ESSENTIALS OF PLANE AND SOLID GEOMETRY

SMITH

In Heaven above

In the Earth beneath

And in the Water under the Earth

In the Works of Nature

In the Works of Man

THE DOMAIN OF GEOMETRY

WENTWORTH-SMITH MATHEMATICAL SERIES

ESSENTIALS OF PLANE AND SOLID GEOMETRY

BY

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PREFACE

Demonstrative geometry is taught for the purpose of giving the student an insight into deductive reasoning, of allowing him to know what it means to prove a statement, of giving him the privilege of "standing upon the vantage ground of truth," of cultivating his habits of independent investigation, of developing his own rules in applied mathematics, and of stimulating his appreciation of the beauties of the science.

In some schools the course of study permits of doing this work thoroughly, while in other schools the pressure upon the curriculum is such as to allow less time than might profitably be used. On this account it is necessary to adjust a textbook so that it may permit of such flexibility in its use as will adapt it to curricula of various kinds. To accomplish this purpose the propositions and corollaries have been limited to those that are actually necessary for the proof of subsequent statements or that are needed for a considerable number of important exercises. The lists of propositions prepared under the authority of the National Committee on Mathematical Requirements and of the College Entrance Examination Board have been followed as closely as the best principles of sequence and selection seem to warrant. The exercises have been carefully selected and have been made so numerous that any school may find abundant material for a long and thorough course, while another school may easily limit the course without destroying the sequence.

In general, the fundamental theorems are given first, ordinarily followed by the fundamental constructions to which the theorems lead. In this way the great basal propositions are so grouped as to command the special attention which they deserve. Indeed, for a brief course in geometry the other propositions, including the numerical work and circle measurement, might be omitted or else referred to informally in the relatively few cases in which they are needed in subsequent proofs.

Among other topics the Supplement contains a treatment of the practical mensuration of plane and solid figures along the lines recognized by the College Entrance Examination Board as furnishing valuable replacement material for some of the more formal work in Books VI–VIII. This feature satisfies a frequent demand for a modern type of training in spatial perception and supplements the logical presentation of the standard propositions.

Among the special features of the work may be mentioned the selection and arrangement of propositions, the simplicity of language and of proofs, the introduction to independent demonstration, the statements of the plan of proof, the applications, the improved typography, and the emphasis secured through the framing of the diagrams.

My long and intimate association with my lamented colleague, George Wentworth, who, unfortunately, died before this book was undertaken, and the life-long influence of the sound principles established by his father, George A. Wentworth, have, I venture to hope, qualified me to write in the spirit which has made the mathematical textbooks bearing the Wentworth name of such inestimable service to more than one generation of teachers and students.

DAVID EUGENE SMITH

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SYMBOLS AND ABBREVIATIONS

The following are the most important symbols used :

$+$	plus	$\angle$	angle
$-$	minus	$\triangle$	triangle
$\times, \cdot$	times	$\square$	rectangle
$\div, \_/, :$	divided by	$\varparallelogram$	parallelogram
$\sqrt{\ }$	square root of	$\odot$	circle
$\sqrt[3]{\ }$	cube root of	st.	straight
$=$	is equal to, equals, is equivalent to	rt.	right
a^2	square of a	$A', A'', A''', \cdots$	A-prime, A-second, A-third, $\cdots$
a^3	cube of a	$A_1, A_2, A_3, \cdots$	A-one, A-two, A-three, $\cdots$
$\cdots$	and so on	Ax.	axiom
$>$	is greater than	Post.	postulate
$<$	is less than	Const.	construction
$\therefore$	therefore	Def.	definition
$\rightarrow$	tends to	Cor.	corollary
$\parallel$	parallel	Iden.	identical
$\perp$	perpendicular		

Symbols of aggregation are used as explained in the text.

There is no generally accepted symbol for "is congruent to." The sign $=$ is commonly employed, the context telling whether equality, equivalence, identity, or congruence is to be understood ; but teachers often use $\cong$, $\simeq$, or $\equiv$ for congruence, and $\backsim$ or $\sim$ for similarity. The symbol $\equiv$ is also used for identity, but is rarely needed in geometry.

There is no generally accepted symbol for "arc." Some teachers recommend using $\widehat{AB}$ for "arc AB," and this symbol has certain advantages.

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PLANE GEOMETRY

INTRODUCTION

I. Common Terms Explained

1. Nature of Geometry. We are now about to begin another branch of mathematics, one not chiefly relating to numbers, although it uses numbers, and not primarily devoted to equations, although it uses them, but one that is concerned principally with the study of forms, such as triangles, parallelograms, and circles. Many facts that are stated in arithmetic and algebra are proved in geometry.

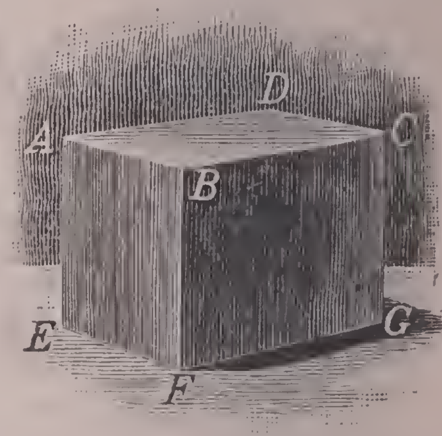
2. Terms already Known. The student already has considerable familiarity with the terms that he will need to use. For example, he has a fairly good idea of such terms as straight line, curve, right angle, acute angle, triangle, square, and circle. In the case of certain of these terms it is unnecessary and even undesirable for the student to give the time and thought essential to the wording of a careful definition.

3. Precise Definitions. In the case of other terms, however, precise definitions are necessary, for the reason that we make use of such definitions in proving certain important statements to be studied later.

Unless the student is specifically told that it is necessary to memorize a definition, it will be sufficient if he is able to use the terms correctly.

4. Surface, Line, Point. The *solid* here shown has six flat *faces*, each being a *rectangle*. It is called a *rectangular solid*.

Statements and terms that should be considered most carefully, although informally, are printed in italic type.



Each of the six flat faces is part of the *surface* of the solid, and each is itself called a *surface*.

If each of these flat faces is so smooth that when a straight ruler lies upon it in any position all points of the ruler touch the surface, the flat face is called a *plane surface* or simply a *plane*.

A surface has length and breadth, but no thickness.

In all such cases, examples in the classroom should be noticed.

In the above figure some of the faces meet in *lines*, and these lines are the *edges* of the solid.

The way in which faces and lines are named will be understood from the statement that the faces *AEFB* and *ABCD* meet in the line *AB*.

A line has length but neither breadth nor thickness.

We may represent a line by a mark, but a mark is really a very thin solid made of chalk, ink, or some other writing material.

We commonly speak of solids, surfaces, and lines as *magnitudes*.

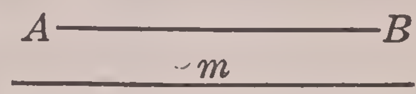
In the above figure the lines *BC* and *CD* meet in the *point C*, a *vertex* of the solid, and one of the eight *vertices*.

A point has position but not size.

A point, a line, a surface, or a solid, or any combination of these, is called a *geometric figure* or simply a *figure*.

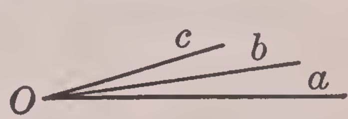
Plane geometry considers figures of which all parts lie in one plane.

5. Lines. The figures AB and m here shown represent *straight lines.* When no misunderstanding is likely to arise, a straight line is called simply a *line.* Thus, we speak of the line AB and the line m, meaning straight lines.



Lines and surfaces are supposed to extend indefinitely far unless the contrary is stated. If we wish to speak of part of a line limited by two points, we call it a *line segment* or simply a *segment.* In this figure, PQ is a line segment, since it is a definite part of the unlimited line l.

If we wish to speak of a line beginning at a certain point O and extending indefinitely, we call it a *ray.* In this figure a, b, c are rays.

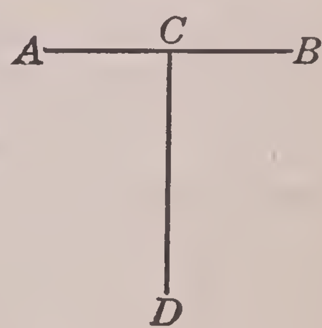


When no misunderstanding is likely to arise, it is customary to use the word "line" instead of "segment" or "ray."

A line of which no part is straight is called a *curve line* or simply a *curve.* The line AB here shown is a curve line.

Two straight-line segments that can be placed one upon another so that their end points coincide are said to be *equal.*

In this figure, $AB = CD$, as may be seen by measuring with compasses. By putting one point of the compasses at C and the other at D, and then, without changing the opening of the compasses, putting one point at A and the other at B, we can transfer CD to AB.



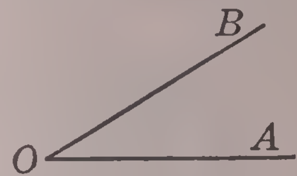
In the line l here shown, AC is the *sum* of AB and BC; that is,

$$AC = AB + BC.$$

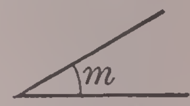
Also, BC is the *difference* between AC and AB; that is,

$$BC = AC - AB.$$

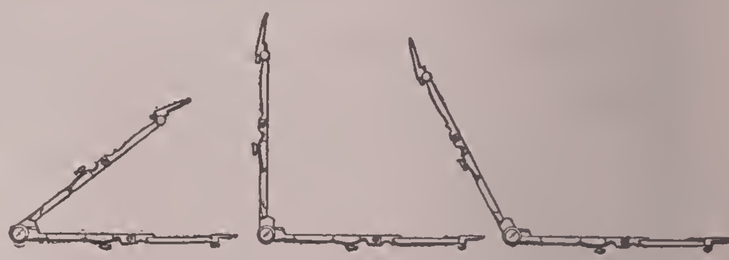
6. Angles. If two rays proceed from the same point, they form an *angle*. In this figure the rays OA and OB form the angle AOB. The *vertex* of this angle is O and the *arms* or *sides* are OA and OB.



When no misunderstanding is likely to arise, an angle may be named by the letter at the vertex or by a small letter within the angle, as in the cases of angles O and m here shown. If three letters are necessary, the middle one represents the vertex, as in the angle AOB above.

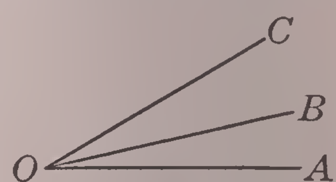


The *size* of an angle depends upon the amount of turning necessary to bring one arm to coincide with the other. Thus, taking these compasses, we see that the first angle is *less* than the second, and that the second is less than the third.



We commonly measure angles in degrees, a right angle being 90°. In the above case the three figures are angles of 40°, 90°, and 120°, approximately.

In this figure, angle AOB is *less* than angle AOC, angle AOC is *greater* than angle AOB, angle AOC is the *sum* of angles AOB and BOC, and angle AOB is the *difference* between angle AOC and angle BOC; that is,



$$\angle AOB < \angle AOC,$$

$$\angle AOC > \angle AOB,$$

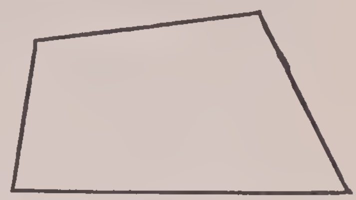
$$\angle AOC = \angle AOB + \angle BOC,$$

and

$$\angle AOB = \angle AOC - \angle BOC.$$

Students are advised to provide themselves with compasses, a ruler, and a protractor for drawing figures.

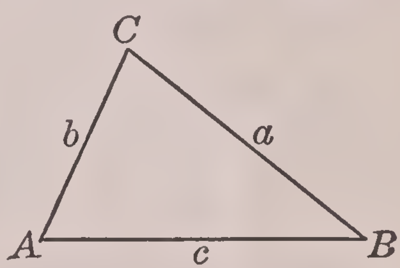
7. Rectilinear Figure. A figure which lies wholly in one plane and which represents a surface that is bounded by segments of straight lines is called a *plane rectilinear figure* or simply a *rectilinear figure.*



The segments are called the *sides* of the figure, and the *adjacent sides* meet in the *vertices* of the figure. The sum of all the sides is the *perimeter* of the figure.

In modern geometry the bounding line is also considered as the figure, and the perimeter as the total *length* of this line. In this book, unless the contrary is stated, only those figures will be considered in which each of the angles within the figure is less than two right angles.

8. Triangle. A rectilinear figure of three sides is called a *triangle.*



A triangle is conveniently lettered as here shown. The small letters represent the sides and correspond to the large letters at the opposite vertices.

The side upon which a triangle or any other rectilinear figure is supposed to stand is considered as the *base* of the figure.

The vertex opposite the base of a triangle is called *the vertex* of the triangle. Although a triangle has three vertices, it has only one that is called *the* vertex.

In the above triangle:

The three vertices are A, B, C, and the three sides are designated as a, b, c or as BC, CA, AB respectively.

The vertex is C and the base is c.

The perimeter is $a+b+c$, or $BC+CA+AB$.

The angles are BAC, CBA, ACB.

The various types of triangles and other common rectilinear figures will be considered later, when the necessity arises.

Exercises. Review of Common Terms

Draw the following figures, writing the name under each:

1. Rectangle.
2. Solid.
3. Curve.
4. Rays.
5. Triangle.
6. Angle.
7. Rectilinear figure.
8. Straight line.
9. Line segment.

Draw a figure representing each of the following:

10. The sum of two line segments.

11. The difference between two line segments.

12. The sum of two angles; of three angles.

13. The difference between two angles.

14. A rectangular solid has how many edges? how many faces? how many vertices?

15. By counting the edges, faces, and vertices of a rectangular solid find the number to be added to the number of edges to equal the sum of the faces and vertices.

This law, which is useful in the study of crystals, holds for all ordinary forms of solids bounded by planes. The student may be interested to try it with a pyramid or any other convenient solid.

16. Use a ruler to find out whether the top of your desk is approximately a plane as described in § 4.

Of course, no such surface is exactly a perfect plane.

17. Draw four angles, a, b, c, d such that $a < b < c < d$.

Consult the table of symbols and abbreviations when symbols are not clearly understood.

18. Draw a curve of such shape that a straight line can cut it in four points and only four.

19. Draw a figure showing the number of points in which one straight line can intersect another.

II. Definitions

9. Nature of Definitions. In §§ 10–22 we shall consider certain definitions which are so important that the student will find it convenient to memorize them, at least in substance, because they are frequently needed in proving other statements.

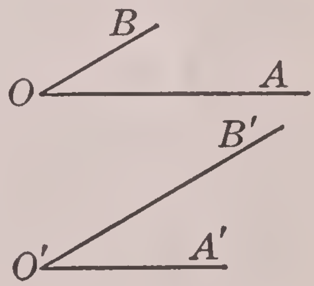
It should be understood that these definitions can be turned around; that is, if we say that certain conditions make a right angle, it follows that a right angle implies these conditions. In other words,

A definition can be inverted.

For example, if the organ of sight is called an eye, then an eye is the organ of sight.

This is mentioned at the present time because the student will occasionally find it convenient to invert a definition.

10. Equal Angles. If either of two angles can be placed on the other so that they coincide, the two are called *equal angles.*

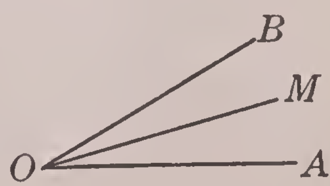


For example, these two angles are equal, all lines being supposed to be indefinitely long. The amount of turning necessary to make one angle is evidently the same as that necessary to make the other.

In speaking of two figures that resemble each other it is often convenient to use primes (′) in lettering one of them. In the above case $\angle A'O'B'$ is read "angle A-prime O-prime B-prime."

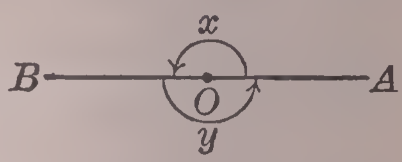
11. Bisector. A point, a line, or a plane that divides a geometric magnitude into two equal parts is called a *bisector* of the magnitude.

For example, M, the *midpoint* of the line AB, is a bisector of the line. Common sense will tell the student the meaning of such simple terms as midpoint.



Similarly, we may have a bisector of an angle; for example, OM bisects the $\angle AOB$ here shown.

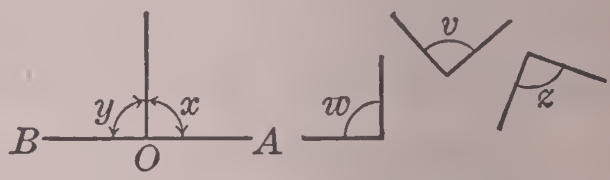
12. Straight Angle. If the arms of an angle extend in opposite directions so as to be in one straight line, the angle is called a *straight angle.*



For example, both x and y in this figure are straight angles, x being formed by turning the arm OA halfway around the vertex O.

A straight angle contains 180°; hence two straight angles contain 360°.

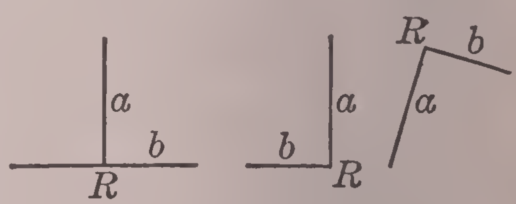
13. Right Angle. Half of a straight angle is called a *right angle.*



For example, x and y are evidently halves of the st. $\angle AOB$ and hence they are right angles; w, v, and z are also right angles.

It follows from the definition that two right angles make a straight angle and that four right angles fill the space about a point.

14. Perpendicular. If one line meets another so as to make a right angle with it, either of the two lines is said to be *perpendicular* to the other.



In each of these figures, R is the vertex of a right angle; hence in each figure, a is perpendicular to b, and b is also perpendicular to a.

The line a is called *a perpendicular* to b, and b a perpendicular to a.

A line that is perpendicular to a line segment and also bisects it is called a *perpendicular bisector* of the segment.

The point R in each figure is called the *foot* of the perpendicular to b, or the foot of the perpendicular to a.

The terms *horizontal*, *vertical*, *oblique*, and *slanting*, referring to lines, are used informally in geometry with the usual meaning with which the student is familiar.

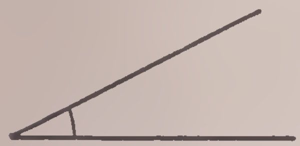
15. Square. A rectilinear figure of four equal sides and four right angles is called a *square.*

This figure is too well known to require illustrating.

The line joining opposite vertices of a square is called the *diagonal*, a term which we shall define later in connection with other figures.

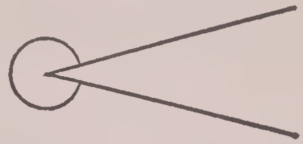
16. Angles further classified. An angle is called

an *acute angle* if it is less than a right angle;
an *obtuse angle* if it is greater than a right angle;
a *reflex angle* if it is greater than a straight angle.



Acute Angle

Obtuse Angle

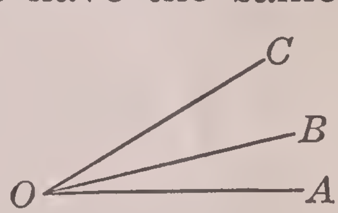


Reflex Angle

Acute and obtuse angles are called *oblique angles,* and each arm is said to be *oblique* to the other arm.

If a wheel turns through more than 180°, each spoke turns through a reflex angle. If it turns through 360°, each spoke turns through a *perigon,* a term occasionally convenient. The wheel may, of course, turn through as many degrees as we please. If we speak of an $\angle O$, however, we mean the $\angle O$ less than 180° unless the contrary is stated.

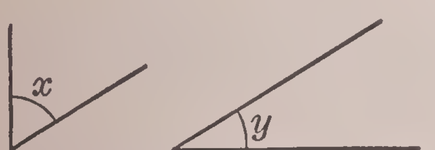
17. Adjacent Angles. Two angles which have the same vertex and a common arm between them are called *adjacent angles.*



For example, in this figure $\measuredangle AOB$ and BOC are adjacent angles.

18. Angles classified by Sums. If the sum of two angles is a right angle, each is called the *complement* of the other, and the two angles are called *complementary angles.*

If the sum of two angles is a straight angle, each is called the *supplement* of the other, and the two angles are called *supplementary angles.*



Complementary Angles

Supplementary Angles

It may be assumed that if the sum of two adjacent angles is a straight angle, their exterior sides form a straight line.

19. Triangles classified as to Sides. A triangle is called an *isosceles triangle* when two of its sides are equal; an *equilateral triangle* when all its sides are equal.

Isosceles Equilateral

The word "equilateral" means equal-sided. It is applied to any figure having equal sides.

An equilateral triangle is a special kind of isosceles triangle.

An isosceles triangle is usually represented as resting on the side which is not equal to either of the other sides. This side is called the *base*, and the opposite vertex is called *the vertex* of the triangle. Ancient writers often spoke of the equal sides as the *legs* of the isosceles triangle, the word "isosceles" meaning equal-legged.

If no two sides of a triangle are equal, the triangle is called a *scalene triangle*, but the term is not commonly used.

20. Triangles classified as to Angles. A triangle is called

a *right triangle* when one angle is a right angle;
an *obtuse triangle* when one angle is an obtuse angle;
an *acute triangle* when all its angles are acute angles;
an *equiangular triangle* when all its angles are equal.

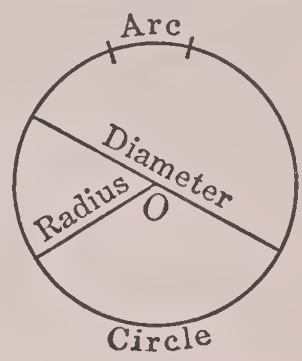
Right Obtuse Acute Equiangular

In a right triangle the side opposite the right angle is called the *hypotenuse*.

The other two sides of a right triangle are often called simply the *sides* when no confusion is likely to arise.

Since ancient writers usually represented the hypotenuse as the base, the other two sides were called the *legs* of the right triangle.

21. Circle. A closed curve lying in a plane and such that all its points are equally distant from a fixed point in the plane is called a *circle*.



When we *draw* a circle we sometimes say that we *describe* a circle. Either word, "draw" or "describe," may be used in this sense. When a circle is drawn with the compasses we often say that we *construct* it.

22. Terms relating to a Circle. The point in the plane from which all points on the circle are equally distant is called the *center* of the circle.

A circle is commonly named by the letter at the center. In the above figure we may designate the circle as the $\odot O$.

Any one of the equal straight-line segments which extend from the center of a circle to the circle itself is called a *radius* (plural "radii").

A straight line through the center and terminated at each end by the circle is called a *diameter*.

It is evident that a diameter is equal in length to two radii.

Any portion of a circle is called an *arc*.

The length of the circle, that is, the distance around the space inclosed, is called the *circumference*.

Formerly the term *circle* was used to mean the part of the plane inclosed, and the bounding line was then called the circumference.

An arc that is half of a circle is called a *semicircle*. The length of a semicircle is called a *semicircumference*.

An arc less than a semicircle is called a *minor arc*; an arc greater than a semicircle is called a *major arc*.

The word "arc" alone may be taken to mean a minor arc.

23. Lines of Elementary Geometry. The straight line and the circle, or parts of such figures, are the only lines used in elementary geometry.

Exercises. Meaning of Terms

1. Draw four right angles in different positions.

All the drawings required on this page may be made freehand or by the aid of a ruler as the teacher may direct. At present the purpose is to fix in mind the meaning of the terms.

2. Draw four lines in different positions and then draw three lines perpendicular to each of the four lines.

3. Draw a horizontal line and a vertical line that intersect. What kind of angle is formed?

4. Draw four acute angles of different sizes.

5. Draw an obtuse angle that is equal to the sum of a right angle and one of the acute angles of Ex. 4.

6. Draw any acute angle and then draw its complement and its supplement.

The protractor may be used advantageously in such cases.

7. Draw three straight lines intersecting by twos. They may determine one point or how many points?

If the word "determine" is not clearly understood, it should be considered in class. We say that *in general* three lines determine three points, meaning that this is the greatest number that they determine, although in special cases, as the student should show, they may determine two points, one point, or no point.

8. Through how many degrees does the minute hand of a clock turn in $\frac{1}{4}$ hr.? in 20 min.? in 45 min.? in $1\frac{1}{2}$ hr.?

9. If a radius $3\frac{5}{8}$ in. long is used in drawing a circle, and if the circumference is $\frac{22}{7}$ times the diameter, find the circumference.

10. If the supplement of $\angle x$ is $4x$, how many degrees are there in each angle?

11. If the complement of $\angle m$ is $3m$, how many degrees are there in each angle?

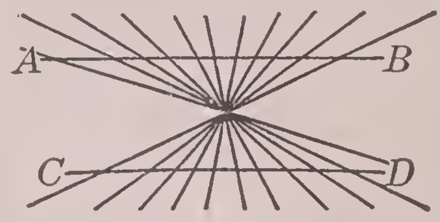
III. Demonstrative Geometry

24. Need for Demonstrative Geometry. In looking at geometric figures we often find that we make mistakes if we judge by appearances. It is partly on this account that we need to demonstrate the truth of our judgments.

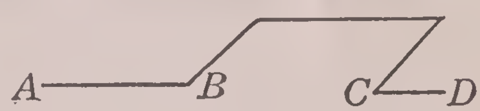
For example, state which is the longer line, *AB* or *XY*, and estimate how many sixteenths of an inch longer it is.

Then test your results by measuring with the compasses or with a carefully marked piece of paper.

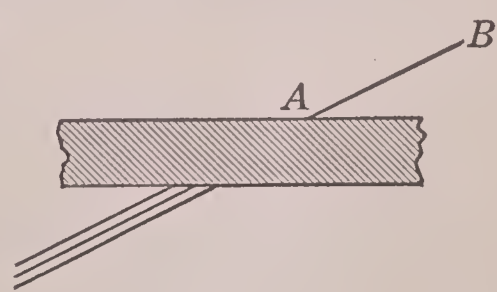
Look at this figure and state whether *AB* and *CD* are both straight lines. If one of them is not a straight line, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.



Look at this figure and state whether the line *AB* will, if prolonged, lie on *CD*. Test your answer by laying a ruler along the line *AB*.



Look at this figure and state which of the three lower lines is *AB* prolonged. Then test your answer by laying a ruler along *AB*.



25. Bases for Proof. The proofs of geometry are based upon certain assumptions known as axioms and postulates. Since these assumptions do not depend merely upon the observation of figures, but upon common sense, they are universally accepted as the foundations upon which we may safely build our work.

26. Axiom. A general statement admitted without proof is called an *axiom*. The following axioms should be memorized; others will be assumed when needed.

All numbers and magnitudes referred to in the axioms are considered as positive.

1. *If equals are added to equals, the sums are equal.*

For example, since $9 = 5 + 4$
and $5 = 3 + 2$
we see at once that $9 + 5 = 5 + 4 + 3 + 2$
or $14 = 14$

Likewise, if $a = 3$ and $b = 7$, then $a + b = 3 + 7 = 10$.

2. *If equals are subtracted from equals, the remainders are equal.*

For example, since $9 = 5 + 4$
and $3 = 2 + 1$
we see at once that $9 - 3 = 5 + 4 - 2 - 1$
or $6 = 6$

Likewise, if $a = 10$ and $x = 3$, then $a - x = 10 - 3 = 7$.

3. *If equals are multiplied by equals, the products are equal.*

For example, since $12 = 15 - 3$
and $2 = 2$
we see at once that $2 \times 12 = 2 \times 15 - 2 \times 3$
that is, $24 = 30 - 6$
or $24 = 24$

Likewise, if $\frac{1}{2}x = 7$, then $x = 2 \times 7 = 14$.

4. *If equals are divided by equals, the quotients are equal.*

For example, since $16 = 9 + 7$
we see at once that $16 \div 4 = (9 + 7) \div 4$
that is, $4 = \frac{9}{4} + \frac{7}{4}$
or $4 = \frac{16}{4} = 4$

The divisor must never be zero, division by zero having no meaning.

5. *A number or magnitude may be substituted for its equal.*

For example, if $a + x = b$ and if $x = y$, then $a + y = b$.

If $b > x$ and if $x = y$, then $b > y$.

If $x = b - a$ and if $y = b - a$, then $x = y$.

The student should make up other examples to illustrate this axiom.

As a special case this axiom is often stated as follows: *Quantities equal to the same quantity are equal to each other.*

The word "quantity" here refers to numbers or magnitudes.

6. *Like powers or like roots of equal numbers are equal.*

That is, if $x = 2$, then $x^2 = 2^2$, or $x^2 = 4$. Also, if $x^3 = 27$, then $x = 3$.

7. *If equals are added to or subtracted from unequals, or if unequals are multiplied or divided by equals, the results are unequal in the same order.*

This means that if $x > y$ and if $a = b$, then

$$x + a > y + b \qquad ax > by$$
$$x - a > y - b \qquad x \div a > y \div b$$

The student should illustrate each of the above cases by numerical examples, using the values $x = 10$, $y = 5$, $a = b = 2$, or others if desired.

If $x < y$ the above signs of inequality will all be reversed.

8. *If unequals are added to unequals in the same order, the sums are unequal in the same order; if unequals are subtracted from equals the remainders are unequal in reverse order.*

If $a > b$, $c > d$, and $x = y$, then $a + c > b + d$, and $x - a < y - b$.

The student should illustrate as in Ax. 7.

9. *If the first of three quantities is greater than the second, and the second is greater than the third, then the first is greater than the third.*

Thus, if $a > b$ and if $b > c$, then $a > c$.

10. *The whole is greater than any of its parts and is equal to the sum of all its parts.*

27. Postulate. In geometry a geometric statement admitted without proof is called a *postulate*. The following postulates of plane geometry should be memorized; others will be assumed when needed.

In considering the postulates, the student should draw a figure to illustrate each one.

1. *One straight line and only one can be drawn through two distinct points.*

This postulate is sometimes more conveniently expressed in one of the following forms:

Two distinct points determine a line.

Two straight lines cannot intersect in more than one point.

For if they intersected in two points, the lines would coincide.

Post. 1 may be given as the authority for any one of the above three statements.

2. *A straight-line segment can be produced to any required length.*

To produce AB is to extend it through B; to produce BA is to extend it through A.

In the figures in this book, lines produced are generally represented by dotted lines, as shown in § 48.

3. *A straight-line segment is the shortest path between two points.*

Since distance in a plane is measured on a straight line, this postulate is sometimes stated as follows: *A straight line is the shortest distance between two points.* More properly speaking, however, distance is the *length* of the line instead of the line itself.

4. *In a plane one and only one circle can be constructed with any given point as center and any given line segment as radius.*

From the definition of a circle and from this postulate we see and may hereafter state that *all radii of the same circle are equal.*

5. *Any figure can be moved without altering its shape or size.*

That is, we may think of a triangle as moved about without any change in shape or size, and similarly for any other figure.

6. *All straight angles are equal and all right angles are equal.*

The second part of the postulate follows from the first, because a right angle is half a straight angle.

7. *A line segment can be bisected, and in one and only one point.*

The student should show the reasonableness of this postulate by means of a figure.

8. *An angle can be bisected, and by one and only one line.*

9. *Angles which have equal complements or equal supplements are equal.*

For example, if the complement of $\angle x$ is 22°, and the complement of $\angle y$ is also 22°, this means that

$$x + 22° = 90°$$

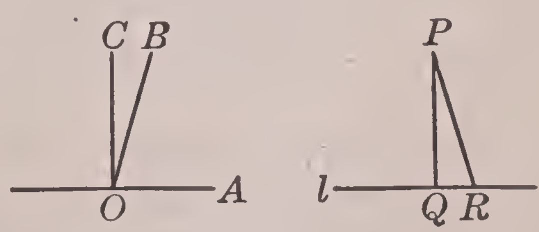
and $$y + 22° = 90°.$$

Then $$x + 22° = y + 22°. \qquad \text{Ax. 5}$$

$$\therefore x = y. \qquad \text{Ax. 2}$$

10. *There is one and only one line which, passing through a given point, is perpendicular to a given line.*

Since a perpendicular to a line makes a right angle with it, and since we cannot, in the first of these figures, have $\angle AOB = \angle AOC$ (Ax. 10), we cannot have two perpendiculars through O.



If a line swings about P as a center, it may be assumed for the moment that there is only one position at which PQ is $\perp$ to l. It is easily proved later that this assumption is true.

As the student proceeds he will find that some of the other postulates, assumed for the present as true, can also be proved.

28. Theorem. A statement which is to be proved is called a *theorem.*

For example, it is stated in arithmetic that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. This statement is one of the most important theorems of plane geometry, and we shall prove it later.

29. Problem. A construction which is to be made so that it shall satisfy certain given conditions is called a *problem.*

For example, it may be required to construct an angle equal to a given angle. This construction will be made in § 106.

30. Proposition. The statement of a theorem to be proved or of a problem to be solved is called a *proposition.*

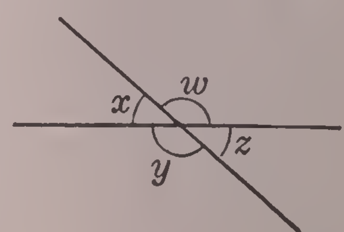
In geometry, therefore, a proposition is either a theorem or a problem. We shall find that the first group of propositions is made up of theorems. After we have proved a number of theorems we shall solve some of the most important problems.

31. Corollary. A statement that follows from another statement with little or no proof is called a *corollary.*

For example, since we admit that all straight angles are equal, it follows as a corollary that all right angles are equal, since a right angle is half a straight angle.

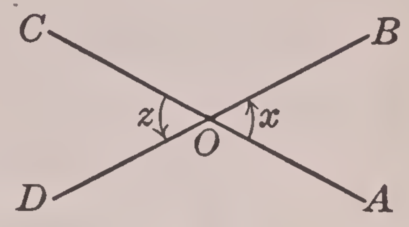
32. How Propositions are Proved. We have said that we are now about to prove our statements in geometry, and we shall first see what is meant by a proof. For this purpose we shall take a simple proposition concerning vertical angles, a term which we must first define.

33. Vertical Angles. When two angles have the same vertex and the sides of one are prolongations of the sides of the other, these angles are called *vertical angles.*



In the figure here shown, x and z are vertical angles, and so are w and y.

34. Study of a Figure. Suppose that we consider the question of vertical angles with respect to the figure here shown. Does there appear to be in the figure any other angle equal to x? If so, which angle is it?



The amount of turning of the ray OA about O to make the $\angle x$ is the same as the amount of turning of what other ray about O to make the $\angle z$?

Then how does the amount of turning necessary to produce any angle compare with the amount of turning necessary to produce its vertical angle?

What does this lead you to infer as to the equality of x and z? as to the equality of any other vertical angles?

Let us now see how we can prove that any angle is equal to its vertical angle by referring to the axioms or postulates instead of considering the amount of turning necessary to produce the two angles.

In the above figure, which angle is the supplement of both x and z?

Then how does the supplement of x compare with the supplement of z?

What does Post. 9 tell us with respect to angles which have equal supplements?

What can then be said about the equality of x and z?

What other two angles in the figure are equal?

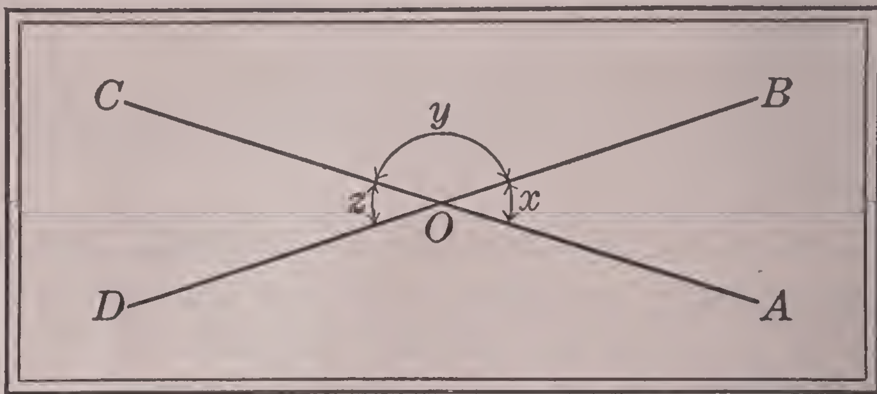
Write and complete the following statement:

If two lines intersect, the vertical $\cdots$.

The student has now seen how to prove a proposition, not by trusting to appearances but by depending only upon a definition and a postulate. The definition was that of the supplement of an angle (§ 18), and the postulate was Post. 9 as mentioned above. In § 35 we shall show how this proof looks when stated more systematically and in proper geometric form.

Specimen Proposition. Vertical Angles

35. Theorem. *If two lines intersect, the vertical angles are equal.*



Given the lines AC and BD intersecting at O and forming $\angle$s x, y, and z as shown.

Prove that $x = z$.

Proof. $x + y =$ a st. $\angle$, § 12

because their arms extend in opposite directions so as to be in one st. line.

Likewise, $y + z =$ a st. $\angle$. § 12

$\therefore y$ is the supplement of x and also of z. § 18

If the sum of two $\angle$s *is a st.* $\angle$, *each is called the supplement of the other.*

$\therefore x = z$, Post. 9

because $\angle$s *which have equal complements or equal supplements are equal.*

36. Nature of a Proof. From § 35 it is seen that there are three steps in proving a theorem: (1) stating *what is given* (sometimes called the *hypothesis*), (2) stating *what is to be proved* (sometimes called the *conclusion*), and (3) giving the *proof*, each statement of which is supported by a definition, an axiom, a postulate, or a proposition previously proved.

BOOK I

RECTILINEAR FIGURES

I. Fundamental Theorems

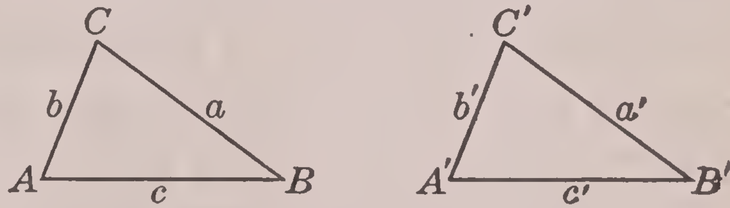
37. Congruent Figures. If two figures have exactly the same shape and size, they are called *congruent figures.*

For example, the two triangles shown below (§ 38) are congruent (cŏn̲′gru-ent) figures, and are said to be congruent. Similarly, two circles with equal radii are congruent.

If two figures can be made to coincide in all their parts, they are congruent figures.

By the parts of a figure we mean the sides, angles, and surface.

38. Corresponding Parts. It is customary to letter the angles of a triangle by capitals arranged about the figure in counterclockwise order; that is, reading about the figure in the direction opposite to that in which the hands of a clock move.



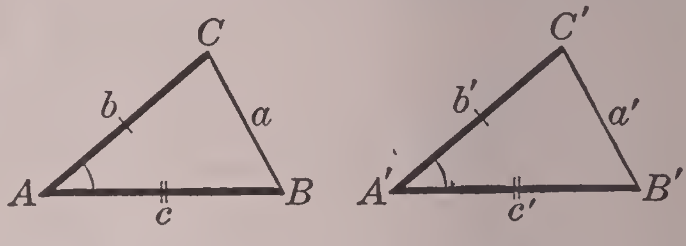
Exceptions to this custom are mentioned later, as occasion arises.

In the triangles shown above, A' *corresponds* to A, B' corresponds to B, C' corresponds to C, a' corresponds to a, and so on; that is, these pairs of parts are respectively equal. It is therefore evident that

In two congruent figures the parts of one figure are equal respectively to the corresponding parts of the other figure.

Some writers speak of corresponding parts as *homologous parts.*

39. Inference as to Congruent Triangles. When we examine two triangles we may easily infer certain facts relating to them. For example, as we look at these triangles, in which $\angle A = \angle A'$, $b = b'$, and $c = c'$, the triangles seem to be congruent. The question is: Are they necessarily congruent?



It aids the eye if we mark the equal corresponding parts in some such way as the one used in the above figures.

In order to aid the beginner, in the figures of Book I the important lines used in the proofs are made heavier than the others and the important angles are appropriately marked. This scheme is not used after Book I.

Teachers will see the objections to the use of colored crayons to designate corresponding parts except, perhaps, in the case of a few propositions. The student should early become familiar with the tools that he will actually use, the black lead pencil and the white crayon.

To prove that the two triangles are congruent let us see if one triangle can be placed upon the other so as to coincide with it. To help us see this clearly we may, if we wish, cut two triangles out of paper.

Suppose that $\triangle ABC$ is placed upon $\triangle A'B'C'$ so that the point A lies on the point A' and c lies along c'; then where does the point B lie, and why?

On what line does b then lie, and why?

Then where must C lie, and why?

Having found where B and C lie, where does a lie?

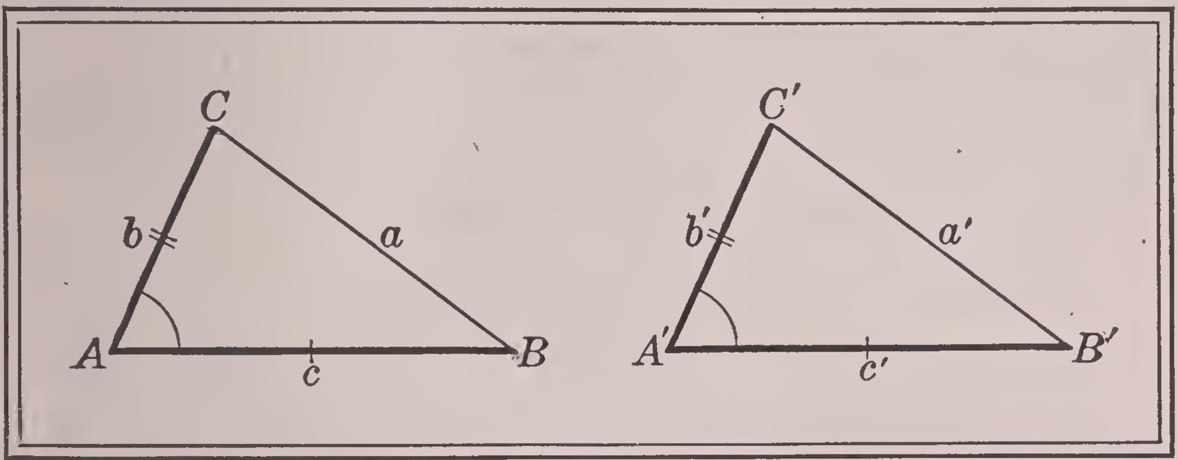
What have we now shown with respect to the coinciding of $\triangle ABC$ with $\triangle A'B'C'$? Are the triangles congruent?

Complete the following statement: *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles* $\cdots$.

The statement and formal proof are given in § 40.

Proposition 1. Two Sides and Included Angle

40. Theorem. *If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another, the triangles are congruent.*



Given the $\triangle ABC$ and $A'B'C'$ with $c=c'$, $b=b'$, and $\angle A=\angle A'$.

Prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

The plan is to place one upon the other and show that they coincide.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A lies on A' and c lies along c', C and C' lying on the same side of c'. Post. 5

Then B lies on B',

because c is given equal to c';

b lies along b',

because $\angle A$ is given equal to $\angle A'$;

and C lies on C',

because b is given equal to b'.

Hence a coincides with a'. Post. 1

One st. line and only one can be drawn through two distinct points.

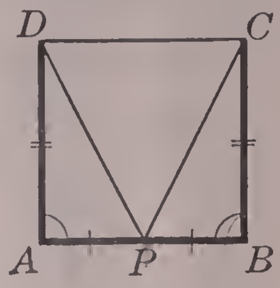
$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$, § 37

by the definition of congruent figures.

This method of proof is called the method of *superposition*.

Exercises. First Congruence Theorem

1. If $ABCD$ is a square and P is the midpoint of AB, prove that $PD = PC$.



The student should write the work in the following form:

Given a square $ABCD$ and P, the midpoint of AB.

Prove that $PD = PC$.

Proof. $AP = BP$,

because P is given as the midpoint of AB.

$AD = BC$.

(*Give the reason from* § *15.*)

$\angle A = \angle B$.

(*Give the reasons from* § *15 and Post. 6.*)

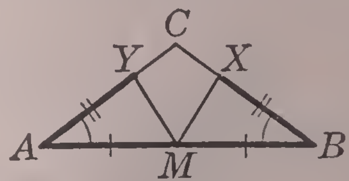
Hence $\cdots$ (*state what follows from* § *40 and give the reason*).

$\therefore PD = PC$.

(*Give the statement at the end of* § *38.*)

When proofs are written on wide sheets of paper, some teachers require students to rule the page vertically in the center and to write the statements on the left side of the line and the full authority for each statement on the right side. Such an arrangement is sometimes convenient, although it is not as concise as the form suggested above, which is used in many standard textbooks.

2. In this figure, if $\angle A = \angle B$, if M bisects AB, and if $AY = BX$, prove that $MY = MX$.



The student should begin the work as follows:

Given $\angle A = \angle B$, M bisecting AB, and $AY = BX$.

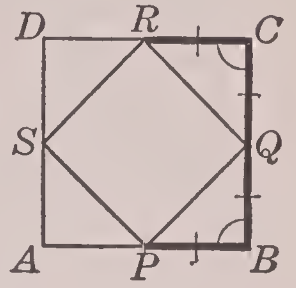
Prove that $MY = MX$.

In the proof the student should see that he can show that $MY = MX$ if he can show that $\triangle AMY$ is congruent to $\triangle BMX$, and that he can show this if $\cdots$, and so on.

When two figures are arranged as above, with the corresponding letters of one in an opposite order from those of the other, it is much better to read one set counterclockwise and the other clockwise, as in the above statement, so as to have the letters correspond more clearly.

3. In the square $ABCD$ the points P, Q, R, S bisect the consecutive sides. Prove that $PQ = QR = RS = SP$.

In this case the student will save time by first proving that $PQ = QR$, beginning as follows:



Given the square $ABCD$ **with** P, Q, R, S **bisecting** AB, BC, CD, DA **respectively.**

Prove that $PQ = QR$.

In the proof show first that

$$AB = BC = CD;$$

then that $PB = BQ = QC = CR$;

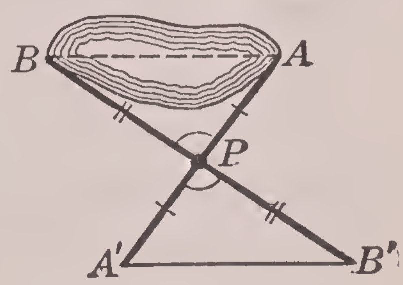
then that $\angle B = \angle C$.

Then show that $\triangle PBQ$ and QCR are congruent.

What follows?

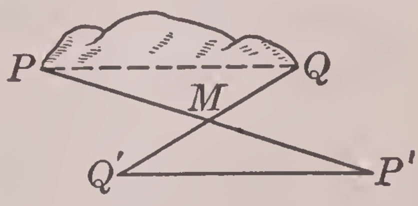
It is now unnecessary to prove the other triangles congruent, for evidently this can be done in precisely the same way. Simply write, "Similarly, the other $\triangle$ are congruent, and hence $PQ = QR = RS = SP$." When such methods of shortening the proof are used, the student must be sure that the cases are exactly similar.

4. Prove that to determine the distance AB across a pond one may sight from A across a post P, place a stake at A' making $PA' = AP$, then sight along BP making $PB' = BP$, and finally measure $A'B'$.



What is given? What is to be proved? Write these statements and then write the proof.

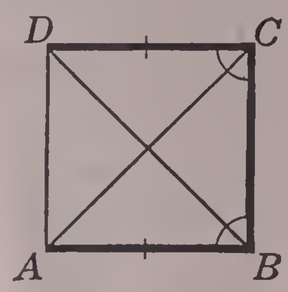
5. Show how to find the distance from a point P west of a hill to a point Q east of the hill, using the figure here shown.



State what measurements you would make on the ground. Then write the proof as in the preceding cases.

In all such cases of outdoor measurement the land on which the triangles are laid out is supposed to be a horizontal plane unless the contrary is stated.

6. In the square $ABCD$ here shown, prove that $AC = BD$.



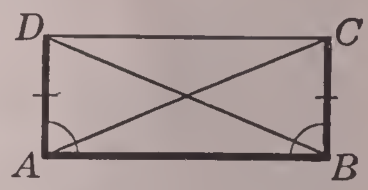
Begin as follows:

Given the square $ABCD$.

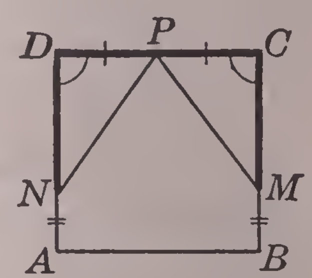
Prove that $AC = BD$.

The student should attack such an exercise by saying to himself, "I can prove this if I can prove that; I can prove that if I can prove this third statement," and so on until he finds something already proved. He should then reverse this process, beginning with a proposition already proved and ending with the statement to be proved.

7. In this figure, $AD = BC$ and each is $\perp$ to AB. What do you infer as to the relation of AC to BD? Prove the correctness of your inference.

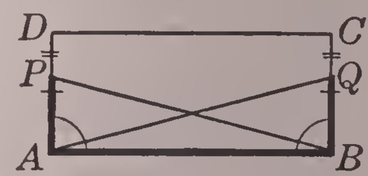


8. If $ABCD$ is a square, if P bisects CD, and if BM is made equal to AN, as shown in the figure, prove that $PM = PN$.



"I can prove this if I can prove that △s —— and —— are congruent. I can prove that these triangles are congruent if . . . ."

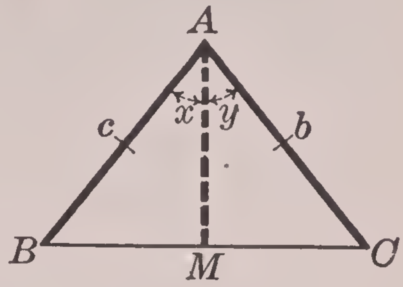
9. In this figure, $AD = BC$, each is $\perp$ to AB, and $DP = CQ$. What do you infer as to the relation of $\angle APB$ to $\angle BQA$ and of PB to QA? Prove the correctness of your inferences.



10. Suppose that it is known that a machine will work if three certain wheels properly gear into three other wheels. Suppose also that it is given that wheel a gears into wheel a', that it can be shown that wheel b gears into wheel b', and that it can then be shown that wheel c gears into wheel c'. What follows?

An occasional exercise like Ex. 10 may be discussed for the sake of training in transferring geometric reasoning to other lines.

