+ETA(I)*DQPRIM(I)
    2 CONTINUE
C
C
      CALL COEFNT (ETA(NO),AMDA,COEF,KSET)
      DQPRIM(NO) = -QPRIM(N)/(2D0*DEL)
      DQPRIM(NO) = COEF*DQPRIM(NO)
      UPPRIM(NO) = 2D0*QPRIM(NO)+ETA(NO)*DQPRIM(NC)
C
      ETA(N1) = 1.0D0
      UPPRIM(N1) = (0D0,0D0)
C
      WRITE (6,10)
C
C     DETERMINE U VECTOR OF LARGEST MAGNITUDE
C
      C = 0.D0
C
      DO 3 I=1,N1
      IF (CDABS(UPPRIM(I)).GT.C) INDEX=I
      IF (CDABS(UPPRIM(I)).GT.C) C=CDABS(UPPRIM(I))
    3 CONTINUE
C
C
      CONST = DCONJG(UPPRIM(INDEX))/C**2
```

```
C
C     NORMALIZE UPPRIMES'S AND SPLIT INTO REAL & IMAGINARY VECTORS
C
C
      DO 4 I=1,N1
      UP(I) = CONST*UPPRIM(I)
      UR(I) = UP(I)
      UI(I) = (0D0,-1D0)*UP(I)
    4 CONTINUE
C
C
C     CONVERT U'S AND ETA'S TO SINGLE PRECISION FCR PLOTG
C
C
      DO 5 I=1,N1
      RAD1(I) = ETA(I)
      UR1(I) = UR(I)
      UI1(I) = UI(I)
      WRITE (6,11) RAD1(I),UR1(I),UI1(I)
    5 CONTINUE
C
C
C     PLOT RESULTS
C
      CALL PLOTG(RAD1,UR1,N1,1,1,1,'RADIUS',6,'PERTURBATION VELOCITY',
     $ 21,0.,1.,-1.,1.,7.,7.)
      CALL PLOTG(RAD1,UI1,N1,2,1,5,'RADIUS',6,'PERTURBATION VELOCITY',
     $ 21,0.,1.,-1.,1.,7.,7.)
      CALL CHART (N,SREY,AR,AI,SGAMMA,SLAMDA)
      CALL PLOT (0.0,0.0,999)
      STOP
C
    6 FORMAT (I2,3D20.10)
    7 FORMAT (F15.7,2(1PD20.10))
    8 FORMAT (I2)
    9 FORMAT (F15.7,2(1PD20.10))
   10 FORMAT ('1')
   11 FORMAT (' ',3F15.7)
      END
C
C
C..................SUBROUTINE COEFNT(ETA,AMDA,COEF,KSET)................
C
C
C     PURPOSE--WHEN AN OFFSET MESH IS USED, THIS SUBROUTINE GENERATES
C              THE COEFFICIENT REQUIRED TO CONVERT DQ/DETA TO DQ/DR AND
C              CONVERTS THE UNIFORM ETA VALUE INTO THE NONUNIFORM R VALUE
C
```