41. Inference as to an Isosceles Triangle. If we examine the isosceles triangle here shown, we can make several inferences; among them, that if



$$b = c,$$

then $$\angle B = \angle C.$$

We have proved one proposition about equal angles (§ 35), but since that referred to vertical angles it does not help us in this case.

We have also proved a proposition about congruent triangles (§ 40), and congruent triangles have equal angles. Possibly we may be able to prove that $\angle B = \angle C$ if we can divide $\triangle ABC$ into two congruent triangles.

In order to use § 40 we must have two sides and the included angle of one triangle equal respectively to two sides and the included angle of another triangle; hence in order to get two equal angles let us suppose that AM is the bisector of $\angle A$ (Post. 8).

Dotted lines are used to represent such auxiliary lines as AM, which are inserted to assist in a proof. In speaking of $\angle A$ we mean the $\angle BAC$, the original angle at A, and so in all similar cases.

Then in $\triangle$ ABM and ACM, what is the relation of c to b?

What is the relation of x to y with respect to size? Why?

What line is the same in $\triangle$ ABM and ACM; that is, what line is *common* to the two triangles?

Then what parts of one triangle have you shown to be equal to what parts of the other triangle?

What can you say as to the congruence of the triangles, and what is the authority for the statement?

What can you say as to the relation of $\angle B$ to $\angle C$?

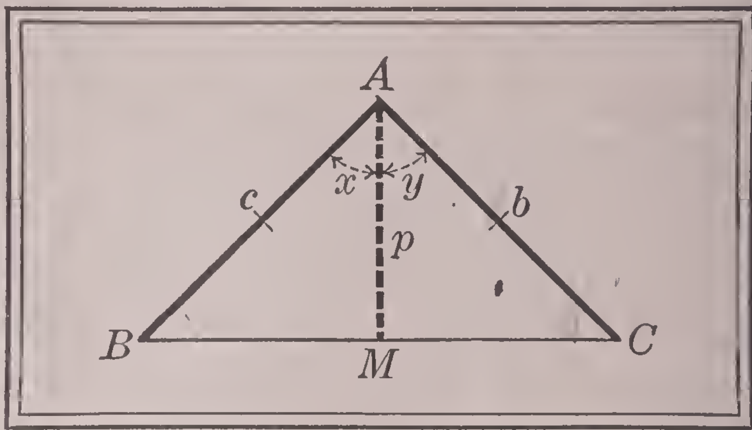
Complete the following statement:

In an isosceles triangle the angles opposite the equal $\cdots$.

The statement and formal proof are given in § 42.

Proposition 2. Isosceles Triangle

42. Theorem. *In an isosceles triangle the angles opposite the equal sides are equal.*



Given the isosceles $\triangle ABC$ with $b = c$.

Prove that $\angle B = \angle C$.

The plan is to prove two $\triangle$s congruent.

Proof. Let p be the bisector of $\angle A$, meeting BC at M. Then in $\triangle$s ABM and ACM it is given that

$$c = b.$$

Further, $x = y$, § 11

because p bisects $\angle A$;

and side p is common to both $\triangle$s.

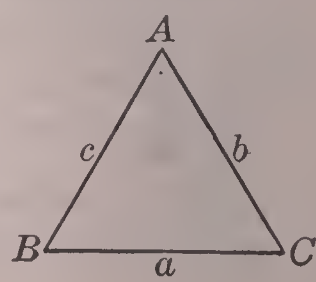
$\therefore \triangle ABM$ is congruent to $\triangle ACM$. § 40

If two sides and the included $\angle$ of one $\triangle$ are equal respectively to two sides and the included $\angle$ of another, the $\triangle$s are congruent.

$\therefore \angle B = \angle C$, § 38

because they are corresponding parts of congruent figures.

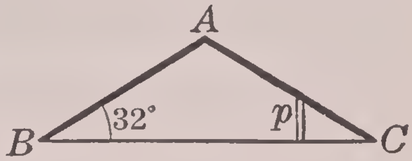
43. Corollary. *An equilateral triangle is equiangular.*



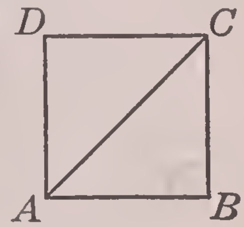
Because $b = c$ (why?), what follows as to $\angle B$ and $\angle C$? Why? Now prove that $\angle A = \angle B$. Why does $\angle A = \angle C$? Write out the full proof.

Exercises. Isosceles Triangles

1. In this figure, which represents the cross section of the attic of a house, it is known that the rafters AB and AC are equal in length. Suppose that we find by measuring that $\angle B = 32°$, but that we cannot conveniently pass the partition p so as to measure $\angle C$. If we are told that $\angle C = 30°$, is the information correct? If not, what should it be? Upon what proposition does the answer depend?

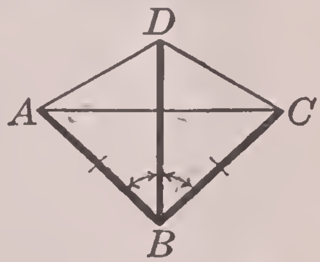


2. This figure represents a square $ABCD$ separated into two triangles by the diagonal AC. Which angles are equal by § 42?



3. In the same figure state which triangles are congruent by § 40, and hence show what other angles are equal besides those found in Ex. 2.

4. In this figure $BA = BC$ and $\angle DBA = \angle DBC$. Prove that $\triangle ACD$ is isosceles.

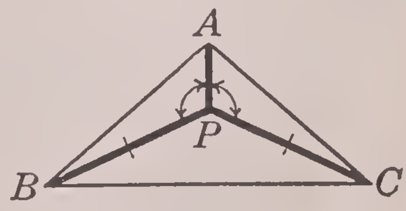


We can prove that $DA = DC$ if we can prove that $\triangle\triangle ABD$ and CBD are congruent. We can prove this if we can show that § 40 applies.

5. In Ex. 4 prove that DB is $\perp$ to AC.

What two angles must be proved equal? In order to prove them equal, what two triangles must be proved congruent?

6. In this figure $PB = PC$ and $\angle APB = \angle APC$. Prove that $\triangle ABC$ is isosceles.

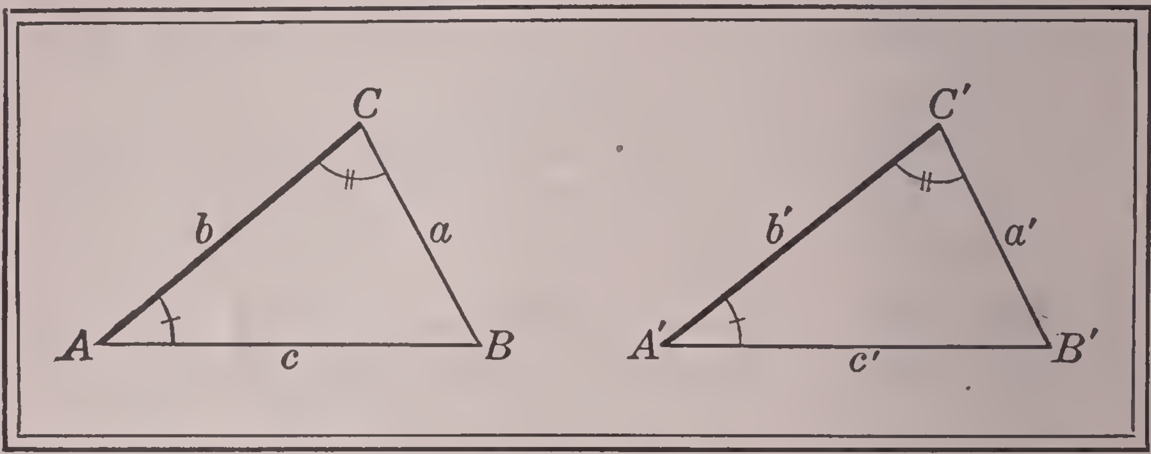


7. In the figure of Ex. 6 make a list of all the pairs of equal angles and prove each statement.

The teacher will find it helpful to introduce such exploring exercises in connection with various other figures, letting the student discover for himself as many relations as possible.

Proposition 3. Two Angles and Included Side

44. Theorem. *If two angles and the included side of one triangle are equal respectively to two angles and the included side of another, the triangles are congruent.*



Given the ≜ ABC and $A'B'C'$ with $\angle A = \angle A'$, $\angle C = \angle C'$, and $b = b'$.

Prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$.

The plan is to place one upon the other and show that they coincide.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$ so that A lies on A' and b lies along b', B and B' lying on the same side of b'. Post. 5

Then C lies on C',

because b is given equal to b';

c lies along c',

because $\angle A$ is given equal to $\angle A'$;

and a lies along a',

because $\angle C$ is given equal to $\angle C'$.

Since B is on a and c, it lies on both a' and c', and so lies on B', the point common to both a' and c'. Post. 1

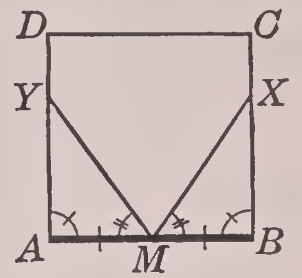
Two st. lines cannot intersect in more than one point.

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$, § 37

by the definition of congruent figures.

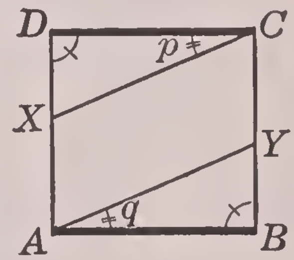
Exercises. Second Congruence Theorem

1. In this figure, $ABCD$ is a square, M is the midpoint of AB, and the lines MX and MY make equal angles with AB. Prove that $\triangle AMY$ is congruent to $\triangle BMX$. What other angles in these triangles are equal, and why?

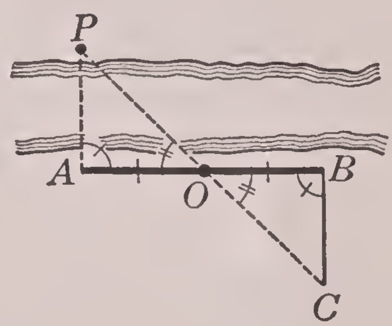


2. In the figure of Ex. 1, what angles of the figure $MXCDY$ are equal, and why?

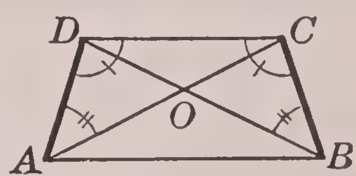
3. In this figure, $ABCD$ is a square and $p = q$. What other angles in the two triangles are equal? What lines are equal? Give the necessary proofs.



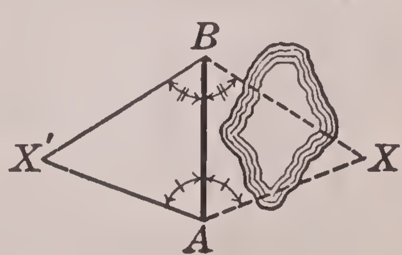
4. Wishing to measure the distance across a river, some boys sighted from A to a point P. They then laid off the line AB at right angles to AP. They placed a stake at O, halfway from A to B, and laid off a perpendicular to AB at B, placing a stake at C on this perpendicular in line with O and P. They then found the width by measuring BC. Prove that they were right.



5. In this figure, $\angle DCB = \angle CDA$, $\angle CBD = \angle DAC$, and $BC = AD$. Find the other equal lines and equal angles and prove that they are equal.

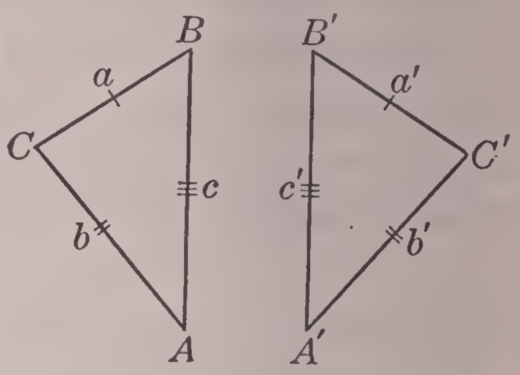


6. Wishing to find the distance BX, some boys measured $\angle XAB$ and $\angle ABX$ with the aid of a protractor. They then made $\angle X'AB = \angle XAB$ and $\angle ABX' = \angle ABX$, thus laying off the $\triangle ABX'$. How could they then find the distance BX? On what proposition does this depend?



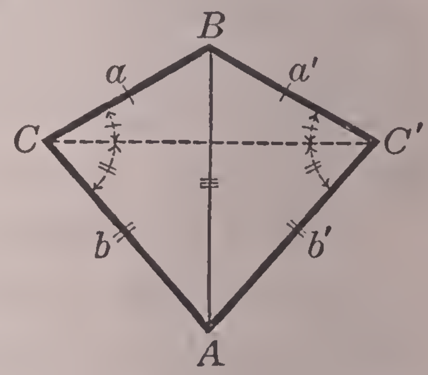
45. Another Inference. Suppose that these two triangles have the three sides of one equal respectively to the three sides of the other; that is, suppose that

$$a = a',$$
$$b = b',$$
and
$$c = c'.$$



From the appearance of the triangles, what do you infer as to their congruence? Would you draw the same inference if the three angles of one were equal respectively to the three angles of the other? Draw figures to illustrate your answer to this second question.

46. Examination of the Inference. In the case in which the three sides of one are equal respectively to the three sides of the other, see if you can give a satisfactory proof by placing $\triangle ABC$ upon $\triangle A'B'C'$, as in §§ 40 and 44. If not, try placing them as here shown, and drawing CC'.



Because $b = b'$, what kind of triangle is $\triangle AC'C$? Therefore what two angles of $\triangle AC'C$ are equal?

Because $a = a'$, what kind of triangle is $\triangle BCC'$? Therefore what two angles of $\triangle BCC'$ are equal?

By adding two pairs of equal angles, what can now be said as to the equality of $\angle C$ and $\angle C'$?

Can you now prove that $\triangle$s ABC and $A'B'C'$ are congruent by using § 40? Try it.

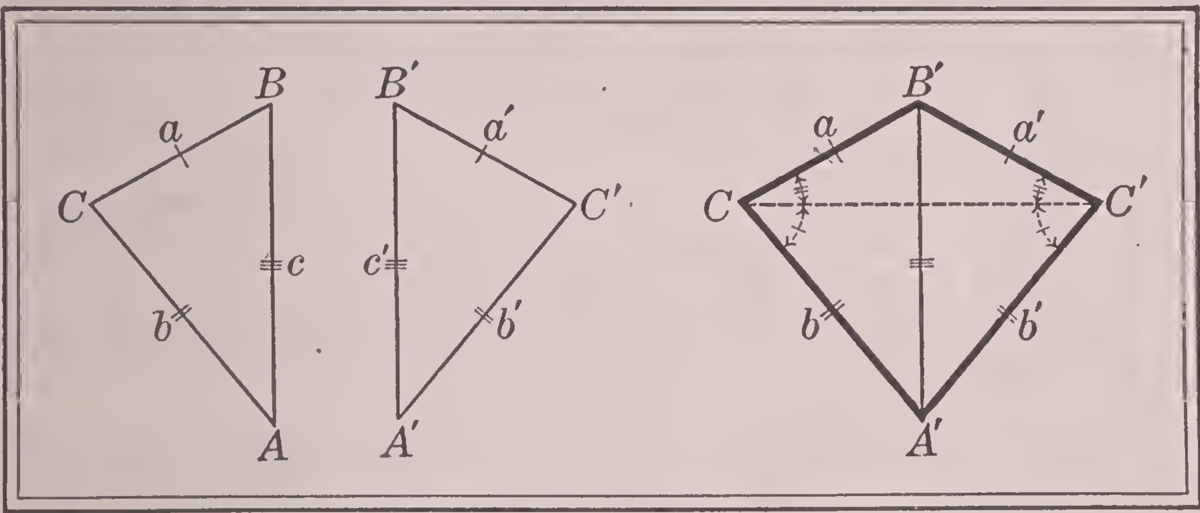
Complete the following statement:

If the three sides of one triangle are....

The statement and formal proof are given in § 47.

Proposition 4. Three Sides

47. Theorem. *If the three sides of one triangle are equal respectively to the three sides of another, the triangles are congruent.*



Given the $\triangle$s ABC and $A'B'C'$ with $a = a'$, $b = b'$, and $c = c'$.

Prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

The plan is to adapt the figure to § 40.

Proof. Place $\triangle ABC$ so that A lies on A', c lies along c', and C and C' lie on opposite sides of $A'B'$. Post. 5

Then B lies on B',

because c is given equal to c′.

Drawing CC', we have $b = b'$, Given

and hence $\angle CC'A' = \angle C'CA'$. § 42

In an isosceles $\triangle$ the $\angle$s opposite the equal sides are equal.

Also, since $a = a'$, Given

we have $\angle B'C'C = \angle B'CC'$. § 42

Adding, $\angle CC'A' + \angle B'C'C = \angle C'CA' + \angle B'CC'$; Ax. 1

that is, $\angle B'C'A' = \angle B'CA' \ (\angle BCA)$.

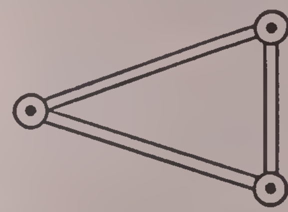
$\therefore$ $\triangle ABC$ is congruent to $\triangle A'B'C'$. § 40

(State the theorem of § 40 as the reason.)

Exercises. Third Congruence Theorem

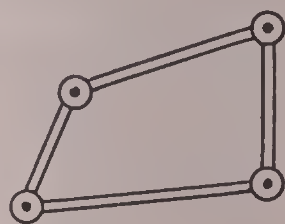
1. By placing three rods of different lengths end to end so as to form a triangle, can you form triangles of different shapes and sizes? State the reason for your answer.

2. Three iron rods are hinged at their ends as shown in this figure. Is the figure thus formed rigid; that is, can its shape be changed? State the reason.



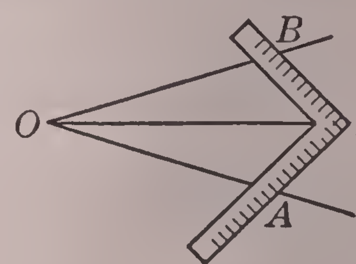
This explains the statement that *a triangle is determined by its three sides.* It also explains why the triangle is called a *unit of rigidity* in bridge building and in steel construction generally.

3. Four iron rods are hinged at their ends as shown in this figure. Is the figure thus formed rigid? If not, state two ways in which, by the addition of a single rod in each case, it can be made rigid. Upon what theorem does this depend?



4. Draw a rough figure of the framework of a bicycle. State the reason or reasons for its rigidity.

5. The following method is sometimes used for bisecting an angle by the aid of a carpenter's square: Place the square as here shown so that the edges shall pass through A and B, two points equidistant from O on the arms of the given $\angle AOB$, and so that $AP = BP$. Then draw OP. Show that OP bisects $\angle AOB$.



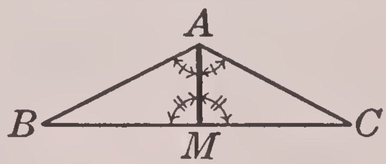
6. If in an equilateral triangle a line is drawn from one vertex to the midpoint of the opposite side, prove that the triangles thus formed are congruent.

Exercises. Review

1. In the $\triangle ABC$ it is given that $AC = BC$ and that CM bisects $\angle C$. Prove that CM bisects AB.

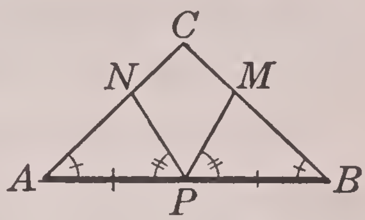
Draw the figure and say: "I can prove this if I can prove ⋯." Always attack an exercise in this way unless you see the proof at once.

2. In this figure it is given that AM bisects $\angle A$ and is also $\perp$ to BC. Prove that $\triangle ABC$ is isosceles.



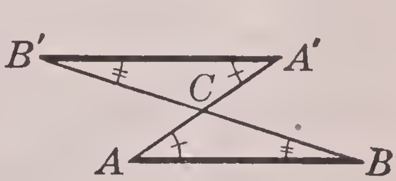
"I can prove this if I can prove ⋯. But I can prove that by § 44." Now reverse the reasoning and write out the proof.

3. In the $\triangle ABC$ it is given that $\angle A = \angle B$, that P bisects AB, and that $\angle NPA = \angle MPB$. Prove that $AN = BM$.



"I can prove this if I can prove that $\triangle APN$ is congruent to $\triangle BPM$. I can prove that because I know ⋯."

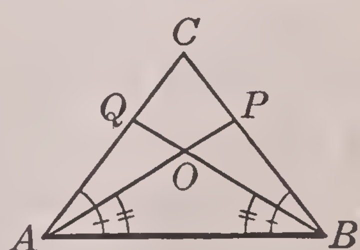
4. In this figure it is given that $\angle A = \angle A'$, $\angle B = \angle B'$, and $AB = A'B'$. Find the other equal lines and equal angles and prove that they are respectively equal.



Remember that BCA' is one of the angles.

5. Prove that a perpendicular to the bisector of an angle forms an isosceles triangle with the arms of the angle.

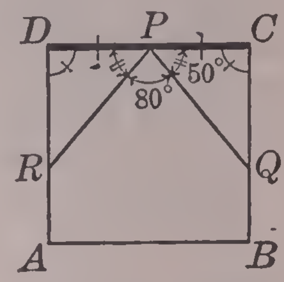
6. In the $\triangle ABC$ it is given that $\angle A = \angle B$ and that AP and BQ are so drawn that $\angle QBA = \angle PAB$. Prove that $BQ = AP$.



"I can prove this if I can prove that $\triangle ABQ$ is congruent to $\triangle BAP$. I can prove that because I know ⋯."

7. In the figure of Ex. 6 state the pairs of equal angles.

8. In the square $ABCD$ it is given that the point P bisects CD and that PQ and PR are so drawn that $\angle QPC = 50°$ and $\angle RPQ = 80°$. Prove that $PQ = PR$.

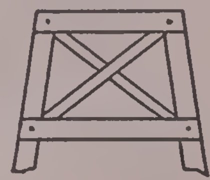


If $\angle QPC = 50°$ and $\angle RPQ = 80°$, express $\angle DPR$ in degrees.

In the △DRP and CQP, what parts are respectively equal, and why?

9. Prove that the line from the vertex of an isosceles triangle to the midpoint of the base is perpendicular to the base.

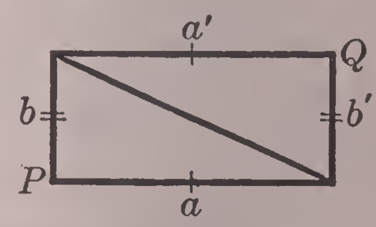
10. In this section of a support for a heavy tank are both cross braces necessary for rigidity? State the reason. If either one is unnecessary, state a reason for having it there.



11. Two isosceles triangles of different heights are constructed on the same base and on the same side of the base. Prove that the line through their vertices bisects the angles at the vertices.

12. In Ex. 11 suppose that the two isosceles triangles are constructed on opposite sides of the base.

13. In this figure $a = a'$ and $b = b'$. Prove that $\angle P = \angle Q$.



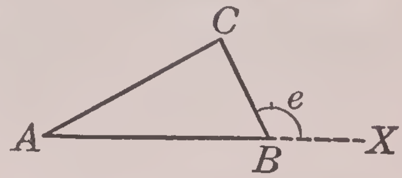
Hereafter the words "prove that" will usually be omitted in the exercises when it is obvious that a proof is required.

14. If from any vertex of a square there are drawn line segments to the midpoints of the two sides not adjacent to the vertex, these line segments are equal.

15. From the propositions already studied write a complete statement of the different conditions under which two triangles are congruent.

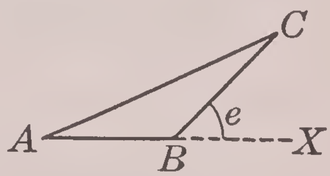
48. Exterior Angle. The angle included by one side of a plane figure and an adjacent side produced is called an *exterior angle* of the figure.

For example, e is an exterior angle of this triangle, and $\measuredangle A$ and C are called the *non-adjacent interior angles.*



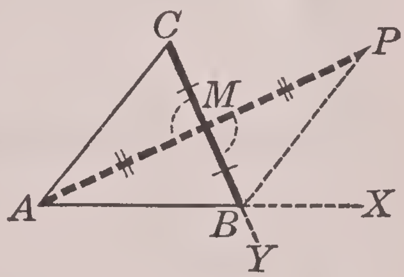
49. Inference as to an Exterior Angle of a Triangle. In the above figure, which seems the larger, e or $\angle A$? e or $\angle C$?

Would your inference be the same if the triangle were of a different shape? Consider, for example, this figure.



We have thus far found no way of proving one angle greater than another, but we have found five different ways of proving one angle equal to another, one in § 35, three in §§ 40, 44, and 47; and one in § 42.

Consider this figure, supposing that M bisects BC, that AM is drawn and is then produced so that $MP = AM$, and that BP is then drawn.



Can you prove that $\triangle BPM$ and CAM are congruent? If so, can you prove that $\angle PBM = \angle ACM$?

Then is $\angle XBC > \angle PBM$? By what axiom is this true?

Then how is $\angle XBC$ related to $\angle C$, and why?

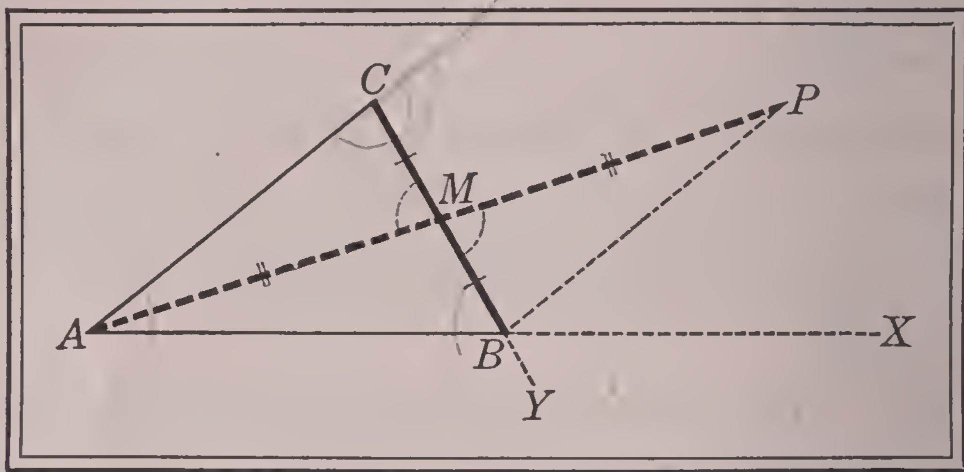
Can you bisect AB and proceed in a similar way to show that $\angle ABY > \angle A$? If so, is $\angle XBC > \angle A$?

The student has now reached the point where he may profitably read the model proofs without such assistance as is given above.

The model proofs should not be memorized, but the student should read the theorems and try to work out the proofs for himself before reading those given in the book. *The complete statement of the authority for each step of the proof should always be given, particularly where the reference number alone is quoted.*

Proposition 5. Exterior Angle of a Triangle

50. Theorem. *An exterior angle of a triangle is greater than either nonadjacent interior angle.*



Given the exterior $\angle XBC$ of the $\triangle ABC$.

Prove that $\angle XBC > \angle C$ *and that* $\angle XBC > \angle A$.

The plan is first to prove that $\angle XBC > \angle PBM$, which is equal to $\angle C$.

Proof. Let M bisect BC. Post. 7

Draw AM and produce it so that $MP = AM$. Posts. 1, 2

Draw BP. Post. 1

The line BP lies within $\angle XBC$, for otherwise AP would cut either AX or AC produced in two points, which is impossible. Post. 1

Then $\angle BMP = \angle CMA$, § 35

$BM = CM$, § 11

and MP was made equal to MA.

Then $\triangle BPM$ is congruent to $\triangle CAM$, § 40

and hence $\angle PBM = \angle C$. § 38

But $\angle XBC > \angle PBM$, Ax. 10

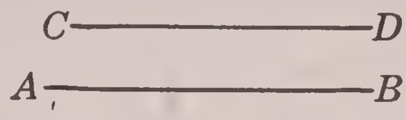
and hence $\angle XBC > \angle C$. Ax. 5

Similarly, $\angle ABY > \angle A$, and hence $\angle XBC > \angle A$.

Draw the figure and give the proof of this last statement.

51. Parallel Lines. Lines which lie in the same plane and cannot meet however far they may be produced are called *parallel lines*, or simply *parallels.*

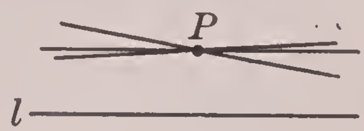
For example, AB and CD are parallel lines. We may think of them as edges of a strip of ribbon. Since the student is already familiar with such lines, further illustrations are not necessary.



It should be observed that in the above definition the words "in the same plane" are essential.

52. Postulate of Parallels. *Through a given point only one line can be drawn parallel to a given line.*

From this figure it seems quite evident that only one of the lines that can be drawn through P can be parallel to l. While this is no proof for the statement, we are probably as convinced that the statement is true as we should be if a proof were given.



53. Transversal. A line which cuts two or more lines is called a *transversal* of those lines.

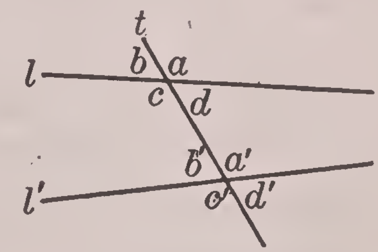
For example, in the figure below, the line t is a transversal of the lines l and l'.

54. Angles made by a Transversal. In the figure given below, it is customary to give special names to certain angles, as follows:

a, b, c', d' are called *exterior angles*;

a', b', c, d are called *interior angles*;

d and b' are called *alternate angles*, and similarly for c and a';

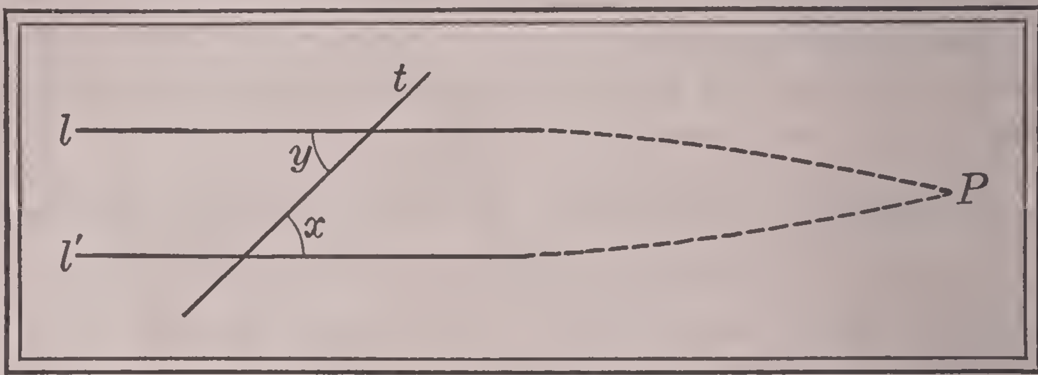


a and a' are called *corresponding angles*, and similarly for b and b', for c and c', and for d and d'.

Sometimes a and c' are called *alternate exterior angles*, and similarly for b and d'; but when alternate angles are mentioned we ordinarily mean *alternate interior angles*; that is, we ordinarily mean d and b' or c and a', and this should be understood in every case unless the contrary is stated.

Proposition 6. Condition of Parallelism

55. Theorem. *When two lines in the same plane are cut by a transversal, if the alternate angles are equal, the two lines are parallel.*



Given the two lines *l*, *l'* in the same plane and cut by the transversal *t* so that the alternate ∠s *x* and *y* are equal.

Prove that l *and* l' *are* ∥.

The plan is to suppose that the lines meet and then to prove that this supposition leads to an impossible result.

Proof. If l and l' are not ∥ they will meet if produced. Suppose that they meet at P.

Then $y > x$. § 50

An exterior ∠ of a △ > either nonadjacent interior ∠.

But this is impossible, because it is given that $y = x$.

Thus the supposition that l and l' are not ∥ leads to an impossible result, and hence l and l' are ∥. § 51

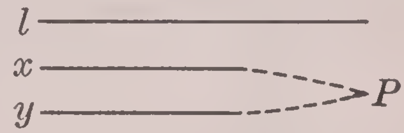
56. Indirect Proof. In the above case we have assumed the proposition to be false and have shown that this leads to an impossible result. We then conclude that the proposition must be true. Such a proof is called an *indirect proof.*

Since the proof excludes all possibilities other than the one stated in the proposition, it is also called a *proof by exclusion.* It was formerly known as the *Reductio ad absurdum,* the "reduction to an absurdity."

57. Corollary. *Two lines in the same plane perpendicular to the same line are parallel.*

Draw the figure. What ∠s in the figure are equal and why? Then by what authority can it be said that the lines are ∥?

58. Corollary. *Two lines in the same plane parallel to a third line are parallel to each other.*



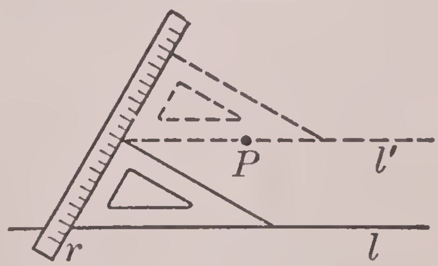
It is given that the lines x and y are both ∥ to the line l.

Then if x and y are not ∥, suppose that they meet at P. If this were possible, how many lines should we have through P ∥ to l? How does § 52 apply?

59. Corollary. *When two lines in the same plane are cut by a transversal, if two corresponding angles are equal or if two interior angles on the same side of the transversal are supplementary, the lines are parallel.*

Draw the figure and show that if two corresponding ∠s are equal or if two interior ∠s on the same side of the transversal are supplementary, two alternate ∠s must be equal, and that § 55 then applies.

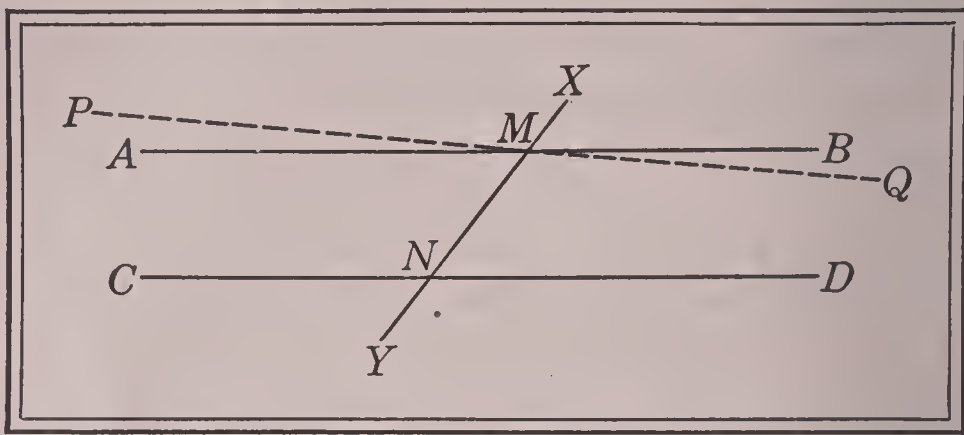
60. Application. In order to draw a line parallel to a given line l and passing through a given point P, a draftsman often uses a celluloid triangle, as here shown. He lays the hypotenuse along the given line l, places a ruler r along one of the sides, and slides the triangle along the ruler until the hypotenuse passes through P. He then draws a line l' along the hypotenuse.



Using this construction, draw a line through a given point and parallel to a given line. State the authority upon which this construction depends. Could another side be used instead of the hypotenuse? Has the side any advantage over the hypotenuse? What other instrument besides a triangle could be used for this purpose?

Proposition 7. Parallels cut by a Transversal

61. Theorem. *If two parallel lines are cut by a transversal, the alternate angles are equal.*



Given AB and CD, two $\parallel$ lines cut by the transversal XY in the points M and N respectively.

Prove that $\angle AMN = \angle DNM.$

The plan is to use an indirect proof.

Proof. Suppose that $\angle AMN$ is not equal to $\angle DNM$, but that a line PQ through M makes $\angle PMN = \angle DNM$.

Then PQ is $\parallel$ to CD. § 55

But this is impossible, § 52

because AB is given as $\parallel$ to CD.

Hence $\angle AMN = \angle DNM.$

62. Corollary. *If two parallel lines are cut by a transversal, the corresponding angles are equal.*

Show that this depends only upon §§ 35 and 61.

63. Corollary. *If two parallel lines are cut by a transversal, the two interior angles on the same side of the transversal are supplementary.*

Show that this depends only on § 61 and certain definitions.

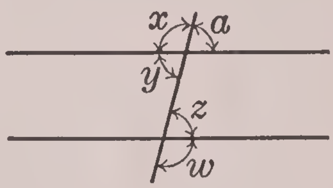
As a special case, *if a line is perpendicular to one of two parallel lines, it is perpendicular to the other also.*

Exercises. Parallel Lines

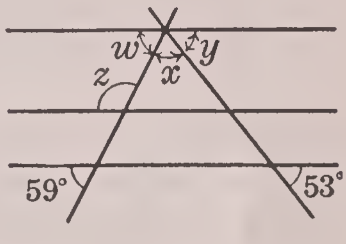
1. *If two parallel lines are cut by a transversal, the alternate exterior angles are equal.*

Exercises which are printed in italics are given as corollaries in some textbooks, and should, therefore, be solved by all students. They are not, however, essential to the logical sequence of the propositions, as they are not used in the proof of subsequent theorems.

2. This figure shows two parallel lines cut by a transversal. Find the values of x, y, z, and w, given that $a = 73°$; given that $a = 78°$.



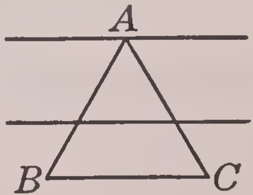
3. Cross arms for electric wires are usually at right angles to the poles. What properties of parallels are illustrated by several cross arms on one pole?



4. In this figure three parallel lines are cut by two transversals, and certain angles are formed as shown. Find the values of w, y, z, and x.

5. A man who is walking southward changes his direction to northwest. Through how many degrees does he turn? If he wishes to walk southward again, through how many degrees must he turn? Draw a figure showing the man's course, and state the proposition upon which your second answer depends.

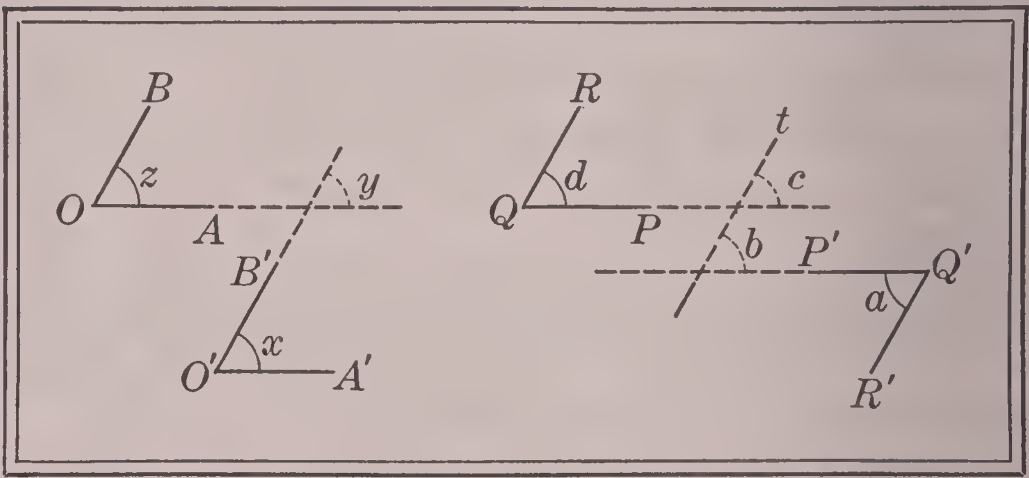
6. In this figure each angle of $\triangle ABC$ is 60°, and two lines have been drawn parallel to the base. What can you discover as to the number of degrees in each of the other angles?



7. Two parallel lines are cut by a transversal so as to make the number of degrees in one interior angle $2x$ and the number of degrees in the other interior angle on the same side of the transversal $x - 30$. Find the value of x.

Proposition 8. Angles with Parallel Arms

64. Theorem. *If two angles have their arms respectively parallel, and if both pairs of parallels extend either in the same direction or in opposite directions from the vertices, the angles are equal.*



Given $\angle$s x and z with arms respectively $\parallel$ and extending in the same direction from the vertices, and $\angle$s a and d with arms respectively $\parallel$ and extending in opposite directions.

Prove that $x = z$ *and that* $a = d$.

The plan is to show that the $\angle$s in each pair are equal to the same $\angle$ or to equal $\angle$s.

Proof. Produce the arms of x and z, thus forming $\angle y$.

Then $x = y = z$. § 62

Produce the arms of a and d, and suppose that t is a transversal $\parallel$ to QR and $Q'R'$, thus forming $\angle$s b and c.

Then $a = b$, § 61

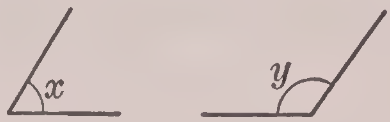
and $b = c = d$. § 62

$\therefore a = d$. Ax. 5

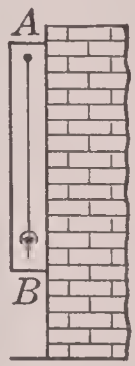
It should be pointed out to the class that the arms of two angles extend in the same direction if the arms are on the same side of a line joining the vertices; otherwise they extend in opposite directions.

Exercises. Review

1. If two angles have their arms respectively parallel, and if one pair of parallels extend in the same direction from the vertices and the other pair extend in opposite directions from the vertices, the angles are supplementary.

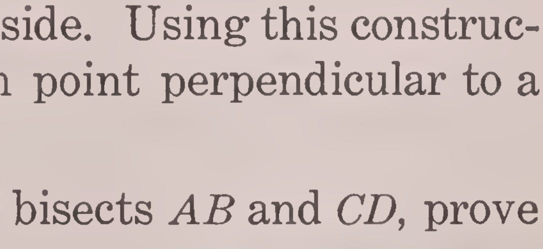


2. A bricklayer often uses the instrument here shown for determining whether a wall is vertical. When the plumb line lies along a line that is parallel to the edge AB, he knows that the wall is vertical. State the geometric principle involved.



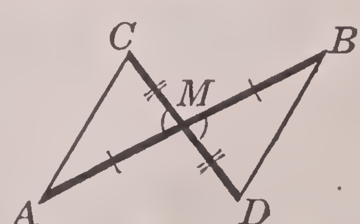
3. In Ex. 2 state the principle involved in the assertion that the plumb line is perpendicular to each line formed by producing the horizontal lines of the brickwork.

4. In order to draw a line perpendicular to a given line l and passing through a given point P, a draftsman lays one side of his triangle along the given line l, places a ruler r along the hypotenuse, and slides the triangle along the ruler until the other side passes through P.



He then draws a line l' along this side. Using this construction, draw a line through a given point perpendicular to a given line. Explain in full.

5. In this figure, given that M bisects AB and CD, prove that AC is ∥ to DB.



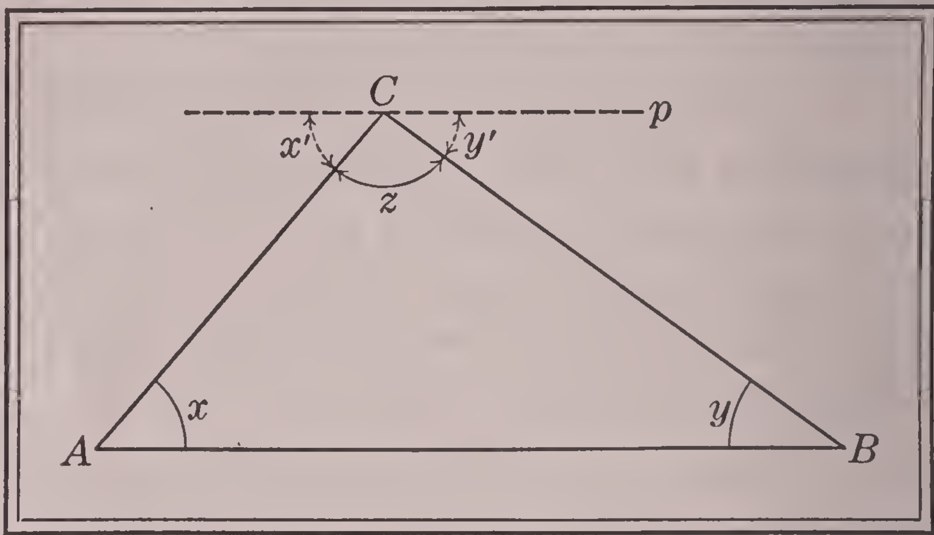
AC is ∥ to DB if what two alternate angles are equal?

These two angles are equal if what two triangles are congruent?

These triangles are congruent according to what proposition?

Proposition 9. Sum of the Angles of a Triangle

65. Theorem. *The sum of the three angles of a triangle is a straight angle.*



Given the $\triangle ABC$ with $\measuredangle s$ x, y, and z.

Prove that $x + z + y = a \text{ st. } \angle.$

Lettering the figure as above, the plan is to show that $x = x'$, $y = y'$, and $x' + z + y' =$ a st. $\angle$. Then it will follow that $x + z + y =$ a st. $\angle$.

Proof. Suppose p to be a line through $C \parallel$ to AB and making $\measuredangle s$ x' and y' as shown. § 52

Then $x' + z + y' =$ a st. $\angle$. § 12

But $x' = x$

and $y' = y$. § 61

Substituting x and y for their equals, x' and y', we have

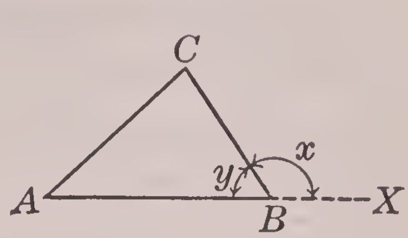
$x + z + y =$ a st. $\angle$. Ax. 5

This proposition is one of the most important in geometry.

In the first statement in the proof it is evident that Ax. 10 is also involved, but such minor statements are usually omitted in proofs. The teacher should call attention to them if necessary.

For students who have never seen this proposition before, it is an interesting exercise to infer its truth by cutting off and fitting together the three angles of a paper triangle.

66. Corollary. *An exterior angle of a triangle is equal to the sum of the two nonadjacent interior angles.*



For $x+y=\text{a st.}\angle,$

and $\angle A+\angle C+y=\text{a st.}\angle.$

$\therefore \angle A+\angle C+y=x+y.$

Subtracting y, we have

$$\angle A+\angle C=x.$$

By subtracting $\angle C$ we see that $\angle A=x-\angle C.$

67. Corollary. *If two angles and a side of one triangle are equal respectively to two angles and the corresponding side of another triangle, the triangles are congruent.*

If the $\measuredangle$ of one are x, y, z, and the $\measuredangle$ of the other are x, y, z', then

$$x+y+z=\text{a st.}\angle$$

and $x+y+z'=\text{a st.}\angle,$ § 65

and hence $x+y+z=x+y+z'.$ Ax. 5

$\therefore z=z'.$ Ax. 2

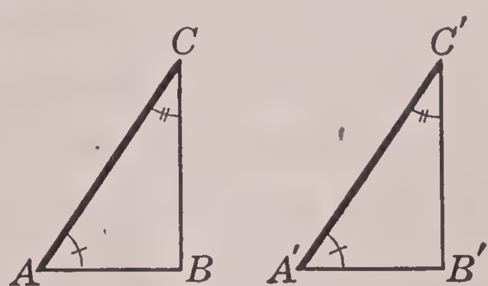
Hence, whatever side is taken, the $\triangle$ are congruent. § 44

68. Corollary. *If the hypotenuse and an adjacent angle of one right triangle are equal respectively to the hypotenuse and an adjacent angle of another, the triangles are congruent.*

Consider the figures here shown, in which $\triangle ABC$ and $A'B'C'$ are rt. $\triangle$ with $\angle A=\angle A'$ and $AC=A'C'$.

Since the rt. $\measuredangle$ are also equal (Post. 6), the third $\measuredangle$ must be equal (§ 65).

We then have



$AC=A'C',$

$\angle A=\angle A',$

and $\angle C=\angle C'.$

Hence the $\triangle$ are congruent by § 44.

It should be observed that this is really a fourth congruence theorem, but it follows so easily from § 65 as to be properly a corollary of this proposition.

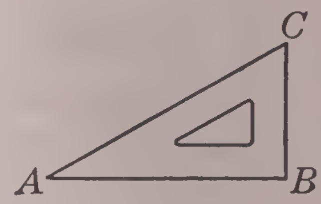
Exercises. Angles of a Triangle

1. If two triangles have the sum of two angles of one equal to the sum of two angles of the other, even though the angles themselves are not respectively equal, the third angles are equal.

2. An equiangular triangle is also equilateral.

3. The sum of the two acute angles of a right triangle is 90°.

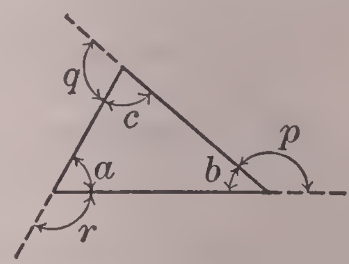
4. In a draftsman's triangle, $\angle B$ is a right angle, as shown in the figure, and $\angle A$ is often 30°. In such a triangle how many degrees are there in $\angle C$?



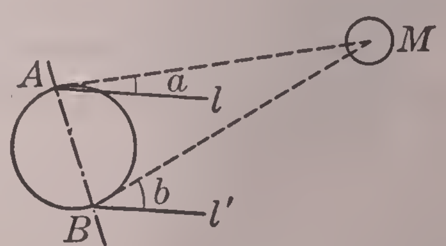
5. If one angle of a right triangle is 37°, what is the size of the other acute angle?

6. Prove § 65 by using the figure in § 66 and supposing that a line is drawn from $B \parallel$ to AC.

7. In this figure, what single angle is equal to $a+c$? To the sum of what angles is q equal? To the sum of what angles is r equal? From these three relations of angles find the number of degrees in $p+q+r$.



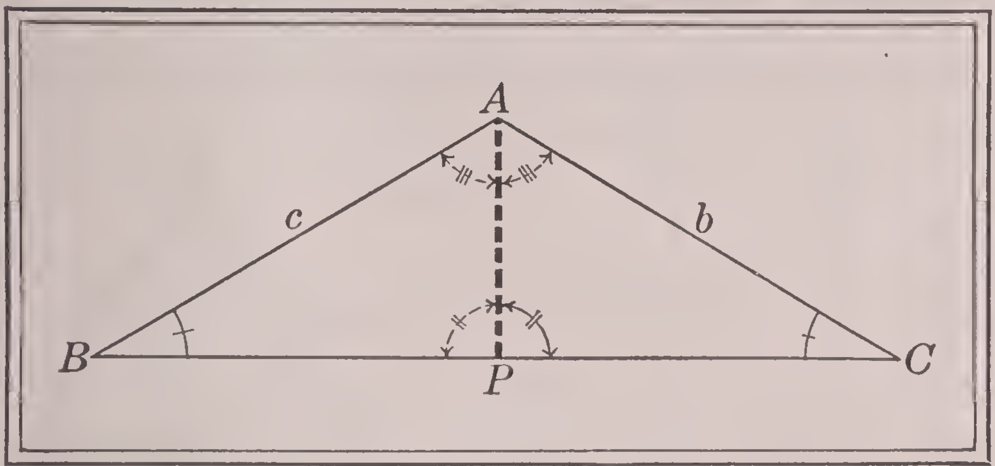
8. In finding the distance of the moon from the earth it is necessary to find first the $\angle AMB$ at the center of the moon, AB being the diameter of the earth. Observations are taken on opposite sides of the earth at A and B. The lines l, l' are $\parallel$, and $\angle$s a and b are accurately measured. Show how, from a and b, to find $\angle M$.



Such figures are necessarily distorted. The details of the finding of $\angle$s a and b need not be considered. We simply assume that these angles can be measured.

Proposition 10. Equal Sides of a Triangle

69. Theorem. *If a triangle has two equal angles, the sides opposite these angles are equal.*



Given the $\triangle ABC$ with $\angle B = \angle C$.

Prove that $b = c$.

The plan is to prove two $\triangle$s congruent.

Proof. Suppose that AP is $\perp$ to BC. Post. 10

The sum of the $\angle$s of $\triangle ABP$ is equal to the sum of the $\angle$s of $\triangle ACP$. § 65, Ax. 5

Then, since $\angle B = \angle C$ Given

and $\angle APB = \angle APC$, Post. 6

because AP is taken as $\perp$ to BC,

we have $\angle BAP = \angle CAP$. Ax. 2

$\therefore \triangle ABP$ is congruent to $\triangle ACP$, § 44

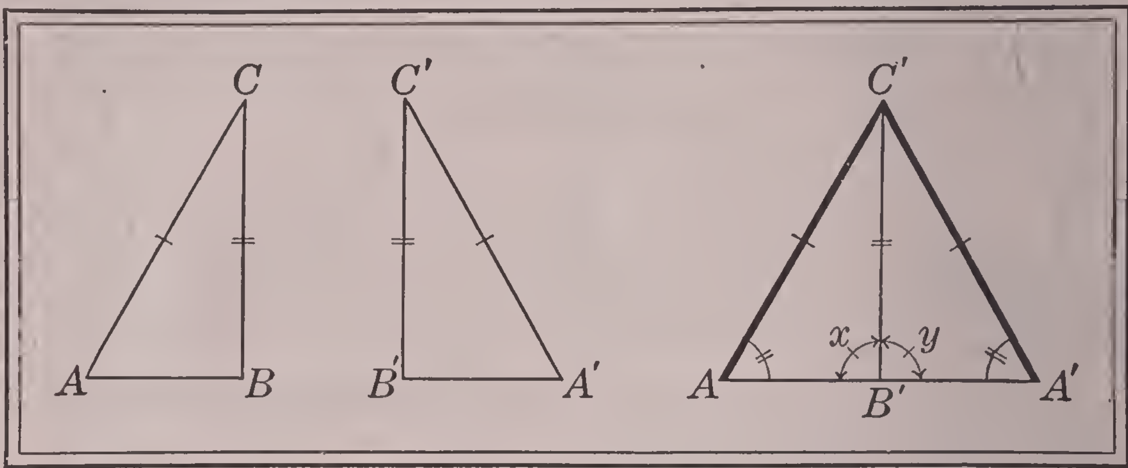
and hence $b = c$. § 38

70. Converse Theorems. It should be observed that § 69 is closely related to § 42. When two theorems are so related that what is given in one is what is to be proved in the other, either theorem is said to be the *converse* of the other.

Because a theorem is true it does not always follow that its converse is true.

Proposition 11. Congruence of Right Triangles

71. Theorem. *If the hypotenuse and a side of one right triangle are equal respectively to the hypotenuse and a side of another, the triangles are congruent.*



Given the rt. $\triangle$ ABC and $A'B'C'$ with hypotenuse $AC =$ hypotenuse $A'C'$ and with $BC = B'C'$.

Prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$.

The plan is first to prove that $\angle A = \angle A'$ and then to apply § 68.

Proof. Place $\triangle ABC$ next to $\triangle A'B'C'$ so that BC lies along $B'C'$, B lies on B', and A and A' lie on opposite sides of $B'C'$. Post. 5

Then C lies on C',

because BC is given equal to $B'C'$.

Also, $x + y =$ a st. $\angle$, § 12

and hence BA lies along $A'B'$ produced. § 18

Since $\triangle AA'C$ is isosceles, § 19

because AC is given equal to $A'C'$,

we have $\angle A = \angle A'$. § 42

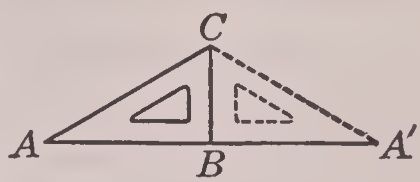
$\therefore \triangle AB'C'$ is congruent to $\triangle A'B'C'$, § 68

and $\triangle ABC$ is congruent to $\triangle A'B'C'$. Ax. 5

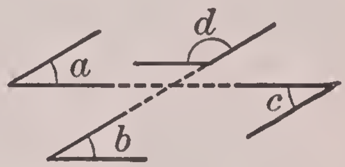
Since the corresponding parts of congruent triangles are equal, Ax. 5 may be applied to congruence.

Exercises. Review

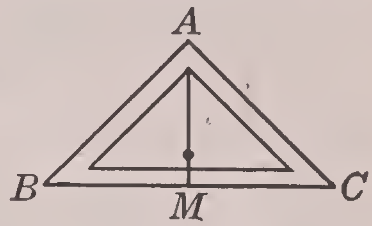
1. The accuracy of the right angle of a draftsman's triangle may be tested by first drawing a line along the side BC with the triangle in the position ABC on a line AA', and then drawing a line along BC with the triangle in the position $A'BC$. State the geometric principle involved.



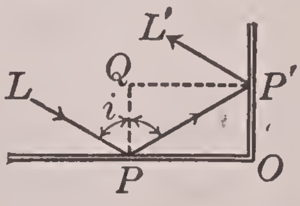
2. Given that the arms of these angles are respectively parallel, prove that d is supplementary to a, to b, and to c.



3. The ancient kind of leveling instrument here shown consists of an isosceles right triangle. When the plumb line cuts the mid-point M of the base BC, the line BC is level. State the geometric principle involved.

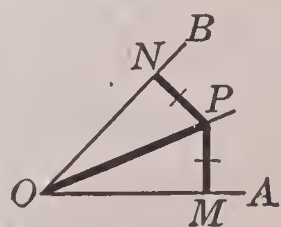


4. If a ray of light LP strikes a mirror OP at P, it is reflected along a line PP' in such a way that $\angle QPL = \angle QPP'$, QP being $\perp$ to OP. If P' is a point on a mirror OP' which is perpendicular to the first mirror, the ray is similarly reflected in a line $P'L'$, QP' being $\perp$ to OP'. Find all the acute angles in the figure in terms of i and show that $P'L'$ is $\parallel$ to PL.

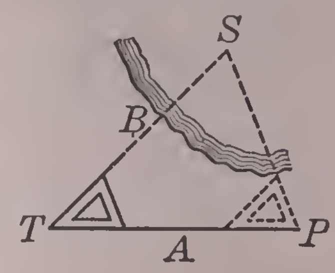


5. Consider Ex. 4 when $\angle O = 60°$; when $\angle O = 30°$.

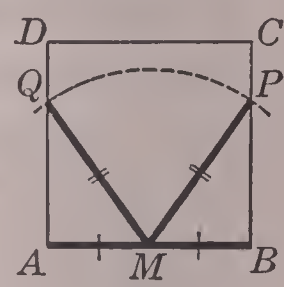
6. Prove that if the $\perp$s PM, PN from the point P to the sides of an $\angle AOB$ are equal, the point P lies on the bisector of $\angle AOB$. Write the general statement of this theorem without using letters as is done here.



7. A method of finding the distance of a ship off shore requires the use of a large wooden isosceles triangle. First stand at T and sight along the sides of the vertical angle of the triangle to the ship S and along the shore on a line TA. Then from a point P on TA sight to T and S along the sides of a base angle of the triangle. Then $TP = TS$. Explain why this is true and show how the distance BS from the shore to the ship can be found.

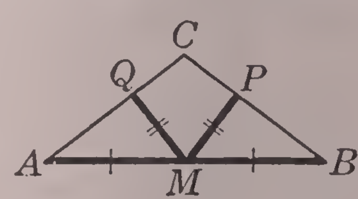


8. $ABCD$ is a square and M is the midpoint of AB. With M as center an arc is drawn, cutting BC at P and AD at Q. Prove that $\triangle MBP$ is congruent to $\triangle MAQ$, and write the general statement of this theorem without making use of letters as is done here.

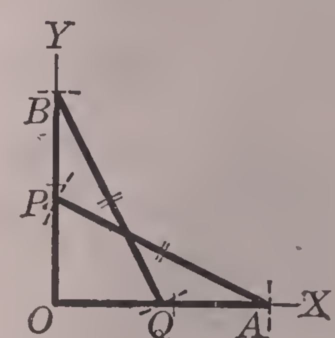


This statement should read, "If an arc drawn with the midpoint of one side of a square as center cuts the two adjacent sides, then the triangles cut off by," and so on.

9. Prove that if the perpendiculars from the midpoint M of the base AB to the sides of the $\triangle ABC$ are equal, then $\angle A = \angle B$. What then follows as to the sides AC and BC? Write the general statement of this theorem without referring to a special figure.



10. Suppose that OY is $\perp$ to OX. With O as center an arc is drawn cutting OX at A and OY at B. Then with A as center an arc is drawn cutting OY at P, and with B as center and the same radius an arc is drawn cutting OX at Q. Prove that $OP = OQ$.



What triangles are congruent by § 71?

72. Quadrilateral. A rectilinear figure of four sides is called a *quadrilateral.* A quadrilateral is called

a *trapezoid* if it has two sides parallel;
a *parallelogram* if it has the opposite sides parallel.

If nonparallel sides of a trapezoid are equal, the figure is said to be *isosceles.* In a trapezoid or a parallelogram the side parallel to the base is called the *upper base,* the base being then called the *lower base.*

Trapezoid Parallelogram Rectangle Rhombus

A parallelogram is called

a *rectangle* if its angles are all right angles;
a *rhombus* if its sides are all equal.

73. Distance. The length of the line segment from one point to another is called the *distance* between the points.

The length of the perpendicular from an external point to a line is called the *distance* from the point to the line.

The length of a perpendicular from one parallel line to another is called the *distance* between the parallels.

74. Height or Altitude. The length of the perpendicular between the bases of a parallelogram or a trapezoid is called the *height* or the *altitude* of the figure.

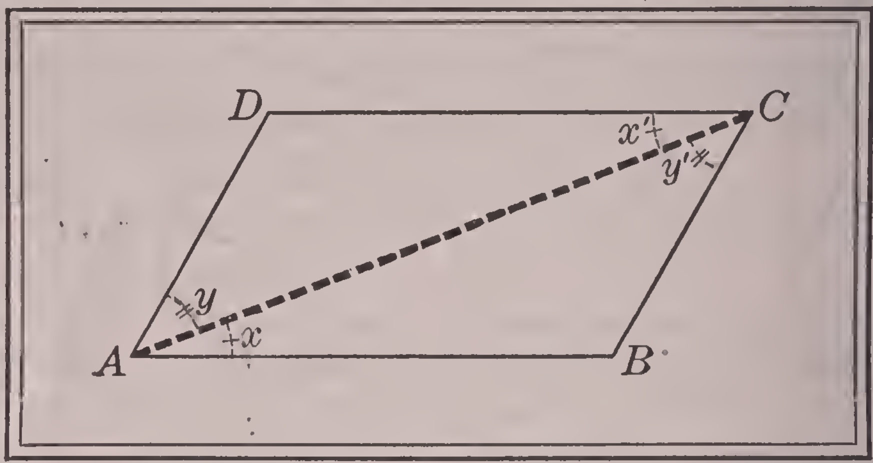
The length of the perpendicular from the vertex of a triangle to the base is called the *height* or the *altitude* of the triangle.

For brevity the perpendicular itself, instead of its length, is often called the *altitude.* The term "altitude" is commonly used in school; the term "height" is commonly used in ordinary conversation.

75. Diagonal. The line segment joining two nonconsecutive vertices of any figure is called a *diagonal* of the figure.

Proposition 12. Opposite Parts of a Parallelogram

76. Theorem. *The opposite sides of a parallelogram are equal and the opposite angles are also equal.*



Given the $\square ABCD$.

Prove that $BC = AD$ *and* $AB = DC$,

and also that $\angle B = \angle D$ *and* $\angle A = \angle C$.

The plan is to prove two $\triangle$s congruent.

Proof. Draw the diagonal AC. Post. 1

Since $x = x'$, $y = y'$, and $AC = AC$, § 61, Iden.

then $\triangle ABC$ is congruent to $\triangle CDA$. § 44

$\therefore BC = AD$, $AB = DC$, and $\angle B = \angle D$. § 38

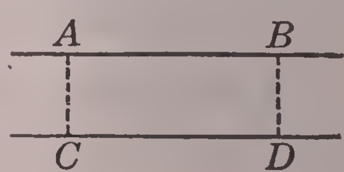
Adding equal $\angle$s, $\angle A = \angle C$. Ax. 1

77. Corollary. *A diagonal divides a parallelogram into two congruent triangles.*

78. Corollary. *Segments of parallel lines cut off by parallel lines are equal.*

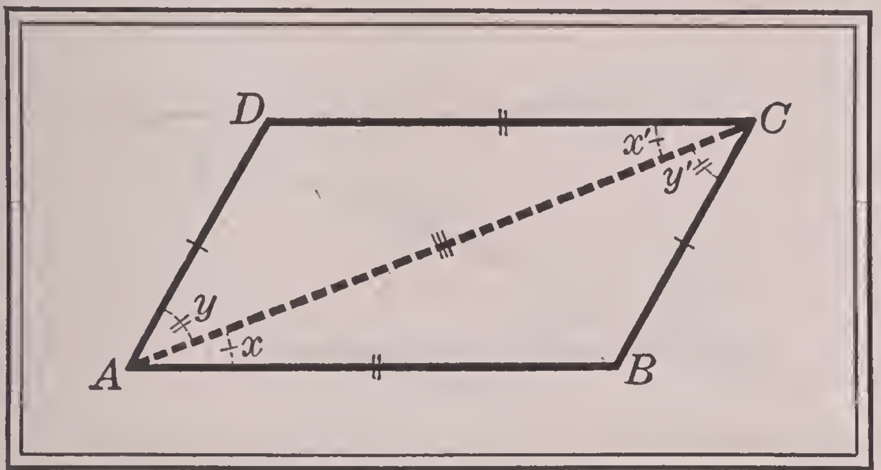
79. Corollary. *Two parallel lines are everywhere equally distant from each other.*

If AB and CD are $\parallel$, what can be said of $\perp$s drawn from *any* points in AB to CD (§ 78), and hence from *all* points?



Proposition 13. First Criterion for a Parallelogram

80. Theorem. *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ with $BC = AD$ and $AB = DC$.

Prove that the quadrilateral $ABCD$ is a $\square$.

The plan is to prove that $x = x'$ and $y = y'$ by congruent $\triangle$, and then to apply § 55.

Proof. Draw the diagonal AC. Post. 1

In the two $\triangle$ it must now be shown that $x = x'$ and $y = y'$.

Since	$BC = AD$	
and	$AB = DC$,	Given
and since	$AC = AC$,	Iden.
we see that	$\triangle ABC$ is congruent to $\triangle CDA$.	§ 47
	$\therefore x = x'$;	§ 38
whence	AB is $\parallel$ to DC.	§ 55
Also,	$y = y'$;	§ 38
whence	BC is $\parallel$ to AD.	§ 55

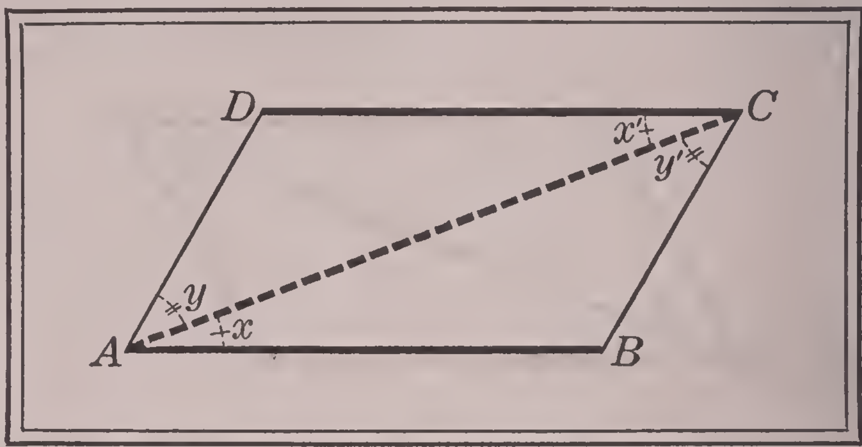
Hence the quadrilateral $ABCD$ is a $\square$. § 72

The proposition is sometimes stated with reference to convex quadrilaterals; but, as stated in § 7, in this book we consider only those rectilinear figures in which each of the angles within the figure is less than two right angles.

PS

Proposition 14. Second Criterion for a Parallelogram

81. Theorem. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



Given the quadrilateral $ABCD$ with AB equal and $\parallel$ to DC.

Prove that the quadrilateral $ABCD$ *is a* ▱.

The plan is to prove that $x = x'$ and $y = y'$, and then to apply § 55.

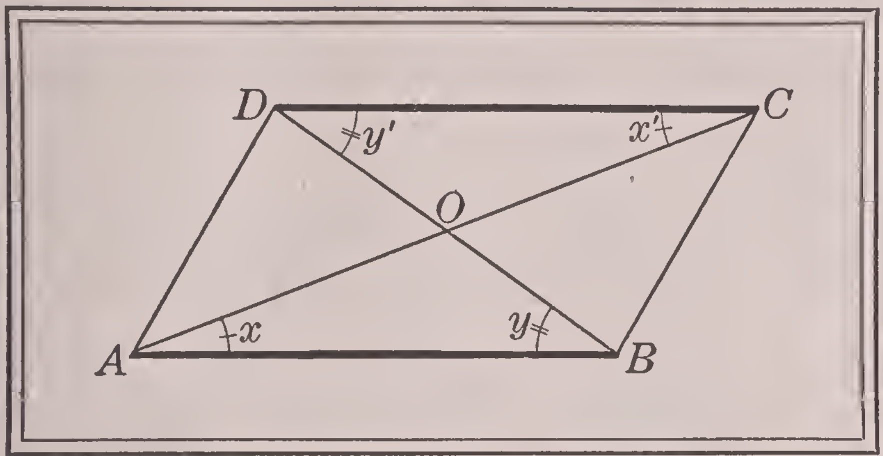
Proof. Draw	the diagonal AC.	Post. 1
Since	$AC = AC$,	Iden.
since	$AB = DC$,	Given
and since	$x = x'$,	§ 61
we see that	$\triangle ABC$ is congruent to $\triangle CDA$.	§ 40
Then	$y' = y$.	§ 38
	$\therefore BC$ is $\parallel$ to AD.	§ 55
Also,	AB is $\parallel$ to DC.	Given
	$\therefore ABCD$ is a ▱.	§ 72

82. Corollary. *If both pairs of opposite angles of a quadrilateral are equal, the figure is a parallelogram.*

The sum of the $\angle$s of the above quadrilateral is the same as the sum of the $\angle$s of the $\triangle$s ABC and CDA; that is, it is 4 rt. $\angle$s (§ 65). Now if $\angle A = \angle C$ and $\angle B = \angle D$, it follows (Ax. 1) that $\angle A + \angle B = \angle C + \angle D$; whence $\angle A + \angle B = \frac{1}{2}$ of 4 rt. $\angle$s $= 2$ rt. $\angle$s. Similarly, $\angle A + \angle D = 2$ rt. $\angle$s. Hence, by § 59, the opposite sides are $\parallel$, and $ABCD$ is a ▱ (§ 72).

Proposition 15. Diagonals of a Parallelogram

83. Theorem. *The diagonals of a parallelogram bisect each other.*



Given the ▱ *ABCD* with the diagonals *AC* and *BD* intersecting at *O*.

Prove that $AO = OC$

and that $BO = OD$.

The plan is to show first that $\triangle ABO$ is congruent to $\triangle CDO$ or that $\triangle BCO$ is congruent to $\triangle DAO$.

Proof. In $\triangle\triangle ABO$ and CDO we have

$AB = CD$. § 76

The opposite sides of a ▱ are equal⋯.

We also have $x = x'$ and $y = y'$. § 61

If two ∥ lines are cut by a transversal, the alternate ∠s are equal.

$\therefore \triangle ABO$ is congruent to $\triangle CDO$. § 44

If two ∠s and the included side of one △ are equal respectively to two ∠s and the included side of another, the △s are congruent.

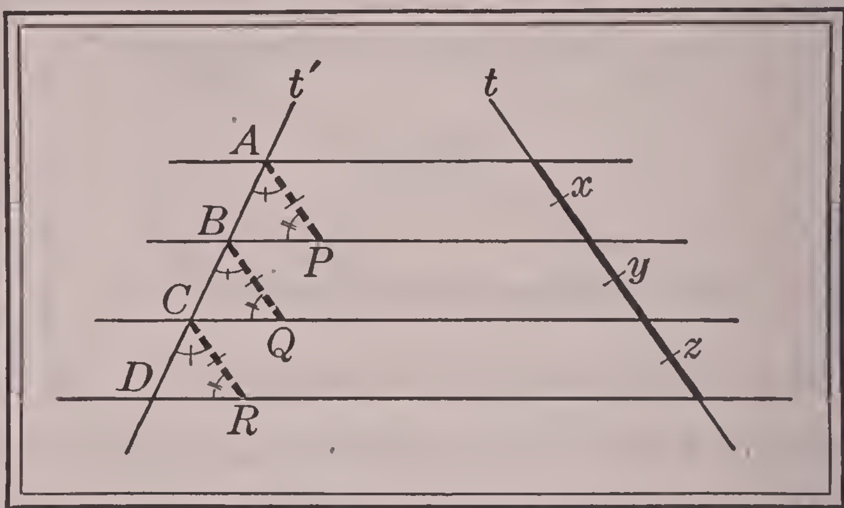
Hence $AO = OC$ and $BO = OD$. § 38

84. Corollary. *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

For then $\triangle ABO$ is congruent to $\triangle CDO$ (§ 40), $x = x'$ (§ 38), and AB is ∥ to DC (§ 55). Similarly, AD is ∥ to BC. Give the proof in full.

Proposition 16. Parallels intercept Equal Segments

85. Theorem. *If three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal.*



Given several $\parallel$s intercepting the equal segments x, y, z on the transversal t and intercepting the segments AB, BC, CD on the transversal t'.

Prove that $AB = BC = CD$.

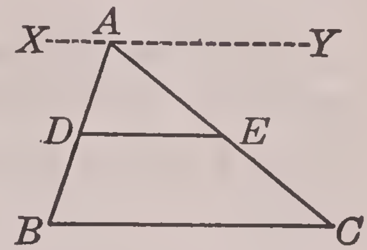
The plan is to prove three $\triangle$ congruent.

Proof. If t is $\parallel$ to t', the proposition is true by § 78.

If t is not $\parallel$ to t', we can evidently prove the theorem if we can show that AB, BC, CD are sides of congruent $\triangle$. This can be done by § 44 if we can prove that $AP = BQ = CR$ and can prove that the $\angle$ including these lines are respectively equal in each case.

Suppose that AP, BQ, CR are each $\parallel$ to t.		§ 52
Since	$AP = x$, $BQ = y$, and $CR = z$,	§ 78
we have	$AP = BQ = CR$.	Ax. 5
Then	$\angle BAP = \angle CBQ = \angle DCR$,	§ 62
and	$\angle APB = \angle BQC = \angle CRD$.	§ 64
Hence	$\triangle ABP$, BCQ, CDR are congruent,	§ 44
and	$AB = BC = CD$.	§ 38

86. Corollary. *If a line parallel to one side of a triangle bisects another side, it bisects the third side also.*



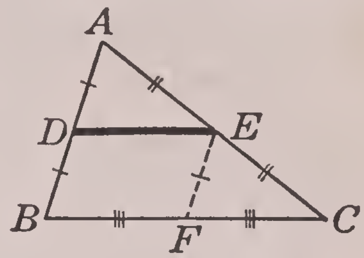
Given the $\triangle ABC$ as shown, with $DE \parallel$ to BC and $BD = DA$.

Prove that $CE = EA$.

In the proof suppose that XY is $\parallel$ to DE. Then show that this is simply a special case of § 85, the two transversals being AB and AC.

The student will find it interesting to take other special cases,—for example, the case in which the transversals cross between the $\parallel$s.

87. Corollary. *The line which joins the midpoints of two sides of a triangle is parallel to the third side and is equal to half the third side.*



Given the $\triangle ABC$ as shown, with $BD = DA$ and $CE = EA$.

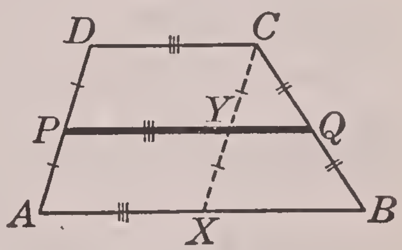
Prove that DE *is* $\parallel$ *to* BC

and that $DE = \frac{1}{2} BC$.

In the proof suppose that EF is $\parallel$ to AB. The corollary is evidently proved if we can prove that $BFED$ is a $\square$ and that $BF = FC$.

Show that a line from $D \parallel$ to BC makes $CE = EA$. Then what follows as to DE and BC? How does EF divide BC?

88. Corollary. *If a line parallel to the base of a trapezoid bisects one of the other sides, it bisects the opposite side and is equal to half the sum of the bases.*



Given the trapezoid $ABCD$ as shown, with $PQ \parallel$ to AB and $AP = PD$.

Prove that $BQ = QC$

and that $PQ = \frac{1}{2}(AB + DC)$.

Proof. Suppose that CX is $\parallel$ to DA. § 52

Then $XY = YC$, and $BQ = QC$. § 85

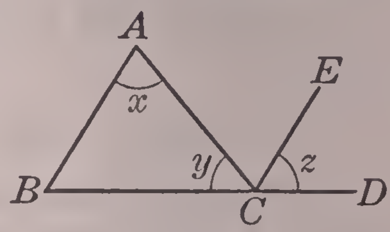
Hence $YQ = \frac{1}{2} XB$. § 87

Also, $PY = AX = DC = \frac{1}{2}(2\,AX) = \frac{1}{2}(AX + DC)$. § 78

$\therefore PY + YQ = \frac{1}{2}(AX + XB + DC) = \frac{1}{2}(AB + DC)$. Axs. 1, 5

Exercises. Review

1. In this figure, B, C, and D are in a straight line. If $x = 73°$, $y = 49°$, and $z = 58°$, prove that CE is $\parallel$ to BA and find the number of degrees in $\angle B$ and in $\angle ECA$.

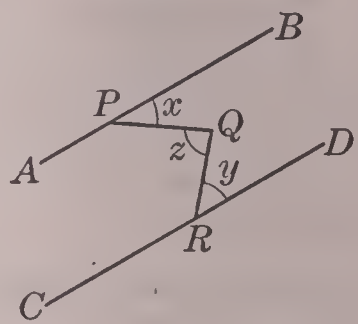


2. In the figure of Ex. 1 suppose that $x = 138°$, $y = 15°$, and $z = 27°$. Prove that CE is $\parallel$ to BA and find the number of degrees in $\angle B$.

The student should sketch a new figure, in which the angles conform approximately to the new measurements.

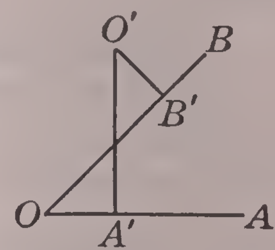
3. In this figure, if $x = 34°$, $y = 49°$, and $z = 83°$, then AB is $\parallel$ to CD.

Produce RQ to meet AB.



4. In the figure of Ex. 3, if it is given that AB is $\parallel$ to CD, then $z = x + y$.

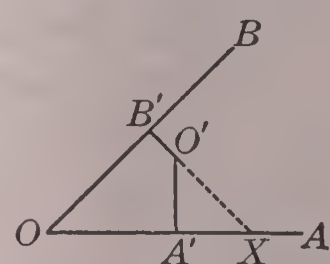
5. In this figure, $O'A'$ is $\perp$ to OA, and $O'B'$ is $\perp$ to OB. Name all the pairs of equal angles in the figure and prove each statement.



6. In the figure of Ex. 5, what other condition would make the two triangles congruent?

7. In Ex. 5 suppose that O' lies within $\angle AOB$, as shown in this figure.

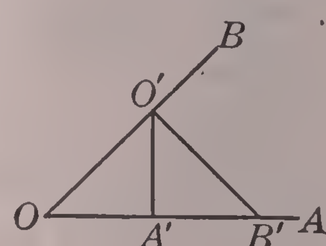
Produce $B'O'$ to meet OA, as at X. Show that the angles of $\triangle XO'A'$ are respectively equal to the angles of $\triangle XOB'$.



8. In Ex. 7 prove that $\angle B'O'A'$ is supplementary to $\angle O$.

9. In Ex. 5 suppose that O' lies on OB, as shown in this figure.

Show that the angles of $\triangle B'O'A'$ are respectively equal to the angles of $\triangle B'OO'$.



89. Polygon. A rectilinear figure of three or more sides is called a *polygon.*

The terms *sides, perimeter, angles, vertices,* and *diagonals* are employed in the usual sense in connection with polygons in general.

90. Polygons classified as to Sides. A polygon is called

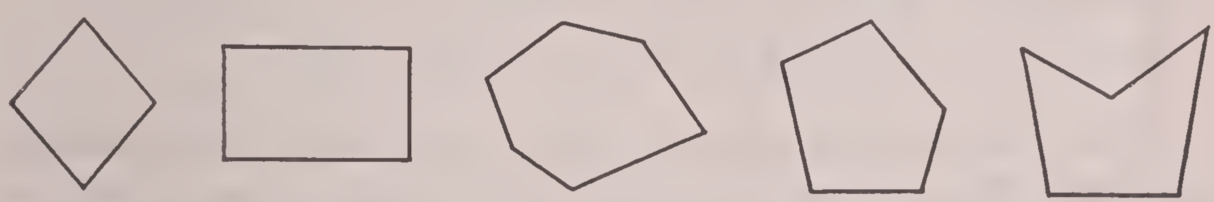
a *triangle* if it has three sides;
a *quadrilateral* if it has four sides;
a *pentagon* if it has five sides;
a *hexagon* if it has six sides.

These names are sufficient for most cases. The next few names in order are *heptagon, octagon, nonagon, decagon, undecagon, dodecagon.*

A polygon is *equilateral* if all its sides are equal.

91. Polygons classified as to Angles. A polygon is

equiangular if all its angles are equal;
convex if each of its angles is less than a straight angle;
concave if it has an angle greater than a straight angle.



Equilateral   Equiangular   Hexagon   Convex   Concave

In a concave polygon, an angle greater than a straight angle is called a *reëntrant angle.* As stated in § 7, when the term *polygon* is used a convex polygon is understood unless the contrary is stated.

92. Regular Polygon. A polygon that is both equiangular and equilateral is called a *regular polygon.*

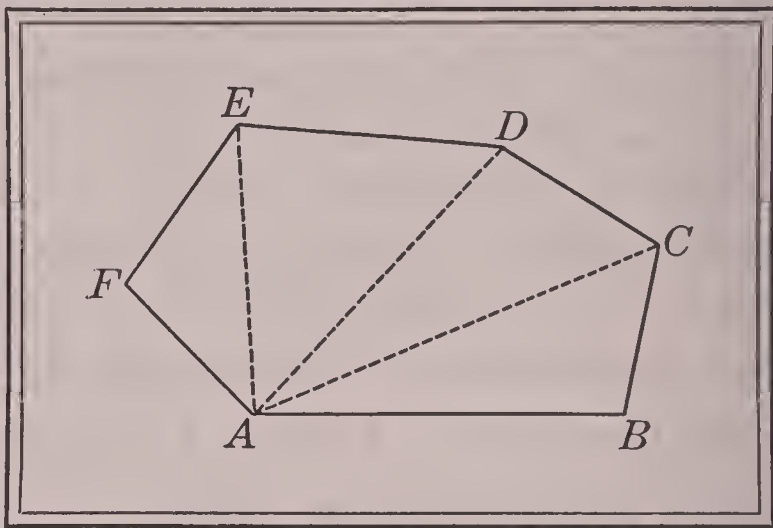
93. Relation of Two Polygons. Two polygons are

mutually equiangular if the angles of the one are equal to the angles of the other, taken in the same order;

mutually equilateral if the sides of the one are equal to the sides of the other, taken in the same order.

Proposition 17. Sum of the Angles of a Polygon

94. Theorem. *The sum of the interior angles of a polygon is as many straight angles less two as the figure has sides.*



Given the polygon *ABCDEF* with *n* sides.

Prove that the sum of the interior ∠s *is* $(n-2)$ *st.* ∠s.

The plan is to cut the figure into △s and apply § 65.

Proof. From any vertex A draw as many diagonals as possible. Then there is a △ for each side except the two adjacent to A. Hence there are $(n-2)$ △s.

The sum of the ∠s of each △ is a st. ∠. § 65

Hence the sum of the ∠s of the $(n-2)$ △s, that is, the sum of the ∠s of the polygon, is $(n-2)$ st. ∠s. Ax. 3

Notice that this proposition includes § 65 as a special case.

95. Corollary. *The sum of the angles of a quadrilateral is two straight angles; and if the angles are all equal, each is a right angle.*

Give brief oral proofs of all such corollaries.

96. Corollary. *Each angle of a regular polygon of n sides is equal to $(n-2)/n$ straight angles.*

Exercises. Review

1. *If the arms of one angle are respectively perpendicular to the arms of another angle, the angles are either equal or supplementary.*

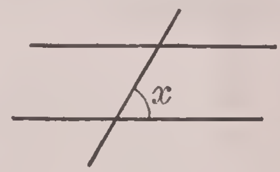
2. *Any two consecutive angles of a parallelogram are supplementary.*

3. If one angle of a triangle is 37° 30′, what is the sum of the other two angles?

4. If the sum of two angles of a triangle is 37° 30′, how many degrees are there in the other angle?

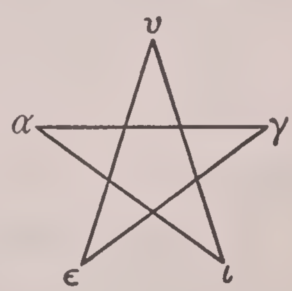
5. If an exterior angle at the base of an isosceles triangle is 98°, find the number of degrees in each angle of the triangle.

6. If the exterior angle at the vertex of an isosceles triangle is 98°, find the number of degrees in each angle of the triangle.

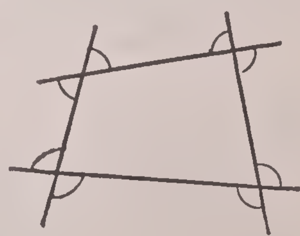


7. In this figure, which shows two parallel lines cut by a transversal, $x = 59°$. How many degrees in each of the other seven angles?

8. Find the sum of the angles at the five points of the usual form of the five-pointed star.



Such a star is sometimes called a *pentagram.* It was used as a badge by the followers of Pythagoras, one of the greatest of the Greek mathematicians, about 525 B.C. At the five points were the Greek letters υ, γ, ι, ϵ, α, the word *ὑγίεια* (hygieia) meaning "health," the single letter ϵ being used for $\epsilon\iota$.

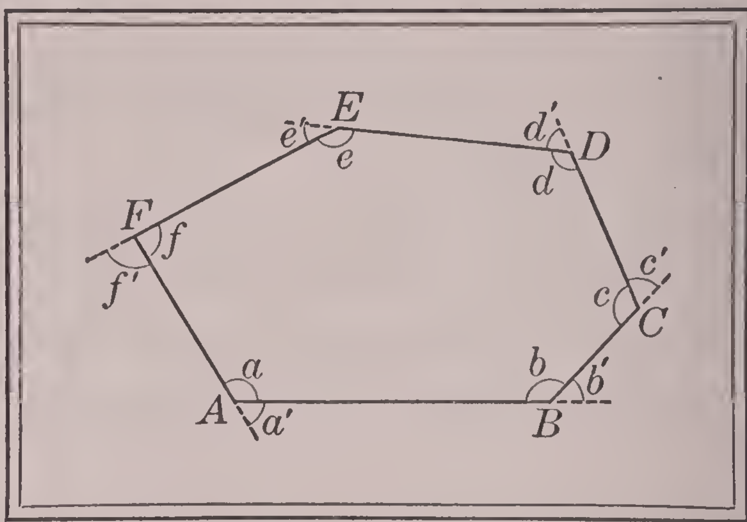


9. Study this figure with respect to the sum of the marked angles, write a theorem concerning it, and prove this theorem.

10. Consider the theorem of Ex. 9 for the special case of the parallelogram.

Proposition 18. Exterior Angles

97. Theorem. *The sum of the exterior angles of a polygon, made by producing each of its sides in succession, is two straight angles.*



Given the polygon *ABCDEF* with its *n* sides produced in succession.

Prove that the sum of the exterior $\angle$s *is* 2 *st.* $\angle$s.

The plan is to take the sum of the interior $\angle$s from n st. $\angle$s.

Proof. Designate the interior $\angle$s by a, b, c, d, e, f, and the corresponding exterior $\angle$s by a', b', c', d', e', f'.

Then, considering each pair of adjacent $\angle$s,

$$a + a' = \text{a st.} \angle,$$

and $$b + b' = \text{a st.} \angle. \qquad \S\, 12$$

In like manner, each pair of adjacent $\angle$s $=$ a st. $\angle$.

Then, since the polygon has n sides and n $\angle$s, the sum of the interior and exterior $\angle$s is n st. $\angle$s. Ax. 3

But the sum of the interior $\angle$s is $(n-2)$ st. $\angle$s § 94

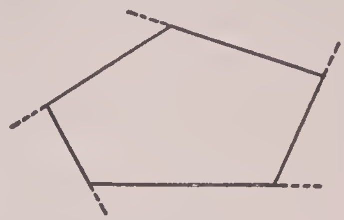
or $$n \text{ st. } \angle\text{s} - 2 \text{ st. } \angle\text{s}.$$

Hence n st. $\angle$s $-$ (n st. $\angle$s $-$ 2 st. $\angle$s) $=$ 2 st. $\angle$s; Ax. 2

that is, the sum of the exterior $\angle$s is 2 st. $\angle$s.

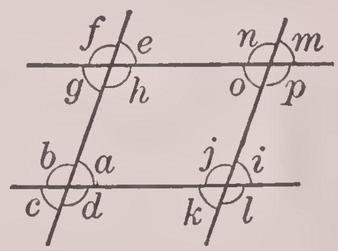
Exercises. Review

1. In making a map of a field a surveyor uses an instrument which enables him to find with equal ease the interior angles and the exterior angles of the field. In order to check his work he may use either § 94 or § 97. Which is the easier for him to use, and why is it easier?



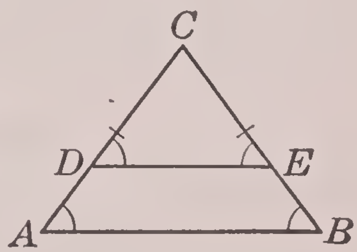
2. In making a map of a field of five sides a surveyor finds that the exterior angles are $20° 30'$, $39° 30'$, $59° 30'$, $35° 30'$, and $24° 30'$. Are his angle measures correct? If all but the last are checked and thus are known to be correct, what is the size of the last angle?

3. This figure represents two pairs of parallel lines. State all the equalities of angles, thus: $a = c = g = e = o = \cdots$. Give the reason in each case.

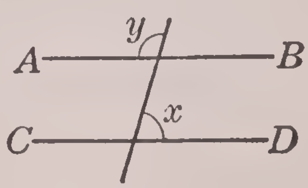


4. In the figure of Ex. 3 state ten pairs of nonadjacent angles which are supplementary; thus: $a + h = 180°$ and $d + e = 180°$.

5. In this figure, given that $AC = BC$ and that DE is $\parallel$ to AB, prove that $CD = CE$. Write a general statement of the theorem.



6. In the figure here shown, $x = 72°$ and $x = \frac{2}{3}y$. Is $AB \parallel$ to CD? Give the proof in full.



7. In the figure of Ex. 6 suppose that $x = 73°$ and $y - x = 32°$. Is AB then $\parallel$ to CD? Give the proof.

8. How many sides has a regular polygon each angle of which is $140°$?

98. Summary of Important Fundamental Theorems. There are many important theorems in geometry, but those which we have thus far studied are used more often than those of any other similar group. We may now summarize the most important of the results as follows:

Conditions of Congruence of Triangles

1. Two sides and included $\angle$ respectively equal.	§ 40
2. Two $\angle$s and included side respectively equal.	§ 44
3. Three sides respectively equal.	§ 47
4. Two $\angle$s and any side respectively equal.	§ 67

Conditions of Congruence of Right Triangles

1. Hypotenuse and an adjacent $\angle$ respectively equal.	§ 68
2. Hypotenuse and a side respectively equal.	§ 71

Conditions of Parallelism

1. Alternate $\angle$s equal.	§ 55
2. Two lines $\perp$ to the same line.	§ 57
3. Two lines $\parallel$ to a third line.	§ 58
4. Corresponding $\angle$s equal.	§ 59
5. Interior $\angle$s on same side supplementary.	§ 59

Transversal Cutting Parallels

1. Alternate $\angle$s are equal.	§ 61
2. Corresponding $\angle$s are equal.	§ 62
3. Interior $\angle$s on same side are supplementary.	§ 63
4. Segments on other transversals are equal.	§ 85

Sums of Angles

1. Of a triangle,	1 st. $\angle$.	§ 65
2. Of a polygon,	$(n-2)$ st. $\angle$s.	§ 94
3. Of a polygon, exterior,	2 st. $\angle$s.	§ 97

II. Fundamental Constructions

99. Construction. When we *construct* a figure we make the figure accurately by the aid of an unmarked ruler and a pair of compasses, which are the only instruments recognized in elementary geometry. When we *draw* a figure we make the figure without the aid of these instruments, but we may use, if we wish, the draftsman's triangle, the protractor, or the T-square, so as to make a neat figure.

In many cases it is immaterial whether we use the word "draw" or the word "construct," as when we speak of drawing a line.

We shall now consider the solution of a few of the most important problems of construction.

100. Nature of a Solution. A solution of a problem has one step that a proof of a theorem does not have.

In proving a theorem we state (1) *what is given*, (2) *what is to be proved*, and (3) the *proof*.

In solving a problem we must state (1) *what is given*; (2) *what is required*, that is, to do some definite thing; (3) the *construction*, that is, how to do it; and (4) the *proof*, showing that the construction explained in step 3 is correct.

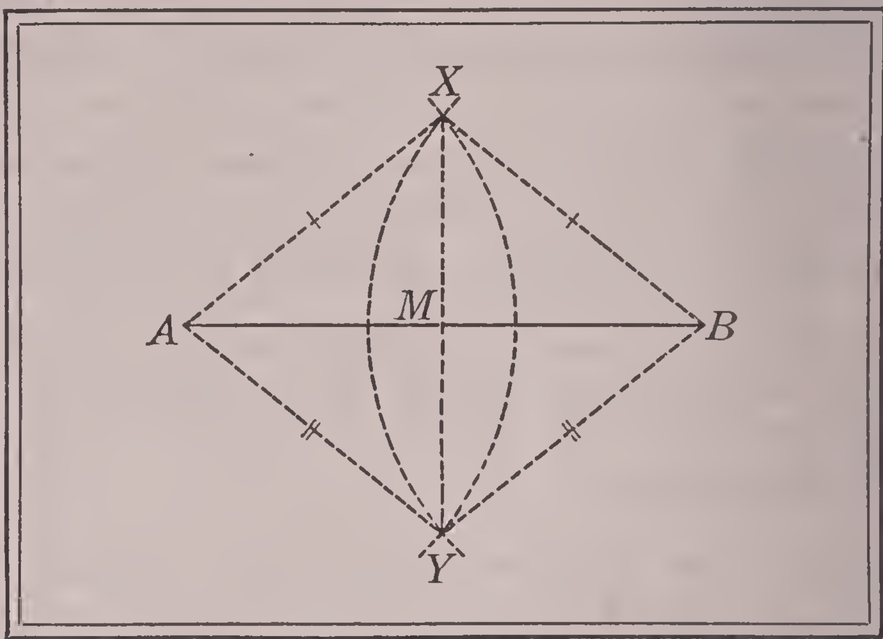
We *prove* a theorem, but we *solve* a problem and then prove that our solution is a correct one.

In the figures for the problems in Book I, given lines are shown as full lines, required lines as heavy black lines, and construction lines and lines produced as dotted lines. (See also the note in § 39.)

101. Discussion of a Problem. Besides the four necessary steps mentioned in § 100, a fifth step may profitably be taken in connection with every problem. This step is the *discussion* of the solution, to see if there are any interesting special cases in which a solution is impossible or in which there is more than one solution. Such discussions are, in general, left to the teacher and students.

Proposition 19. Bisecting a Line Segment

102. Problem. *Bisect a given line segment.*



Given the line segment AB.

Required to bisect AB.

The plan is to construct two congruent $\triangle$s.

Construction. With A and B as centers and with any convenient radius construct two arcs that intersect. Post. 4

A convenient radius in many cases is AB itself.

Designate the points of intersection of the arcs as X and Y.

Draw the st. line XY and designate the point where it cuts the given line segment as M. Post. 1

Then XY bisects AB at M; that is, $AM = BM$.

Proof. Draw AX, BX, AY, BY. Post. 1

Since $\triangle AYX$ is congruent to $\triangle BYX$, § 47

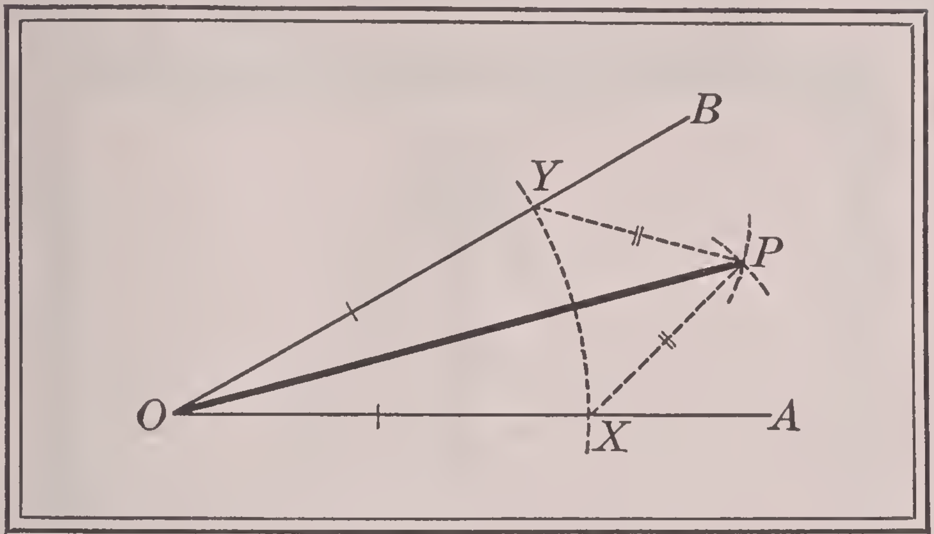
we have $\angle AXY = \angle BXY$. § 38

$\therefore \triangle AMX$ is congruent to $\triangle BMX$. § 40

The student has here the essential features of the proof. He should now give the steps in full.

Proposition 20. Bisecting an Angle

103. Problem. *Bisect a given angle.*



Given the $\angle AOB$.

Required to bisect $\angle AOB$.

The plan is to construct two congruent $\triangle$s.

Construction. With O as center and any convenient radius describe an arc cutting OA at X and OB at Y. Post. 4

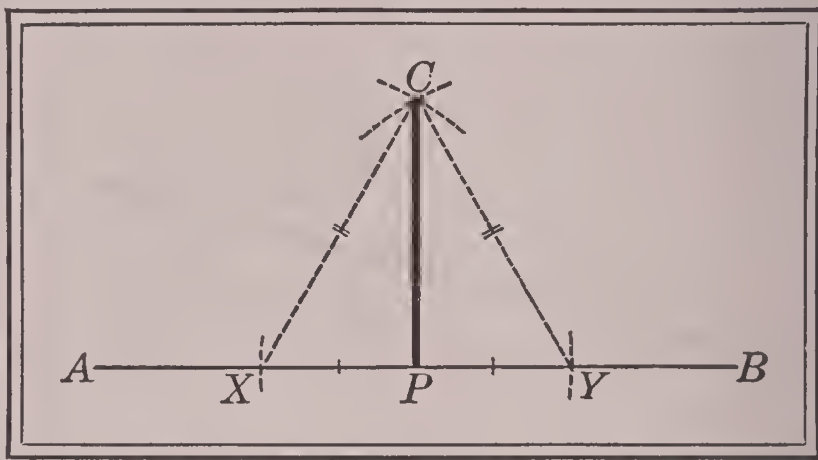
With X and Y as centers and with a radius greater than half the line segment from X to Y, construct intersecting arcs and designate their point of intersection as P. Post. 4

A convenient radius may be found by placing one point of the compasses on X and the other on Y.