```
C
C     EXAMPLE OF THE CALLING ARGUMENT
C
C           CALL COEFNT(ETA,AMDA,COEF,KSET)
C
C
C     DESCRIPTION OF PARAMETERS
C
C        ETA -THE VALUE OF THE INDEPENDENT VARIABLE REPLACING R
C             IN THE NONUNIFORM MESH FOR THE STATION BEING COMPUTED.
C
C        AMDA-THE NONDIMENSIONAL MESH PARAMETER FOR THE PARTICULAR
C             SET OF DATA BEING PLOTTED.
C
C        COEF-THE VALUE OF (DR/DETA)**-1 NEEDED TC CONVERT DQ/DETA TO
C             DQ/DR.
C
C        KSET-OFFSET PARAMETER WHICH IS EQUAL TO -1 FOR WALL OFFSET,
C             0 FOR UNIFORM MESH AND 1 FOR AXIS OFFSET.
C
C
C     OTHER SUBROUTINES NEEDED
C
C        NONE
C
C.......................................................................
C
C
      SUBROUTINE COEFNT (ETA,AMDA,COEF,KSET)
C
C
      IMPLICIT REAL*8(A-H,O-Z)
C
      TETA = ETA
      IF (AMDA.LE.1D-10) GO TO 3
      IF (KSET.EQ.1) TETA=1DO-TETA
      CNST = DTANH(AMDA)/AMDA
      ETAP = AMDA*TETA
      COEF = CNST*(DCOSH(ETAP))**2
      IF (AMDA.GE.1.D-10) CONST=1DO/DTANH(AMDA)
      IF (KSET) 1,3,2
C
    1 ETA = CONST*DTANH(ETAP)
      RETURN
C
    2 ETA = 1.DO-CONST*DTANH(ETAP)
      RETURN
C
```

```
    3 COEF = 100
      RETURN
      END
C
C
C...............SUBROUTINE CHART(N,SREY,AR,AI,SGAMMA,SLAMDA)............
C
C     PURPOSE
C
C        TO LABEL THE GRAPH WITH INFORMATION PERTAINING TO THE PLOT
C
C
C     EXAMPLE OF THE CALLING ARGUMENT
C
C        CALL CHART(N,SREY,AR,AI,SGAMMA,SLAMDA)
C
C
C     DESCRIPTION OF PARAMETERS
C
C        THE PARAMETERS ARE SELF-EXPLANATORY AND MUST BE IN SINGLE
C           PRECISION FOR PLOTTING.
C
C
C     OTHER SUBROUTINES NEEDED
C
C        ONLY BUILT-IN VERSATEC PLOTTING FUNCTIONS NEWPEN,SYMBOL &
C           NUMBER.  NOTE THAT THESE ROUTINES MAY ONLY BE
C           ACCESSED WHEN RUNNING UNDER 'FORTCLGW'.
C
C
C.......................................................................
C
C
      SUBROUTINE CHART (N,SREY,AR,AI,SGAMMA,SLAMDA)
C
C
      X0 = 2.5
      Y0 = 6.5
      HT = 0.15
      HT1 = 0.7*HT
      DELY1 = .08+HT
      DELY2 = .065+HT1
      DELX = .1
C
C     GRAPH TITLE
C
      CALL NEWPEN (2)
      CALL SYMBOL(X0,Y0,HT,'NORMALIZED PERTURBATION VELOCITY',0.,32)
```