Draw	OP.	Post. 1
Then	OP bisects $\angle AOB$.	
Proof. Draw	PX and PY.	Post. 1
Since	$OX = OY$,	Post. 4
since	$PX = PY$,	Const.
and since	$OP = OP$,	Iden.
we see that	$\triangle OXP$ is congruent to $\triangle OYP$.	§ 47
	$\therefore \angle XOP = \angle YOP$.	§ 38

Proposition 21. Perpendicular through Internal Point

104. Problem. *Through a given point on a given straight line construct a perpendicular to the line.*



Given the line *AB* and the point *P* on *AB*.

Required through P to construct a ⊥ to AB.

The plan is to construct two congruent ⚠.

Construction. By drawing arcs, make $PX = PY$. Post. 4

With X as center and XY as radius construct an arc, and with Y as center and the same radius construct another arc intersecting the first arc at C. Post. 4

Draw PC, which is the required ⊥. Post. 1

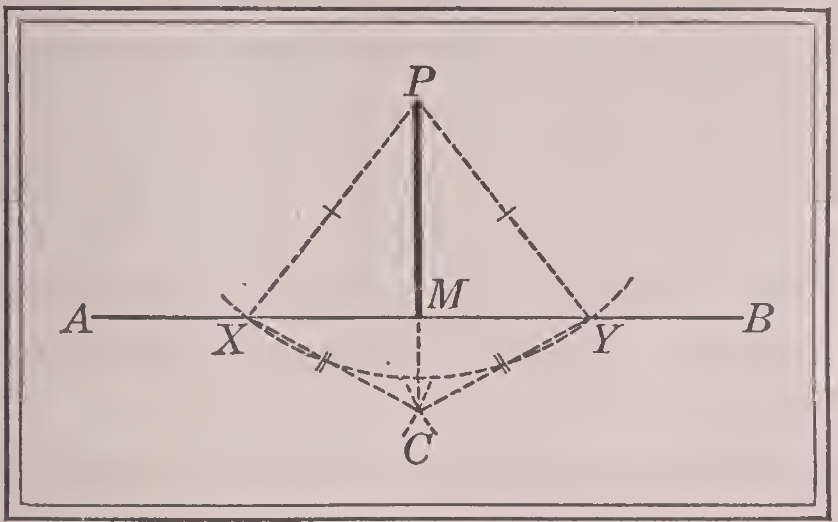
Proof. Draw CX and CY. Post. 1

Since we used the same radius in constructing the intersecting arcs, we have

	$CX = CY$.	Const.
Also	$PX = PY$,	Post. 4
and	$CP = CP$.	Iden.
	$\therefore \triangle XPC$ is congruent to $\triangle YPC$,	§ 47
and	$\angle CPX = \angle CPY$.	§ 38
	$\therefore \angle CPX$ is a rt. $\angle$,	§ 13
and	PC is $\perp$ to AB.	§ 14

Proposition 22. Perpendicular through External Point

105. Problem. *Through a given point outside a given straight line construct a perpendicular to the line.*



Given the line *AB* and the point *P* not on *AB*.

The plan is to construct two congruent △.

Required through P to construct a ⊥ to AB.

Construction. With P as center and a radius sufficiently long construct an arc cutting AB at X and Y. Post. 4

Such a radius can easily be found by simply placing one point of the compasses on P and the other on any point below AB.

With X and Y as centers and a radius sufficiently long construct two arcs intersecting at C below AB. Post. 4

Such a radius may be any length greater than half of XY.

Draw PC. Post. 1

Let M be the point of intersection of PC and AB.

Then PM is the required ⊥.

Proof. Draw PX, PY, CX, CY. Post. 1

Then $\triangle PXC$ is congruent to $\triangle PYC$. § 47

Now write out the full proof, which should show that $\triangle XMP$ is congruent to $\triangle YMP$ by § 40.

Exercises. Constructions

1. Draw a line segment $3\frac{1}{2}$ in. long and bisect this line segment by measuring. Then bisect it by § 102 and thus test the accuracy of your measurement.

2. By the aid of a protractor draw and bisect an angle of 60°. Then bisect the angle by § 103.

3. Draw a line AB, take a point P not on AB, and through P draw a perpendicular to AB by means of a draftsman's triangle. Then through P construct a perpendicular to AB by the method of § 105, and thus check the accuracy of the drawing.

In ordinary practice, either of these methods is satisfactory.

4. Draw a line AB, take a point P on the line, and through P draw a perpendicular to AB by means of a draftsman's triangle. Then construct a perpendicular as in § 104.

5. Write a statement about the relative sizes of the halves of equal line segments; of the halves of equal angles; of the halves of equal circles; of the halves of any equal magnitudes. Draw a diagram to illustrate each statement.

6. Write a statement about the result of adding equal line segments to equal line segments; of adding equal angles to equal angles. Draw a diagram to illustrate each statement.

7. How many degrees are there in an angle that is equal to half its complement? to half its supplement?

8. How many degrees are there in an angle that is equal to 10° more than its complement? to 20° less than its complement? to 30° less than half its complement?

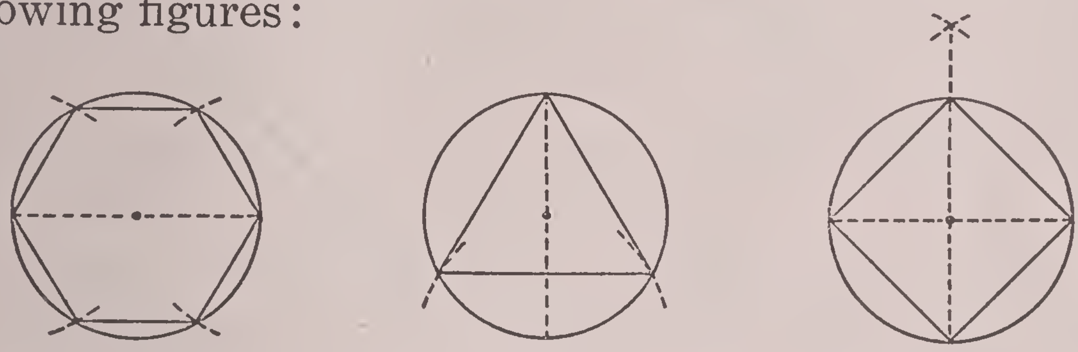
9. Construct a line segment equal to the sum of two given line segments; to the difference between two given line segments.

Construct angles of the following sizes:

10. 45°. **11.** 22° 30′. **12.** 11° 15′. **13.** 135°. **14.** 157° 30′.

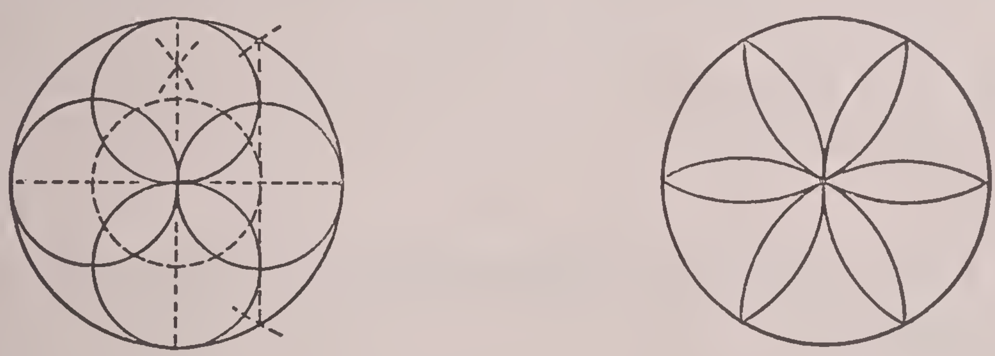
15. Construct a square 2 in. on a side. If the figure is correctly constructed the two diagonals are equal. Check the work by measuring the diagonals with the compasses.

16. By the use of compasses and ruler construct the following figures:



The lines made of short dashes show how to locate the points needed in drawing the figure. They should be erased after the figure is completed unless the teacher directs that they be retained to show how the construction was made.

17. By the use of compasses and ruler construct the following figures:

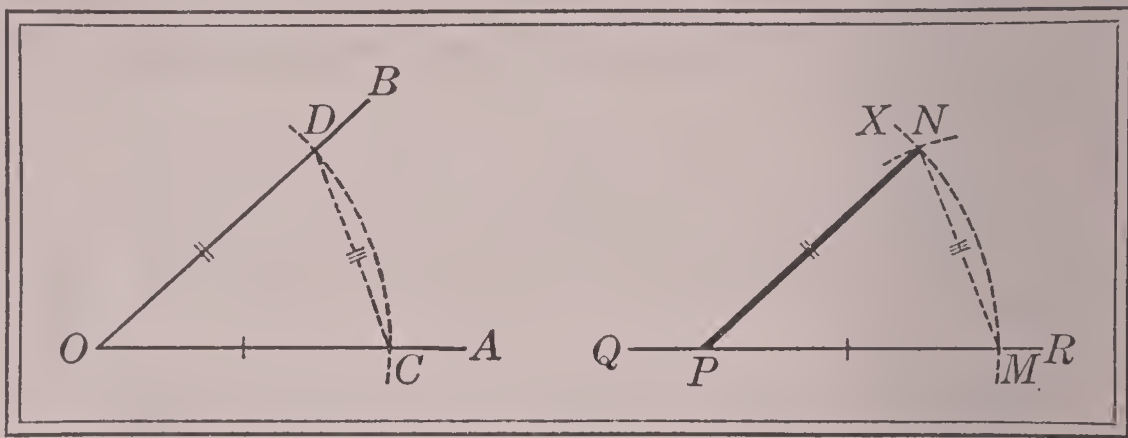


In the figures in Exs. 16 and 17 it should be noticed that the radius of a circle may be used to draw arcs which shall divide the circle into six equal parts.

18. By the use of compasses and ruler construct four original designs similar in nature to those of Ex. 17. Try to make the designs as varied as possible.

Proposition 23. Constructing Equal Angles

106. Problem. *From a given point on a given line construct a line which shall make with the given line an angle equal to a given angle.*



Given the $\angle AOB$ and the point P on the line QR.

Required from P to draw a line making with the line QR an $\angle$ equal to $\angle AOB$.

The plan is to construct two congruent $\triangle$.

Construction. With O as center and any radius describe an arc cutting OA at C and OB at D. Post. 4

With P as center and the same radius describe an arc MX, cutting QR at M. Post. 4

Draw CD. Post. 1

With M as center and CD as radius describe an arc cutting the arc MX at N. Post. 4

Draw PN. Post. 1

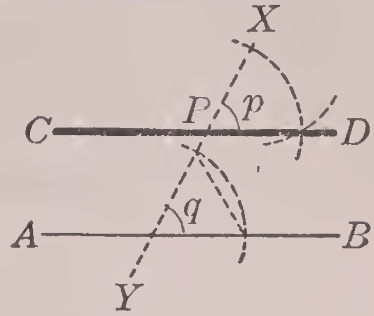
Then PN is the required line.

Proof. Draw MN. Post. 1

Now prove that $\triangle$ OCD and PMN are congruent by § 47.

This method of constructing equal angles is more nearly accurate than the method of drawing by the aid of a protractor.

107. Corollary. *Through a given external point construct a line parallel to a given line.*



Let P be the given external point and AB the given line.

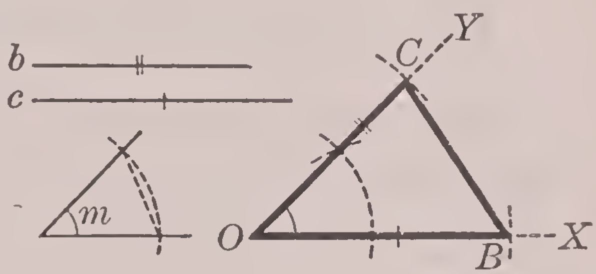
Draw any line XPY through P, cutting AB as in the figure.

At P construct $p = q$, and draw DPC.

The line CD is the required line.

Write the construction in the usual form and give the proof.

108. Corollary. *Given two sides and the included angle of a triangle, construct the triangle.*



Let b and c be the given sides and m the given $\angle$.

Construct $\angle XOY = m$.

On OX mark off with the compasses $OB = c$, and on OY mark off $OC = b$.

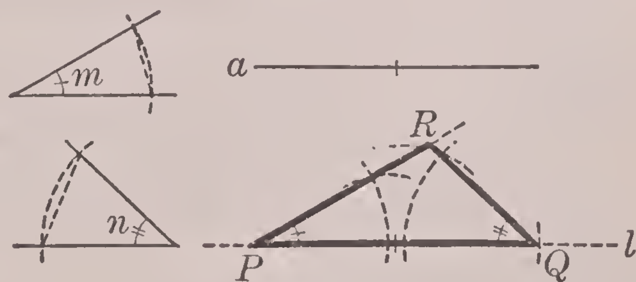
Draw BC.

Then $\triangle OBC$ is the required $\triangle$.

Write the construction in the usual form and give the proof.

Of course the $\triangle$ may be turned over, giving another appearance, but such cases, if thought important, are left to the consideration of the class.

109. Corollary. *Given a side and two angles of a triangle, construct the triangle.*



Let a be the given side and m and n the given $\angle$s.

Then if the side is included by the $\angle$s, mark off with the compasses on any line l the segment $PQ = a$.

At P construct an $\angle$ equal to m, and at Q construct an $\angle$ equal to n.

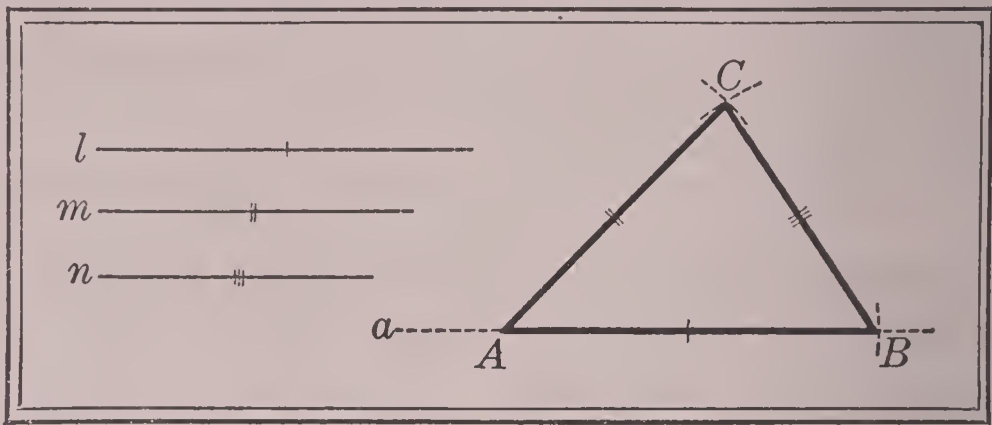
Then $\triangle PQR$ in the figure is the required $\triangle$.

Write the construction in the usual form and give the proof.

If the side is not included by the $\angle$s, find the third $\angle$ by means of § 65 and then proceed as above.

Proposition 24. Triangle with Given Sides

110. Problem. *Construct a triangle with its sides equal respectively to three given line segments.*



Given the line segments l, m, n.

Required to construct a $\triangle$ *with sides equal to* l, m, n.

The plan is to draw two arcs which shall determine the $\triangle$.

Construction. Draw a line a with the ruler and on it mark off with the compasses a line segment $AB = l$.

With A as center and m as radius draw an arc; with B as center and n as radius draw another arc cutting the first arc at C. Post. 4

Draw AC and BC. Post. 1

Then ABC is the required $\triangle$.

Proof. $AB = l$, $AC = m$, and $BC = n$. Const.

The discussion (§ 101) should disclose any special cases.

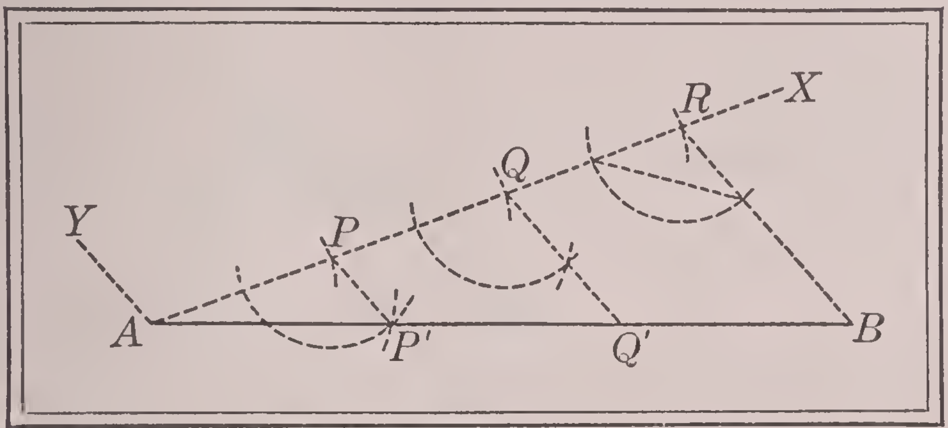
111. Corollary. *Given one of the sides, construct an equilateral triangle.*

In this case, and similarly in § 112, the student should perform the construction, and then write out the construction and the proof in proper geometric form.

112. Corollary. *Given the base and one of the two equal sides, construct an isosceles triangle.*

Proposition 25. Dividing a Line Segment

113. Problem. *Divide a given line segment into a given number of equal parts.*



Given the line segment AB.

Required to divide AB into a given number of equal parts.

The only proposition thus far studied that relates to equal segments on a line is the one concerning a transversal cutting ∥s (§ 85). The plan is, therefore, to bring this problem under that theorem.

Construction. From A draw the line AX, making any convenient $\angle$ with AB. Post. 1

Take any convenient length and, by describing arcs, apply it to AX as many times as is indicated by the number of parts (say three) into which AB is to be divided. Post. 4

From R, the last point thus found, draw RB. Post. 1

From the points P, Q by which AX was divided into equal parts, construct PP' and QQ' ∥ to RB. § 107

These lines divide AB into equal parts as required.

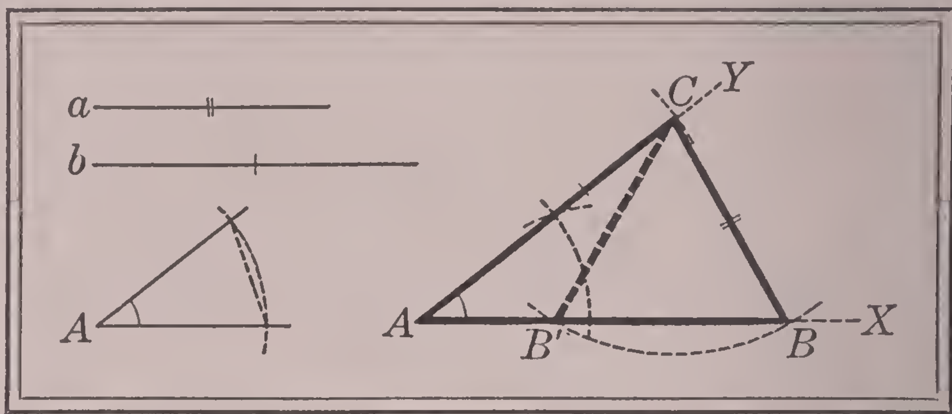
Proof. Construct AY ∥ to BR. § 107

Since the ∥s AY, $P'P$, $Q'Q$, BR were constructed so as to cut off equal segments on AX, they cut off the equal segments AP', $P'Q'$, $Q'B$ on AB. § 85

This method is more nearly accurate than trying to divide AB by measuring its length with a ruler.

Proposition 26. Two Sides and One Angle

114. Problem. *Given two sides of a triangle and the angle opposite one of them, construct the triangle.*



Given a and b, two sides of a $\triangle$, and A the $\angle$ opposite a.

Required to construct the $\triangle$.

The plan is to determine the $\triangle$ by means of arcs.

Construction. CASE 1. *If* $a < b$.

On a line AX construct $\angle XAY = \angle A$. § 106

On AY take $AC = b$. Post. 4

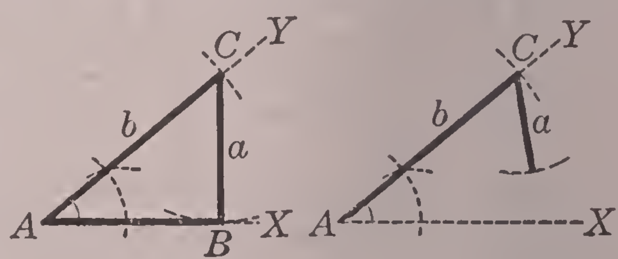
With C as center and a as radius construct an arc intersecting the line AX at B and B'. Post. 4

Draw BC and $B'C$, thus completing the $\triangle$. Post. 1

Then both $\triangle ABC$ and $\triangle AB'C$ satisfy the conditions.

This is called the *ambiguous case.*

Except for students specializing in mathematics, § 114 may be omitted.

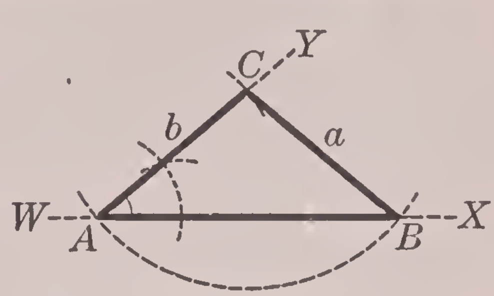


For the present we shall assume that if $a < b$ there are, in general, two constructions as stated. If a is equal to the $\perp$ from C to AX, it is evident that there is but one construction, the rt. $\triangle ABC$, as shown in the figure at the left.

If a is less than the $\perp$ from C to AX, it is apparent that there is no $\triangle$, as shown in the figure at the right.

CASE 2. *If* $a = b$.

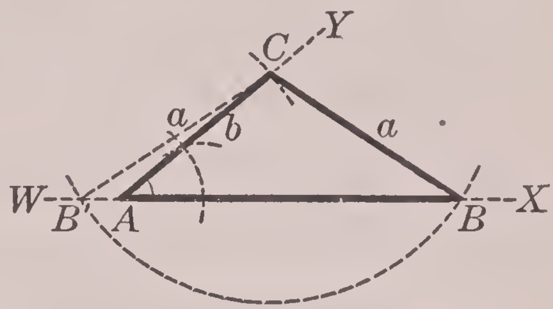
If the given $\angle A$ is acute and $a = b$, the arc constructed from C as center with radius a apparently cuts the line WX at the points A and B. There is, however, but one $\triangle$; namely, the isosceles $\triangle ABC$.



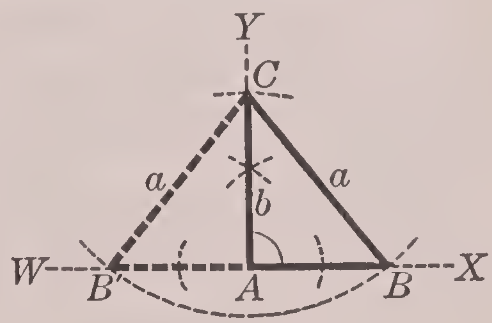
If A is a rt. $\angle$ or an obtuse $\angle$, there is no $\triangle$ when $a = b$, for a $\triangle$ cannot have two rt. $\angle$s or two obtuse $\angle$s (§ 65).

CASE 3. *If* $a > b$.

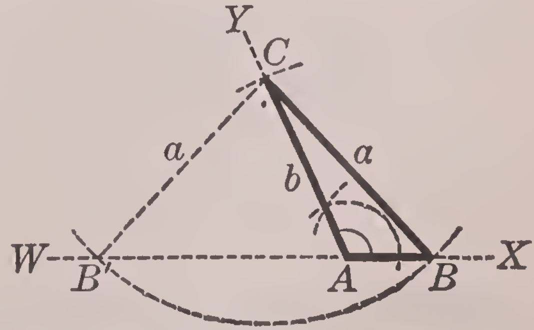
If the given $\angle A$ is acute, the arc constructed from C cuts the line WX on opposite sides of A at the points B and B'. Then $\triangle ABC$ satisfies the conditions, but $\triangle AB'C$ does not, for it does not contain the acute $\angle A$. There is then only one $\triangle$ that satisfies the conditions.



If the given $\angle A$ is a rt. $\angle$, the arc constructed from C cuts the line WX on opposite sides of A at the points B and B', and we have two congruent rt. $\triangle$s that satisfy the conditions.



If the given $\angle A$ is obtuse, the arc constructed from C cuts the line WX on opposite sides of A at the points B and B'; but only the $\triangle ABC$ satisfies the conditions.



The proofs of these statements are given later, but since this proposition will not be used in proving any theorems, it is permissible to use them here in discussing the problem.

Exercises. Review of Constructions

1. Divide a given line segment into four equal parts.

2. Construct an equilateral triangle of given perimeter.

3. Through a given point draw a line which shall make equal angles with the two sides of a given angle.

4. Through a given point draw two lines which shall form with two intersecting lines two isosceles triangles.

5. Construct a triangle with its three angles respectively equal to the three angles of a given triangle.

By first constructing an equilateral triangle and then bisecting certain angles construct angles of:

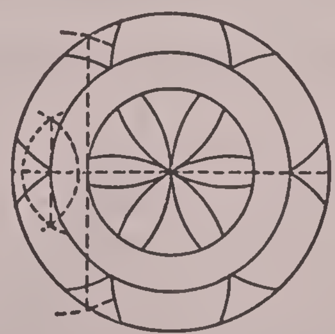
6. $30°$. **7.** $15°$. **8.** $7° 30'$. **9.** $\frac{1}{3}$ of a rt. $\angle$.

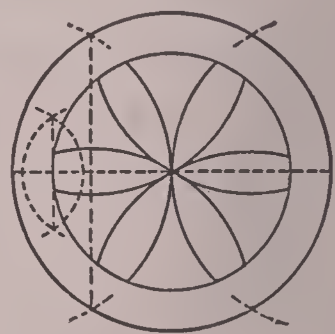
10. Construct an isosceles triangle with its base equal to one third of one of the equal sides.

11. Construct an isosceles right triangle.

12. Construct an isosceles triangle with one of the base angles $60°$. What other special name can you give to the triangle? Prove that your answer is correct.

13. By the use of compasses and ruler construct the following figures (see Ex. 16, page 73):



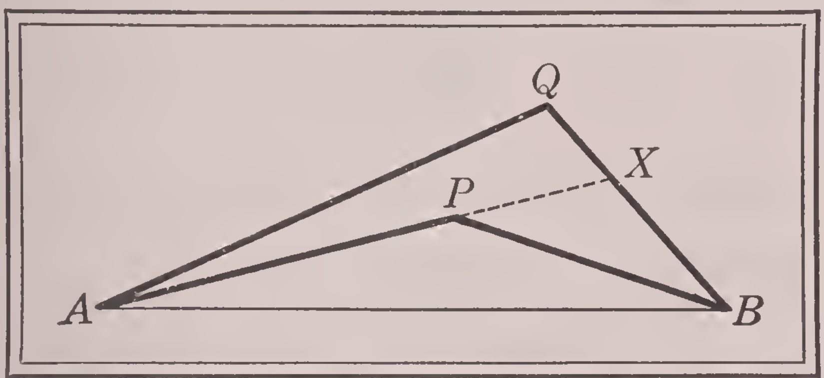


In such figures artistic patterns may be made by coloring various portions of the drawings. In this way designs are made for oilcloth, for stained-glass windows, for colored tiles, and for other decorations.

III. Inequalities

Proposition 27. Unequal Sums of Lines

115. Theorem. *The sum of two line segments from a given external point to the extremities of a given line segment is greater than the sum of two other line segments similarly drawn but included by them.*



Given the line segment AB and the segments from the external points Q, P to A and B.

Prove that $AQ + QB > AP + PB$.

The plan is to show that $AQ + QB > AX + XB > AP + PB$.

Proof. Produce AP to meet QB as at X. Post. 2

Then $AQ + QX > AP + PX$. Post. 3

Likewise, $PX + XB > PB$. Post. 3

Adding these inequalities, we have

$$AQ + QX + PX + XB > AP + PX + PB. \quad \text{Ax. 8}$$

Substituting QB for its equal, $QX + XB$, we have

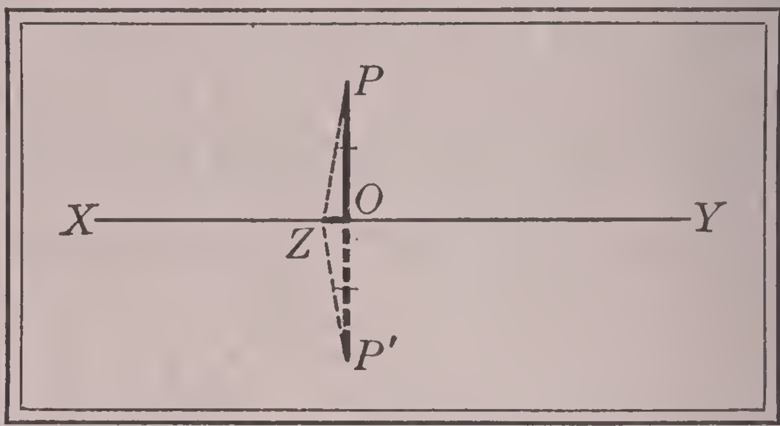
$$AQ + QB + PX > AP + PX + PB. \quad \text{Ax. 5}$$

$$\therefore AQ + QB > AP + PB. \quad \text{Ax. 7}$$

It may be asked why AP produced meets BQ at any point whatever. Such discussions, of little significance at this stage, are left to the teacher to initiate if thought desirable.

Proposition 28. Perpendicular from an External Point

116. Theorem. *One and only one perpendicular can be constructed to a given line from a given external point.*



Given a line XY and an external point P.

Prove that one and only one $\perp$ can be constructed from P to XY.

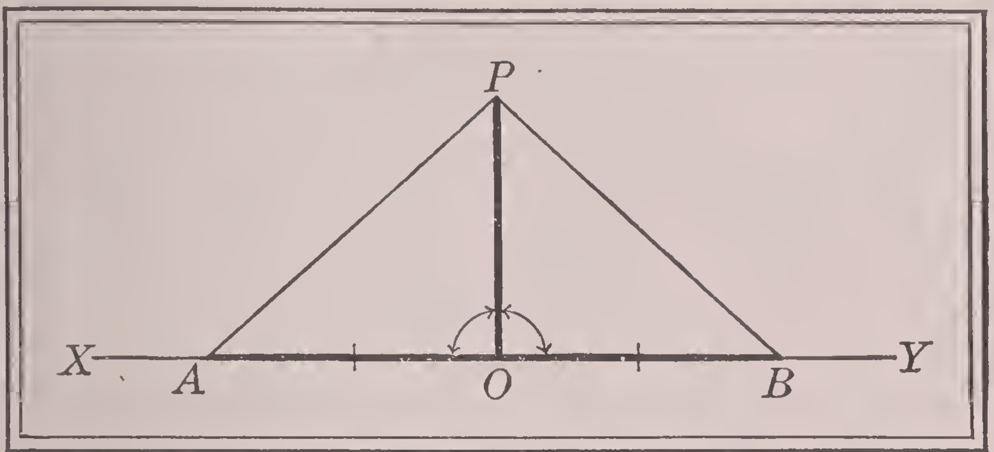
The plan is to show that if two lines from P are $\perp$ to XY, then Post. 1 is violated.

Proof. One $\perp$ to XY, as PO, can be constructed. § 105
Let PZ be any other line from P to XY. Post. 1
Produce PO to P', making $OP' = OP$. Post. 2
Draw $P'Z$. Post. 1
Since POP' is a st. line, PZP' is not a st. line. Post. 1
Hence $\angle P'ZP$ is not a st. $\angle$. § 12
Since $\angle$s POZ and $P'OZ$ are rt. $\angle$s, § 14
we have $\angle POZ = \angle P'OZ$. Post. 6
Hence $\triangle OPZ$ is congruent to $\triangle OP'Z$, § 40
so that $\angle OZP = \angle OZP'$. § 38
$\therefore \angle OZP$, the half of $\angle P'ZP$, is not a rt. $\angle$. § 13
Hence PZ is not $\perp$ to XY, § 14
and PO is the only $\perp$ to XY.

We may now cease to depend upon part of Post. 10.

Proposition 29. A Perpendicular and Equal Obliques

117. Theorem. *If two line segments drawn from a point on a perpendicular to a given line cut off on the given line equal segments from the foot of the perpendicular, the line segments are equal and make equal angles with the perpendicular.*



Given $PO \perp$ **to** XY**, and** PA **and** PB **two lines cutting off from** O **on** XY **the equal segments** OA **and** OB**.**

Prove that $PA = PB$,

and that $\angle APO = \angle BPO$.

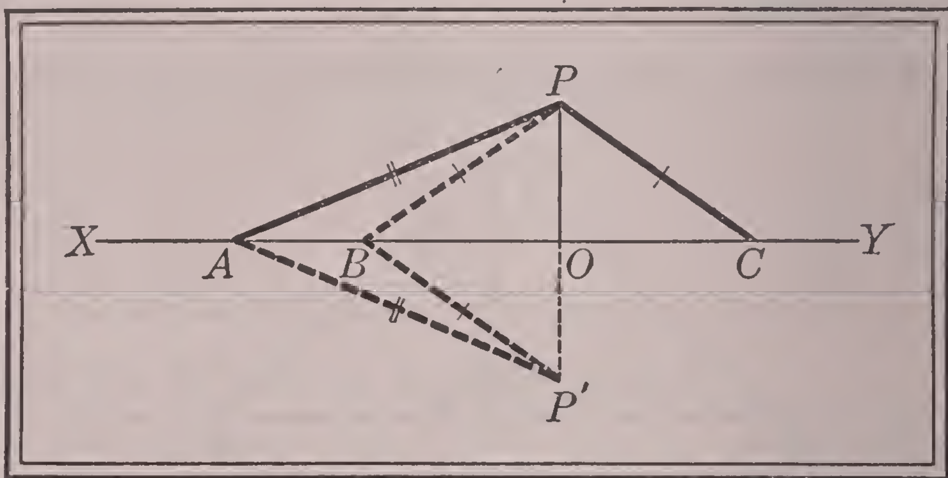
The plan is to prove that the $\triangle$s AOP and BOP are congruent.

Proof. Since	PO is $\perp$ to XY,	Given
we see that	$\angle$s POA and POB are rt. $\angle$s.	§ 14
	$\therefore \angle POA = \angle POB$.	Post. 6
Also,	$OA = OB$,	Given
and	$PO = PO$.	Iden.
Hence	$\triangle AOP$ is congruent to $\triangle BOP$.	§ 40
	$\therefore PA = PB$,	
and	$\angle APO = \angle BPO$.	§ 38

While not dealing directly with inequalities, §§ 116 and 117 are related to the theory, as is shown later.

Proposition 30. A Perpendicular and Unequal Obliques

118. Theorem. *If two line segments drawn from a point on a perpendicular to a given line cut off on the given line unequal segments from the foot of the perpendicular, the line segment more remote is the greater.*



Given $PO \perp$ to XY and two lines PA, PC drawn from P to XY so that $OA > OC$.

Prove that $PA > PC$.

The plan is to show that $PA > PB$, which is equal to PC.

Proof. Take	$OB = OC$ and draw PB.	Post. 1
Then	$PB = PC$.	§ 117
Produce	PO to P', making $OP' = OP$.	Post. 2
Draw	$P'A$ and $P'B$.	Post. 1
Then	$PA = P'A$ and $PB = P'B$.	§ 117
But	$PA + P'A > PB + P'B$,	§ 115

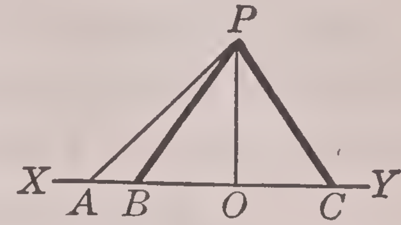
because PP' is a line segment to the ends of which we have drawn segments from A and B.

$\therefore 2\,PA > 2\,PB$, Ax. 5

because we may substitute PA for $P'A$, and PB for $P'B$.

Hence	$PA > PB$,	Ax. 7
and	$PA > PC$.	Ax. 5

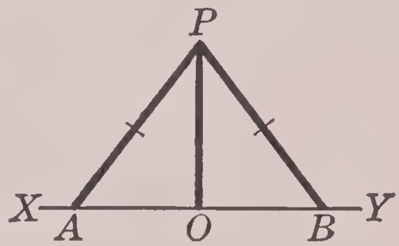
119. Corollary. *Only two equal obliques can be drawn from a given point to a given line.*



Let PA, PB, PC be three obliques and let PO be $\perp$ to XY.

Then to suppose that $PA = PB = PC$ is to contradict § 118, where it was proved that $PA > PC$.

120. Corollary. *Equal obliques from a point to a line cut off equal segments from the foot of the perpendicular from the point to the line.*



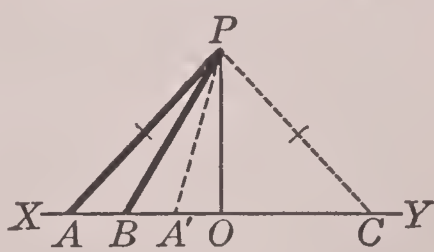
Given $PO \perp$ **to** XY **and** $PA = PB$.

Prove that $OA = OB$.

Proof. In $\triangle AOP$ and BOP we see that

	$\angle POA$ and POB are rt. $\angle$s, *because PO is given as $\perp$ to XY.*	§ 14
	$\therefore \triangle AOP$ and BOP are rt. $\triangle$s.	§ 20
Also,	$PA = PB$,	Given
and	$PO = PO$.	Iden.
	$\therefore \triangle AOP$ is congruent to $\triangle BOP$,	§ 71
and	$OA = OB$.	§ 38

121. Corollary. *If two unequal line segments are drawn from a point to a line, the greater cuts off the greater segment from the foot of the perpendicular from the point to the line.*



In this figure, in which PO is $\perp$ to XY and $PA > PB$, it is impossible that A should lie between B and O. For if A should be at A', then PA (that is, PA') would be less than PB (§ 118), which is contrary to what is given. Further, A cannot fall on B, for then $PA = PB$, which is also contrary to what is given.

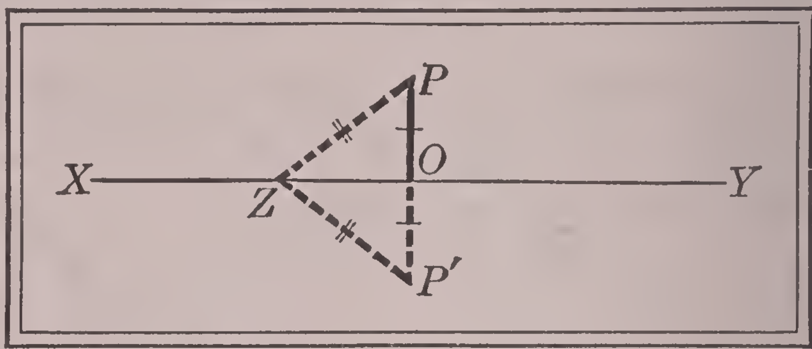
Thus A cannot lie on B or between B and O. Hence the greater segment PA cuts off the greater segment on XY from O.

Similarly, if PA lies on the right of PO, as at PC, then, since $PA = PC$, we see that $OA = OC$ (§ 120), so that $OC > OB$.

Since we have covered all possible cases, the corollary is true.

Proposition 31. Perpendicular Shortest Line

122. Theorem. *The perpendicular is the shortest line segment that can be constructed to a given line from a given external point.*



Given PO, **the** $\perp$ **from an external point** P **to the line** XY.

Prove that PO *is the shortest line from* P *to* XY.

The plan is to show that PO is shorter than any other line.

Proof. Let PZ be any other line segment from P to XY.

Produce PO to P', making $OP' = OP$. Post. 2

Draw $P'Z$. Post. 1

Since XY is given $\perp$ to PP', then $PZ = P'Z$. § 117

Then $PZ + P'Z = 2\,PZ$,

and $PO + P'O = 2\,PO$. Axs. 5, 10

But $PO + P'O < PZ + P'Z$. Post. 3

Hence $2\,PO < 2\,PZ$, Ax. 5

and $PO < PZ$; Ax. 7

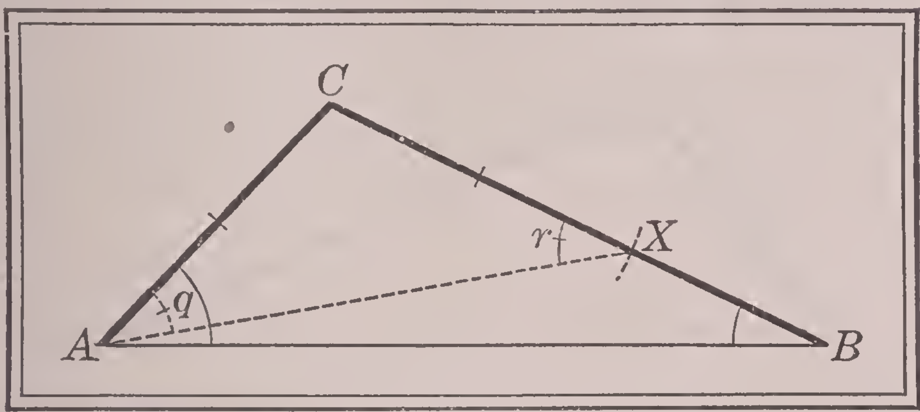
that is, PO is the shortest line from P to XY.

123. Corollary. Conversely, *the shortest line segment to a given line from an external point is the perpendicular from the point to the line.*

For if PO is the shortest line segment, it must be $\perp$ to XY. Otherwise we should have a line segment from P to XY shorter than the $\perp$, which is impossible (§ 122).

Proposition 32. Angles of a Triangle

124. Theorem. *If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*



Given the $\triangle ABC$ with $CB > CA$.

Prove that $\angle BAC > \angle B$.

The plan is to show that $\angle BAC > q = r > \angle B$.

Proof. Because $CB > CA$ we may suppose that CX can be marked off with the compasses on CB so that $CX = CA$.

Draw AX. Post. 1

Then $\triangle AXC$ is isosceles. § 19

Then, in the figure, $q = r$, § 42

because in an isosceles △ the ∠s opposite the equal sides are equal.

But $r > \angle B$, § 50

because an exterior ∠ of a △ > either nonadjacent interior ∠.

Also, $\angle BAC > q$. Ax. 10

Substituting r for its equal, q, we have

$\angle BAC > r$. Ax. 5

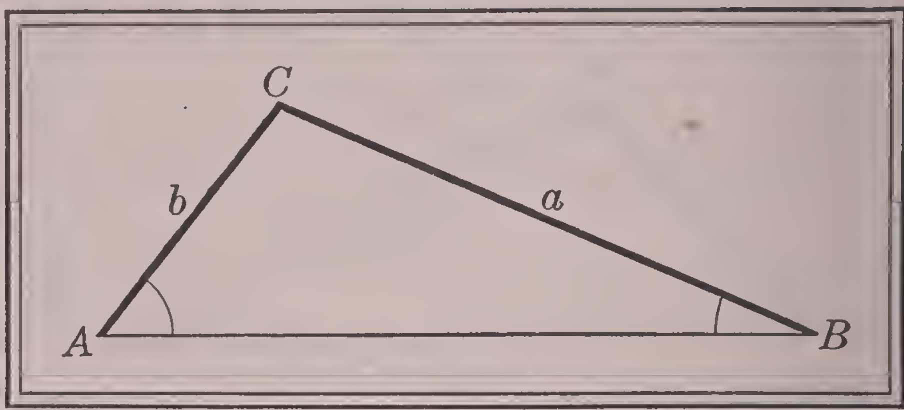
Since $r > \angle B$, Proved

then $\angle BAC > \angle B$. Ax. 9

If the first of three quantities > the second, and the second > the third, then the first > the third.

Proposition 33. Sides of a Triangle

125. Theorem. *If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater.*



Given the $\triangle ABC$ with $\angle A > \angle B$.

Prove that $a > b$.

The plan is to show that other suppositions lead to an impossibility.

Proof. Now a is either equal to b, less than b, or greater than b.

If $a = b$,

then $\angle A = \angle B$. § 42

And if $a < b$,

that is, if $b > a$,

then $\angle B > \angle A$. § 124

Both these conclusions are contrary to the fact that

$\angle A > \angle B$. Given

Hence it follows that $a > b$.

This is another example of an indirect proof (§ 56). We suppose that the statement to be proved is false, that is, that $a = b$ and that $b > a$, and we show that these suppositions lead to impossibilities; namely, that $\angle A = \angle B$ or $\angle B > \angle A$, when we know that $\angle A > \angle B$. Accordingly, we conclude that the theorem is true.

Exercises. Inequalities

1. *The sum of any two sides of a triangle is greater than the third side, and the difference between any two sides is less than the third side.*

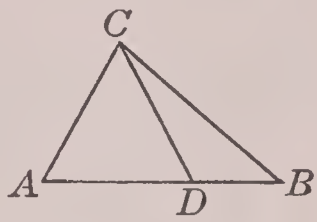
Use Post. 3 for the first statement and Ax. 7 for the second.

State in what cases it is possible to form triangles with rods of the following lengths, and give the reason:

2. 2 in., 3 in., 4 in.
3. 3 in., 4 in., 7 in.
4. 6 in., 7 in., 9 in.
5. 7 in., 10 in., 20 in.
6. 8 in., $9\frac{1}{2}$ in., 18 in.
7. $9\frac{3}{4}$ in., $10\frac{1}{2}$ in., 20 in.

8. In this figure prove that $AB+BC>AD+DC$.

Why is $DB+BC>DC$? What is the result of adding AD to these unequals?



9. In the figure of Ex. 8 suppose that $CA=CB$, and prove that $CD<CB$. Write a theorem based upon this fact.

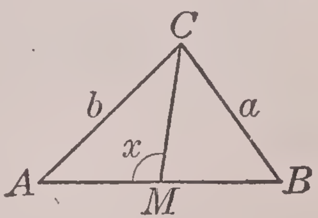
The theorem may begin as follows: The line segment joining the vertex of an isosceles triangle to any point on the base is less than....

10. The hypotenuse of a right triangle is greater than either of the other sides.

11. Prove § 122 by the use of § 125. Is this legitimate?

It is legitimate in case § 122 was not used directly or indirectly in the proof of § 125; otherwise it is not legitimate.

12. In this figure, given that x is an obtuse angle and that M is the midpoint of AB, prove that $a<b$.

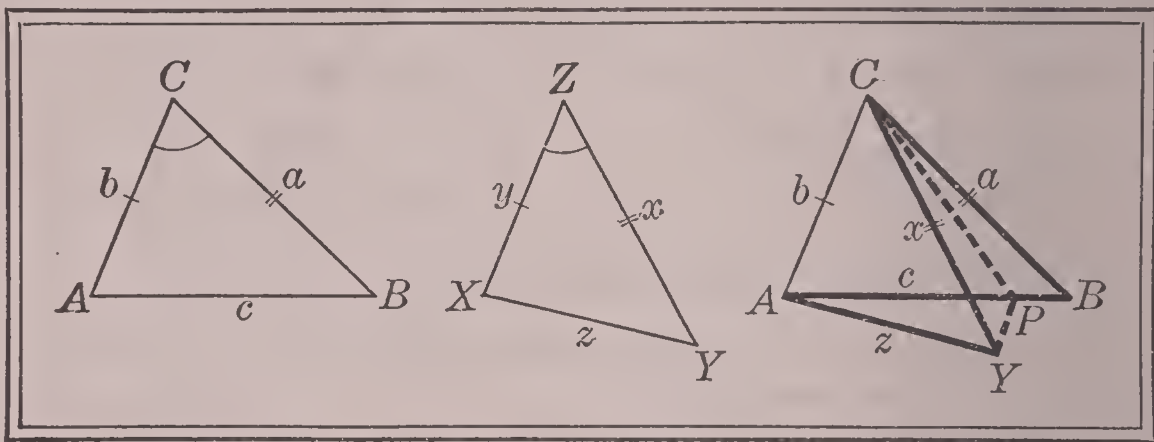


Draw a perpendicular from C to AB.

13. On the base AB of a quadrilateral $ABCD$ the point P is taken. Prove that the perimeter of the quadrilateral is greater than the perimeter of $\triangle PCD$.

Proposition 34. Unequal Angles of Triangles

126. Theorem. *If two sides of one triangle are equal respectively to two sides of another, but the included angle of the first triangle is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.*



Given the $\triangle$ ABC and XYZ with $b = y$, $a = x$, and $\angle C > \angle Z$.

Prove that $c > z$.

In the figure, the plan is to show that $AP + PB = AP + PY > z$.

Proof. Place the $\triangle$ so that Z coincides with C, y lies along b, and Y lies on the same side of AC as B. Post. 5

Then since $y = b$, X lies on A, and since $\angle Z < \angle C$, x lies within $\angle ACB$.

Let CP bisect $\angle YCB$ and draw YP. Posts. 8, 1

Then, since $CP = CP$, CY is given equal to CB, and $\angle YCB$ is bisected, we see that

$\triangle PYC$ is congruent to $\triangle PBC$. § 40

$\therefore PY = PB$. § 38

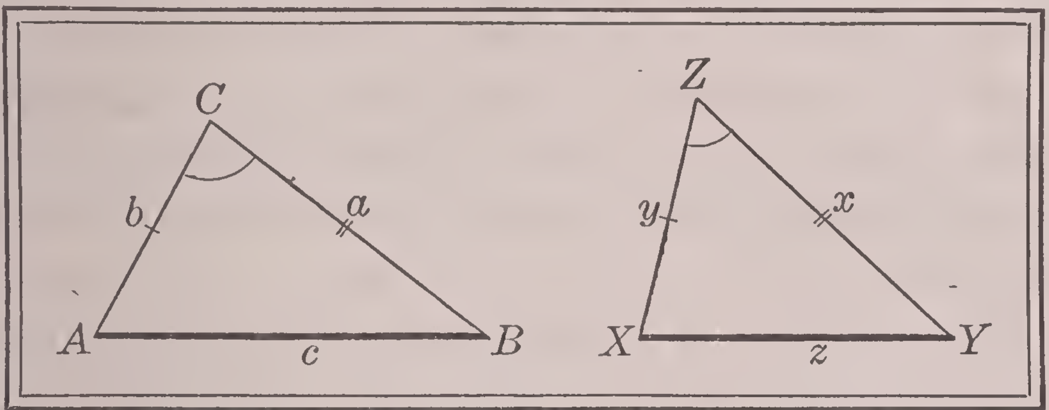
Now $AP + PY > AY$. Post. 3

$\therefore AP + PB > AY$, Ax. 5

and hence $AB > AY$, or $c > z$. Ax. 10

Proposition 35. Unequal Sides of Triangles

127. Theorem. *If two sides of one triangle are equal respectively to two sides of another, but the third side of the first triangle is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.*



Given the $\triangle ABC$ and XYZ with $b = y$, $a = x$, and $c > z$.