```
      XO = XO+7.*DELX                                                   CHAR 440
      YO = YO-DELY1                                                     CHAR 450
      CALL SYMBOL(XO,YO,HT,'FOR THE CASE N = 0',0.,18)                  CHAR 460
C                                                                       CHAR 470
C     MESH VALUE                                                        CHAR 480
C                                                                       CHAR 490
      CALL NEWPEN (1)                                                   CHAR 500
      XO = XO+4.*DELX                                                   CHAR 510
      YO = YO-DELY1                                                     CHAR 520
      SN = FLOAT(N)                                                     CHAR 530
      CALL SYMBOL(XO,YO,HT1,'NMESH  = ',C.,9)                           CHAR 540
      CALL NUMBER (999.,999.,HT1,SN,0.,-1)                              CHAR 550
C                                                                       CHAR 560
C     REY VALUE                                                         CHAR 570
C                                                                       CHAR 580
      YC = YO-DELY2                                                     CHAR 590
      CALL SYMBOL(XO,YO,HT1,'REY     = ',0.,9)                          CHAR 600
      CALL NUMBER (999.,999.,HT1,SREY,0.,-1)                            CHAR 610
C                                                                       CHAR 620
C     ALPHA VALUE                                                       CHAR 630
C                                                                       CHAR 640
      X1 = XO+11.*DELY2                                                 CHAR 650
      YO = YO-DELY2                                                     CHAR 660
      CALL SYMBOL(XO,YO,HT1,'ALPHA  = ',0.,9)                           CHAR 670
      CALL NJMBER (999.,999.,HT1,AR,0.,1)                               CHAR 680
      CALL NUMBER (X1,YO,HT1,AI,0.,1)                                   CHAR 690
C                                                                       CHAR 700
C     GAMMA RL* VALUE                                                   CHAR 710
C                                                                       CHAR 720
      YO = YO-DELY2                                                     CHAR 730
      CALL SYMBOL(XO,YO,HT1,'GAMMA* = ',0.,9)                           CHAR 740
      CALL NUMBER (999.,999.,HT1,SGAMMA,0.,4)                           CHAR 750
C                                                                       CHAR 760
C     LAMBDA VALUE                                                      CHAR 770
C                                                                       CHAR 780
      YO = YO-DELY2                                                     CHAR 790
      CALL SYMBOL(XO,YO,HT1,'LAMBDA = ',0.,9)                           CHAR 800
      CALL NUMBER (999.,999.,HT1,SLAMDA,0.,1)                           CHAR 810
C                                                                       CHAR 820
C     SYMBOL LEGEND                                                     CHAR 830
C                                                                       CHAR 840
      YO = 1.75                                                         CHAR 850
      CALL SYMBOL(XO,YO,HT1,'OCTAGON = U(REAL)',0.,17)                  CHAR 860
      YO = YO-DELY2                                                     CHAR 870
      CALL SYMBOL(XO,YO,HT1,'DIAMOND = U(IMAG)',0.,17)                  CHAR 880
      RETURN                                                            CHAR 890
      END                                                               CHAR 900
C
```

```
C
C.......................................................................
C
C
C      THE FOLLGWING CARDS COMPRISE THE DATA DECK FOR PROGRAM EIGFCN.
C
C         /*
C         //GO.SYSIN DD *
C                 .
C                 .
C         DATA DECK FROM ONE RUN OF PROGRAM PIPEO (MODENO = 1)
C                 .
C                 .
C         /*
C
C
C.......................................................................
```

```
//STB$CONT JOB (1719,0947,AX74),'SMC 1882',TIME=2
// EXEC FORTCLGW
//FORT.SYSIN DD *
C.......................................................................
C
C     PROGRAM STBCONT
C
C     PURPOSE
C
C        TO GENERATE A CONTOUR PLOT CONTAINING LINES OF INCIPIENT,
C        CRITICAL & FULLY DEVELOPED INSTABILITY USING DATA GENERATED
C        BY PROGRAM PIPEO (MODENO = 2).  THE PLOT IS GENERATED ON THE
C        NPS VERSATEC PLOTTER USING SUBROUTINE PLOTG WITH ALPHA REAL
C        ON THE X-AXIS AND ALPHA IMAGINARY ON THE Y-AXIS.
C
C.......................................................................
C
      DIMENSION X1(200), X2(200), X3(200), Y1(200), Y2(200), Y3(200)
      COMMON /ARAY/ G1(41,41),AR1(41),AI1(41)
      DATA X1,X2,X3,Y1,Y2,Y3/1200*0.0/
C
C     SET INITIAL VALUES
C
      READ (5,9) NDIM
      XMIN = -.5
      XMAX = 0.
      YMIN = 0.
      YMAX = 10.
C
C     THE NEXT 3 VALUES MUST BE SET BY THE USER PRIOR TO RUNNING
C     THE PROGRAM.
C
      SN = 47.
      SREY = 4000.
      SLAMDA = 0.0
C
C     READ STABILITY MAP VALUES OUTPUT BY PROGRAM PIPEO (MODENO =2)
C
C
      DO 1 I=1,NDIM
C
      DO 1 J=1,NDIM
      READ (5,10) AR1(I),AI1(J),G1(I,J)
    1 CONTINUE
C
C
C     COMPUTE POINTS FOR INCIPIENT,CRITICAL & FULLY DEVELOPED
C     INSTABILITY CURVES.
```

```
C
      CALL SEARCH (-1,X1,Y1,NPLT1,NDIM)
      CALL SEARCH (0,X2,Y2,NPLT2,NDIM)
      CALL SEARCH (1,X3,Y3,NPLT3,NDIM)
C
C     JUMP TO PLOT LABEL ROUTINE IF NO INCIPIENT POINTS
C
      IF (NPLT1) 8,8,2
C
C     IF POINTS COMPUTED, NEW PAGE AND WRITE THEM OUT
C
    2 WRITE (6,11)
C
      DC 3 I=1,NPLT1
      WRITE (6,12) X1(I),Y1(I)
    3 CONTINUE
C
C
C     PLOT INCIPIENT INSTABILITY POINTS
C
      CALL PLOTG(X1,Y1,NPLT1,1,0,1,'ALPHA REAL',1C,'ALPHA IMAGINARY',15,
     $ XMIN,XMAX,YMIN,YMAX,7.,7.)
C
C     LEGEND FOR INCIPIENT SYMBOL
C
      CALL NEWPEN (1)
      CALL SYMBOL(1.3,0.7,.1,'OCTAGON  = INCIPIENT INSTABILITY',0.,32)
C
C     JUMP TO PLOT LABEL ROUTINE IF NO CRITICAL POINTS
C
      IF (NPLT2) 8,8,4
C
C     IF  POINTS COMPUTED, NEW PAGE AND PRINT THEM OUT
C
    4 WRITE (6,11)
C
      DO 5 I=1,NPLT2
      WRITE (6,12) X2(I),Y2(I)
    5 CONTINUE
C
C
C     PLOT CRITICAL POINTS
C
      CALL PLOTG(X2,Y2,NPLT2,2,0,2,'ALPHA REAL',10,'ALPHA IMAGINARY',15,
     $ XMIN,XMAX,YMIN,YMAX,7.,7.)
C
C     LEGEND FOR CRITICAL SYMBOL
C
```

```
      CALL NEWPEN (1)
      CALL SYMBOL(1.3,.53,.1,'TRIANGLE = CRITICAL INSTABILITY',0.,31)
C
C     JUMP TO PLOT LABEL ROUTINE IF NO FULLY DEVELOPED POINTS
C
      IF (NPLT3) 8,8,6
C
C     IF  POINTS COMPUTED, NEW PAGE AND PRINT THEM OUT
C
    6 WRITE (6,11)
C
      DC 7 I=1,NPLT3
      WRITE (6,12) X3(I),Y3(I)
    7 CONTINUE
C
C
C     PLOT FULLY DEVELOPED POINTS
C
      CALL PLOTG(X3,Y3,NPLT3,3,0,5,'ALPHA REAL',10,'ALPHA IMAGINARY',15,
     $ XMIN,XMAX,YMIN,YMAX,7.,7.)
C
C     LEGEND FOR FULLY DEVELOPED SYMBOL
C
      CALL NEWPEN (1)
      CALL SYMBOL(1.3,.36,.1,'DIAMOND  = FULLY DEVELOPED INSTABILITY',
     $ C.,38)
C
C     LABEL THE PLOT
C
    8 CALL CHART (SN,SREY,SLAMDA)
      CALL PLOT (0.,0.,999)
      STOP
C
    9 FORMAT (I2)
   10 FORMAT (3E20.10)
   11 FORMAT ('1')
   12 FORMAT (' ',2E20.10)
      END
C
C
C......................SUBROUTINE SEARCH(NCASE,X,Y,NDIM)................
C
C     PURPOSE
C
C        TO SCAN THE STABILITY MAP FOR CHANGES OF SIGN WITH RESPECT
C        TO A SPECIFIED STABILITY VALUE AND GENERATE AN ARRAY OF X,Y
C        POINTS DEFINING A CONTOUR OF THE SPECIFIED STABILITY.
C
```