Prove that $\angle C > \angle Z$.

The plan is to show that other suppositions lead to an impossibility.

Proof. Now $\angle C$ is either equal to $\angle Z$, less than $\angle Z$, or greater than $\angle Z$.

If $\angle C = \angle Z$,

then $\triangle ABC$ is congruent to $\triangle XYZ$, § 40

because it then has two sides and the included $\angle$ equal respectively to two sides and the included $\angle$ of $\triangle XYZ$;

and $c = z$. § 38

And if $\angle C < \angle Z$,

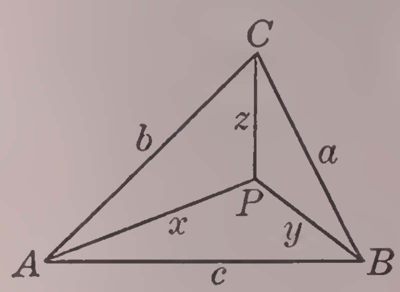
then $c < z$. § 126

Neither conclusion can be true, because $c > z$. Given

$\therefore \angle C > \angle Z$.

Exercises. Review

1. The point P within the $\triangle ABC$ is connected with A, B, C by the line segments x, y, z as shown in this figure. Then $a+b$ is greater than the sum of what two line segments? What proposition proves your statement?

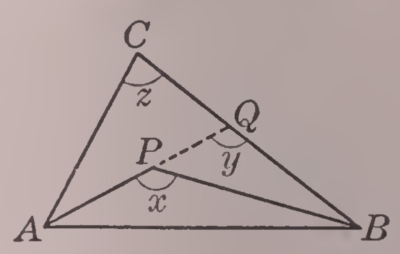


2. In Ex. 1, $b+c$ is greater than what sum, and $c+a$ is greater than what other sum?

3. In the figure of Ex. 1 write three similar inequalities, beginning with $x+y>c$, add the three inequalities, and see what interesting result you can find relating to $x+y+z$ and $a+b+c$.

4. Draw a figure showing how many exterior angles a triangle may have and find their sum in degrees.

5. In the angles of this figure how does x compare with y? State the reason. How does y compare with z, and why? Then how does x compare with z, and why? Write a theorem beginning, "If from a point within a triangle lines are drawn to any two vertices, the angle formed by these lines is greater than . . .".



6. Draw a rectilinear figure of four sides, and produce one of the sides to form an exterior angle. State your inference as to the relation of the size of this exterior angle to that of any of the nonadjacent interior angles. Discuss each possibility in full.

7. The angles of a certain quadrilateral are so related that the second is twice the first, the third three times the first, and the fourth four times the first. How many degrees are there in each angle?

IV. Attacking Originals

128. General Suggestions. Various important suggestions for attacking those exercises which are often called *originals* have already been given in connection with the exercises themselves. These will now be summarized:

1. *Draw the figure carefully, but do not stop to construct it unless there seems to be some special need for doing so.*

A proof is often unnecessarily difficult simply because the figure is carelessly or incorrectly drawn.

2. *Draw as general figures as possible.*

For example, if you wish to prove a proposition about any triangle, do not take a triangle that is isosceles, right, or equilateral.

3. *After drawing the figure, state precisely what is given and precisely what is to be proved.*

Many of the difficulties of geometry come from failing to keep in mind *precisely* what is given and *precisely* what is to be proved. Draw no extra lines unless it is necessary.

4. *Now see if the proof is at once clear. If it is not, say: "I can prove this if I can prove that; I can prove that if I can prove . . ."; and so on until you reach a proved proposition. Then reverse your reasoning.*

5. *If two line segments are to be proved equal, try to prove them corresponding sides of congruent triangles, sides of an isosceles triangle, opposite sides of a parallelogram, or segments between parallels which cut equal segments from another transversal.*

6. *If two angles are to be proved equal, try to prove them alternate or corresponding angles of parallel lines, corresponding angles of congruent triangles, base angles of an isosceles triangle, or opposite angles of a parallelogram.*

7. *Try the indirect method (§ 56) as a last resort.*

129. Synthetic Method. The method of proof in which known truths are put together in order to obtain a new truth is called the *synthetic method.*

This method is used in proving most of the theorems of geometry. The proposition usually suggests some propositions already proved, and from these we proceed to the proof required.

130. Analytic Method. The method of attack which asserts that a proposition under consideration is true if another proposition is true, and so on, step by step, until a known truth is reached, is called the *analytic method.*

This is the method referred to in the fourth suggestion in § 128. It is the one which the student should use if he does not at once see the proof.

131. Concurrent Lines. If two or more lines pass through the same point they are called *concurrent lines.*

The word "concurrent" is from two Latin words meaning "running together." Since two lines are generally concurrent, the term is commonly used in connection with three or more lines.

132. Median. A line segment from any vertex of a triangle to the midpoint of the opposite side is called a *median* of the triangle.

The term is occasionally employed with reference to a trapezoid to mean the line segment joining the midpoints of the two nonparallel sides, but it is rarely needed for this purpose.

133. Trisect. To divide any geometric magnitude into three equal parts is to *trisect* it.

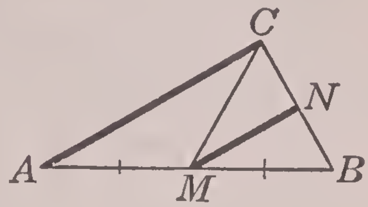
Exercises. Review

1. How many sides are there in a regular polygon each of whose angles is 175°?

2. If a side and an angle of one isosceles triangle are equal respectively to the corresponding side and angle of another isosceles triangle, the triangles are congruent.

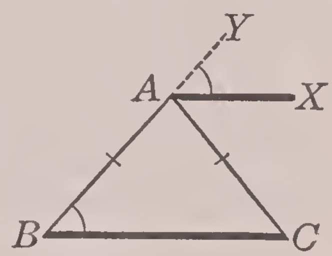
3. Given the rt. $\triangle ABC$ with $\angle B = 2\angle A$, with M the midpoint of the hypotenuse AB, and with $MN \parallel$ to AC, as shown in the figure. Give the authority for each of the following statements:

(1) $BN = NC$. (5) $\angle A = 30°$.
(2) MN is $\perp$ to BC. (6) $\angle B = 60°$.
(3) $MB = MC$. (7) $MB = BC$.
(4) $\angle A + \angle B = 90°$. (8) $AB = 2\,BC$.



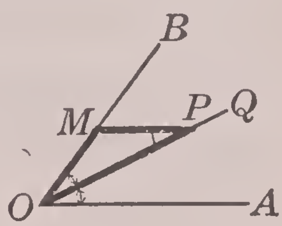
4. The bisector of an exterior angle of an isosceles triangle, formed by producing one of the equal sides through the vertex, is parallel to the base.

"I can prove that AX is $\parallel$ to BC if I can prove that $\measuredangle$ —— and —— are equal. I can prove these angles equal if I can prove that $\angle CAY$ is twice $\angle$ —— of the $\triangle ABC$."

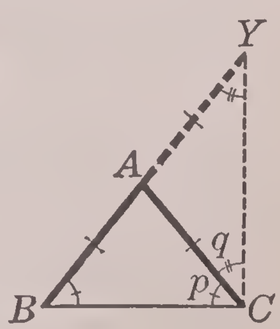


5. If the line drawn from the vertex of a triangle to the midpoint of the base is equal to half the base, the angle at the vertex is a right angle.

6. If through any point in the bisector of an angle a line is drawn to either side of the angle parallel to the other side, the triangle thus formed is isosceles.



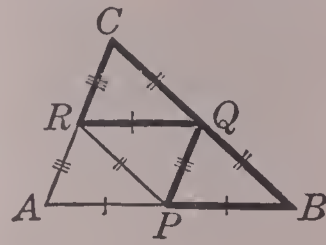
7. If one of the equal sides of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the side produced to the nearer end of the base is perpendicular to the base.



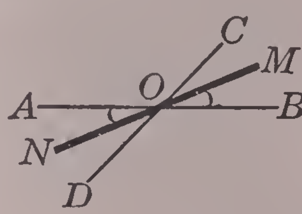
"I can prove that $\angle YCB$ is a right angle if I can prove that it is equal to the sum of $\measuredangle$ —— and —— of $\triangle BCY$. I can prove that it is equal to this sum if I can prove that $p = \angle$—— and $q = \angle$——." Now reverse this reasoning and write out the proof in full.

8. Through any point P on the line AB an intersecting line is drawn, and from any two points on this line equidistant from P perpendiculars are drawn to AB or AB produced. Prove that these perpendiculars are equal.

9. The bisectors of two supplementary adjacent angles are perpendicular to each other.

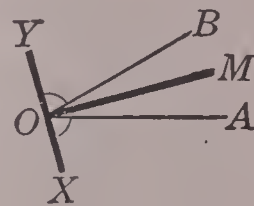


10. The lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.



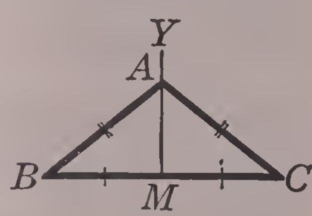
11. The bisectors of two vertical angles are in the same straight line.

12. The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.



13. If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.

14. The bisector of the angle at the vertex of an isosceles triangle bisects the base and is perpendicular to the base.



15. The perpendicular bisector of the base of an isosceles triangle is concurrent with the equal sides and bisects the angle at the vertex.

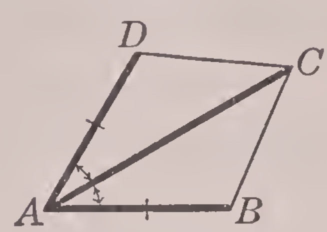
16. If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.

17. Any point on the bisector of the angle at the vertex of an isosceles triangle is equidistant from the ends of the base.

Take any point P on AM in the figure of Ex. 15 and show that $PB=PC$.

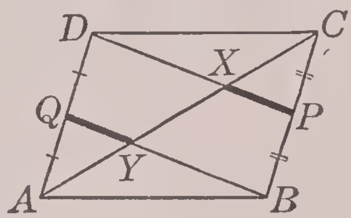
Exercises. Equal Lines

1. In an isosceles triangle the medians drawn to the equal sides are equal.

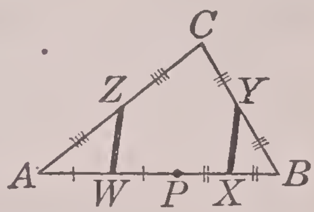


2. If the sides AB and AD of a quadrilateral $ABCD$ are equal, as shown in this figure, and if the diagonal AC bisects the angle at A, then $BC = DC$.

3. If a line segment is terminated by two parallel lines, and if another line segment is drawn through the midpoint of the first and is terminated by the parallels, the second segment is bisected by the first.

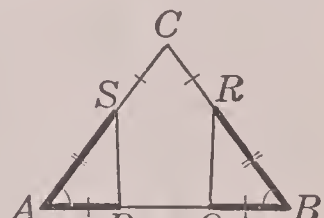


4. In a $\square ABCD$ the line BQ bisects AD, and DP bisects BC. Prove that BQ and DP trisect AC.



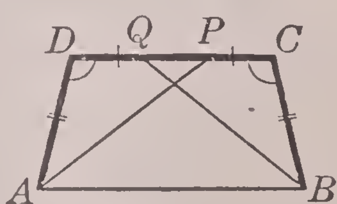
5. If on the base AB of a $\triangle ABC$ any point P is taken, and the lines AP, PB, BC, and CA are bisected by W, X, Y, and Z respectively, then $XY = WZ$.

6. In the square $ABCD$, if CD is bisected by Q, and if P and R are taken on AB so that $AP = BR$, then $PQ = RQ$.



7. In this figure, if $AC = BC$, and if $AP = BQ = CS = CR$, then $PS = QR$.

8. If from the vertex and the midpoints of the equal sides of an isosceles triangle lines are drawn perpendicular to the base, they divide the base into four equal parts.



9. In this figure, if AB is $\parallel$ to DC, if $\angle C = \angle D$, and if $CP = DQ$, then $AP = BQ$.

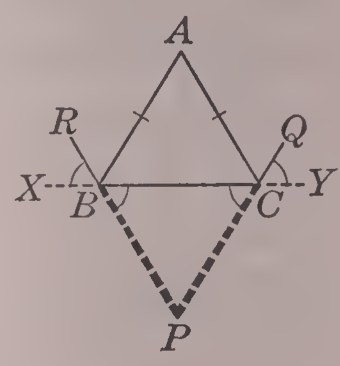
Produce AD and BC to intersect. Then how can it be shown that $AD = BC$?

Exercises. Equal Angles

1. If the angles at the vertices of two isosceles triangles coincide, what can be said of the bases? Prove it.

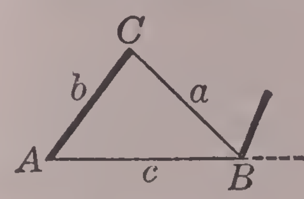
2. The bisectors of the equal angles of an isosceles triangle form with the base another isosceles triangle.

3. In this figure, if $AB = AC$, and if CQ and BR bisect the $\measuredangle$s YCA and XBA respectively, the triangle formed by producing QC and RB is isosceles.



4. The bisectors of any two angles of an equilateral triangle form an angle equal to any exterior angle.

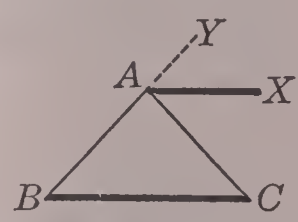
5. In which direction must the side b of a $\triangle ABC$ be produced so as to intersect the bisector of the opposite exterior angle?



Consider the three cases $\angle A < \angle C$, $\angle A = \angle C$, $\angle A > \angle C$.

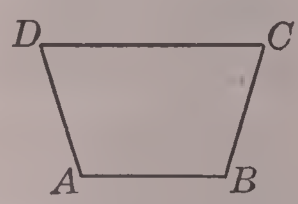
6. A line drawn parallel to the base of an isosceles triangle makes equal angles with the sides or the sides produced.

7. If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.



8. If through the three vertices of an isosceles triangle lines are drawn parallel to the opposite sides, they form an isosceles triangle.

9. In the figure here shown, if $AD = BC$, and $\angle A = \angle B$, then DC is $\parallel$ to AB.



10. If a line drawn at right angles to AB, the base of an isosceles $\triangle ABC$, cuts AC at P and BC produced at Q, then $\triangle PCQ$ is isosceles.

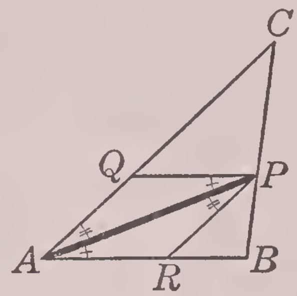
Exercises. Congruence

1. *If two sides and the included angle of one parallelogram are equal respectively to two sides and the included angle of another, the parallelograms are congruent.*

This proposition is occasionally required in courses of study. In proving it the method of § 40 should be used.

2. If in a $\triangle ABC$ a perpendicular is drawn from B to the bisector of $\angle A$, meeting this bisector at X and AC or AC produced at Y, then $BX = XY$.

3. If through any point equidistant from two parallel lines two lines are drawn cutting the parallels, they intercept equal segments on these parallels.

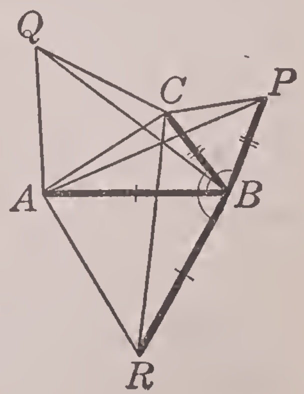


4. If, from the point where the bisector of an angle of a triangle meets the opposite side, lines are drawn parallel to each of the other sides, the segments of these lines cut off by the sides are equal.

5. The diagonals of a square are perpendicular to each other and bisect the angles of the square.

6. If two line segments bisect each other at right angles, any point on either segment is equidistant from the ends of the other segment.

7. If either diagonal of a parallelogram bisects one of the angles, the sides of the parallelogram are all equal.



8. On the sides of any $\triangle ABC$ the equilateral $\triangle$s BPC, CQA, ARB are constructed. Prove that $AP = CR = BQ$.

How can we prove that $\triangle ABP$ is congruent to $\triangle RBC$ and that $\triangle ARC$ is congruent to $\triangle ABQ$? Does proving these facts establish the proposition?

Exercises. Sums of Angles

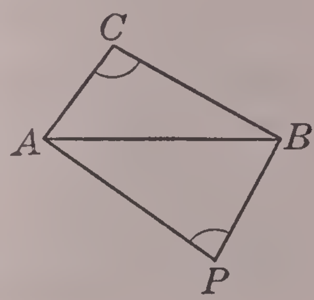
1. An exterior angle of an acute triangle or of a right triangle cannot be acute.

2. If the sum of two angles of a triangle is equal to the third angle, the triangle is a right triangle.

3. If the line joining any vertex of a triangle to the midpoint of the opposite side divides the triangle into two isosceles triangles, the original triangle is a right triangle.

4. If the angles at the vertices of two isosceles triangles are supplements one of the other, the base angles of the one are complements of those of the other.

5. If from the ends of the base AB of a $\triangle ABC$ perpendiculars to the other two sides are drawn, meeting at P, then $\angle P$ is the supplement of $\angle C$.

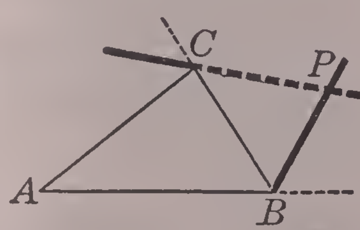


Here AP is $\perp$ to AC and BP is $\perp$ to BC. Consider the case in which AP is $\perp$ to BC and BP is $\perp$ to AC.

6. The bisectors of two consecutive angles of a parallelogram are perpendicular to each other.

7. If two sides of a quadrilateral are parallel, and the other two sides are equal but not parallel, the sums of the opposite angles are equal.

8. If the exterior angles at B and C of any $\triangle ABC$ are bisected by lines meeting at P, then $\angle P + \frac{1}{2}\angle A =$ a rt. $\angle$.



9. The opposite angles of the quadrilateral formed by the bisectors of the interior angles of any quadrilateral are supplementary.

10. The angles of a quadrilateral are x, $2x$, $2x$, and $3x$. How many degrees are there in each angle?

Exercises. Inequalities

1. In the $\triangle ABC$ the $\angle A$ is bisected by a line meeting BC at D. Prove that $BA > BD$, and that $CA > CD$.

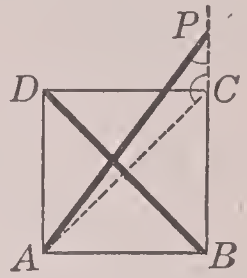
While less important than the suggestions given in § 128, the following will be found helpful:

If one angle is to be proved greater than another, try to show that it is an exterior angle of a triangle, or an angle opposite the greater side of a triangle.

If one line is to be proved greater than another, try to show that it is opposite the greater angle of a triangle.

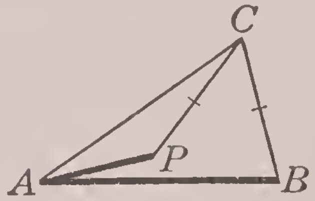
2. If AD is the longest side and BC is the shortest side of the quadrilateral $ABCD$, then $\angle B > \angle D$ and $\angle C > \angle A$.

3. If a line is drawn from the vertex A of a square $ABCD$ so as to cut CD and to meet BC produced in P, then $AP > DB$.



4. If the angle between two adjacent sides of a parallelogram is increased, the length of the sides remaining unchanged, the diagonal from the vertex of this angle is diminished.

5. If a point P is taken within a $\triangle ABC$ such that $CP = CB$, as shown in this figure, then $AB > AP$.

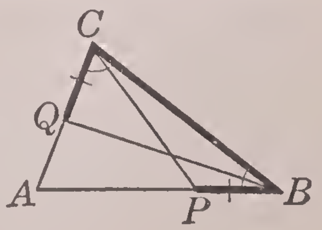


6. In a quadrilateral $ABCD$, if $AD = BC$ and $\angle C < \angle D$, then $AC > BD$.

7. In Ex. 6 prove that $\angle B > \angle A$.

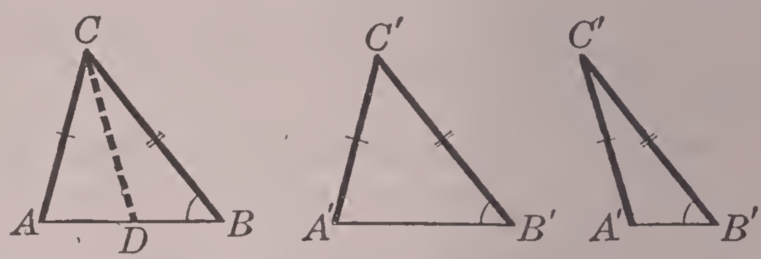
8. In a pentagon $ABCDE$ it is given that $\angle A = \angle B < \angle C$. Can you make any inference as to the equality or inequality of AC, BD, and BE? Explain your answer.

9. In the $\triangle ABC$, if $AB > AC$ and if on AB and AC respectively BP is taken equal to CQ, then $BQ > CP$.



Exercises. Triangles

1. If two triangles have two sides of one equal respectively to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are either equal or supplementary, and if equal the two triangles are congruent.



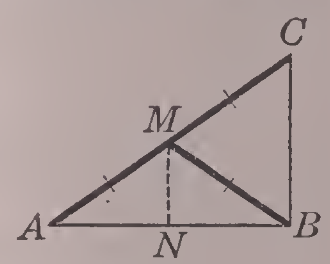
Using superposition, as in § 40, and placing the corresponding parts in the usual way, since $\angle B' = \angle B$, then $B'A'$ lies along what line? Then A' lies on A or on some other point of BA, as D. If A' lies on A, are $\triangle A'B'C'$ and ABC congruent?

If A' lies on D, are $\triangle A'B'C'$ and DBC congruent?

Since $CD = C'A' = CA$, what is the relation of $\angle A$ to $\angle CDA$? of $\angle CDA$ to $\angle BDC$? of $\angle A$ to $\angle BDC$?

The triangles are congruent under what conditions with respect to $\angle s\ B$ and B'? with respect to $\angle s\ A$ and A'?

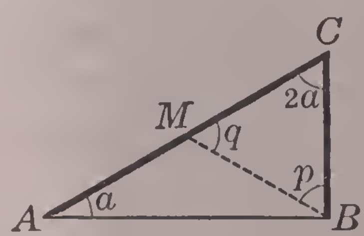
2. The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.



We have to prove that $AM = BM$, that $CM = BM$, or that BM is half of a line segment that is equal to AC.

This may be proved in several ways. Probably the simplest way with this figure is to prove certain triangles congruent. Another way would be to adapt the figure to § 83.

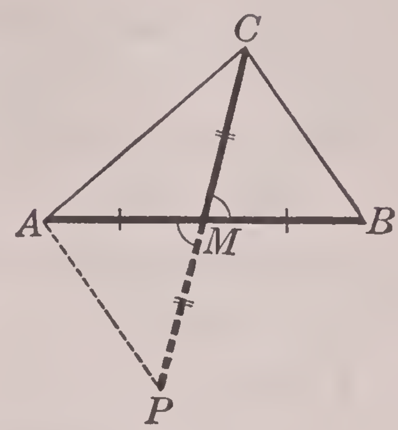
3. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side.



This is the familiar 30°-60° right triangle used by draftsmen.

If $AM = CM$, then $AM = BM = CM$, as in Ex. 2. The exercise then reduces to proving that $\triangle BCM$ is equilateral by proving that $p = 2a = 60° = q$.

4. A median of a triangle is less than half the sum of the two adjacent sides.



The student should attack this exercise by analysis, beginning as follows:

Given CM, a median of the $\triangle ABC$.

Prove that $CM < \frac{1}{2}(BC + CA)$.

"I can show that $CM < \frac{1}{2}(BC + CA)$ if I can show that $2\,CM < BC + CA$.

This suggests producing CM by its own length to P and drawing AP.

Now	$CP = 2\,CM$,	
and I can show that	$2\,CM < BC + CA$	
if I can show that	$CP < BC + CA$.	
But	$CP < AP + CA$.	Post. 3
Hence	$CP < BC + CA$	
if I can show that	$BC = AP$,	

and this is true if $\triangle MBC$ is congruent to $\triangle MAP$."

Now complete the analysis, then reverse the reasoning, and write out the proof in full.

5. The diagonals of a rhombus form four right angles.

6. The perpendiculars from two opposite vertices of a parallelogram drawn to the diagonal determined by the other vertices are equal.

7. From the vertex A of a $\triangle ABC$ the line AD is drawn $\perp$ to BC. Consider the following statements and tell which ones are true in general. Then tell what other conditions must be given in order that the other statements shall be true:

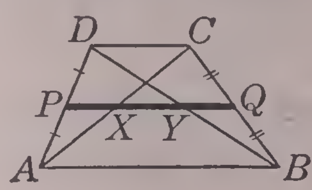
(1) $BD = BC$.
(2) $AD < \frac{1}{2}(AB + AC)$.
(3) $\angle ADB > \angle B$.
(4) Either $\angle CDA < \angle B$ or $CDA < \angle C$.

Exercises. Review

1. Make a list of the numbered propositions in Book I, stating under each the previous propositions upon which it depends either directly or indirectly.

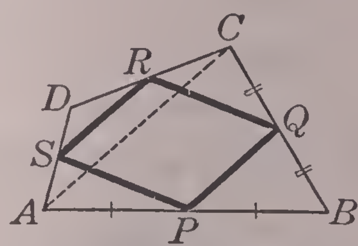
2. Make another list of the numbered propositions, stating under each the subsequent propositions in Book I which depend upon it.

3. The line joining the midpoints of the nonparallel sides of a trapezoid passes through the midpoints of the two diagonals.



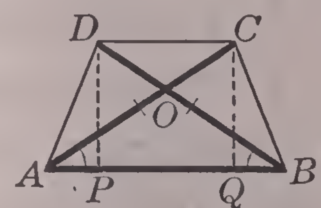
How is PQ related to AB and DC? Why? Since PQ bisects AD and BC, how does it divide AC and BD? Why?

4. The lines joining the midpoints of the consecutive sides of any quadrilateral form a parallelogram.



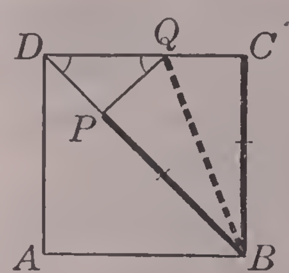
How are PQ and SR related to AC?

5. If the diagonals of a trapezoid are equal, the trapezoid is isosceles.



Construct DP and $CQ \perp$ to AB. How is $\triangle AQC$ related to $\triangle BPD$? Why? Then how is $\angle QAC$ related to $\angle PBD$? Then how is $\triangle ABD$ related to $\triangle BAC$?

6. If, from the diagonal BD of a square $ABCD$, BP is cut off equal to BC, and PQ is constructed $\perp$ to BD, meeting the side CD at Q, then $PD = PQ = QC$.

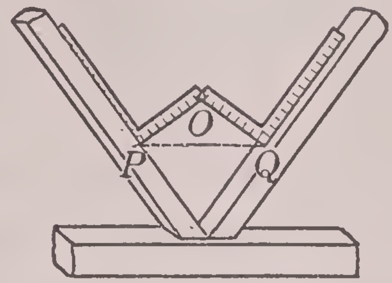


How is rt. $\triangle BQP$ related to rt. $\triangle BQC$? Why? How many degrees are there in $\angle PDQ$ and in $\angle PQD$? Then how is PD related to PQ? Why?

7. Study Ex. 6 for the case of $BP = \frac{1}{2} BD$, and state and prove the resulting proposition.

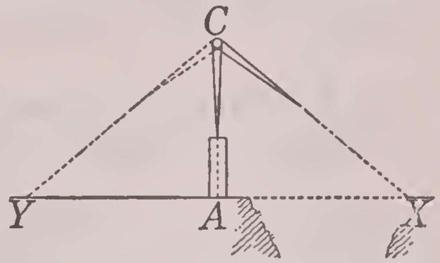
Exercises. Applications

1. In order to put in a brace which shall join two converging beams and make equal angles with them, a carpenter places two steel squares as here shown, so that $OP = OQ$. Prove that PQ makes equal angles with the beams.

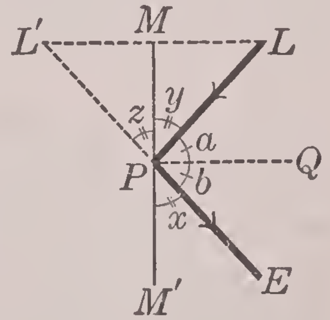


2. In what other way can you construct the line PQ in Ex. 1 so that it shall make equal angles with the beams?

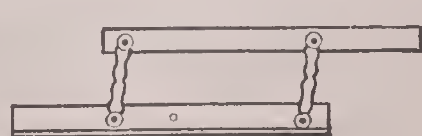
3. Wishing to measure the distance AX in this figure, a boy placed a pair of compasses C on top of a post A so that one leg was vertical and the other pointed to X. He then turned the compasses around, keeping the angle fixed and the leg on the post vertical, and sighted along the other leg to Y. He then measured AY and thus found the distance AX. Explain the principle involved.



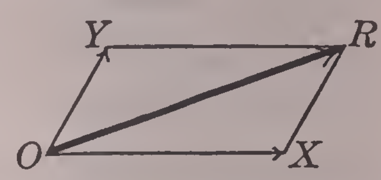
4. In the figure, MM' represents a mirror and PQ is $\perp$ to MM' at P. If a ray of light LP from a light L strikes the mirror at P, it is reflected to the eye at E in such a way that $a = b$. The line LL' is $\perp$ to MM', and EPL' is a straight line. Prove that $x = y$ and that $y = z$, and explain why the light appears to be at the same distance behind the mirror, at L', that it really is in front of it, at L.



5. This figure represents a *parallel ruler* which is used for drawing parallel lines. Explain how it may be used, and state the theorem upon which its principle depends.

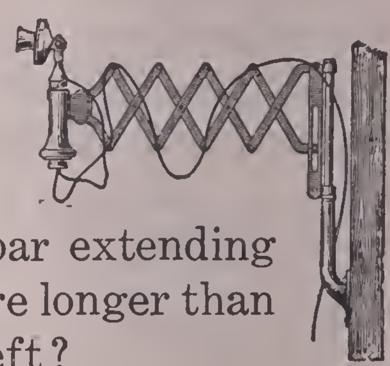


6. It is proved in physics that two forces acting on an object O have the same effect as a single force known as their *resultant*. If, to scale, we let OX represent a force of 300 lb. pulling in the direction OX, and OY a force of 150 lb. pulling in the direction OY, the resultant is represented by the diagonal OR of the $\square OXRY$. By measuring OR and $\angle XOR$ we can find the magnitude and the direction of the resultant. Using a protractor and ruler, find the magnitude and the direction of the resultant of OX and OY in the above case.

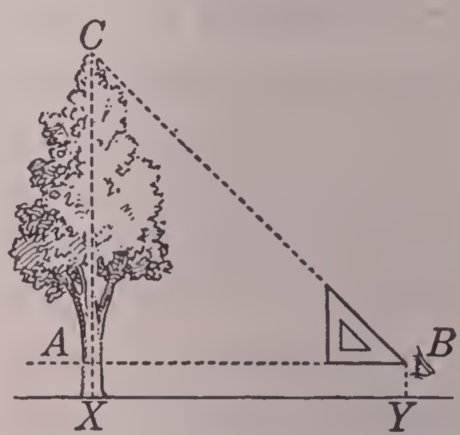


7. Two forces at right angles to each other are exerted upon an object. One force is 500 lb. to the right and the other is 800 lb. upward. Find the resultant as in Ex. 6.

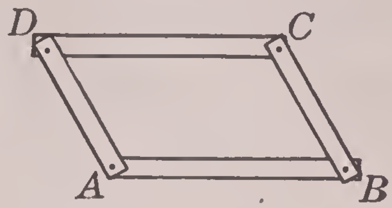
8. Explain geometrically why this telephone extends horizontally when it is pulled out. State each proposition involved in the answer. What would be the effect on the direction if each bar extending from the top downward to the right were longer than each bar extending downward to the left?



9. To ascertain the height of a tree or of the school building, fold a piece of paper so as to make an angle of 45°, or take a draftsman's 45-degree triangle; then walk back from the tree until the top is seen at an angle of 45° with the ground, being careful to hold the base of the triangle level. In the figure prove that $AB = AC$, and hence that $CX = AB + BY$, where BY is the height of the observer's eye above the ground.

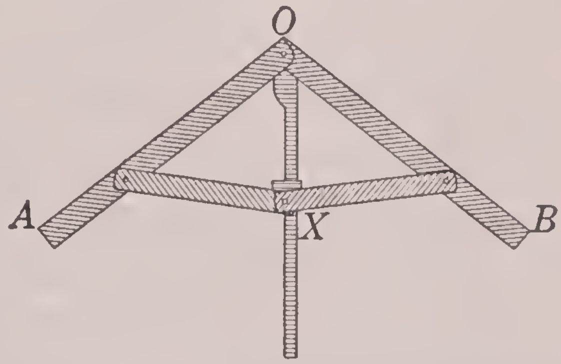


10. This figure represents four hinged rods with $AB = DC$ and $AD = BC$. As the angles change, does the figure continue to be a parallelogram? Upon what theorem does this depend?



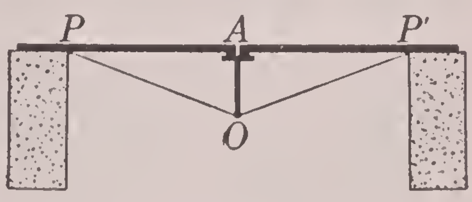
11. In the figure of Ex. 10, if $\angle A$ is $125°$, how large are $\measuredangle$s B, C, and D?

12. Explain how this instrument, in which the joint X can be moved along the rod OX, is used to bisect an angle with the sides of which the arms OA and OB can be made to coincide. State the propositions on which the explanation depends.

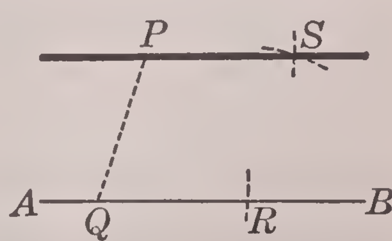


On account of the joints and other mechanical features of such an instrument this method of bisection is not so nearly accurate as the construction given in § 103.

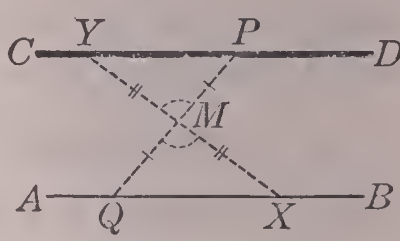
13. The simple bridge construction here shown is occasionally used. The beams PA and $P'A$ rest on the perpendicular support OA in the center of the bridge. The rods OP and OP' are fastened at O, P, and P'. Show by means of the congruence of certain triangles that the point O always remains directly beneath A. Why will the bridge support a weight?



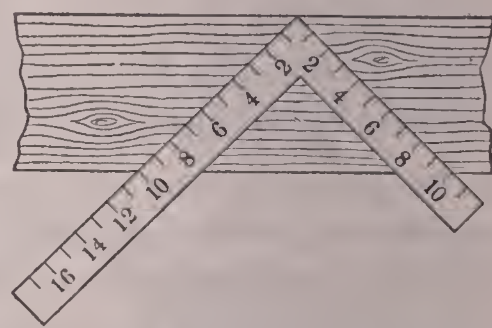
14. In laying out a tennis court it is desired to run a line through a point $P \parallel$ to AB. This is a convenient method: Stretch a tape from P to any point Q on AB; then with Q as center swing the tape to cut AB at R; with P and R as centers and the same radius as before mark arcs intersecting at S, and draw a line through P and S. Prove that PS is the line required.



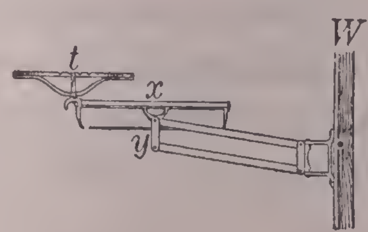
15. In laying out a tennis court, another way of running a line through a point $P \parallel$ to AB is as follows: A line is drawn from P to Q, any point on AB, and the midpoint M of PQ is found with a tape. Then from another point X on AB the tape is stretched through M, and a point Y is found such that $MY = XM$. Then CD, drawn through Y and P, is $\parallel$ to AB.



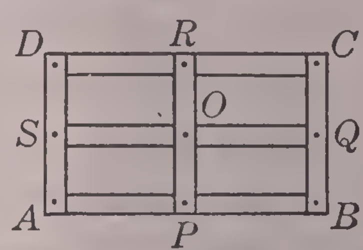
16. A board 8 in. wide is to be sawed into five strips of equal width. In order to draw the lines for sawing, a carpenter lays his steel square as here shown, placing the corner on one edge and the 10-inch mark on the other, and marks the board at the divisions 2, 4, 6, 8 on the square. He then moves the square along the board and repeats the process. Prove that lines drawn through the corresponding marks satisfy the requirements.



17. A dentist's working table, in which bar x is fastened to bar y at right angles and table t is fixed parallel to bar x, is attached to a vertical wall W as shown in this figure. State in full the geometric proof that table t is always horizontal.



18. This figure represents six hinged rods in which all the angles are right angles and P, Q, R, S bisect AB, BC, CD, DA respectively. Prove that the figure can be pulled into different shapes, the angles then ceasing to be right angles, but that all the quadrilaterals will still continue to remain parallelograms.



BOOK II

THE CIRCLE

I. Fundamental Theorems

134. Properties of a Circle. From the definitions in § 22 and from a study of the figure we see that a circle has certain properties, among which are the following:

1. *All radii of the same circle or of equal circles are equal.*

2. *All circles with equal radii are equal.*

3. *All diameters of the same circle or of equal circles are equal.*

4. *If a straight line intersects a circle in one point, it intersects it in two points and only two.*

5. *If two circles intersect in one point, they intersect in two points and only two.*

6. *A point is within, on, or outside a circle according as its distance from the center is less than, equal to, or greater than a radius.*

7. *A diameter bisects the circle and the surface inclosed, and conversely.*

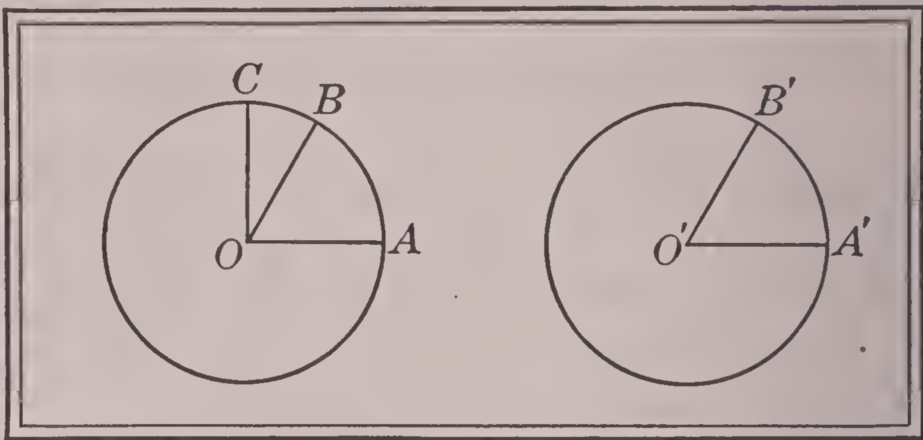
These statements may be taken as postulates and referred to as properties of a circle, although they are capable of proof.

135. Central Angle. If the vertex of an angle is at the center of a circle and the sides are radii of the circle, the angle is called a *central angle.*

An angle is said to *intercept* any arc cut off by its sides, and the arc is said to *subtend* the angle. Preferably, we speak of the arc as *having* a central angle, and conversely.

Proposition 1. Equal Angles have Equal Arcs

136. Theorem. *If two central angles of the same circle or of equal circles are equal, the angles have equal arcs; and if two central angles are unequal, the greater angle has the greater arc.*



Given the equal ⓢ O and O' with central $\angle AOB =$ central $\angle A'O'B'$ and with central $\angle AOC >$ central $\angle A'O'B'$.

Prove that arc $AB =$ arc $A'B'$ and that arc $AC >$ arc $A'B'$.

The best plan is to place one figure on the other.

Proof. Place ⊙ O on ⊙ O' so that $\angle AOB$ coincides with its equal, $\angle A'O'B'$. Post. 5

In the case of the same ⊙ simply swing one $\angle$ about O.

Then A lies on A' and B on B'. § 134, 1

$\therefore$ arc AB coincides with arc $A'B'$, § 21

because all points of each are equidistant from O'.

Since $\angle AOC > \angle A'O'B'$ and $\angle AOB = \angle A'O'B'$, Given

we have $\angle AOC > \angle AOB$. Ax. 5

$\therefore OC$ lies outside $\angle AOB$, § 6

and hence arc $AC >$ arc AB. Ax. 10

But arc $AB =$ arc $A'B'$, Proved

and hence arc $AC >$ arc $A'B'$. Ax. 5

Proposition 2. Equal Arcs have Equal Angles

137. Theorem. *If two arcs of the same circle or of equal circles are equal, the arcs have equal central angles; and if two minor arcs are unequal, the greater arc has the greater central angle.*

Given the equal $\circledS$ O and O' with arc $AB =$ arc $A'B'$, minor arc $AC >$ minor arc $A'B'$, and the central $\angle$s AOB, $A'O'B'$, AOC.

Prove that $\angle AOB = \angle A'O'B'$ *and that* $\angle AOC > \angle A'O'B'$.

The best plan, as in § 136, is that of superposition.

Proof. Using the figure of § 136, place $\odot O$ on $\odot O'$ so that OA shall lie on its equal, $O'A'$, and the arc AB on its equal, the arc $A'B'$. Post. 5

Then OB coincides with $O'B'$. Post. 1

$\therefore \angle AOB = \angle A'O'B'$, § 10

thus proving the first part of the theorem.

Since arc $AC >$ arc $A'B'$, Given

we have arc $AC >$ arc AB, Ax. 5

because arc $A'B'$ is given equal to arc AB;

and hence OB lies within $\angle AOC$,

because otherwise we could not have arc $AC >$ arc AB.

$\therefore \angle AOC > \angle AOB$, Ax. 10

and hence $\angle AOC > \angle A'O'B'$, Ax. 5

thus proving the second part of the theorem.

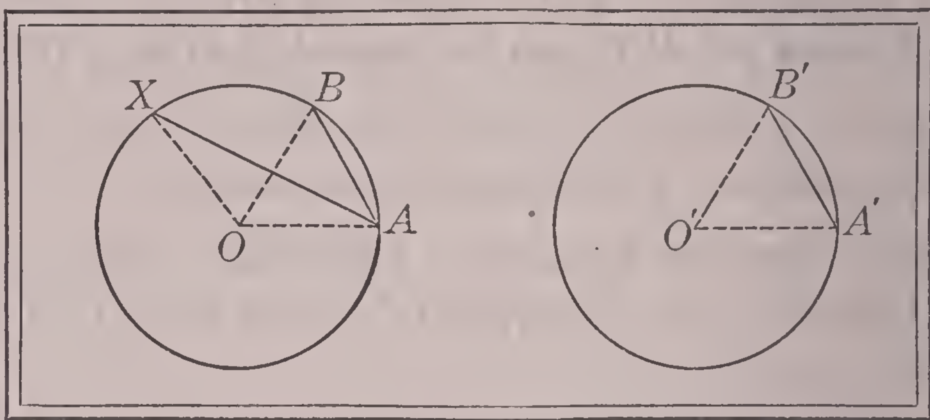
This proposition is the converse (§ 70) of the one in § 136.

138. Chord. A straight line that has its ends on a circle is called a *chord* of the circle.

A chord is said to *subtend* the arcs that it cuts from a circle, but it is more simple to speak of the chord of the arc. Unless the contrary is stated, the chord is to be considered as belonging to the minor arc.

Proposition 3. Equal Arcs have Equal Chords

139. Theorem. *If two arcs of the same circle or of equal circles are equal, the arcs have equal chords; and if two minor arcs are unequal, the greater arc has the greater chord.*



Given the equal Ⓢ O and O' with arc $AB =$ arc $A'B'$ and with minor arc $AX >$ minor arc $A'B'$.

Prove that chord $AB =$ chord $A'B'$

and that chord $AX >$ chord $A'B'$.

The plan is to show that two $\triangle$ are congruent and that the greater chord is opposite the greater $\angle$ of a $\triangle$.

Proof. Draw radii to A, B, X, A', B'. Post. 1

Since $OA = O'A'$ and $OB = O'B'$, § 134, 1

and $\angle AOB = \angle A'O'B'$, § 137

we see that $\triangle OAB$ is congruent to $\triangle O'A'B'$, § 40

and hence chord $AB =$ chord $A'B'$, § 38

thus proving the first part of the theorem.

Then in $\triangle OAX$ and $O'A'B'$ we have

$OA = O'A'$ and $OX = O'B'$, § 134, 1

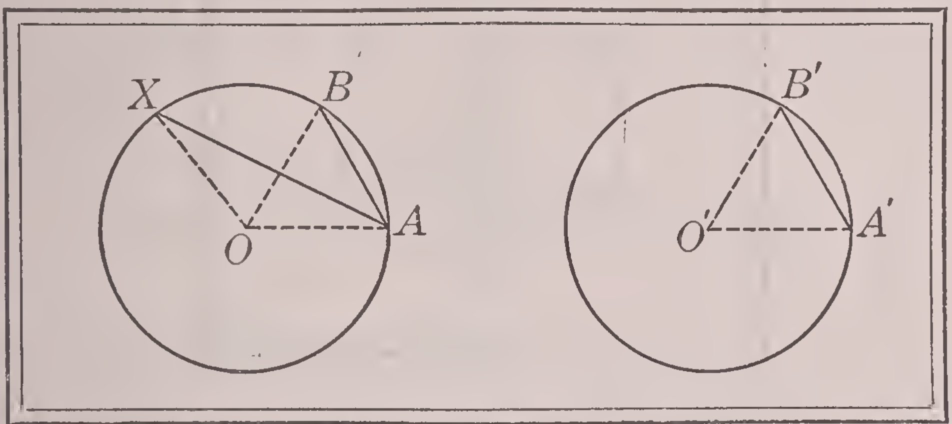
while $\angle AOX > \angle A'O'B'$. § 137

$\therefore$ chord $AX >$ chord $A'B'$, § 126

thus proving the second part of the theorem.

Proposition 4. Equal Chords have Equal Arcs

140. Theorem. *If two chords of the same circle or of equal circles are equal, the chords have equal arcs; and if two chords are unequal, the greater chord has the greater minor arc.*



Given the equal ⓢ O and O' with chord $AB =$ chord $A'B'$ and with chord $AX >$ chord $A'B'$.

Prove that arc $AB =$ arc $A'B'$

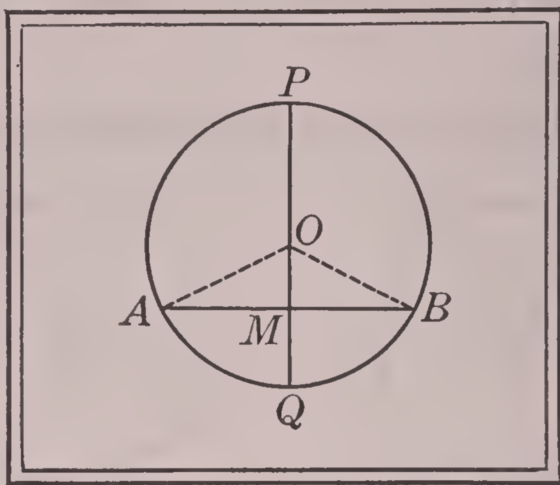
and that arc $AX >$ arc $A'B'$.

The plan is to show that the equal chords have equal central $\angle$s and that the central $\angle$ of the greater chord is opposite the greater side of a $\triangle$.

Proof. Draw	radii to A, B, X, A', B'.	Post. 1
Since	$OA = O'A'$ and $OB = O'B'$,	§ 134, 1
and	chord $AB =$ chord $A'B'$,	Given
we see that	$\triangle OAB$ is congruent to $\triangle O'A'B'$.	§ 47
	$\therefore \angle AOB = \angle A'O'B'$,	§ 38
and hence	arc $AB =$ arc $A'B'$.	§ 136
Then in $\triangle$s OAX and $O'A'B'$ we have		
	$OA = O'A'$ and $OX = O'B'$,	§ 134, 1
while	chord $AX >$ chord $A'B'$.	Given
	$\therefore \angle AOX > \angle A'O'B'$,	§ 127
and	arc $AX >$ arc $A'B'$.	§ 136

Proposition 5. Diameter Perpendicular to a Chord

141. Theorem. *If a diameter is perpendicular to a chord, it bisects the chord and its two arcs.*



Given the ⊙ O with a diameter $PQ \perp$ to a chord AB at M.

Prove that $AM = BM$,

that arc AQ = arc BQ,

and that arc AP = arc BP.

The plan is to prove first that two $\triangle$ are congruent.

Proof. Draw radii to A and B. Post. 1

Since PQ is given $\perp$ to AB, $\triangle AMO$ and BMO are rt. $\triangle$. § 20

Then since $OM = OM$, Iden.

and $OA = OB$, § 134, 1

$\triangle AMO$ is congruent to $\triangle BMO$. § 71

$\therefore AM = BM$. § 38

Likewise, $\angle AOQ = \angle BOQ$, § 38

and $\angle POA = \angle POB$; Post. 9

hence arc AQ = arc BQ, and arc AP = arc BP. § 136

142. Corollary. *If a diameter bisects a chord which is not itself a diameter, it is perpendicular to the chord.*

Show that § 47 applies.

143. Corollary. *The perpendicular bisector of a chord passes through the center of the circle and bisects the arcs of the chord.*

How many $\perp$ bisectors of the chord are possible? Then with what line must the $\perp$ bisector coincide (§ 141)? Complete the proof.

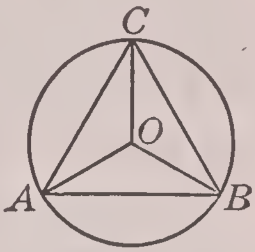
Exercises. Chords and Arcs

1. The greater of two unequal major arcs has the shorter chord.

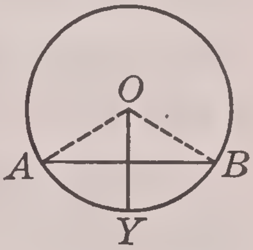
Prove that this follows from § 139.