```
C      SAMPLE OF THE CALLING ARGUMENT
C
C         CALL SEARCH(NCASE,X,Y,NDIM)
C
C      DESCRIPTION OF PARAMETERS
C
C         NCASE - DEFINES THE INSTABILITY CASE (INCIPIENT,CRITICAL OR
C                 FULLY DEVELOPED) TO BE USED WHEN GENERATING THE X,Y
C                 ARRAYS.
C                     NCASE = -1   INCIPIENT INSTABILITY CRITERION
C                     NCASE =  0   CRITICAL INSTABILITY CRITERION
C                     NCASE =  1   FULLY DEVELOPED INSTABILITY CRITERION
C
C         X     - THE ARRAY OF ALPHA REAL COORDINATES DEFINING THE
C                 LOCATION OF THE POINTS OF SPECIFIED INSTABILITY.
C
C         Y     - THE ARRAY OF ALPHA IMAGINARY COORDINATES DEFINING THE
C                 LOCATION OF THE POINTS OF THE SPECIFIED INSTABILITY.
C
C         NDIM  - THE ORDER OF THE MAP ARRAY.
C
C      OTHER ROUTINES NEEDED
C
C         STATEMENT FUNCTION CRIT AND SUBROUTINE INTERP.
C
C.......................................................................
       SUBROUTINE SEARCH (NCASE,X,Y,K,NDIM)
       DIMENSION X(500),  Y(500)
       COMMON /ARAY/ G(41,41),AR(41),AI(41)
C
C      DEFINE THE STATEMENT FUNCTION CRIT(NCASE,ALPHA)
C
       CRIT(NCASE,ALPHA) = FLOAT(NCASE)*ABS(ALPHA)
C
C
       K = 0
       MDIM = NDIM-1
C
C      SEARCH FOR SIGN CHANGES BY COLUMN & INTERPOLATE FOR
C      ALPHA IMAGINARY AT WHICH SIGN CHANGE OCCURS
C
C
       DO 5 I=1,NDIM
C
       DO 5 J=1,MDIM
       IF (G(I,J)-CRIT(NCASE,AR(I))) 2,4,1
     1 IF (G(I,J+1)-CRIT(NCASE,AR(I))) 3,3,5
     2 IF (G(I,J+1)-CRIT(NCASE,AR(I))) 5,3,3
```

```
    3 Y1 = G(I,J)-CRIT(NCASE,AR(I))
      Y2 = G(I,J+1)-CRIT(NCASE,AR(I))
      CALL INTERP (AI(J),AI(J+1),Y1,Y2,AIVAL)
      K = K+1
      X(K) = AR(I)
      Y(K) = AIVAL
      GO TO 5
    4 K = K+1
      X(K) = AR(I)
      Y(K) = AI(J)
    5 CONTINUE
C
C
C     SEARCH FOR SIGN CHANGE BY ROWS AND INTERPOLATE FOR ALPHA
C     REAL AT WHICH SIGN CHANGE OCCURS
C
C
      DO 10 I=1,NDIM
C
      DO 10 J=1,MDIM
      IF (G(J,I)-CRIT(NCASE,AR(J))) 7,9,6
    6 IF (G(J+1,I)-CRIT(NCASE,AR(J+1))) 8,8,10
    7 IF (G(J+1,I)-CRIT(NCASE,AR(J+1))) 10,8,8
    8 Y1 = G(J,I)-CRIT(NCASE,AR(J))
      Y2 = G(J+1,I)-CRIT(NCASE,AR(J+1))
      CALL INTERP (AR(J),AR(J+1),Y1,Y2,ARVAL)
      K = K+1
      X(K) = ARVAL
      Y(K) = AI(I)
      GO TO 10
    9 K = K+1
      X(K) = AR(J)
      Y(K) = AI(I)
   10 CONTINUE
C
      RETURN
      END
C
C
C.......................SUBROUTINE INTERP(X1,X2,Y1,Y2,X3)................
C
C     PURPOSE
C
C        TO LINEARLY INTERPCLATE FOR THE POINT OF ACTUAL SIGN
C        CHANGE (X3) BETWEEN TWO POINTS (Y1 & Y2) OF OPPOSITE
C        SIGN. THE X-COORDINATES OF Y1 & Y2 ARE X1 & X2, RESPECTIVELY.
C
C     SAMPLE OF THE CALLING ARGUMENT
```

```
C
C        CALL  INTERP(X1,X2,Y1,Y2,X3)
C
C     DESCRIPTION OF  PARAMETERS
C
C        X1  & X2  - X-COORDINATES OF POINTS Y1 & Y2 RESPECTIVELY.
C
C        Y1  & Y2  - TWO POINTS OF OPPOSITE SIGN FOR WHICH THE POINT
C                    OF ACTUAL SIGN CHANGE (Y = 0) IS TO BE INTERPOLATED.
C
C        X3        - THE VALUE OF X FOR WHICH Y = 0.
C
C     OTHER ROUTINES NEEDED
C
C        NONE
C
C.........................................................................
C
      SUBROUTINE INTERP (X1,X2,Y1,Y2,X3)
      X3 = (X2*Y1-X1*Y2)/(Y1-Y2)
      RETURN
      END
C
C
C.............................BLOCK DATA..................................
C
C     PURPOSE
C
C        TO INITIALIZE COMMON ARRAYS G1,AR1 & AI1 TO ZERC.
C
C     SAMPLE OF CALLING ARGUMENT
C
C        NONE
C
C     DESCRIPTION OF PARAMETERS
C
C        G1   - THE MAP OF STABILITY VALUES GENERATED BY PIPEO.
C               EACH ELEMENT OF G1 IS A VALUE OF GAMMA* CORRESPONDING
C               TO A SPECFIC VALUE OF THE REAL AND IMAGINARY PARTS OF
C               THE WAVE NUMBER, ALPHA.
C
C        AR1  - THE LINEAR ARRAY OF X-COORDINATES OF THE STABILITY MAP
C               (THE REAL PART OF THE WAVE NUMBER, ALPHA).
C
C        AI1  - THE LINEAR ARRAY OF Y-COORDINATES OF THE STABILITY MAP
C               (THE IMAGINARY PART OF THE WAVE NUMBER, ALPHA).
C
C     OTHER ROUTINES NEEDED
```

```
C
C         NONE
C
C...........................................................................
C
      BLOCK DATA
      COMMON /ARAY/ G1(41,41),AR1(41),AI1(41)
      DATA G1,AR1,AI1/1681*0.0,82*0.0/
      END
C
C
C................SUBROUTINE CHART(SN,SREY,SLAMDA)...........................
C
C     PURPOSE
C
C        TO LABEL THE CONTOUR PLOT
C
C     SAMPLE OF THE CALLING ARGUMENT
C
C        CALL CHART(SN,SREY,SLAMDA)
C
C     DESCRIPTION OF PARAMETERS
C
C        SN     - THE NUMBER OF INTERIOR MESH POINTS USED  FOR THE
C                 STABILITY CONTOUR MAP BEING PLOTTED.
C
C        SREY   - REYNOLDS NUMBER
C
C        SLAMDA - THE NONUNIFORM MESH PARAMETER APPLICABLE TO THE
C                 DATA BEING PLOTTED.
C
C     OTHER ROUTINES NEEDED
C
C        ONLY BUILT-IN VERSATEC PLOTTING FUNCTIONS NEWPEN,SYMBOL &
C        NUMBER.  NOTE THAT THESE ROUTINES MAY ONLY BE ACCESSED WHEN
C        RUNNING UNDER 'FORTCLGW'.
C
C...........................................................................
C
      SUBROUTINE CHART (SN,SREY,SLAMDA)
C
C
      X0 = 2.5
      Y0 = 6.5
      HT = 0.15
      HT1 = 0.7*HT
      DELY1 = .08+HT
      DELY2 = .065+HT1
```