2. The greater of two unequal chords has the shorter major arc.

Prove that this follows from § 140.



3. If $\triangle ABC$ is an equilateral triangle, find the number of degrees in the central $\angle$s AOB, BOC, COA and in the arcs AB, BC, CA. State the reason in each case.

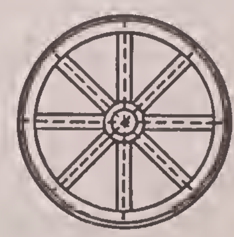


4. If a radius bisects an arc it bisects the chord of that arc.

5. If a radius bisects a chord which is not a diameter, it bisects its central angle.

6. If a diameter bisects a chord which is not itself a diameter it bisects the two arcs of the chord.

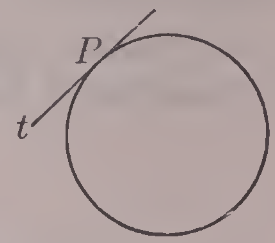
7. The line bisecting the two arcs which have the same chord is the perpendicular bisector of the chord.



8. If a wheel has eight spokes, spaced equally, how many degrees are there in each of the eight small arcs thus formed? State the reason involved in the answer.

9. The chord of half an arc is greater than half the chord of the whole arc.

144. Tangent. An unlimited straight line which touches a circle at only one point is said to be *tangent* to the circle. Such a line is called *a tangent* to the circle.

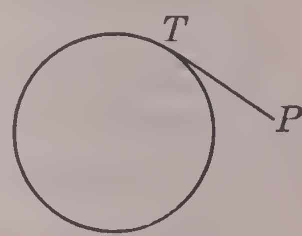


For example, in this figure t is tangent to the circle at the point P.

The word "tangent" is from the Latin word *tangere*, to touch. Hence we may say that a line touches a circle instead of saying that it is tangent to the circle. If a line is tangent to a circle, the circle is also said to be tangent to the line.

The point at which a tangent touches the circle is called the *point of tangency* or *point of contact.*

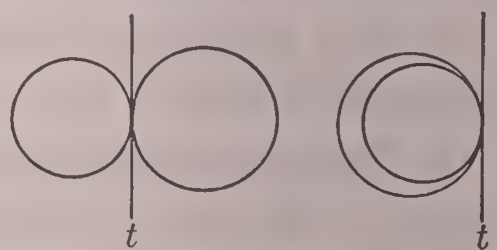
Although a tangent is unlimited in length, when we speak of a tangent from an external point to a circle we mean the segment between the point and the circle.



For example, the tangent from P to the circle here shown is the segment PT.

145. Tangent Circles. Two circles which are both tangent to the same line at the same point are called *tangent circles.*

Circles are said to be *tangent externally* or *tangent internally* according as they lie on opposite sides or on the same side of the tangent line.



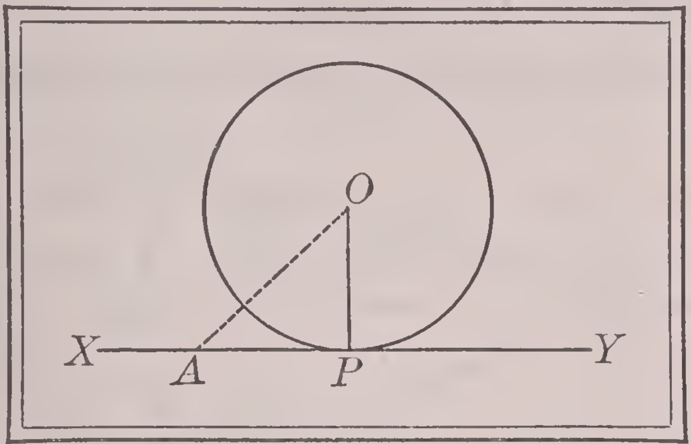
For example, in the first of these figures the circles are tangent externally, and in the second figure they are tangent internally.

The point of contact of two tangent circles with the tangent line is called the *point of contact* or *point of tangency* of the circles.

In the first of the two figures just above, the line t is called a *common internal tangent*, and in the second a *common external tangent.*

Proposition 6. Condition of Tangency

146. Theorem. *If a line is perpendicular to a radius at its end on the circle, the line is tangent to the circle.*



Given the ⊙ *O* with the line *XY* ⊥ to the radius *OP* at *P*.

Prove that *XY is tangent to the* ⊙.

The plan is to show that all points on XY except P are outside the ⊙.

Proof. Let A be any point on XY except P, and draw OA.

Then $OA > OP$, § 122

and hence A is outside the ⊙. § 134, 6

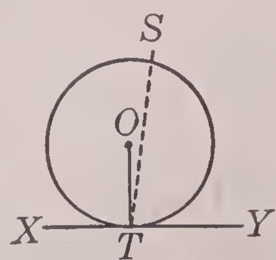
Then every point on XY except P lies outside the ⊙,

and hence XY is tangent to the ⊙. § 144

147. Corollary. *If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.*

Since every point on XY except P is outside the ⊙, then OP is the shortest line segment from O to XY. Hence $\angle OPX$ is a rt. $\angle$ (§ 123).

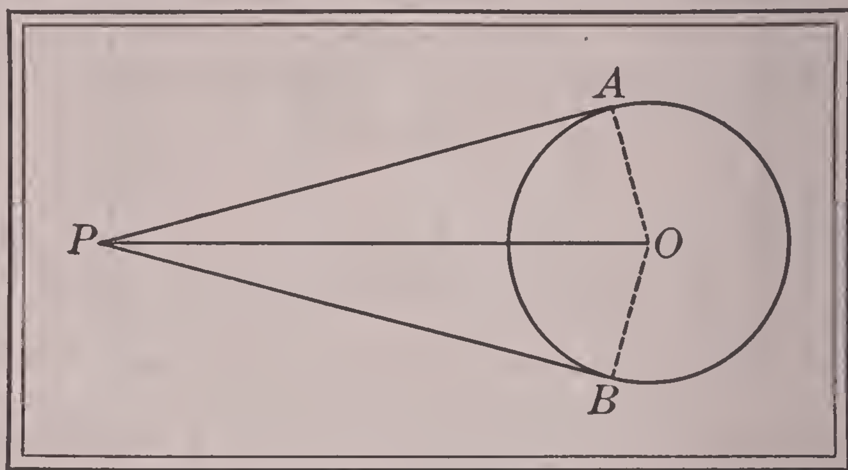
148. Corollary. *If a line is perpendicular to a tangent at the point of contact, it passes through the center of the circle.*



A radius OT is ⊥ to a tangent at T (§ 147). If a ⊥, say TS, constructed to XY at T, did not coincide with this radius, we should have two ⊥s to XY at the same point T, which is impossible (Post. 10).

Proposition 7. Lengths of Tangents

149. Theorem. *The tangents to a circle from an external point are equal and make equal angles with the line joining the point to the center.*



Given PA and PB, tangents from an external point P to the $\odot$ O, and also given PO, the line joining P to O.

Prove that $PA = PB$

and that $\angle OPA = \angle OPB$.

The plan is to prove that two △ are congruent.

Proof. Draw the radii OA, OB. Post. 1

Now PA is $\perp$ to OA,

and PB is $\perp$ to OB, § 147

because if a line is tangent to a $\odot$, it is $\perp$ to the radius drawn to the point of contact.

$\therefore$ △ APO and BPO are rt. △. § 20

Then in △ APO and BPO we have

$PO = PO$, Iden.

and $OA = OB$. § 134, 1

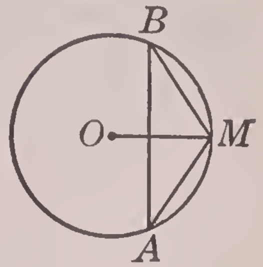
$\therefore \triangle APO$ is congruent to $\triangle BPO$. § 71

Hence $PA = PB$

and $\angle OPA = \angle OPB$. § 38

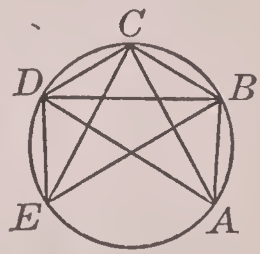
Exercises. Review

1. *A perpendicular from the center of a circle to a tangent passes through the point of contact.*

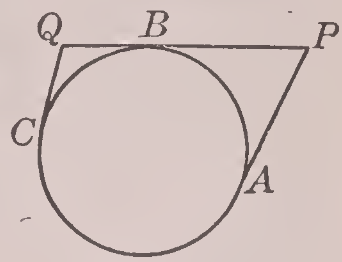


2. In this circle the chords AM and BM are equal. Prove that M bisects the arc AB and that the radius OM bisects the chord AB.

3. If P is a point on a circle such that it is equidistant from two radii OA and OB, then P bisects the arc AB.

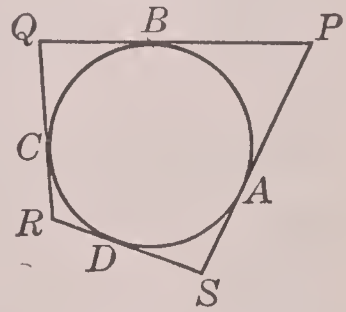


4. If five points A, B, C, D, E are so placed on a circle that AB, BC, CD, DE are equal chords, then AC, BD, CE are equal chords, and AD and BE are also equal chords.



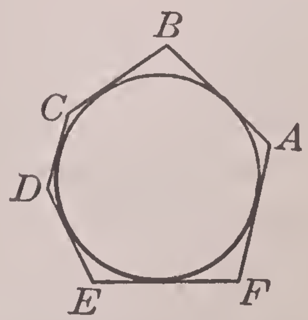
5. If tangents to a circle at the points A, B, C meet in P and Q, as here shown, then $AP + QC = PQ$.

Apply § 149 twice.



6. If a quadrilateral has each side tangent to a circle, the sum of one pair of opposite sides equals the sum of the other pair.

In this figure show that $SP + QR = PQ + RS$. Apply § 149 four times.



7. The hexagon here shown has each side tangent to the circle. Prove that

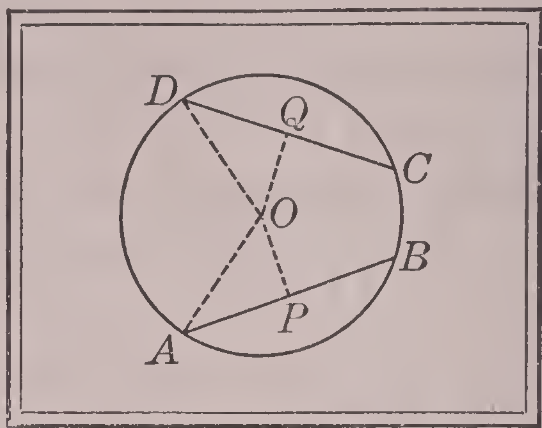
$$AB + CD + EF = BC + DE + FA.$$

8. If a quadrilateral has each side tangent to a circle and if the vertices are joined to the center, the sum of the angles at the center opposite any two opposite sides is equal to a straight angle.

PS

Proposition 8. Equal Chords

150. Theorem. *Equal chords of the same circle or of equal circles are equidistant from the center.*



Given the ⊙ O with chord $AB =$ chord CD.

Prove that AB *and* CD *are equidistant from* O.

The plan is to prove that two △ are congruent.

Proof. Let OP be ⊥ to AB and let OQ be ⊥ to CD. § 116

Draw	OA and OD.	Post. 1
Then	△ OAP and ODQ are rt. △	§ 20
with	$AP = \frac{1}{2} AB$ and $DQ = \frac{1}{2} CD$.	§ 141
Since	$AB = CD$,	Given
then	$AP = DQ$.	Ax. 4
Also,	$OA = OD$.	§ 134, 1
	∴ $\triangle OAP$ is congruent to $\triangle ODQ$,	§ 71
and hence	$OP = OQ$;	§ 38
that is,	AB and CD are equidistant from O.	§ 73

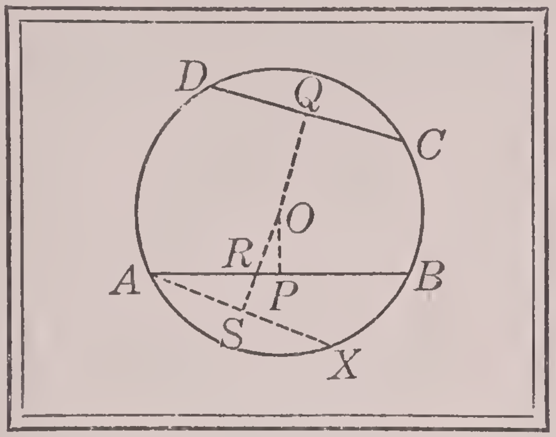
Although equal ⊙ are mentioned here and in several subsequent theorems, it is evidently necessary to consider only a single ⊙.

151. Corollary. *Chords that are equidistant from the center of a circle are equal.*

If $OP = OQ$, then, since $OA = OD$, we have two congruent rt. △ (§ 71). Then $AP = DQ$ (§ 38), and hence $AB = CD$ (§ 141 and Ax. 3).

Proposition 9. Unequal Chords

152. Theorem. *The less of two chords of the same circle or of equal circles is more remote from the center.*



Given the $\odot$ O with chord $CD <$ chord AB.

Prove that CD *is more remote from* O *than* AB.

In the figure the plan is to prove that $OR > OP$ and that $OS > OR$.

Proof. Since $CD < AB$, Given

we have arc $CD <$ arc AB. § 140

Suppose that arc $AX =$ arc CD, and draw AX. Post. 1

Then $AX = CD$. § 139

Let the $\perp$s from O upon AB, CD, AX be OP, OQ, OS respectively, and designate the intersection of OS and AB as R. § 116

Then $OS = OQ$. § 150

Equal chords ⋯ are equidistant from the center.

Also, $OR > OP$. § 122

The $\perp$ is the shortest line ⋯ from a given external point.

But $OS > OR$, Ax. 10

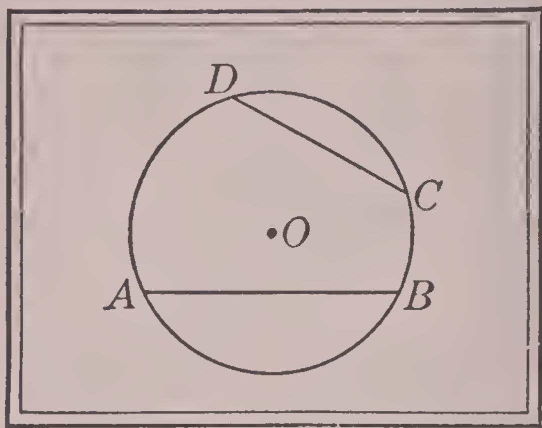
so that $OS > OP$, Ax. 9

and hence $OQ > OP$; Ax. 5

that is, CD is more remote from O than AB. § 73

Proposition 10. Chords Unequally Distant

153. Theorem. *If two chords of the same circle or of equal circles are unequally distant from the center, the chord more remote is the shorter.*



Given the $\odot$ O with two chords, AB and CD, such that CD is more remote from O.

Prove that $CD < AB$.

The plan is to show that any other possibility violates § 152 or § 150.

Proof. Now CD must be greater than AB, equal to AB, or less than AB.

If	$CD > AB$,	
then	AB is more remote from O.	§ 152
If	$CD = AB$,	
then	AB and CD are equidistant from O.	§ 150
But	CD is more remote from O.	Given
	$\therefore CD < AB$.	

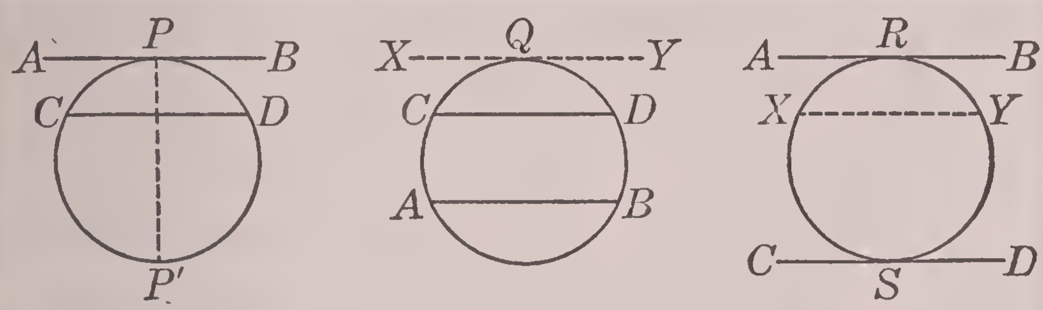
This proposition is the converse of the one in § 152. The student has probably concluded that where we have only three possible conditions, as we do in §§ 150 and 152, the converses are always true.

154. Corollary. *A diameter of a circle is greater than any other chord.*

For no other chord can be as near the center.

Proposition 11. Parallels and Arcs

155. Theorem. *If two parallel lines intersect a circle or are tangent to it, they intercept equal arcs.*



Case 1 Case 2 Case 3

1. Given a ⊙ with *AB*, a tangent at *P*, ∥ to a chord *CD*.

Prove that arc $CP =$ arc DP.

The plan is to show first that certain arcs are equal by § 141.

Proof. Let PP' be ⊥ to AB at P. Post. 10

Then PP' is a diameter (§ 148), and is also ⊥ to CD (§ 63).

Hence arc $CP =$ arc DP. § 141

2. Given a ⊙ with *AB* and *CD*, two ∥ chords.

Prove that arc $AC =$ arc BD.

Proof. Let XY, a tangent at Q, be ∥ to CD. § 52

Then XY is ∥ to AB. § 58

∴ arc $AQ =$ arc BQ, and arc $CQ =$ arc DQ. Case 1

Hence arc $AC =$ arc BD. Ax. 2

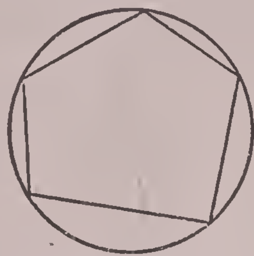
3. Given a ⊙ with *ARB* and *CSD*, two ∥ tangents.

Prove that arc $RXS =$ arc RYS.

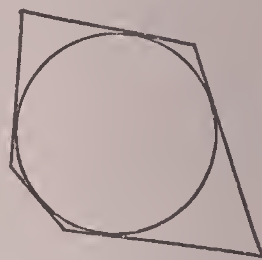
Proof. Let chord XY be ∥ to AB. § 52

Now complete the proof by § 58, Case 1 (above), and Ax. 1.

156. Inscribed and Circumscribed Polygons. If the sides of a polygon are all chords of a circle, the polygon is said to be *inscribed* in the circle; if the sides are all tangents, the polygon is said to be *circumscribed* about the circle.



Inscribed Polygon



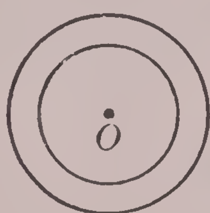
Circumscribed Polygon

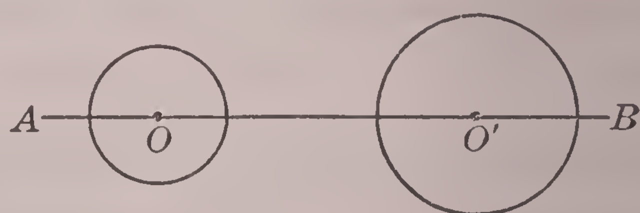
The circle is said to be *circumscribed* about the inscribed polygon and to be *inscribed* in the circumscribed polygon.

157. Concentric Circles. Two circles which have the same center are said to be *concentric*.

For example, the two circles in the first figure below are concentric.

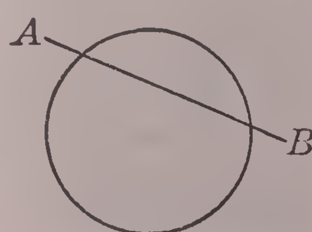
158. Line of Centers. The line determined by the centers of two circles is called the *line of centers*.





For example, in the second figure above, the line AB is the line of centers of the ⓢ O and O'.

159. Secant. A straight line which intersects a circle is called a *secant*.

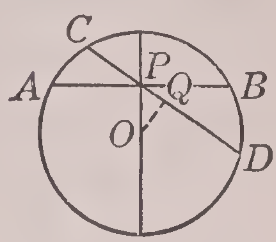


In this figure the line AB is a secant.

It is readily inferred from the figure that a secant can intersect a circle in only two points, and the student should notice that this is further evidence of the truth of the statement given in § 134, 4. This property of a circle will be stated and proved later as a corollary (§ 192); but until then § 134, 4, may be assumed as a postulate.

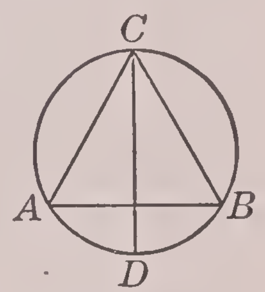
Exercises. Review

1. If an equilateral triangle and a square are inscribed in a circle, the sides of the square are more remote from the center than the sides of the triangle.



2. The shortest chord that can be drawn through a given point within a circle is the one which is perpendicular to the diameter through the point.

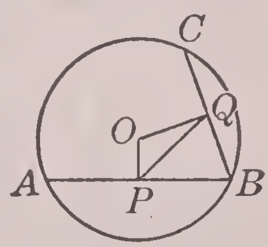
Show that any other chord CD, through P, is nearer O than is AB.



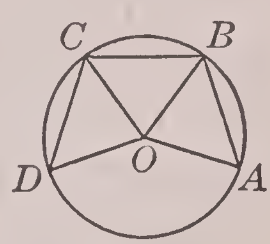
3. In this figure, if the diameter CD bisects the arc AB, then $\angle CBA = \angle CAB$.

What kind of triangle is $\triangle ABC$?

4. In two concentric circles it is given that MN is a diameter of the larger circle and PQ an intersecting diameter of the smaller circle. Prove that P, M, Q, and N are the vertices of a parallelogram.

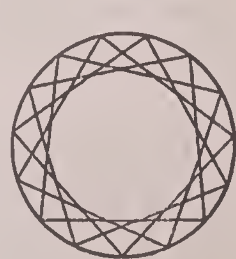


5. In this figure arc $AB >$ arc BC and OP and OQ are perpendiculars from the center upon AB and BC respectively. Prove that $\angle QPO > \angle PQO$.



6. Three equal chords AB, BC, CD are taken end to end, and the radii OA, OB, OC, OD are drawn. Prove that $\angle AOC = \angle BOD$ and state any other pairs of equal angles.

7. All equal chords of a circle are tangent to a concentric circle.

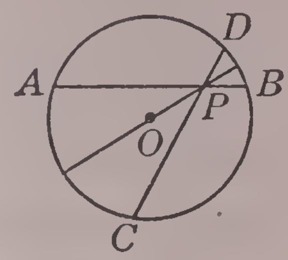


8. If a number of equal chords are drawn in this circle, the figure gives the impression of a second circle inside the first and concentric with it. Explain the reason.

9. If two circles are concentric, chords of the larger circle that are tangents to the smaller are equal.

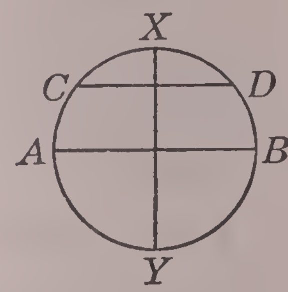
10. Two equal circles cut two equal chords from a secant drawn parallel to the line of centers.

11. If two intersecting chords make equal angles with the diameter through their point of intersection, the chords are equal.

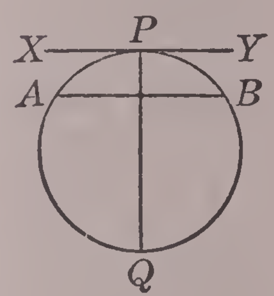


12. If two equal chords intersect, the segments of one are equal respectively to the segments of the other.

13. In this figure, XY is a diameter $\perp$ to the $\parallel$ chords AB and CD, arc $BD = 40°$, and arc $DX = 50°$. How many degrees are there in the arcs XC, CA, AY, and YB?

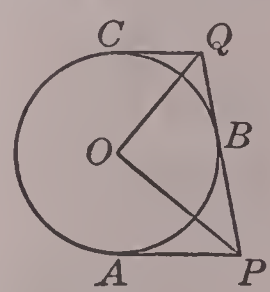


14. In this figure, XY is tangent to the circle at P, the chord AB is $\perp$ to the diameter PQ, and the arc $AQ = 125°$. How many degrees are there in arc BP?



15. If from any number of points on the larger of two concentric circles tangents are drawn to the smaller circle, these tangents are equal.

16. In this figure, AP and CQ are parallel tangents which are cut by a third tangent QP. If O is the center of the circle, prove that $\angle POQ = 90°$.

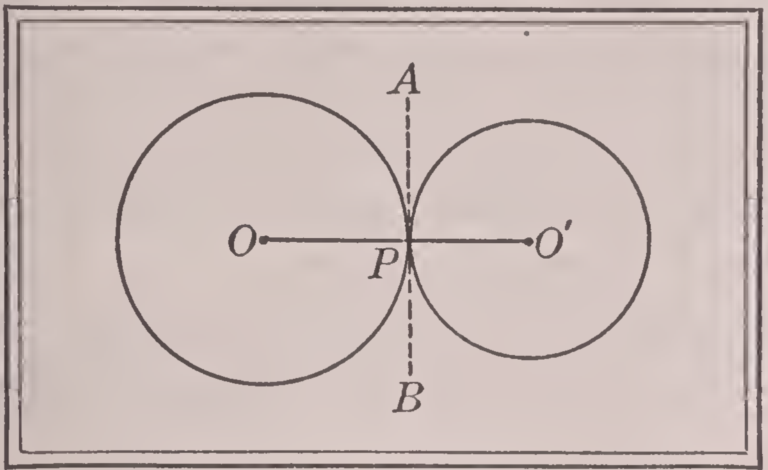


What is the relation of the $\measuredangle QPA$ and PQC? How do OP and OQ divide these angles? Now consider the angles of the $\triangle PQO$.

17. If AB is a diameter of a circle with center O, and if BC is any chord from B, then a radius OP which is $\parallel$ to BC and lies within $\angle CBA$ bisects the arc CA.

Proposition 12. Tangent Circles

160. Theorem. *If two circles are tangent to each other, the line of centers passes through the point of contact.*



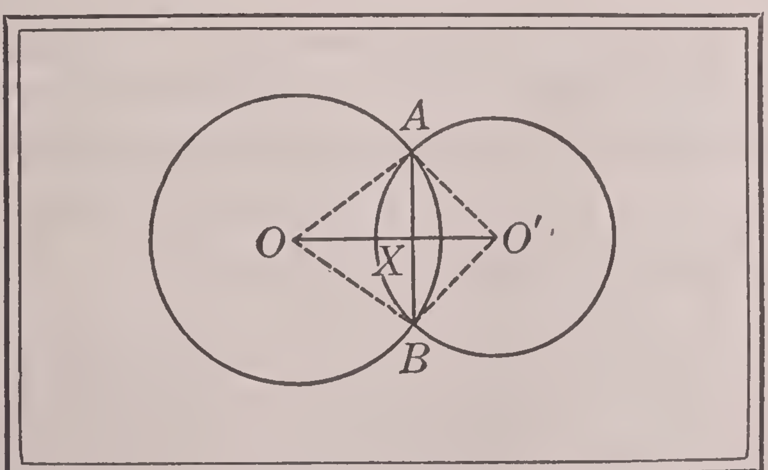
Given the ⓢ O and O' tangent at P and the line of centers OO'.

Prove that P *is on the line of centers.*

The plan is to show that OO' is $\perp$ to the common tangent, which is left for the student to prove. Although §§ 160 and 161 are not required in standard courses, they have many interesting applications.

Proposition 13. Line of Centers

161. Theorem. *If two circles intersect, the line of centers is the perpendicular bisector of their common chord.*



The proof of this proposition is left for the student.

Exercises. Review

Describe the relative position of two circles if the line segment joining the centers is related to the radii as stated in Exs. 1–3, and illustrate each case by a figure:

1. The segment is greater than the sum of the radii.

2. The segment is equal to the sum of the radii.

3. The segment is less than the sum but greater than the difference between the radii.

4. If two circles are tangent externally, the tangents to them from any point of the common internal tangent are equal.

5. If two circles tangent externally are tangent to a line AB at A and B, their common internal tangent bisects AB.

6. The line drawn from the center of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord which joins the points of contact.

7. The diameters of two circles are 8.15 in. and 6.22 in. respectively. Find the distance between the centers of the circles if they are tangent externally. Find the distance between the centers of the circles if they are tangent internally.

8. Three circles of diameters 2.4 in., 1.8 in., and 2.1 in. are tangent externally, each to the other two. Find the perimeter of the triangle formed by joining the centers.

9. If two circles tangent externally at P are tangent to a line AB at A and B, then $\angle BPA = 90°$.

10. If two radii of a circle, at right angles to each other, when produced are cut at A and B by a tangent to the circle, the other tangents from A and B are parallel to each other.

162. Measure. The number of times a quantity contains a unit of the same kind is called the *numerical measure* of the quantity, or simply its *measure.*

For example, the numerical measure of the length of a room in feet is the number of times the length contains the unit of length, 1 ft.

163. Commensurable Magnitudes. Two magnitudes of the same kind which can both be expressed as integers in terms of a common unit are called *commensurable magnitudes.*

For example, $2\frac{1}{4}$ sq. ft. and 3 sq. ft. are commensurable, for $\frac{1}{4}$ sq. ft. is contained 9 times in the first and 12 times in the second. In this case the common unit taken was $\frac{1}{4}$ sq. ft.; but any unit fraction, say $\frac{1}{8}$, of this unit is also a common unit.

Any common unit used in measuring two or more commensurable magnitudes is called a *common measure* of the magnitudes. Each of the magnitudes is called a *multiple* of any common measure.

164. Incommensurable Magnitudes. Two magnitudes of the same kind which cannot both be expressed in integers in terms of a common unit are called *incommensurable magnitudes.*

The diagonal and the side of a square are, as we shall later prove, incommensurable lines. We also have incommensurable numbers such as 2 and $\sqrt{3}$, for there is no number which is contained in both of these numbers without a remainder.

165. Ratio. The quotient of the numerical measures of two magnitudes expressed in terms of a common unit is called the *ratio* of the magnitudes.

Thus, if a room is 20 ft. by 35 ft., the ratio of the width to the length is 20 ft. ÷ 35 ft., or $\frac{20}{35}$, which reduces to $\frac{4}{7}$. Here the common unit is 1 ft.

The ratio of a to b is written $\frac{a}{b}$, or $a : b$, as in arithmetic and algebra. While we shall ordinarily use the first form, the form $a : b$ is sometimes convenient. Thus the ratio of 20° to 30°, which is $\frac{20}{30}$, or $\frac{2}{3}$, may also be written $2 : 3$.

166. Incommensurable Ratio. The ratio of two incommensurable magnitudes is called an *incommensurable ratio.*

Although the exact value of such a ratio cannot be expressed by an integer, a common fraction, or a decimal fraction of a limited number of places, it may be expressed approximately. For example, $\sqrt{2} = 1.41421356 \cdots$, which is greater than 1.414213 but less than 1.414214, and therefore differs from either by less than 0.000001.

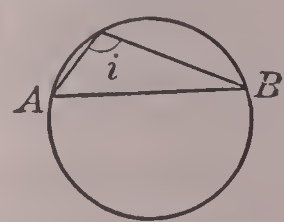
By carrying the decimal further an approximate value may be found that will differ from the ratio by less than a billionth, a trillionth, or *any other assigned value.* That is, *for practical purposes all ratios are commensurable.*

For the present we shall consider only the ratios of commensurable geometric magnitudes. For the incommensurable cases see the optional work in §§ 515–517.

167. Segment. A portion of a plane bounded by an arc of a circle and its chord is called a *segment* of the circle.

In the figure of § 168 the part above the chord AB is a *minor segment* of the circle, and the part below is a *major segment.*

168. Inscribed Angle. An angle with its vertex on a circle and with chords for its arms is called an *inscribed angle.*

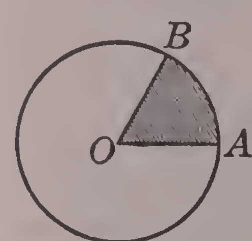


In the figure here shown, i is an inscribed angle.

An angle is said to be *inscribed in a segment* if its vertex is on the arc of the segment and its arms pass through the ends of the arc.

In the figure above, i is inscribed in the minor segment.

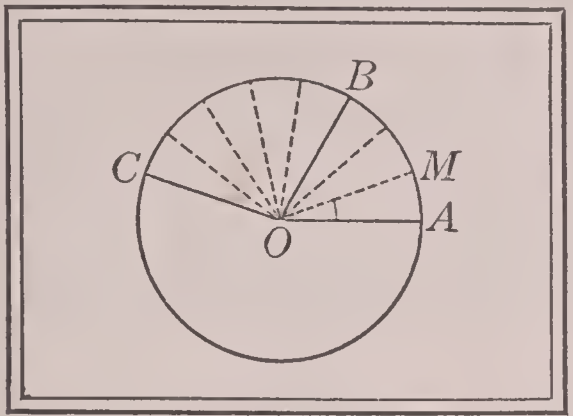
169. Sector. A portion of a plane bounded by two radii and the arc of the circle which is cut off by the radii is called a *sector.*



In this figure the shaded portion AOB is a sector of the circle. If AB is a quarter of the circle, it and its sector are each called a *quadrant.*

Proposition 14. Central Angles

170. Theorem. *Two central angles of the same circle or of equal circles have the same ratio as their arcs.*



Given the $\odot$ O with the central $\angle$s AOB and BOC.

Prove that $$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}.$$

The plan is to assume that the $\angle$s and their arcs are commensurable.

Proof. Suppose that some $\angle AOM$ is contained 3 times in $\angle AOB$ and 5 times in $\angle BOC$. § 163

Then $$\frac{\angle AOB}{\angle BOC}=\frac{3\cdot\angle AOM}{5\cdot\angle AOM}=\frac{3}{5}.$$ § 165

Construct $\angle$s equal to $\angle AOM$ as shown. § 106

Then the arcs of these $\angle$s are equal, § 136

and $$\frac{\text{arc } AB}{\text{arc } BC}=\frac{3\cdot\text{arc } AM}{5\cdot\text{arc } AM}=\frac{3}{5}.$$ § 165

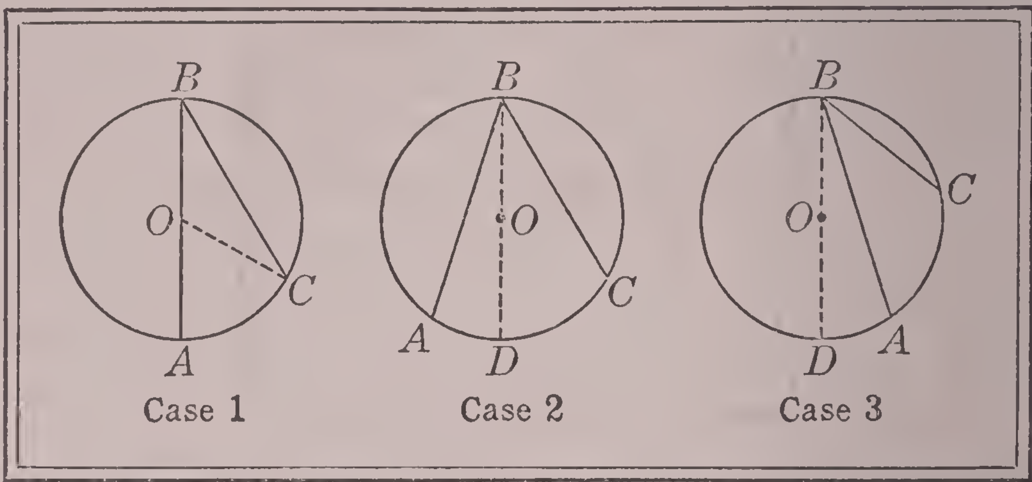
Hence $$\frac{\angle AOB}{\angle BOC}=\frac{\text{arc } AB}{\text{arc } BC}.$$ Ax. 5

The proof is the same if any other numbers are used.

171. Angle and Arc Measure. Since the central angles contain the same number of units as their arcs, the angles and their arcs have the same numerical measure. Briefly stated, *a central angle is measured by its arc.*

Proposition 15. Inscribed Angle

172. Theorem. *An inscribed angle is measured by half its intercepted arc.*



Given the $\odot O$ with the inscribed $\angle B$ intercepting arc AC.

Prove that $\angle B$ *is measured by* $\frac{1}{2}$ *arc* AC.

In the first figure the plan is to show that $\angle B = \frac{1}{2} \angle AOC$.

Proof. 1. If O is on AB, draw OC. Post. 1

Then since $OC = OB$, § 134, 1

we have $\angle B = \angle C$. § 42

Then since $\angle B + \angle C = \angle AOC$, § 66

we have $2 \angle B = \angle AOC$; Ax. 5

whence $\angle B = \frac{1}{2} \angle AOC$. Ax. 4

Since $\angle AOC$ is measured by arc AC, § 171

then $\frac{1}{2} \angle AOC$ is measured by $\frac{1}{2}$ arc AC. Ax. 4

$\therefore \angle B$ is measured by $\frac{1}{2}$ arc AC. Ax. 5

2. If O lies within $\angle B$, draw the diameter BD. Post. 1

Then $\angle ABD$ is measured by $\frac{1}{2}$ arc AD,

and $\angle DBC$ is measured by $\frac{1}{2}$ arc DC. Case 1

$\therefore \angle ABD + \angle DBC$ is measured by $\frac{1}{2}$ arc $(AD + DC)$; Ax. 1

that is, $\angle B$ is measured by $\frac{1}{2}$ arc AC. Ax. 10

3. If O lies outside $\angle B$, draw the diameter BD. Post. 1

Then $\angle DBC$ is measured by $\frac{1}{2}$ arc DC,

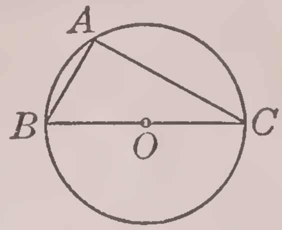
and $\angle DBA$ is measured by $\frac{1}{2}$ arc DA. Case 1

$\therefore \angle DBC - \angle DBA$ is measured by $\frac{1}{2}$ arc $(DC - DA)$; Ax. 2

that is, $\angle B$ is measured by $\frac{1}{2}$ arc AC. Ax. 5

It should be observed that the expression "$\angle B$ is measured by $\frac{1}{2}$ arc AC" is only a shortened form for the expression "The measure of $\angle B =$ the measure of $\frac{1}{2}$ arc AC." Furthermore, "measure of $\frac{1}{2}$ arc AC" is equivalent to "$\frac{1}{2}$ the measure of arc AC," and hence the first expression is really an equation and the axioms of equations may be applied.

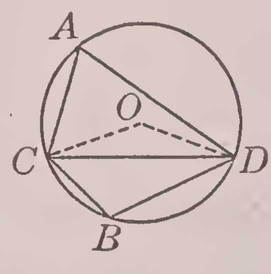
173. Corollary. *An angle inscribed in a semicircle is a right angle.*



Show that $\angle A$ is half the central st. $\angle BOC$.

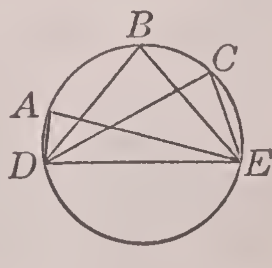
Instead of proving the corollary in this way, it may be shown that $\angle A$ is measured by half of what arc? It is then what kind of $\angle$?

174. Corollary. *An angle inscribed in a segment greater than a semicircle is an acute angle, and an angle inscribed in a segment less than a semicircle is an obtuse angle.*



In giving the proof draw the radii OC, OD. Then show that $\angle A$ is half the $\angle COD$. Finally, show that $\angle B$ is half the reflex $\angle DOC$ (§ 16), which is greater than a st. $\angle$.

175. Corollary. *Angles inscribed in the same segment or in equal segments are equal.*



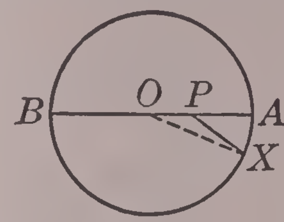
Show that each of the $\measuredangle A$, B, C is half the same central $\angle$.

176. Corollary. *If a quadrilateral is inscribed in a circle, the opposite angles are supplementary.*

Consider $\measuredangle A$ and B in the figure of § 174. Their sum is measured by half the sum of what two arcs? Give the proof in full.

Exercises. Review

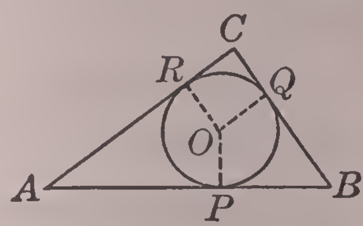
1. The shorter segment of the diameter through a given point within a circle is the shortest line that can be drawn from that point to the circle.



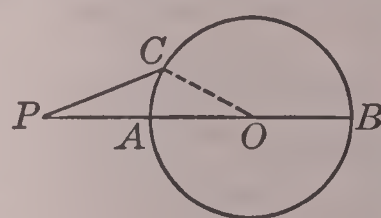
Let P be the given point. Prove that PA is shorter than any other line PX from P to the circle.

2. The longer segment of the diameter through a given point within a circle is the longest line that can be drawn from that point to the circle.

3. The diameter of the circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other two sides

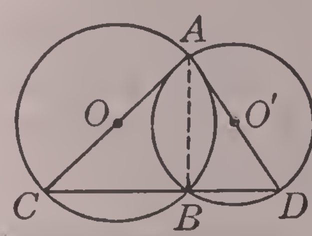


4. A line from a given point outside a circle passing through the center contains the shortest line segment that can be drawn from that point to the circle.

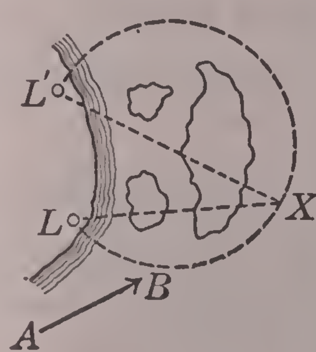


Let P be the point and O the center. How does $PC + CO$ compare with PO?

5. Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line which joins the ends of the diameters passes through the other point of intersection.

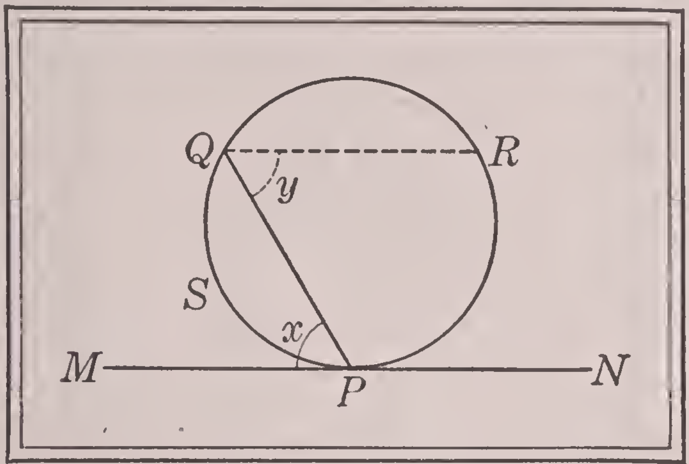


6. The captain of a ship sailing along the course AB is informed by his chart that the *horizontal danger angle* ($\angle L'XL$) for a reef lying off the coast near two lighthouses L and L' is 30°. How can the captain avoid the reef and where should he change his course?



Proposition 16. Tangent and Chord

177. Theorem. *An angle formed by a tangent and a chord drawn from the point of contact is measured by half its intercepted arc.*



Given a ⊙ with the tangent *MN*, through *P*, and the chord *PQ* making the ∠ *x*.

Prove that x *is measured by* $\frac{1}{2}$ *arc PSQ.*

In the figure the plan is to show that $x = y$, that the arcs QSP and PR are equal, and then to apply § 172.

Proof. Suppose that chord QR is ∥ to MN, thus forming $\angle y$ in the figure. § 52

Then $x = y$, § 61

and arc PSQ = arc PR. § 155

Also, y is measured by $\frac{1}{2}$ arc PR. § 172

$\therefore x$ is measured by $\frac{1}{2}$ arc PSQ. Ax. 5

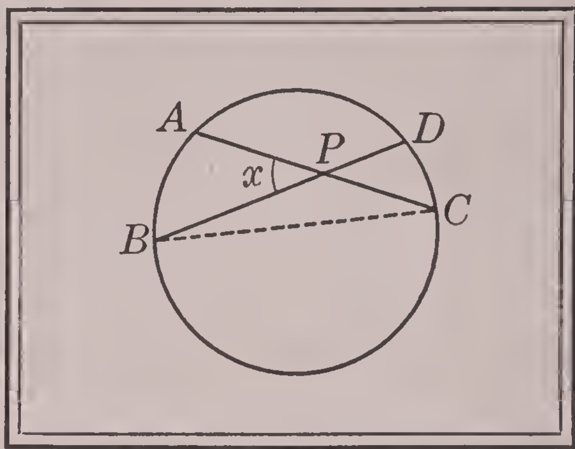
It may be shown that $\angle NPQ$ in the above figure is measured by $\frac{1}{2}$ arc PRQ. This is done by showing that the st. $\angle NPM$ is measured by half the entire ⊙, and that if we subtract x and $\frac{1}{2}$ arc QSP, we have left $\angle NPQ$ and $\frac{1}{2}$ arc PRQ.

It is instructive to consider the arc by which x is measured as PQ swings about P, first when PQ is ⊥ to MN and then when PQ lies along PN so that x is a st. ∠.

PS

Proposition 17. Two Chords

178. Theorem. *An angle formed by two chords intersecting within a circle is measured by half the sum of its intercepted arc and that of its vertical angle.*



Given a $\odot$ with $\angle x$ formed by the chords AC and BD.

Prove that x is measured by $\frac{1}{2}$(arc AB + arc CD).

The plan is to show that $x = \angle C + \angle B$, and then refer to § 172.

Proof. Draw BC. Post. 1

Then $x = \angle C + \angle B$. § 66

An exterior $\angle$ of a $\triangle$ is equal to the sum of the two nonadjacent interior $\angle$s.

Also, $\angle C$ is measured by $\frac{1}{2}$ arc AB,

and $\angle B$ is measured by $\frac{1}{2}$ arc CD. § 172

$\therefore x$ is measured by $\frac{1}{2}$(arc AB + arc CD). Ax. 1

It is interesting to discuss this theorem along the following lines:

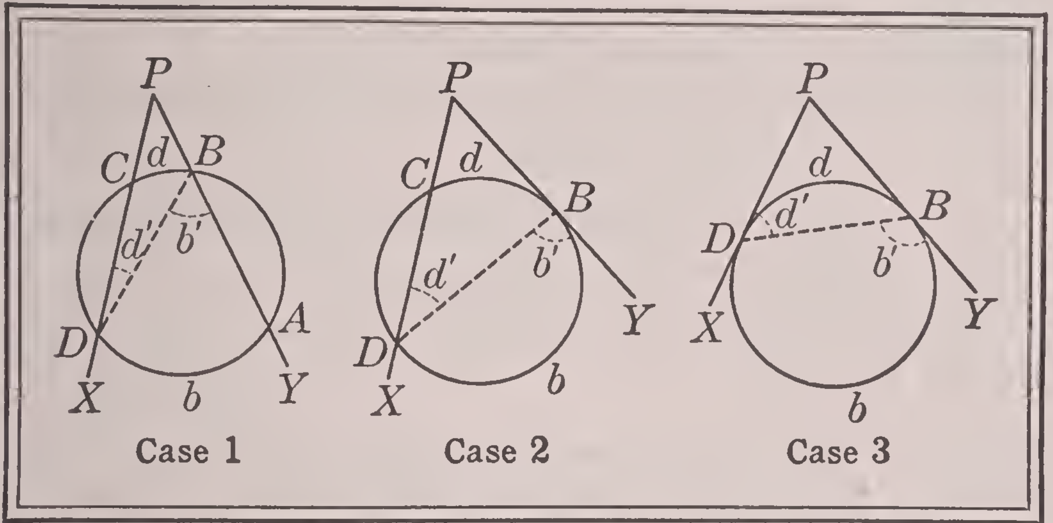
If P is the vertex of $\angle x$, and if we move P to the center of the $\odot$, to what previous proposition does this one reduce?

If P is on the $\odot$, as at D, to what previous proposition does this one then reduce?

Suppose that the point P remains as in the figure, and that the chord AC swings about P as a pivot until it coincides with the chord BD. What can then be said of the measure of $\angle$s APB and CPD? What can be said as to the measure of $\angle$s DPA and BPC?

Proposition 18. Two Secants

179. Theorem. *An angle formed by two secants, by a secant and a tangent, or by two tangents drawn to a circle from an external point is measured by half the difference between its intercepted arcs.*



Given two lines PX and PY from an external point P, cutting off on a ⊙ two arcs b and d such that $b > d$.

Prove that $\angle P$ *is measured by* $\frac{1}{2}(b-d)$.

The plan is to show that $\angle P = b' - d'$, and then to apply §§ 172, 177.

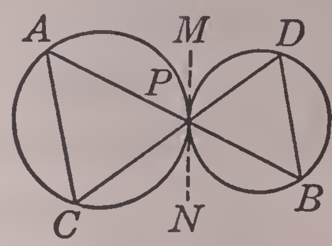
Proof. In the figures as lettered above, we have an angle formed by two secants (Case 1), by a secant and a tangent (Case 2), and by two tangents (Case 3).

In each figure draw	BD.	Post. 1
In each case, since	$\angle P + d' = b'$,	§ 66
we have	$\angle P = b' - d'$.	Ax. 2
Then	b' is measured by $\frac{1}{2}b$,	
and	d' is measured by $\frac{1}{2}d$.	§§ 172, 177
	$\therefore \angle P$ is measured by $\frac{1}{2}(b-d)$.	Ax. 5

If the secant PY swings around to tangency, it becomes the tangent PB, and Case 1 becomes Case 2. If PX also swings around to tangency, it becomes the tangent PD, and Case 2 becomes Case 3.

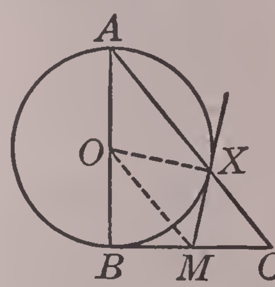
Exercises. Measure of Angles

1. If two circles are tangent externally and if two line segments drawn through the point of contact are terminated by the circles, the chords which join the ends of these lines are parallel.