```
      DELX =  .1                                                        CHAR 380
C                                                                       CHAR 390
C     GRAPH TITLE                                                       CHAR 400
C                                                                       CHAR 410
      CALL NEWPEN (2)                                                   CHAR 420
      CALL SYMBOL(XO,YO,HT,'STABILITY CONTOUR PLOT',0.,22)              CHAR 430
      XO = XO+3.*DELX                                                   CHAR 440
      YC = YO-DELY1                                                     CHAR 450
      CALL SYMBOL(XO,YO,HT,'FOR THE CASE N = 0',0.,18)                  CHAR 460
C                                                                       CHAR 470
C     MESH VALUE                                                        CHAR 480
C                                                                       CHAR 490
      CALL NEWPEN (1)                                                   CHAR 500
      XO = XO+4.*DELX                                                   CHAR 510
      YO = YO-DELY1                                                     CHAR 520
      CALL SYMBOL(XO,YO,HT1,'NMESH  = ',0.,9)                           CHAR 530
      CALL NUMBER (999.,999.,HT1,SN,0.,-1)                              CHAR 540
C                                                                       CHAR 550
C     REY VALUE                                                         CHAR 560
C                                                                       CHAR 570
      YO = YO-DELY2                                                     CHAR 580
      CALL SYMBOL(XO,YO,HT1,'REY    = ',0.,9)                           CHAR 590
      CALL NUMBER (999.,999.,HT1,SREY,0.,-1)                            CHAR 600
C                                                                       CHAR 610
C                                                                       CHAR 620
C     LAMBDA VALUE                                                      CHAR 630
C                                                                       CHAR 640
      YC = YO-DELY2                                                     CHAR 650
      CALL SYMBOL(XO,YO,HT1,'LAMBDA = ',0.,9)                           CHAR 660
      CALL NUMBER (999.,999.,HT1,SLAMDA,0.,1)                           CHAR 670
C                                                                       CHAR 680
C     STABILITY AREA LABELS                                             CHAR 690
C     NOTE - SINCE THE SHAPE OF THE CURVE VARIES WITH                   CHAR 700
C            EACH SET OF INPUT DATA, THE COORDINATES OF THE FOLLCWING   CHAR 710
C            LABELS MUST BE ADJUSTED FOR EACH SPECIFIC PLOT.            CHAR 720
C                                                                       CHAR 730
      CALL NEWPEN (2)                                                   CHAR 740
      CALL SYMBOL(4.0,4.5,HT1,'SUPERCRITICAL',0.,13)                    CHAR 750
      CALL SYMBOL(5.6,4.5,HT1,'SUBCRITICAL',0.,11)                      CHAR 760
      CALL SYMBOL(6.9,4.5,HT1,'STABLE',0.,6)                            CHAR 770
      YC = 4.5-DELY2                                                    CHAR 780
      CALL SYMBOL(4.0,YO,HT1,' INSTABILITY',0.,12)                      CHAR 790
      CALL SYMBOL(5.6,YO,HT1,'INSTABILITY',0.,11)                       CHAR 800
C                                                                       CHAR 810
      RETURN                                                            CHAR 820
      END                                                               CHAR 830
C
C
```

```
C.......................................................................
C
C      THE FOLLOWING CARDS CCMPRISE THE CATA DECK FOR PROGRAM STBCONT.
C
C         /*
C         //GO.SYSIN DD *
C                 .
C                 .
C         DATA DECK FROM ONE RUN OF PROGRAM PIPEO (MODENO = 2)
C                 .
C                 .
C         /*
C
C.......................................................................
```

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c.1 Investigation of
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and results for wave
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