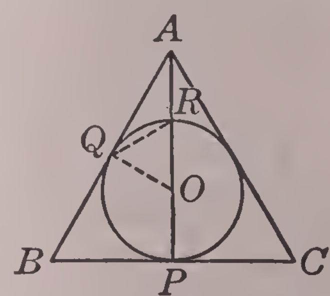
This can be proved if it can be shown that $\angle A$ equals what angle? To what two angles can these angles be proved equal by § 177? Are those angles equal?

2. If one side of a right triangle is the diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other side of the triangle.



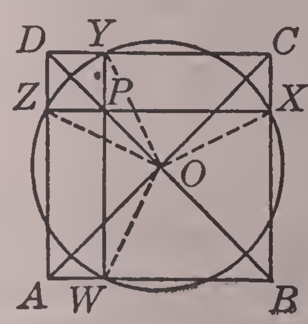
If OM is $\parallel$ to AC, then because $BO = OA$, what is the relation of BM to MC? The proposition therefore reduces to proving that OM is $\parallel$ to what line of $\triangle ABC$? This can be proved if $\angle BOM$ can be shown equal to what angle?

3. The radius of the circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.



To prove this we must show that AR equals what line segment? It looks as if AR might equal QR, and QR equal OR. Is there any way of proving $\triangle ORQ$ equilateral? of proving $\triangle AQR$ isosceles?

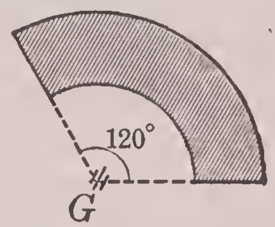
4. If two lines are drawn parallel to the sides through any point on a diagonal of a square, the points where these lines meet the sides lie on the circle whose center is the point of intersection of the diagonals of the square.



It can be shown that $OY = OZ$ if what two triangles are congruent? How can you prove these triangles congruent? Then how can you prove that $OY = OX$ and that $OX = OW$?

II. Loci

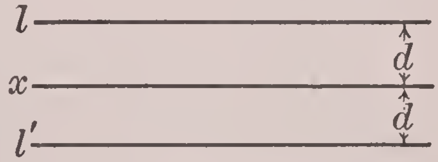
180. Meaning of Locus. The Latin word for place is *locus*, the plural of which is *loci* (usually pronounced lō′sī in mathematics). In speaking of the place where certain points lie, it is often convenient to speak of it as the locus of the points. For example, if a gun at G in this figure can be turned through an angle of 120°, and if the projectile will fall at some point between 5000 yd. and 9000 yd., depending upon the angle at which the gun is elevated, the locus of the points at which the projectile may fall is a certain region which is represented by the shaded part of the figure.



While it is proper to represent a locus as a surface, a portion of space, a line, or even a point, it is the custom in plane geometry to study only those loci which are lines. All the definitions and discussions of loci in Book II refer to loci in a plane.

The following statements concerning loci are so evident that they may be treated as postulates:

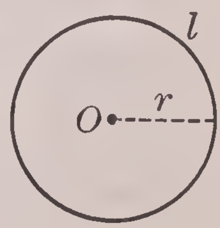
1. *The locus of points at a given distance d from a given line x is a pair of lines, l and l', parallel to x and at the distance d from it.*



It is then said that any point on l or l' *satisfies the condition* that it is at the distance d from x.

Instead of speaking of the *locus of points* that satisfy a given condition, we may speak of the *locus of a point* as satisfying the condition.

2. *The locus of points equidistant from a given point is a circle whose center is the given point.*



Since the circle is a very obvious locus, the subject of loci is considered in Book II.

181. Proof of a Locus. To prove that a certain line or group of lines is the locus of points that satisfy a given condition, it is necessary and sufficient to prove two things:

1. *Every point on the line or lines satisfies the given condition;*

2. *Every point which satisfies the given condition lies on the line or lines.*

If we can prove this for any point whatsoever, that is, not merely for some special point, it is evidently true for every point.

One of the best ways of determining a locus is to take on paper a number of points which satisfy the given condition and then try to determine on what line or lines they lie.

Exercises. Loci in a Plane

State without proof the following loci:

1. The locus of the tip of the hour hand of a watch.

2. The locus of the center of the hub of an automobile wheel as the car moves straight ahead on a level road.

3. The locus of the tips of a pair of shears as they open, provided the bolt which holds the blades together remains always fixed in one position.

4. The locus of the center of a circle that rolls around another circle, inside or outside, and always just touches it.

Draw the following loci, but give no proofs:

5. The locus of points $\frac{1}{2}$ in. below the base of a given $\triangle ABC$, and also of points $\frac{1}{2}$ in. above the base.

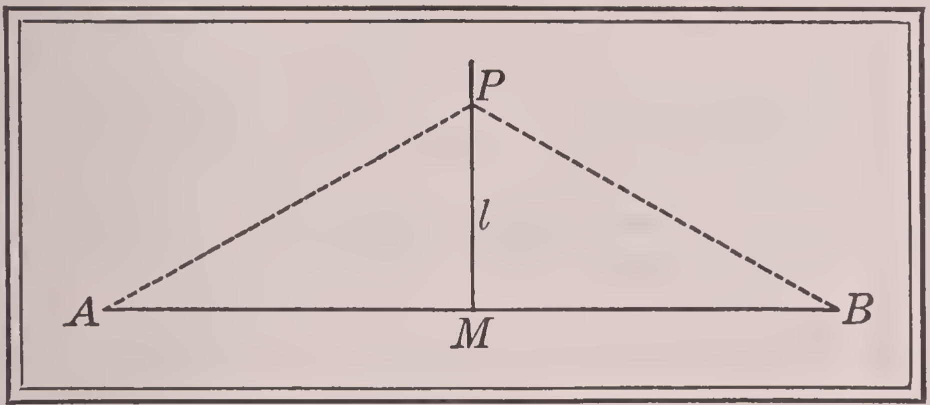
6. The locus of points $\frac{1}{4}$ in. from a given line AB.

7. The locus of points $\frac{3}{4}$ in. from a given point O.

8. The locus of points $\frac{1}{4}$ in. outside the circle drawn with a given point O as center and a radius of 1 in.

Proposition 19. Perpendicular Bisector

182. Theorem. *The locus of points equidistant from two points is the perpendicular bisector of the line segment joining them.*



Given two points A and B, and l, the $\perp$ bisector of AB.

Prove that every point on l is equidistant from A and B and that every point equidistant from A and B lies on l.

The plan is first to apply § 117.

Proof. From P, any point on l, draw PA, PB. Post. 1

Then $PA = PB$. § 117

This proves the first part of the theorem.

Let P be any point in the plane such that $PA = PB$.

Suppose that PM bisects $\angle APB$. Post. 8

Then $\triangle AMP$ is congruent to $\triangle BMP$. § 40

$\therefore AM = BM$, and the $\angle$s at M are equal. § 38

Hence the $\angle$s at M are rt. $\angle$s, § 13

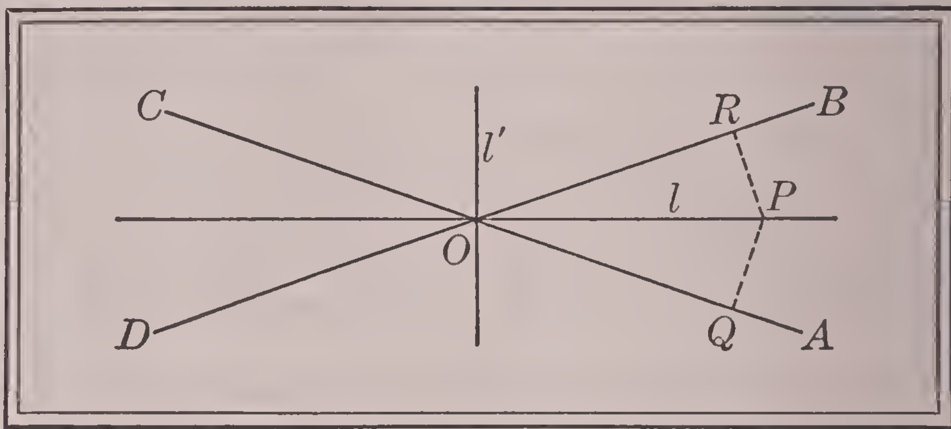
and PM is $\perp$ to AB. § 14

Since there is only one point of bisection (Post. 7), and since only one $\perp$ can be constructed at M (Post. 10), PM is the $\perp$ bisector of AB; that is, P lies on l.

This proves the second part of the theorem.

Proposition 20. Bisector of an Angle

183. Theorem. *The locus of points equidistant from two given intersecting lines is a pair of lines which bisect the angles formed by them.*



Given two lines AC and BD intersecting at O, and l and l', the bisectors of $\angle$s AOB and BOC respectively.

Prove that every point on l or l' is equidistant from AC and BD and that every point that is equidistant from AC and BD is on l or l'.

The plan is to prove the two statements of § 181.

Proof. Let P be any point on l,

and let PQ be $\perp$ to AC and PR be $\perp$ to BD. § 116

Since $\angle AOB$ is bisected by l, Given

then rt. $\triangle OQP$ is congruent to rt. $\triangle ORP$. § 68

$\therefore PQ = PR$, or P is equidistant from AC and BD. § 38

Let P be any point in the plane such that $\perp PQ = \perp PR$.

Draw PO. Post. 1

Then rt. $\triangle OQP$ is congruent to rt. $\triangle ORP$. § 71

$\therefore \angle AOP = \angle BOP$; § 38

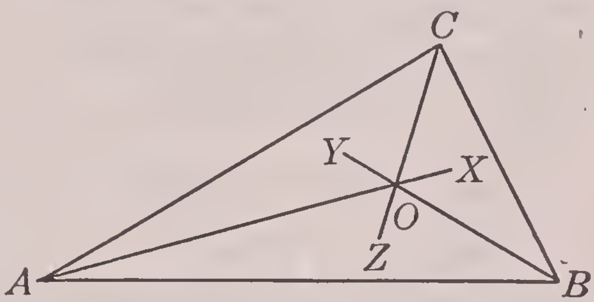
that is, P lies on the bisector of $\angle AOB$, or on l. Post. 8

Evidently both parts of the same proof hold for l' and $\angle BOC$.

184. Incenter of a Triangle. There are four propositions relating to loci which are often given as exercises; but because of their special interest they are here given more prominence, although their inclusion as fundamental propositions is optional. The first of these propositions relates to the bisectors of the angles of a triangle.

Theorem. *The bisectors of the angles of a triangle meet in a point equidistant from the three sides.*

In giving the proof the student should first show that the bisectors of ∡ A and B intersect as at O. Then prove that O is equidistant from AC and AB, also from BC and AB, and hence from AC and BC. Then by § 183 prove that O lies on the bisector CZ.



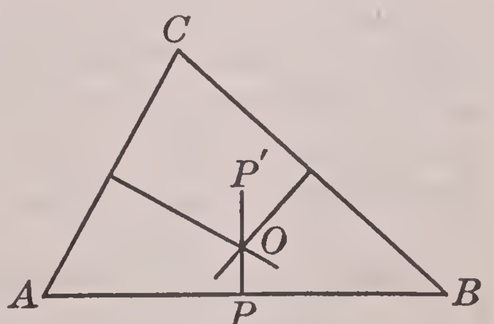
This point is called the *incenter* of the triangle.

The reason for using this term will appear in § 193, where the problem which gives this theorem its importance is considered.

185. Circumcenter of a Triangle. The second proposition in this group relates to the perpendicular bisectors of the sides of a triangle.

Theorem. *The perpendicular bisectors of the sides of a triangle meet in a point equidistant from the vertices.*

In giving the proof, show that the ⊥ bisectors of the two sides BC and CA intersect as at O. Then prove that O is equidistant from B and C, also from A and C, and hence from A and B. Then prove that O lies on the ⊥ bisector PP'.

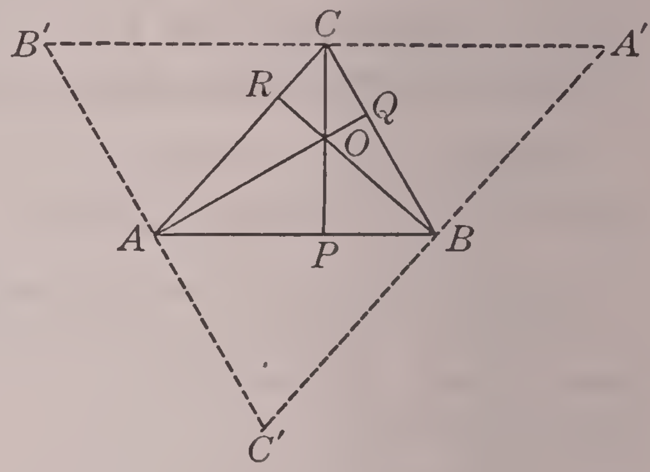


This point is called the *circumcenter* of the triangle.

The reason for using this term will appear in the problem of § 188.

186. Orthocenter of a Triangle. The third proposition relates to the altitudes (§ 74) of a triangle.

Theorem. *The altitudes of a triangle meet in a point.*



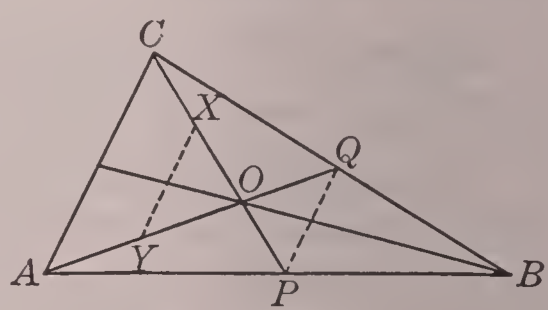
In giving the proof let the altitudes be AQ, BR, and CP. Through A, B, C draw $B'C'$, $C'A'$, and $A'B'$ ∥ to CB, AC, and BA respectively. Then prove that $C'A = BC = AB'$. What is the relation of AQ to $B'C'$? In the same way prove that BR and CP are the ⊥ bisectors of the other sides of the $\triangle A'B'C'$. Then apply § 185.

This point is called the *orthocenter* of the triangle.

The prefix "ortho-" means straight, and this center is found by drawing lines from the vertices straight (perpendicular) to the sides.

187. Centroid of a Triangle. The last proposition relates to the medians (§ 132) of a triangle.

Theorem. *The medians of a triangle meet in a point which is two thirds of the distance from each vertex to the midpoint of the opposite side.*



In giving the proof let any two medians, as AQ and CP, meet as at O. Then if Y is the midpoint of AO and X of CO, prove that YX and PQ are ∥ to AC and equal to $\frac{1}{2} AC$. Then prove that $AY = YO = OQ$, and that $CX = XO = OP$. Hence any median cuts off any other median two thirds of its length from the vertex.

This point is called the *centroid* of the triangle.

The syllable "-oid" means like, so that the word "centroid" means centerlike. This point is the center of gravity of the triangle.

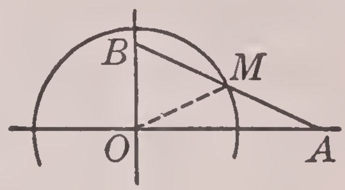
Exercises. Circular and Straight-Line Loci

1. *The locus of the vertex of a right triangle which has a given hypotenuse as base is the circle constructed upon this hypotenuse as diameter.*

2. *The locus of the vertex of a triangle which has a given base and a given angle at the vertex is the arc which forms with the base a segment of a circle in which the given angle may be inscribed.*

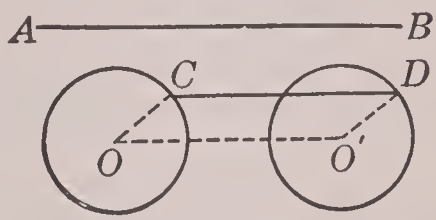
3. Two forts are placed 28 mi. apart on opposite sides of a harbor entrance. Each fort has a gun with a range of 16 mi. Draw a plan showing the area which can be exposed to the fire of both guns, using a scale of $\frac{1}{16}$ in. to a mile.

4. A straight rod AB moves so that its ends constantly touch two fixed rods which are perpendicular to each other. Find the locus of its midpoint M.



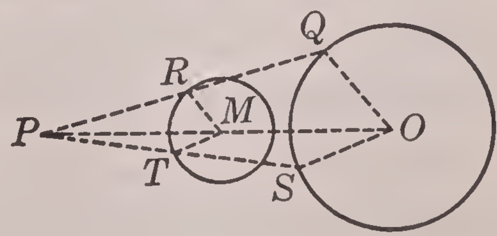
5. Show how to locate a light equidistant from two intersecting streets and 48 ft. from the point of intersection, as shown in the figure.

6. A line moves so that it remains parallel to a given line and so that one end lies on a given circle. Find the locus of the other end.



7. A circle of center O and radius r' rolls around a fixed circle of radius r, always touching the fixed circle. What is the locus of O? Prove it.

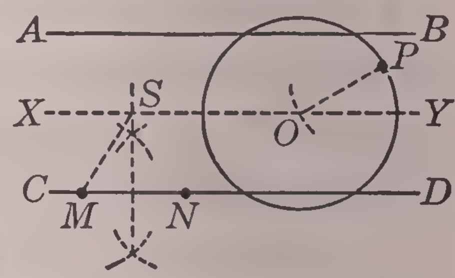
8. Find the locus of the midpoint of a line segment drawn from a given external point to a given circle.



9. A water main has a gate located at a point 9 ft. from a certain lamp-post which stands on the edge of a straight sidewalk. The gate is placed 4 ft. from the edge of the walk, toward the street. Draw a plan showing every possible position of the gate and state the principles involved.

10. During a war a man buried some valuables. He remembered that they were buried north of an east-and-west line which joined two trees 140 ft. apart, and that the point was 80 ft. from the eastern tree and 100 ft. from the western tree. Draw a plan to the scale of 20 ft. = 1 in., indicate the point where the valuables were buried, and state the geometric principles involved.

11. Find the locus of the center of a circle that passes through a given point between two parallels and cuts equal chords of a given length from them.

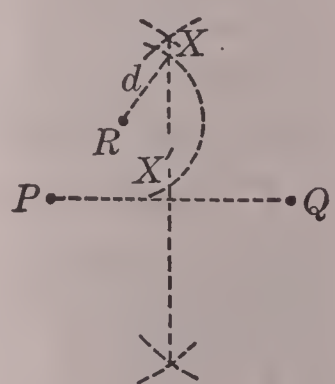


Let P be the given point, AB, CD the given parallels, and MN the given length.

Since the circle cuts equal chords from two parallels, what must be the relative distance of its center from each? Then what line is one locus for O, the center of the circle?

Construct the perpendicular bisector of MN, cutting XY at S. How does SM compare with the radius of the circle? What is then another locus for O? How can we then find O so as to satisfy the given conditions?

12. Find the locus of a point equidistant from two given points P, Q and at a given distance d from a third given point R.



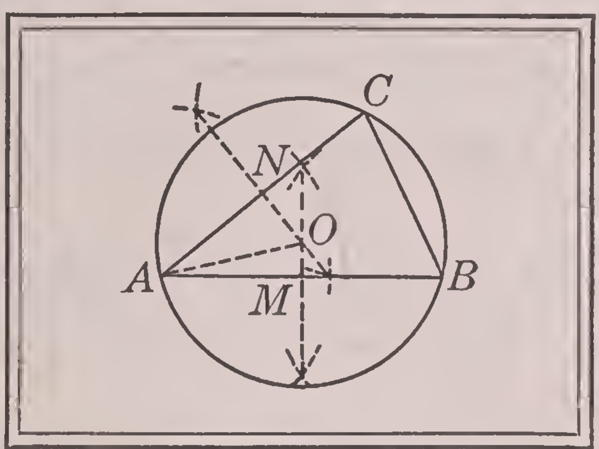
13. Find the locus of the center of a circle that has a given radius and passes through two given points.

14. What is the locus of the midpoints of a number of parallel chords of a circle? Prove it.

III. Fundamental Constructions

Proposition 21. Circle about a Triangle

188. Problem. *Circumscribe a circle about a given triangle.*



Given the △ *ABC*.

Required to circumscribe a ⊙ *about* △*ABC*.

The plan is to show that the intersection of the ⊥ bisectors of two sides of the △ is the center of the required ⊙.

Construction. Construct the ⊥s bisecting the sides *AB* and *AC* as at *M* and *N* respectively. §§ 102, 104

These ⊥s meet, as at *O*, or else are ∥. If *ON* is ∥ to *OM*, then *ON* is ⊥ to *AB* (§ 63), and hence *AB* is ∥ to *AC* (§ 57). But this is impossible, since *AB* and *AC* form ∠*A* (§ 6).

With *O* as center and *OA* as radius, construct a ⊙. Post. 4

Then ⊙ *ABC* is the required ⊙.

Proof. *O* is equidistant from *A* and *B*,

and *O* is equidistant from *A* and *C*. § 182

∴ *O* is equidistant from *A*, *B*, *C*.

Hence the ⊙ *O* passes through *A*, *B*, *C*. § 134, 6

189. Corollary. *Given a circle or an arc, find the center of the circle.*

Take three points on the ⊙ or arc and apply § 188.

190. Corollary. *Through any three given points not lying in a straight line one circle and only one can pass.*

The points may be considered as the vertices of a △, and hence a ⊙ can pass through them (§ 188).

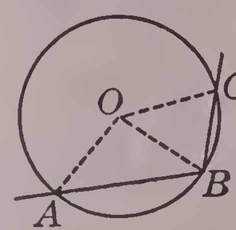
Since the points are not in a st. line (given), points equidistant from A, B, and C in the figure of § 188 must lie on MO and NO (§ 182). Since these lines can intersect in only one point, O, only one ⊙ is possible.

191. Corollary. *Two distinct circles can have at most two points in common.*

Because if they have three points in common, they will coincide (§ 190).

192. Corollary. *A straight line can intersect a circle in at most two points.*

This corollary, which is essentially § 134, 4, is introduced at this point because of its analogy to § 191.



Suppose that the st. line ABC can intersect the ⊙ O in A, B, C. Then $OA = OB = OC$ (§ 134, 1).

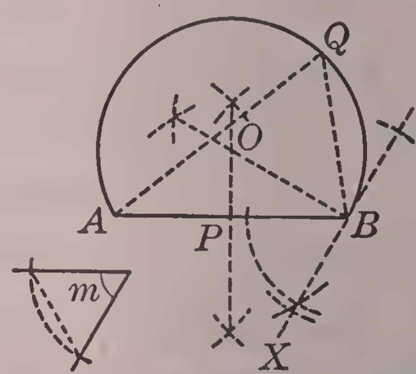
Hence $\angle OCB = \angle CBO$ and $\angle OBA = \angle BAO$ (§ 42), and thus each is less than a rt. $\angle$ (§ 65).

Hence if the supposition were true, we should have three equal obliques from O to ABC; but this is impossible (§ 119).

Exercises. Constructions

1. *Bisect a given arc.*

2. *Upon a given line segment as a chord, construct a segment of a circle in which a given angle may be inscribed.*



Proceed as follows:

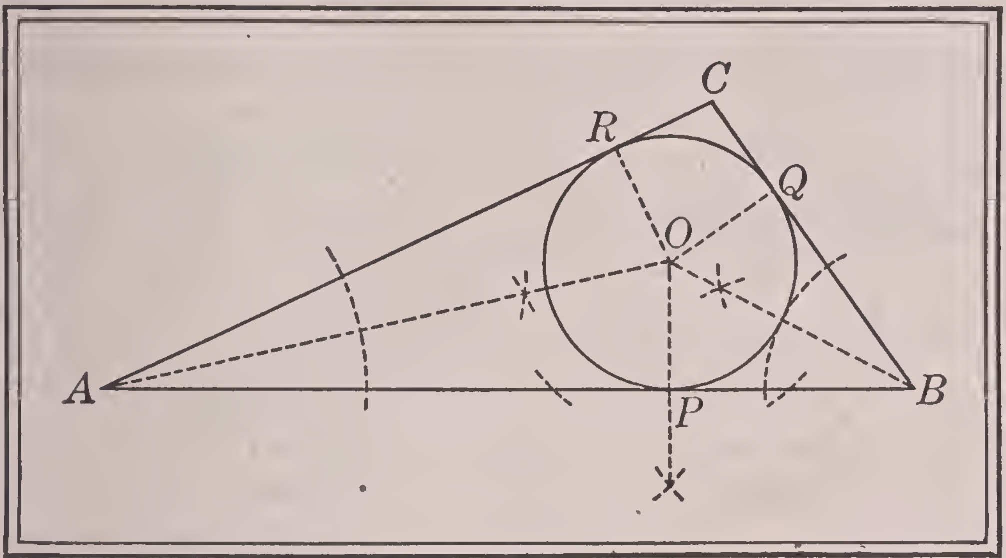
Given the line segment AB and the $\angle m$.

Required on AB as a chord to construct a segment of a ⊙ in which $\angle m$ may be inscribed.

Construction. Construct $\angle ABX = m$ (§ 106). The rest of the construction is readily inferred from the figure.

Proposition 22. Circle in a Triangle

193. Problem. *Inscribe a circle in a given triangle.*



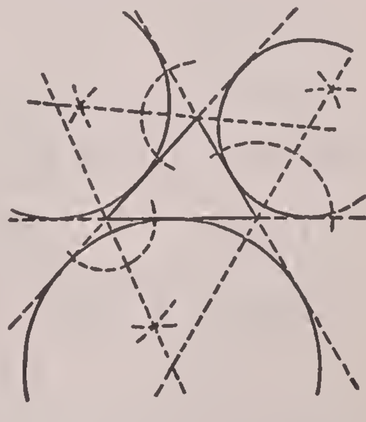
Given the $\triangle ABC$.

Required to inscribe a $\odot$ *in* $\triangle ABC$.

The plan is to show that the intersection of the bisectors of two $\measuredangle$s of the $\triangle$ is the center of the required $\odot$.

The construction and proof, which are suggested by the figure, are left for the student.

194. Centers of a Polygon. The center of a circle circumscribed, if possible, about a polygon is called the *circumcenter* of the polygon.

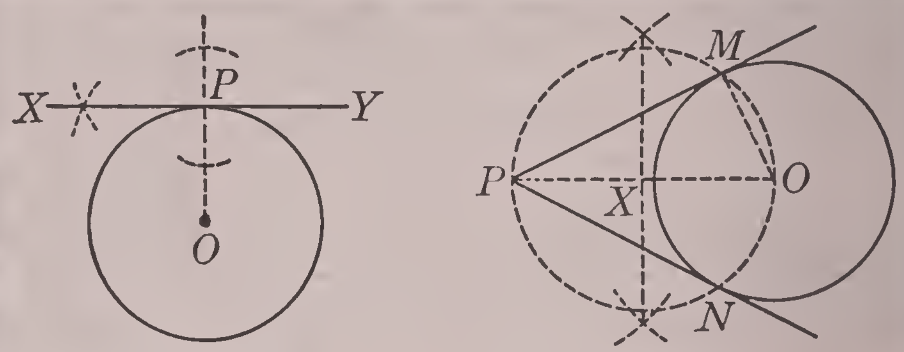


The center of a circle inscribed, if possible, in a polygon is called the *incenter* of the polygon.

The intersections of the bisectors of the exterior angles of a triangle are the centers of three circles, each of which is tangent to one side of the triangle and to the other two sides produced. These three circles are called *escribed circles*, and their centers are called the *excenters* of the triangle.

Proposition 23. Constructing a Tangent

195. Problem. *Through a given point construct a tangent to a given circle.*



Case 1 Case 2

Given the point *P* and the ⊙ *O*.

Required through P to construct a tangent to the ⊙.

The plan is to construct a line which shall make a rt. ∠ with a radius.

Construction. 1. If P is on the ⊙, draw OP. Post. 1

At P construct $XY \perp$ to OP. § 104

Then XY is the required tangent.

2. If P is outside the ⊙, draw OP. Post. 1

Bisect OP, as at X. § 102

With X as center and XP as radius, construct a ⊙ intersecting ⊙ O as at M and N, and draw PM. Posts. 4, 1

Then PM is the required tangent.

Proof. 1. Since XY is $\perp$ to OP, Const.

XY is tangent to the ⊙ at P. § 146

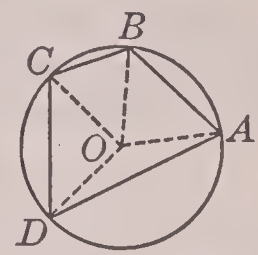
2. Drawing OM, $\angle PMO$ is a rt. ∠. § 173

$\therefore PM$ is tangent to the ⊙ at M. § 146

In like manner we may prove that PN is tangent to the ⊙.

Exercises. Constructions

1. If two opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.

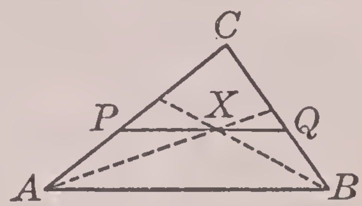


Apply § 188 to constructing a circle through A, B, C.

Prove that if the circle does not pass through D also, $\angle D$ is greater than or less than some other angle that is supplementary to $\angle B$, which is impossible.

2. In a $\triangle ABC$ construct $PQ \parallel$ to the base AB and cutting the sides in P and Q so that $PQ = AP + BQ$.

Assume for the moment that the problem is solved.



Then AP must equal some part of PQ, as PX, and BQ must equal QX.

But if $AP = PX$, what must $\angle PXA$ equal?

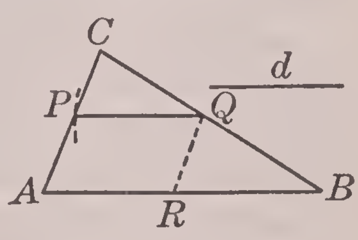
Since PQ is $\parallel$ to AB, what does $\angle PXA$ equal?

Then why must $\angle BAX = \angle XAP$?

Similarly, what about $\angle QBX$ and $\angle XBA$?

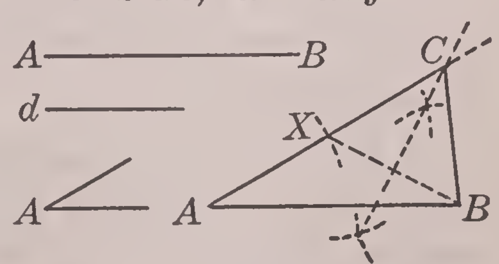
Now reverse the process. What should we do to $\measuredangle A$ and B in order to fix X? Then how shall PQ be constructed? Now give the proof.

3. Construct a line intersecting two sides of a triangle and parallel to the third side, such that the part intercepted between the two sides has a given length.



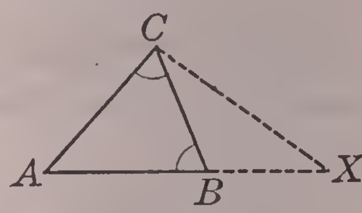
If $PQ = d$ and if QR is $\parallel$ to PA, what does AR equal? Then what two constructions must you make in order to locate Q?

4. Construct a triangle, given one side, an adjacent angle, and the difference between the other two sides.



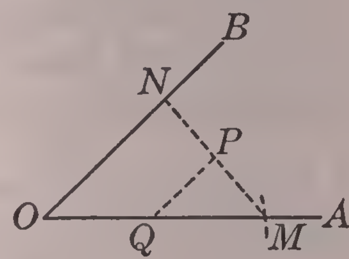
If AB, $\angle A$, and the difference d between AC and BC are given, what points in this figure are determined? Can XB be constructed? What kind of triangle is XBC? How can the vertex C of the triangle be located?

5. Given two angles of a triangle and the sum of two sides, construct the triangle.



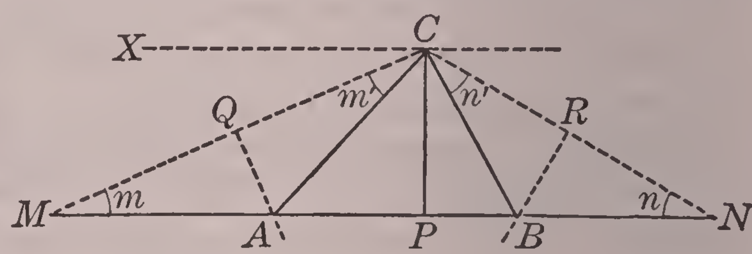
Can the third angle be found? Assume the problem solved. If $AX = AB + BC$, what kind of triangle is BXC? What does $\angle CBA$ equal? Is $\angle X$ known? How can C be fixed?

6. Through a given point P between the arms of an $\angle AOB$ construct a line terminated by the arms of the angle and bisected at P.



If $PM = PN$, and PQ is $\parallel$ to BO, is $OQ = QM$?

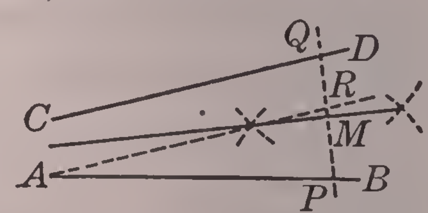
7. Given the perimeter of a triangle, one angle, and the altitude from the vertex of the given angle, construct the triangle.



Assume for the moment that the problem is solved, as shown in this figure, in which ABC is the required triangle, MN the given perimeter, $\angle ACB$ the given angle, CP the given altitude, $AM = AC$, and $BN = BC$. By a study of the figure we shall be led to the following solution:

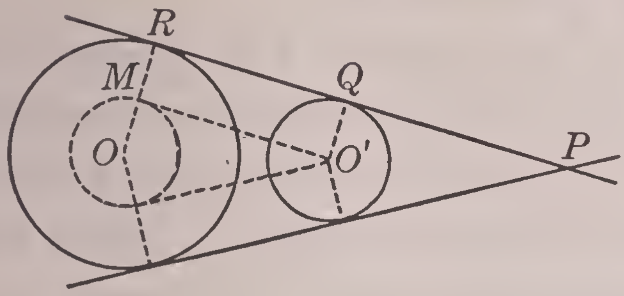
As in Ex. 2, page 148, on MN construct a segment of a circle in which $\angle MCN$, which is found by the analysis to be equal to $90° + \frac{1}{2}\angle ACB$, may be inscribed. Construct $XC \parallel$ to MN at the distance CP and cutting the arc of the circle at C. Then the vertices A and B are on the perpendicular bisectors of CM and CN.

8. Construct a line that would bisect the angle formed by two lines if those lines were produced to meet.



If AB and CD are the given lines, and if they could be produced to meet, then the bisector of the angle between them would be the perpendicular bisector of PQ, a line which makes equal angles with the given lines. How can we construct PQ so as to make $\angle P = \angle Q$?

9. Construct a common tangent to two given circles.



If the centers are O and O' and the radii r and r', the tangent QR in the left-hand figure seems to be ∥ to $O'M$, a tangent from O' to a circle whose radius is $r-r'$. What does this suggest?

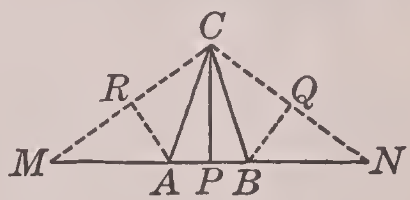
In general, there are four common tangents, but circles tangent externally and internally and intersecting circles should be considered.

Construct an isosceles triangle, given:

10. The base and the angle at the vertex.

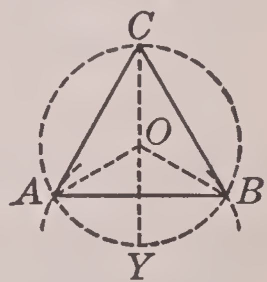
11. The base and the radius of the circumscribed circle.

12. The perimeter and the altitude.



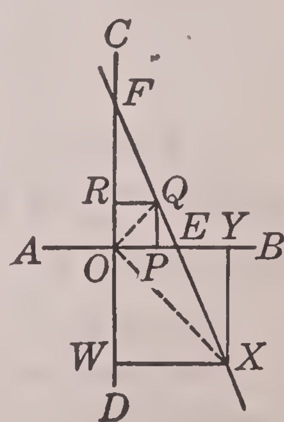
In this figure ABC is the required triangle and MN the given perimeter. Then the altitude CP passes through the midpoint of MN, and the △ MAC and NBC are isosceles.

13. Construct an equilateral triangle, given the radius of the circumscribed circle.



14. Construct a rectangle, given one side and the angle between the diagonals.

15. Given two perpendicular lines AB and CD intersecting in O, and a line intersecting these perpendiculars in E and F, construct a square, one of whose angles shall coincide with one of the right angles at O, and such that the vertex of the opposite angle of the square shall lie on EF.

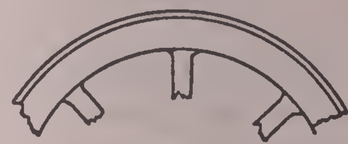


Notice the two solutions.

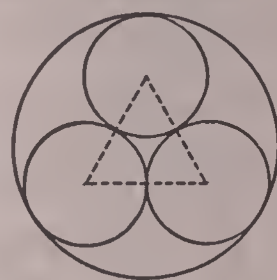
Exercises. Applications

1. Two pulleys of radii 1 ft. 6 in. and 2 ft. 3 in. respectively are connected by a belt which runs straight between the points of tangency. If the centers of the pulleys are 6 ft. apart, construct the figure, using the scale of 1 in. = 1 ft.

2. Given a portion of the tire of a wheel, show how to determine the center and to reproduce the tire of the wheel in a drawing.



3. Construct this design, making the figure twice this size.

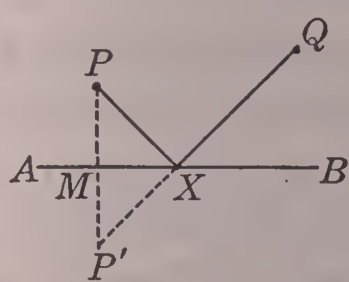


First construct the equilateral triangle. Then construct the small circles with half the side of the triangle as a radius. Then find the radius of the circumscribed circle.

4. A circular window in a church has a design similar to the accompanying figure. Construct the design, making the figure twice this size.

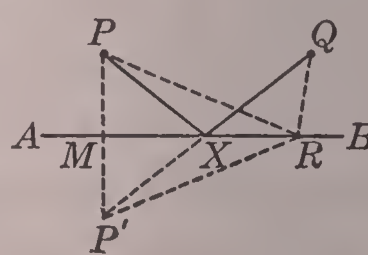
This design is made from the figure of Ex. 3.

5. From two given points P and Q construct lines which shall meet on a given line AB and make equal angles with AB.



Since $\angle BXQ$ must be equal to $\angle AXP$, then $\angle MXP' = \angle MXP$. If PP' is $\perp$ to AB, so that $MP' = MP$, and if $P'Q$ is drawn, what follows?

6. Find the shortest possible path from a point P to a line AB and thence to a point Q.



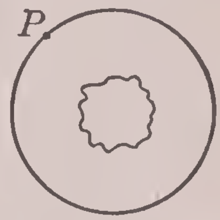
If $\angle PXA = \angle QXB$, is $PX + XQ < PR + RQ$?

This problem shows that a ray of light is reflected in the shortest possible path.

Exercises. Review

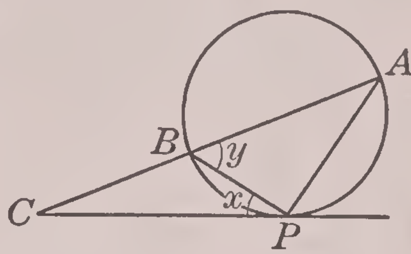
1. Make a list of the numbered propositions in Book II, stating under each the propositions in Books I and II upon which it depends either directly or indirectly.

2. Make another list of the numbered propositions, stating under each the propositions in Book II which depend upon it.



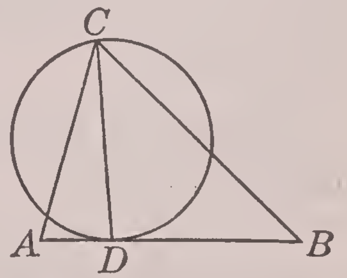
3. Show how to construct a tangent to this circle at the point P, the center of the circle not being accessible.

4. In this figure it is given that $x = 34°$ and $y = 56°$. Find the number of degrees in each of the other angles and determine whether or not AB is a diameter of the circle.



5. In a circle with center O the chord AB is drawn so that $\angle BAO = 31°$. How many degrees are there in $\angle AOB$?

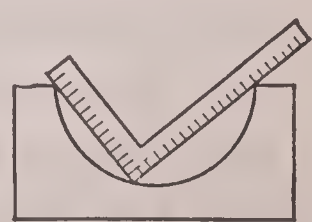
6. In this figure it is given that $\angle B = 44°$, $\angle A = 76°$, and $\angle BDC = 95°$. Find the number of degrees in each of the other angles, and determine whether or not CD is a diameter.



7. In a circle with center O the chord AB is drawn so that $\angle BAO = 35°$. On either arc AB a point P is taken and joined to A and B. What is the size of $\angle APB$?

8. Find the locus of the midpoint of a chord formed by a secant from a given external point to a given circle.

9. Show how a carpenter's square may be used to determine whether or not the curve in this casting is a perfect semicircle. State the geometric principle involved.

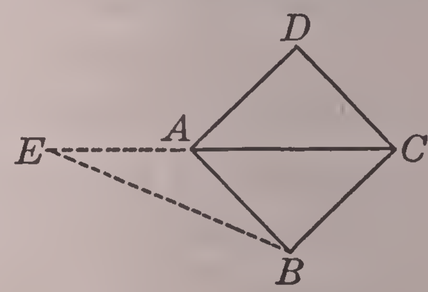


10. In a circle with center O, OM and ON are constructed $\perp$ to the chords AB and CD respectively, and it is known that $\angle NMO = \angle ONM$. Prove that $AB = CD$.

11. Two circles intersect at A and B, and a secant drawn through A cuts the circles at C and D. Prove that $\angle DBC$ does not change in size, however the secant is drawn.

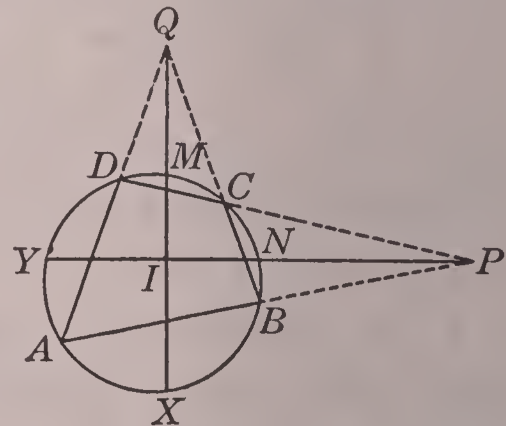
12. Let A and B be two fixed points on a given circle, and M and N the ends of any diameter. Find the locus of the point of intersection of the lines AM and BN.

13. Given the sum of the diagonal and one side of a square, construct the figure.



Assuming the problem solved, produce the diagonal CA, making $AE = AB$. Then CE is the given sum and $\angle ACB = \angle BAC = 45°$. Why? Find the value of $\angle E$. Reversing the reasoning, construct $\triangle E$ and ECB on EC.

14. If the opposite sides of an inscribed quadrilateral are produced to intersect, the bisectors of the angles at the points thus found intersect at right angles.



Referring only to arcs instead of chords, we have

$$AX - MD = XB - CM,$$

and $$YA - BN = DY - NC.$$

$$\therefore YX + NM = MY + XN.$$

Hence $\angle YIX = \angle XIN$.

How does this prove the proposition? Discuss the impossible case.

Construct a right triangle, given:

15. The median and the altitude upon the hypotenuse.

16. The hypotenuse and the altitude upon the hypotenuse.

17. Construct a triangle, given one side, an adjacent angle, and the sum of the other sides.

BOOK III

PROPORTION AND SIMILARITY

I. Fundamental Theorems

196. Proportion. An expression of equality between two ratios is called a *proportion.*

Preferably, a proportion is written in the more familiar fractional form, as follows:

$$\frac{a}{b}=\frac{c}{d}.$$

For convenience in printing, however, the form $a:b=c:d$, or $a/b=c/d$, is often used. All three forms have the same meaning, and each is read "a is to b as c is to d," or "the ratio of a to b is equal to the ratio of c to d."

197. Terms. In a proportion the four quantities compared are called the *terms.* The first and third terms are called the *antecedents*; the second and fourth terms, the *consequents.* The first and fourth terms are called the *extremes*; the second and third terms, the *means.*

Thus, in the proportion $a:b=c:d$, a and c are the antecedents, b and d the consequents, a and d the extremes, b and c the means.

Such names were of more value before algebra came into common use than they are at present.

In the case of $a:b=b:c$, the term b is called *the mean proportional* between a and c.

There is only one positive mean proportional between two numbers, and hence we speak of *the* mean proportional, as above.

198. Algebraic Relations. Since we are treating of the numerical measures of lines, we shall treat all ratios algebraically. The following laws should be understood :

1. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $ad=bc$.

For we may multiply each of the given equals by bd. Ax. 3

2. *If* $\frac{a}{b}=\frac{a}{d}$, *then* $d=b$; *and if* $\frac{a}{b}=\frac{c}{b}$, *then* $a=c$.

For $ad=ab$, by the first law, and hence $d=b$; or $ab=cb$, and hence $a=c$. Ax. 4

3. *If* $ad=bc$, *then* $\frac{a}{b}=\frac{c}{d}$.

For we may divide each of the given equals by bd. Ax. 4

4. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a}{c}=\frac{b}{d}$.

For we may multiply each of these equals by $\frac{b}{c}$. Ax. 3

5. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{b}{a}=\frac{d}{c}$.

For we may divide $1=1$, member for member, by these equals. Ax. 4

6. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a+b}{b}=\frac{c+d}{d}$.

For we may add 1 to each of these equals, giving $\frac{a+b}{b}=\frac{c+d}{d}$. Ax. 1

7. *If* $\frac{a}{b}=\frac{c}{d}$, *then* $\frac{a-b}{b}=\frac{c-d}{d}$.

For we may subtract 1 from each of these equals. Ax. 2

8. *If* $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}=\cdots=r$, *then* $\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=r=\frac{a}{b}$.

For $a=br$, $c=dr$, $e=fr$, $g=hr$, $\cdots$, and hence

$$a+c+e+g+\cdots=r(b+d+f+h+\cdots); \qquad \text{Ax. 1}$$

whence $$\frac{a+c+e+g+\cdots}{b+d+f+h+\cdots}=r=\frac{a}{b}. \qquad \text{Ax. 4}$$

Exercises. Algebraic Relations

Prove the following as in § 198 or by referring to § 198:

1. In any proportion the product of the extremes is equal to the product of the means.

2. If the two antecedents of a proportion are equal, the two consequents are equal.

3. If the product of two quantities is equal to the product of two others, either two may be made the extremes of a proportion in which the other two are made the means.

4. If four quantities are in proportion, they are in proportion by *alternation*; that is, the first term is to the third as the second term is to the fourth.

5. If four quantities are in proportion, they are in proportion by *inversion*; that is, the second term is to the first as the fourth term is to the third.

6. If four quantities are in proportion, they are in proportion by *composition*; that is, the sum of the first two terms is to the second term as the sum of the last two terms is to the fourth term.

7. If four quantities are in proportion, they are in proportion by *division*; that is, the difference between the first two terms is to the second term as the difference between the last two terms is to the fourth term.

8. In a series of equal ratios the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

9. If $a:b=c:d$, then $a^3:b^3=c^3:d^3$.

10. If $a:b=b:c$, then $a:c=a^2:b^2$.

11. If $a:b=b:c$, then $b=\sqrt{ac}$.

We shall consider only positive numbers unless the contrary is stated.

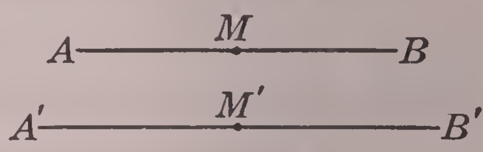
199. Nature of the Quantities in a Proportion. Although we may have ratios of lines, of areas, or of other geometric magnitudes, we treat all the terms of a proportion as positive numbers.

If b and d are lines or solids, for example, we cannot multiply each member of $\frac{a}{b}=\frac{c}{d}$ by bd, as in § 198, 1, because we cannot think of multiplying by a solid.

Hence *when we speak of the product of two geometric magnitudes, we mean the product of the numbers which represent the magnitudes when they are expressed in terms of a common unit.*

200. Proportional Line Segments. If we have two line segments AB and $A'B'$ and if M and M' are their respective midpoints, then $AM:MB=1$, and $A'M':M'B'=1$, and hence

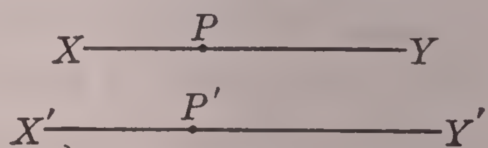
$$\frac{AM}{MB}=\frac{A'M'}{M'B'}.$$



This is evidently true whatever may be the lengths of AB and $A'B'$.

In like manner, if we have two line segments XY and $X'Y'$, we may divide XY at P and $X'Y'$ at P' in such a way that

$$\frac{XP}{PY}=\frac{X'P'}{P'Y'}.$$



When we divide two line segments in such a way as to have the parts form a proportion like this one, we say that the line segments are *divided proportionally.*

If P is on the line AB and is between A and B, it *divides AB internally*; if it is not between A and B, it *divides AB externally.*

In this figure, P divides AB internally in the ratio 1 : 2, and P' divides AB externally in the ratio 1 : 2. That is, $AP:PB=AP':P'B=1:2$.

Exercises. Proportion

Express the following ratios in their simplest forms:

1. $10:12$.
2. $8a:12a$.
3. $\dfrac{32x}{48x}$.
4. $a:a^2$.
5. $6m^2:9m^3$.
6. $\dfrac{a+b}{a^2-b^2}$.
7. $\frac{2}{3}:\frac{3}{4}$.
8. $\frac{3}{4}:\frac{2}{3}$.
9. $\dfrac{a^2-b^2}{a-b}$.
10. $a:a^2+ab$.
11. $a^2+ab:a$.
12. $\dfrac{a^2+2a+1}{a+1}$.

Given the proportion $a:b=c:d$, prove the following:

13. $a:d=bc:d^2$.
14. $1:b=c:ad$.
15. $ad:b=c:1$.
16. $ma:b=mc:d$.
17. $ma:nb=mc:nd$.
18. $a-1:b=bc-d:bd$.
19. $a+1:1=bc+d:d$.
20. $1:bc=1:ad$.

21. Divide a line segment 4.2 in. long into two parts which shall have the ratio $1:2$.

22. Divide a line segment 3.6 in. long into two parts such that the ratio of the shorter part to the whole segment shall be $4:5$.

23. What is the ratio of half a right angle to one eighth of a straight angle?

Given the proportion $a:b=b:c$, prove the following:

24. $c:b=b:a$.
25. $a:c=b^2:c^2$.
26. $(b+\sqrt{ac})(b-\sqrt{ac})=0$.
27. $ac-1:b-1=b+1:1$.

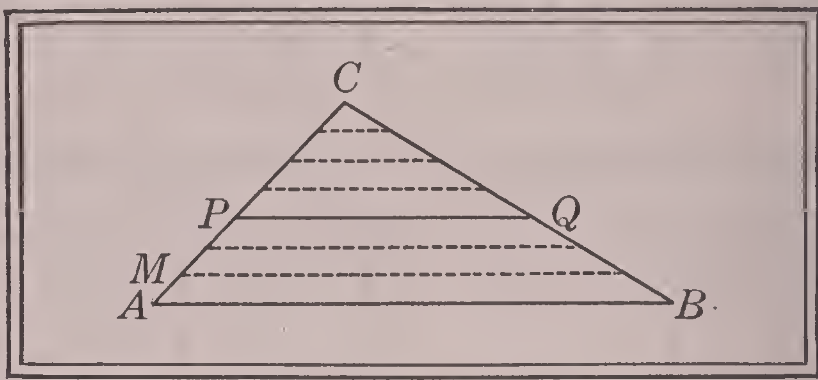
Find the value of x in each of the following:

28. $2:8=x:12$.
29. $3:5=x:9$.
30. $7:x=x:28$.
31. $1:1+x=x-1:3$.

Certain exercises on this page, such as Exs. 1–20 and 24–31, are introduced merely for the purpose of accustoming the student to the use of ratios and proportions. They are not needed in geometry, and may therefore be omitted if desired.

Proposition 1. Sides of a Triangle

201. Theorem. *If through two sides of a triangle a line is constructed parallel to the third side, it divides the two sides proportionally.*



Given the $\triangle ABC$ with $PQ \parallel$ to AB.

Prove that $$\frac{AP}{PC} = \frac{BQ}{QC}.$$

Proof. Assuming AP and PC commensurable, let some segment AM be contained 3 times in AP and 4 times in PC. § 163

Then $$\frac{AP}{PC} = \frac{3 \cdot AM}{4 \cdot AM} = \frac{3}{4}.$$ § 165

At the several points of division on AP and PC construct lines $\parallel$ to AB. § 107

These lines divide BC into 7 equal parts, of which BQ contains 3 parts and QC contains 4 parts. § 85

Then, $$\frac{BQ}{QC} = \frac{3}{4}.$$ § 165

Hence $$\frac{AP}{PC} = \frac{BQ}{QC}.$$ Ax. 5

The proof is evidently the same if any other numbers are used.

For the incommensurable case (§ 166) see § 516.

Since the student is now so far advanced as to be able to state for himself the plan of attack, it is no longer given as part of the printed proof. The student, however, should give it as part of his proof.

202. Corollary. *One side of a triangle is to either of its segments cut off by a line parallel to the base as the third side is to its corresponding segment.*

In the figure of § 201, $\frac{AP}{PC}=\frac{BQ}{QC}$. § 201

Adding 1 to each member of this proportion, we have

$$\frac{AP}{PC}+1=\frac{BQ}{QC}+1, \qquad \text{Ax. 1}$$

or $$\frac{AP+PC}{PC}=\frac{BQ+QC}{QC};$$

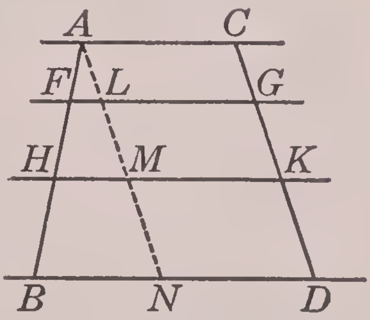
whence $$\frac{AC}{PC}=\frac{BC}{QC}. \qquad \text{Ax. 10}$$

We may also begin with $\frac{PC}{AP}=\frac{QC}{BQ}$, § 198, 5

add 1 as above, and end with $\frac{AC}{AP}=\frac{BC}{BQ}$.

203. Corollary. *Three or more parallel lines cut off proportional segments on any two transversals.*

Construct $AN \parallel$ to CD. § 107



Then $AL=CG$,

$LM=GK$,

and $MN=KD$. § 78

Now $\frac{AF}{FH}=\frac{AL}{LM}$. § 201

Hence $\frac{AF}{FH}=\frac{CG}{GK}$, Ax. 5

or $\frac{AF}{CG}=\frac{FH}{GK}$. § 198, 4

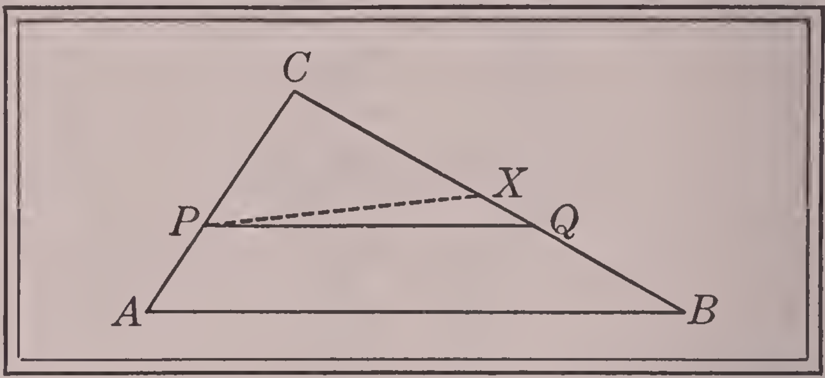
That is, the first two segments of AB are proportional to the first two segments of CD. Similarly, the other segments are proportional. This is indicated as follows:

$$\frac{AF}{CG}=\frac{FH}{GK}=\frac{HB}{KD}=\cdots.$$

The student should also consider the case in which AB and CD intersect between AC and BD.

Proposition 2. Converse of § 201

204. Theorem. *If a line divides two sides of a triangle proportionally from a vertex, it is parallel to the third side.*



Given the $\triangle ABC$ with PQ so drawn that $\frac{AP}{PC} = \frac{BQ}{QC}$.

Prove that PQ *is* $\parallel$ *to* AB.

Proof. Taking the indirect method (§ 56), suppose that

PQ is not $\parallel$ to AB.

From P construct some other line, as PX, $\parallel$ to AB. § 107

Then $$\frac{AC}{PC} = \frac{BC}{XC}.$$ § 202

But $$\frac{AP}{PC} = \frac{BQ}{QC}.$$ Given

Hence $$\frac{AP + PC}{PC} = \frac{BQ + QC}{QC},$$ § 198, 6

or $$\frac{AC}{PC} = \frac{BC}{QC}.$$ Ax. 10

Hence $$\frac{BC}{QC} = \frac{BC}{XC},$$ Ax. 5

and $$QC = XC.$$ § 198, 2

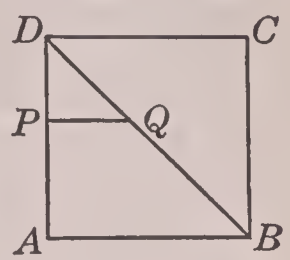
$\therefore$ PQ and PX must coincide. Post. 1

But PX is $\parallel$ to AB. Const.

$\therefore$ PQ, which coincides with PX, is $\parallel$ to AB. § 52

Exercises. Proportional Lines

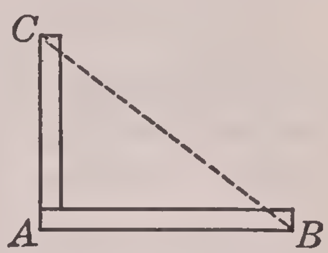
1. In the figure of § 203 given that $AF = 2$ in., $FH = 3$ in., and $CK = 6$ in. Find the length of CG.



2. If a side of this square is 10 in., the diagonal DB is 14.14 in. long. If $DP = 4$ in. and PQ is ∥ to AB, what is the length of DQ?

3. The sides of a triangle are 6 in., 8 in., and 10 in. respectively. A line parallel to the 8-inch side cuts the 6-inch side 2 in. from the vertex of the largest angle. Find the lengths of the segments of the 10-inch side.

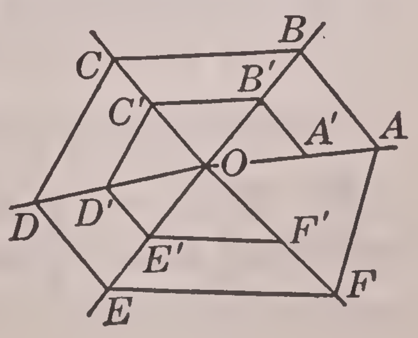
4. Two joists 6 in. wide are fitted together at right angles, as here shown. The distance from A to B is 16 ft., that from A to C is 12 ft., and that from B to C is 20 ft. In fitting another joist along the dotted line BC the carpenter has to saw off the ends of the first joists on the slant. Find the length of the slanting cut across the upright piece; across the horizontal piece.



5. From any point P the lines PA, PB, PC are drawn to the vertices of a $\triangle ABC$ and are bisected respectively by A', B', and C'. Prove that $\angle CBA = \angle C'B'A'$.

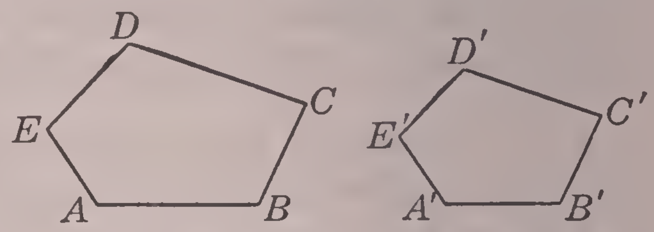
6. From any point P within the quadrilateral $ABCD$ lines are drawn to the vertices A, B, C, D and are bisected by A', B', C', D'. Prove that $\angle CBA = \angle C'B'A'$.

7. If a spider, in making its web, makes $A'B'$ ∥ to AB, $B'C'$ ∥ to BC, $C'D'$ ∥ to CD, $D'E'$ ∥ to DE, and $E'F'$ ∥ to EF, and then runs a line from F' ∥ to FA, will it strike the point A'? Prove it.



First show that $OA' : A'A = OF' : F'F$, and then use § 204.

205. Similar Polygons. Polygons that have their corresponding angles equal and their corresponding sides proportional are called *similar polygons.*



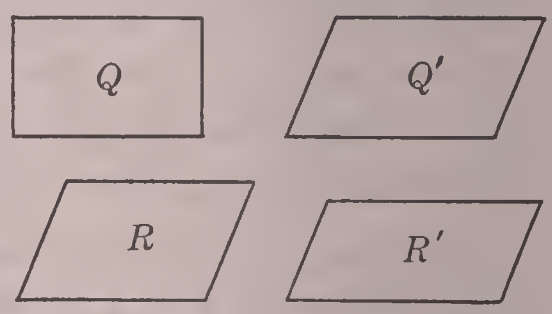
Thus, the polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

$$\angle A = \angle A', \angle B = \angle B', \angle C = \angle C', \angle D = \angle D', \angle E = \angle E',$$

and if
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}.$$

Instead of saying that two polygons are similar, it is frequently said that they have the same shape, or that they are the same figure drawn to different scales. Familiar illustrations of similar polygons are given by maps or by photographs of buildings.

In the figures here shown, Q and Q' are not similar, for although the corresponding sides are proportional, the corresponding angles are not equal; neither are the figures R and R' similar, even though the corresponding angles are equal, for the corresponding sides are not proportional.



As will be shown in §§ 208–214, in the case of triangles either condition implies the other, but this is not true of other figures.

206. Corresponding Line Segments. In similar polygons those line segments that are similarly situated with respect to the equal angles are called *corresponding line segments*, or simply *corresponding lines.*

Corresponding lines are occasionally called *homologous lines.*

207. Ratio of Similitude. The ratio of any two corresponding line segments in similar polygons is called the *ratio of similitude* of the polygons.

Exercises. Review

1. If a pendulum swinging from the point O cuts two parallel lines at the varying points P and Q respectively, the ratio $OP:OQ$ remains the same whatever may be the position of the pendulum.

2. Through a fixed point P a line is drawn cutting a fixed line at X. The line segment PX is then divided at Y so that the ratio $PY:YX$ always remains the same. Find the locus of the point Y as X moves along the fixed line.

3. Given that $3:x=x:27$, find the value of x.

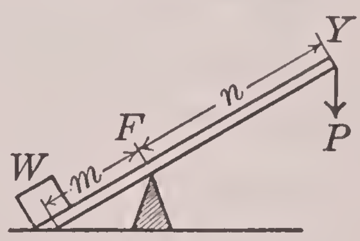
4. Given that $x:8=32:x$, find the value of x.

5. From the definition of a square, prove that two squares are always similar.

6. From what you have proved concerning equilateral triangles, can you state that two equilateral triangles are always similar? Give the reasons.

7. Divide a line segment 5.4 in. long into two parts which shall have the ratio of 4 to 5; of 8 to 10; of 2 to $2\frac{1}{2}$.

8. The law of levers states that $mW=nP$, where W, as in this figure, is the weight; m, the distance from the weight to the fulcrum F; P, the power applied; and n, the distance from the power to the fulcrum. State this equation in the form of a proportion.

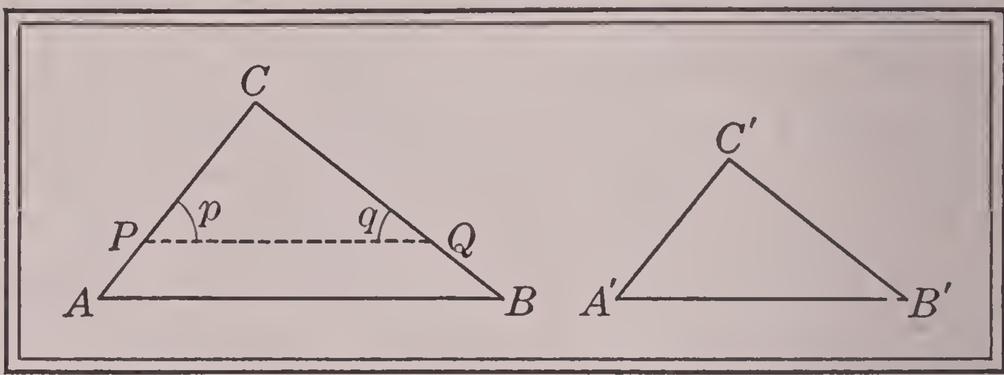


9. From the point P on the side CA of the $\triangle ABC$ parallels are drawn to the other sides, meeting AB in Q and BC in R. Prove that $AQ:QB=BR:RC$.

10. In the $\triangle ABC$ the points P and Q are taken on the sides CA and BC so that $AP:PC=BQ:QC$. Then a line AR is drawn $\parallel$ to PB, meeting CB produced in R. Prove that $CQ:CB=CB:CR$.

Proposition 3. Mutually Equiangular Triangles

208. Theorem. *Two mutually equiangular triangles are similar.*



Given the $\triangle ABC$ and $A'B'C'$ with $\angle A, B, C$ equal to $\angle A', B', C'$ respectively.

Prove that $\triangle ABC$ *is similar to* $\triangle A'B'C'$.

Proof. Since the $\triangle$ are given as mutually equiangular, we have only to prove that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}. \qquad \S\ 205$$

Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle C'$ coincides with its equal, $\angle C$, and $A'B'$ takes the position PQ. Post. 5

Then, in the figure, $p = \angle A$, Given

and hence PQ is $\parallel$ to AB. § 59

Then $\frac{AC}{PC} = \frac{BC}{QC}$ (§ 202), or $\frac{AC}{A'C'} = \frac{BC}{B'C'}$. Ax. 5

Similarly, by placing $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle B'$ coincides with its equal, $\angle B$, we can prove that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'};$$

whence $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$. Ax. 5

$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$. § 205

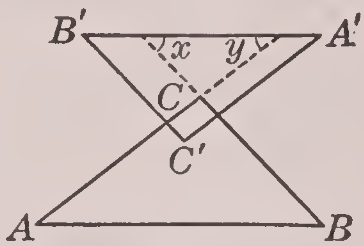
209. Corollary. *If two angles of one triangle are equal respectively to two angles of another, the triangles are similar.*

Since in each $\triangle$ the sum of the $\angle$s is 2 rt. $\angle$s (§ 65), and since two $\angle$s of one $\triangle$ are given equal to two $\angle$s of the other, the third $\angle$s are equal (Ax. 2); that is, the $\triangle$s are mutually equiangular. Hence the $\triangle$s are similar (§ 208).

210. Corollary. *If an acute angle of one right triangle is equal to an acute angle of another, the triangles are similar.*

Since the rt. $\angle$s are also equal (Post. 6), the $\triangle$s have two $\angle$s of one equal respectively to two $\angle$s of the other. Hence the $\triangle$s are similar (§ 209).

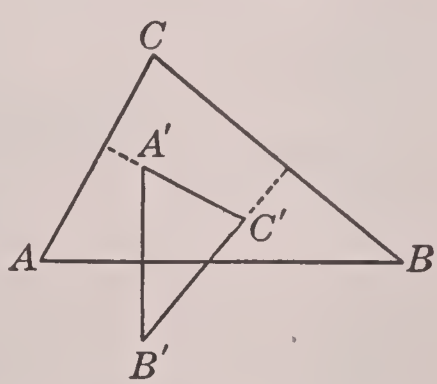
211. Corollary. *If two triangles have their sides respectively parallel to one another, the triangles are similar.*



In this figure how can it be proved that $\angle B = \angle B'$ and that $\angle A = \angle A'$? Is this sufficient to prove the corollary?

Although §§ 211 and 212 are interesting corollaries of § 208, they are not needed in subsequent propositions. Hence they may be treated as exercises or omitted if desired. These corollaries are required in some courses of study and are often given in examinations.

212. Corollary. *If two triangles have their sides respectively perpendicular to one another, the triangles are similar.*

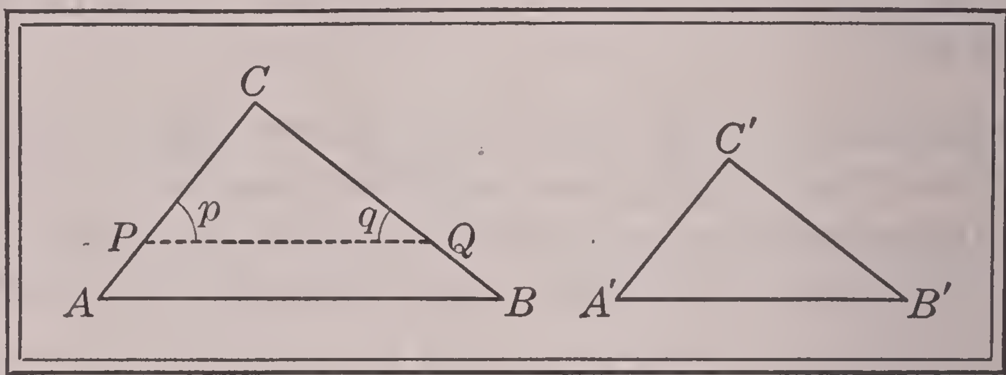


In this figure, what $\angle$ is the complement of $\angle B$? of $\angle B'$? Are these two complements equal? Does this prove that $\angle B' = \angle B$? Since $A'B'$ is $\perp$ to AB and $A'C'$ is $\perp$ to AC, what can you say about the other two $\angle$s of the quadrilateral formed by these lines? What other $\angle$ is a supplement of one of these $\angle$s? Can it then be proved that $\angle A' = \angle A$? Is this sufficient to prove the corollary?

When, as in this case, a figure becomes somewhat complicated, it is well to recall this fact: *The corresponding sides of similar triangles are opposite the corresponding and mutually equal angles*, and conversely.

Proposition 4. Angle and Proportional Sides

213. Theorem. *If two triangles have an angle of one equal to an angle of the other and the including sides proportional, the triangles are similar.*



Given the $\triangle ABC$ and $A'B'C'$ with $\angle C = \angle C'$ and with $\frac{CA}{C'A'} = \frac{CB}{C'B'}$.

Prove that $\triangle ABC$ *is similar to* $\triangle A'B'C'$.

Proof. Place $\triangle A'B'C'$ upon $\triangle ABC$ so that $\angle C'$ coincides with its equal, $\angle C$, $A'B'$ taking the position PQ. Post. 5

Now $$\frac{CA}{C'A'} = \frac{CB}{C'B'};$$ Given

that is, $$\frac{CA}{CP} = \frac{CB}{CQ}.$$ Ax. 5

Hence $$\frac{CA - CP}{CP} = \frac{CB - CQ}{CQ},$$ § 198, 7

or $$\frac{PA}{CP} = \frac{QB}{CQ}.$$ Ax. 5

$\therefore PQ$ is $\parallel$ to AB. § 204

Then $\angle A = p$, and $\angle B = q$. § 62

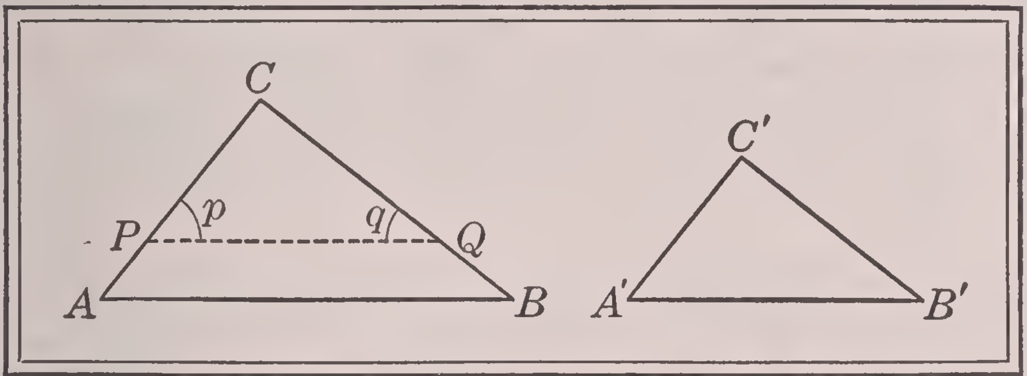
Also, $\angle C = \angle C'$. Given

Hence $\triangle ABC$ is similar to $\triangle PQC$; § 208

that is, $\triangle ABC$ is similar to $\triangle A'B'C'$. Ax. 5

Proposition 5. Proportional Sides

214. Theorem. *If two triangles have their sides respectively proportional, they are similar.*



Given the $\triangle s\ ABC$ and $A'B'C'$ with $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CA}{C'A'}$.

Prove that $\triangle ABC$ *is similar to* $\triangle A'B'C'$.

Proof. On CA take $CP = C'A'$ and on CB take $CQ = C'B'$.

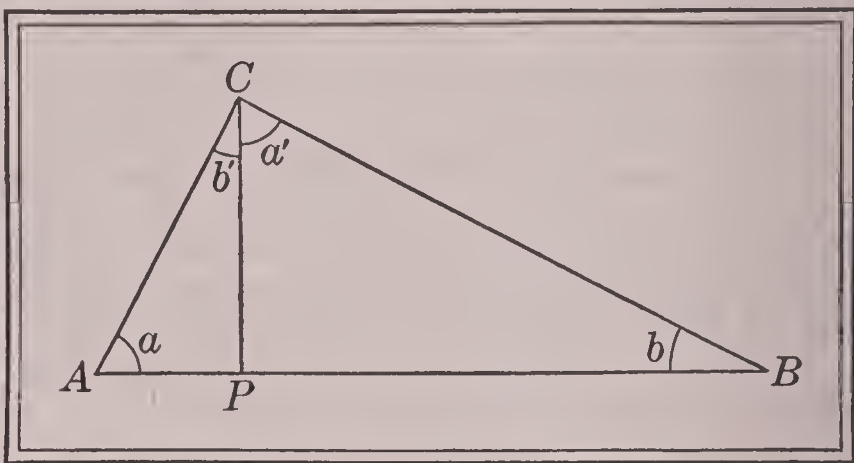
Draw PQ. Post. 1

When it is desired to give a considerable number of steps on a single page, the fraction form of the proportion may be replaced by the form used below.

Now	$CA : C'A' = BC : B'C'$,	Given
and, since	$CP = C'A'$, and $CQ = C'B'$,	Const.
then	$CA : CP = CB : CQ$.	Ax. 5
	$\therefore \triangle s\ ABC$ and PQC are similar.	§ 213
Then	$CA : CP = AB : PQ$;	§ 205
that is,	$CA : C'A' = AB : PQ$.	Ax. 5
But	$CA : C'A' = AB : A'B'$.	Given
	$\therefore AB : PQ = AB : A'B'$,	Ax. 5
and	$PQ = A'B'$.	§ 198, 2
Hence	$\triangle s\ PQC$ and $A'B'C'$ are congruent.	§ 47
But	$\triangle ABC$ is similar to $\triangle PQC$.	Proved
	$\therefore \triangle ABC$ is similar to $\triangle A'B'C'$.	Ax. 5

Proposition 6. Right Triangle

215. Theorem. *The perpendicular from the vertex of the right angle of a right triangle to the hypotenuse divides the triangle into two triangles which are similar to the given triangle and to each other.*



Given the rt. $\triangle ABC$ with $CP \perp$ to the hypotenuse AB.

Prove that $\triangle$s ABC, ACP, CBP *are similar.*

Proof. Lettering the figure as shown, since a is common to rt. $\triangle$s ACP and ABC, these $\triangle$s are similar. § 210

Likewise, $\triangle CBP$ is similar to $\triangle ABC$. § 210

Hence $\triangle$s ACP and CBP are each mutually equiangular with the given $\triangle ABC$. § 205

Then the three $\triangle$s are mutually equiangular, Ax. 5

and hence the $\triangle$s are similar. § 208

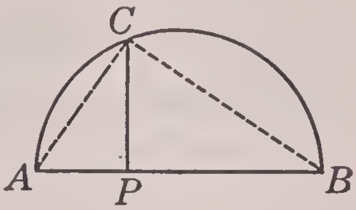
216. Corollary. *The perpendicular from the vertex of the right angle to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse.*

Since $\triangle$s ACP and CBP are similar, § 215

we have $AP : CP = CP : PB$, § 205

and hence the $\perp$ CP is the mean proportional between the segments AP and PB. § 197

217. Corollary. *The perpendicular from any point on a circle to a diameter of the circle is the mean proportional between the segments of the diameter.*



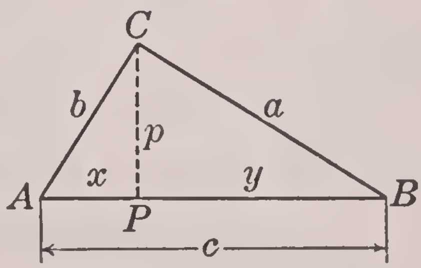
Since $\angle ACB$ is a rt. $\angle$ (§ 173), $\triangle ABC$ is a rt. $\triangle$, and hence § 216 applies.

218. Corollary. *The square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

This means that the square of the numerical measure of the hypotenuse is equal to the sum of the squares of the numerical measures of the two sides.

This is the most celebrated single proposition in geometry, and on account of its great importance we shall prove it again, by another method, in § 252. This theorem was known for special cases as early as the third millennium B.C., but it is thought to have been first proved by Pythagoras, a famous Greek mathematician, about 525 B.C.

In the rt. $\triangle ABC$, in which $\angle C$ is the rt. $\angle$, let the $\perp$ p from C to AB form the segments x and y as here shown. Then a simple proof, based on § 215, is as follows:



Since the three $\triangle$s are similar, § 215

we have $\frac{c}{a} = \frac{a}{y}$, and $\frac{c}{b} = \frac{b}{x}$. § 205

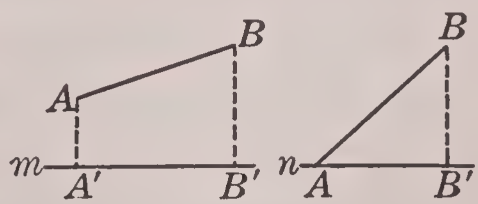
Hence $a^2 = cy$,

and $b^2 = cx$. § 198, 1

Then $a^2 + b^2 = c(x + y)$. Ax. 1

$\therefore a^2 + b^2 = c \cdot c = c^2$. Ax. 5

219. Projection. If from the ends of a given line segment perpendiculars are constructed to a given line, the segment thus formed on the given line is called the *projection* of the given segment upon the line.



Thus $A'B'$ and AB' in these figures are the projections of AB upon the lines m and n respectively.

Exercises. Similar Triangles

1. *If a perpendicular is drawn from the vertex of the right angle of a right triangle to the hypotenuse, each of the other sides is the mean proportional between the hypotenuse and the projection of that side upon it.*

2. *The squares of the two sides of a right triangle are proportional to the projections of the sides upon the hypotenuse.*

In the figure of § 215, $\overline{AC}^2 = AB \cdot AP$, and $\overline{BC}^2 = AB \cdot BP$. Why?

Hence $$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \cdot AP}{AB \cdot BP} = \frac{AP}{BP}.$$

3. *The square of the hypotenuse and the square of either side of a right triangle are proportional to the hypotenuse and the projection of that side upon it.*

In the figure of § 215, $\overline{AB}^2 = AB \cdot AB$, and $\overline{AC}^2 = AB \cdot AP$.

Then $$\frac{\overline{AB}^2}{\overline{AC}^2} = \frac{AB \cdot AB}{AB \cdot AP} = \frac{AB}{AP}.$$

4. *If a perpendicular is drawn from any point on a circle to a diameter, the chord from that point to either end of the diameter is the mean proportional between the diameter and its segment adjacent to the chord.*

5. *Perpendiculars drawn from any corresponding vertices of two similar triangles to the opposite sides have the same ratio as any two corresponding sides.*

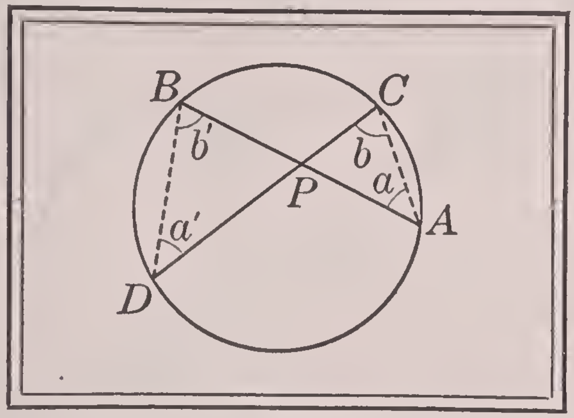
6. Find the length of the hypotenuse of a right triangle of which the two sides including the right angle are 37 in. and $49\frac{1}{3}$ in. respectively.

7. Find the other side of a right triangle of which the hypotenuse is 17 in. and one side is 10.2 in.

8. From the three similar triangles in the figure of § 215 it is possible to write a large number of proportions. Write twelve of them.

Proposition 7. Intersecting Chords

220. Theorem. *If two chords of a circle intersect, the product of the segments of either one is equal to the product of the segments of the other.*



Given a ⊙ with the chords *AB* and *CD*, intersecting at *P*.

Prove that $PA \cdot PB = PC \cdot PD$.

Proof. Draw AC and BD. Post. 1

Then in the figure, as lettered above,

$$a = a',$$ § 172

because each of these ∡ is measured by ½ arc CB;

and $$b = b',$$ § 172

because each of these ∡ is measured by ½ arc DA.

∴ △ CPA and BPD are similar, § 209

and hence $$\frac{PA}{PD} = \frac{PC}{PB}.$$ § 205

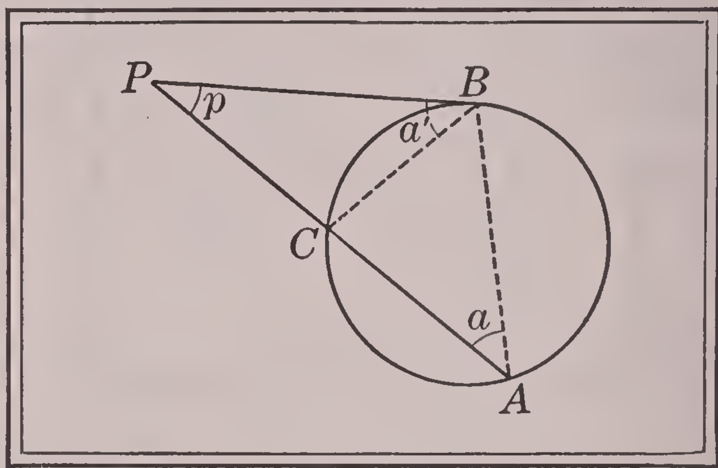
$$\therefore PA \cdot PB = PC \cdot PD.$$ § 198, 1

221. Secant to a Circle. When we speak of a *secant from an external point to a circle* it is understood that we mean the segment of the secant which lies between the given external point and the second, or more remote, point of intersection of the secant and the circle.

Thus in the figure of § 222 we may speak of PA as such a secant.

Proposition 8. Secant and Tangent

222. Theorem. *If from a point outside a circle a secant and a tangent are drawn, the tangent is the mean proportional between the secant and its external segment.*



Given a secant PA and the tangent PB drawn to the $\odot ABC$ from the external point P.

Prove that $$\frac{PA}{PB}=\frac{PB}{PC}.$$

Proof. Draw AB and BC. Post. 1

Now a is measured by $\frac{1}{2}$ arc BC, § 172

and a' is measured by $\frac{1}{2}$ arc BC. § 177

$\therefore a=a'$. Ax. 5

Then $\triangle PAB$ is similar to $\triangle PBC$. § 209

Hence $\frac{PA}{PB}=\frac{PB}{PC}$. § 205

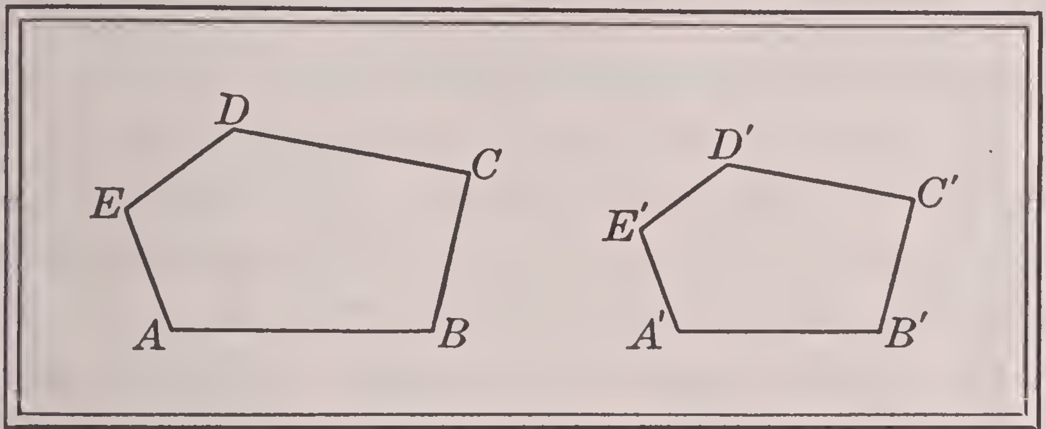
223. Corollary. *If from a point outside a circle two or more secants are drawn, the product of any secant and its external segment is equal to the product of any other secant and its external segment.*

Since $PA : PB = PB : PC$ (§ 222), then $PA \cdot PC = \overline{PB}^2$ (§ 198, 1).

Moreover, since PB always remains the same (§ 149), $PA \cdot PC$ always remains the same.

Proposition 9. Ratio of Perimeters

224. Theorem. *The perimeters of two similar polygons have the same ratio as any two corresponding sides.*



Given the two similar polygons *ABCDE* and *A'B'C'D'E'* with perimeters *p* and *p'* respectively.

Prove that $$\frac{p}{p'} = \frac{AB}{A'B'}.$$

Proof. $$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}. \qquad \S\ 205$$

Then $$\frac{AB + BC + CD + DE + EA}{A'B' + B'C' + C'D' + D'E' + E'A'} = \frac{AB}{A'B'}. \qquad \S\ 198, 8$$

Hence $$\frac{p}{p'} = \frac{AB}{A'B'}. \qquad \text{Ax. } 5$$

The proof is evidently the same whatever the number of sides of the polygons.

Exercises. Review

1. *If two chords intersect within a circle, their segments are reciprocally proportional.*

This means, for example, that, in the figure of § 220, $PA : PD$ is equal to the *reciprocal* of $PB : PC$; that is, it is equal to $PC : PB$.

2. Discuss § 220 when P is on the circle; when P is outside the circle.

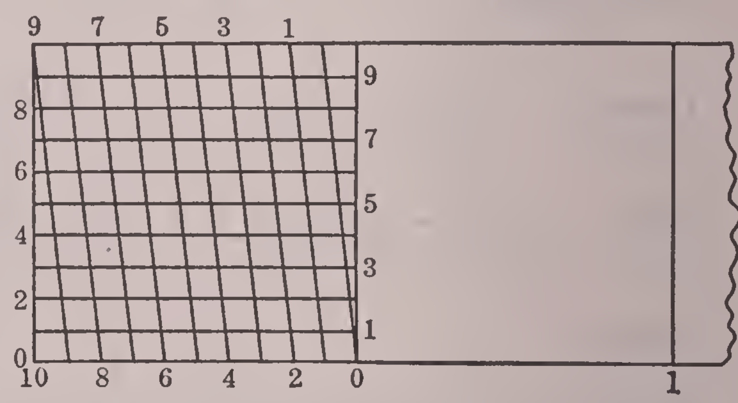
3. If two parallels are cut by three transversals which meet in a point, the corresponding segments of the parallels are proportional.

4. The base and altitude of a triangle are 30 in. and 14 in. respectively. If the corresponding base of a similar triangle is $7\frac{1}{2}$ in., find the corresponding altitude.

5. The point P is any point on the arm OX of the $\angle XOY$, and from P a $\perp$ PQ is constructed to OY. Prove that for any position of P on OX the ratio $OP : PQ$ remains the same, and the ratio $PQ : OQ$ also remains the same.

6. In drawing a map of a triangular field with sides of 75 rd., 60 rd., and 50 rd. respectively, the longest side is made 2 in. long. How long are the other two sides made?

7. This figure represents part of a diagonal scale sometimes used by draftsmen. Between vertical lines 1 and 0 the distance is 1 in.; between vertical line 1 and the intersection of diagonal line 2 with horizontal line 0 it is 1.2 in.; between vertical line 1 and the intersection of diagonal line 0 with horizontal line 8 it is 1.08 in.; and so on. Show how to measure 1.5 in.; 1.25 in.; 1.03 in.; 1.67 in.; 1.79 in. Upon what proposition does this depend?



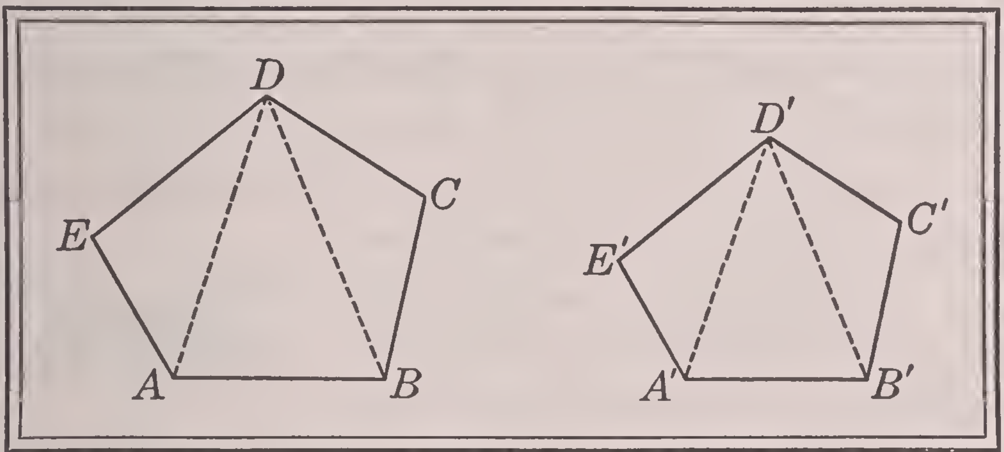
8. In the similar $\triangle ABC$ and $A'B'C'$, $AB = 3$ in., $BC = 3\frac{1}{2}$ in., $CA = 4$ in., and $A'B' = 4\frac{1}{2}$ in. Find $B'C'$ and $C'A'$.

9. The perimeter of an equilateral triangle is 72 in. Find the side of an equilateral triangle of half the altitude.

10. The hypotenuse of a right triangle is 98.5 mm. and one side is 78.8 mm. Find the other side.

Proposition 10. Separating Similar Polygons

225. Theorem. *If two polygons are similar, they can be separated into the same number of triangles, similar each to each and similarly placed.*



Given two similar polygons *ABCDE* and *A'B'C'D'E'*.

Prove that ABCDE and A'B'C'D'E' can be separated into the same number of △*, similar each to each and similarly placed.*

Proof. Draw $DA, D'A'$ and $DB, D'B'$, Post. 1

thus separating each polygon into three △ similarly placed.

Since $\angle E = \angle E'$,

and $DE : D'E' = EA : E'A'$, § 205

we see that △ DEA and $D'E'A'$ are similar. § 213

In like manner, △ DBC and $D'B'C'$ are similar.

Furthermore, $\angle A = \angle A'$,

and $\angle DAE = \angle D'A'E'$. § 205

Subtracting, $\angle BAD = \angle B'A'D'$. Ax. 2

Now $DA : D'A' = EA : E'A'$,

and $AB : A'B' = EA : E'A'$. § 205

Then $DA : D'A' = AB : A'B'$. Ax. 5

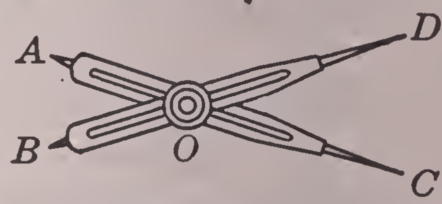
$\therefore$ △ DAB and $D'A'B'$ are similar. § 213

Exercises. Review

1. The sides of a polygon are 3 in., 3 in., 4 in., 4 in., and 6 in. respectively. Find the perimeter of a similar polygon whose longest side is 9 in.

2. In drawing a map to the scale of 1:100,000, what lengths, to the nearest 0.01 in., should be taken for the sides of a rectangular county 30 mi. long and 20 mi. wide?

3. By adjusting the screw at O, the lengths OA and OC of these *proportional compasses*, and the corresponding lengths OB and OD, may be varied proportionally. The distance AB is what part of CD when $OA = 4\frac{1}{2}$ in. and $OC = 7\frac{1}{2}$ in.? when $OA = 5.25$ in. and $OC = 6.75$ in.?



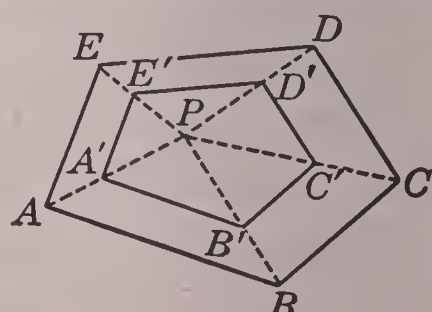
4. A baseball diamond is a square 90 ft. on a side. What is the distance, to the nearest 0.1 ft., from first base to third base?

5. Find a formula for the height of an equilateral triangle of perimeter p.

6. Find the lengths of the sides of an isosceles triangle of perimeter 39 in. if the ratio of one of the equal sides to the base is $\frac{5}{8}$.

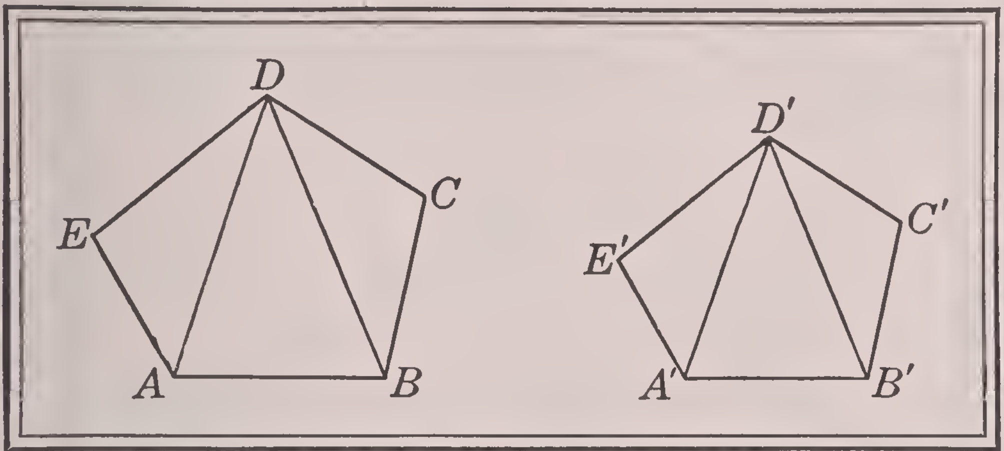
7. Find the length and the height of a rectangle of perimeter 64 in. if the ratio of these dimensions is $\frac{5}{3}$.

8. Within the polygon $ABCDE$ here shown any point P is joined to the vertices. Beginning at a point A' on AP lines are drawn so that $A'B'$ is ∥ to AB, $B'C'$ is ∥ to BC, $C'D'$ is ∥ to CD, and $D'E'$ is ∥ to DE. Prove that a line $E'A'$ is ∥ to EA and that the two polygons are similar.



Proposition 11. Condition of Similarity

226. Theorem. *If two polygons are composed of the same number of triangles, similar each to each and similarly placed, the polygons are similar.*



Given two polygons $ABCDE$ and $A'B'C'D'E'$ composed of the $\triangle DEA, DAB, DBC$ similar respectively to the $\triangle D'E'A', D'A'B', D'B'C'$, and similarly placed.

Prove that $ABCDE$ *is similar to* $A'B'C'D'E'$.

Proof. Since $\angle DAE = \angle D'A'E'$,

and since $\angle BAD = \angle B'A'D'$, § 205

because the $\triangle$ *are given similar,*

we have $\angle BAE = \angle B'A'E'$. Ax. 1

Similarly, $\angle CBA = \angle C'B'A'$,

and $\angle EDC = \angle E'D'C'$.

Also, $\angle C = \angle C'$,

and $\angle E = \angle E'$. § 205

Hence the polygons are mutually equiangular.

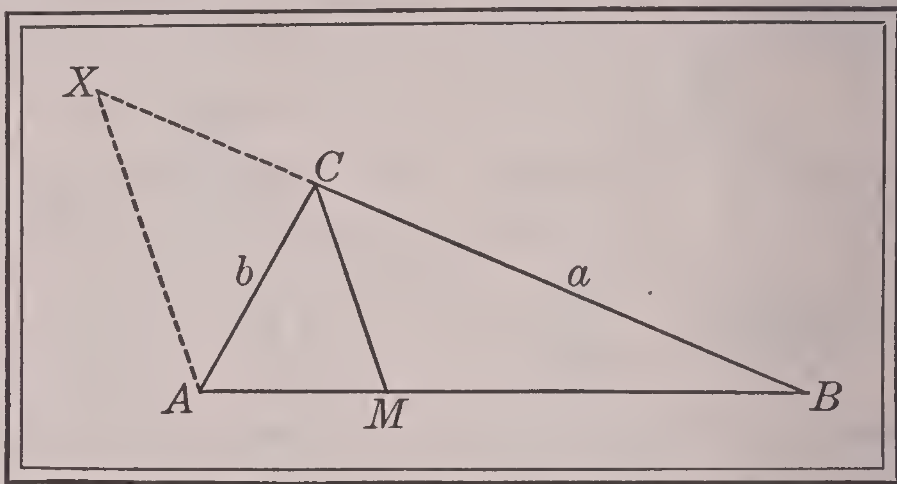
Also, $\dfrac{DE}{D'E'} = \dfrac{EA}{E'A'} = \dfrac{DA}{D'A'} = \dfrac{AB}{A'B'} = \dfrac{DB}{D'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$, § 205

because the $\triangle$ *are given similar.*

Hence the polygons are similar. § 205

Proposition 12. Bisector of an Interior Angle

227. Theorem. *The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides.*



Given the bisector of $\angle C$ of the $\triangle ABC$, meeting AB at M.

Prove that $$\frac{AM}{MB}=\frac{b}{a}.$$

Proof. From A construct a line $\parallel$ to MC. § 107

Then this line must meet BC produced, § 52

because CM and CB cannot both be $\parallel$ to it.

Let this line meet BC produced at X.

Then $AM:MB=XC:a$. § 201

Also, $\angle ACM=\angle CAX$, § 61

and $\angle MCB=\angle X$. § 62

But $\angle ACM=\angle MCB$. § 11

$\therefore \angle CAX=\angle X$, Ax. 5

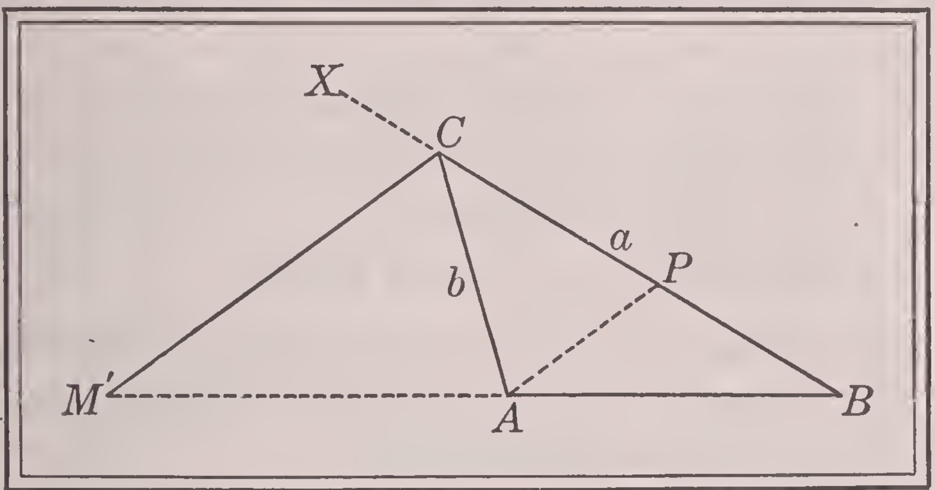
and hence $XC=b$. § 69

Substituting b for XC in the above proportion,

$$\frac{AM}{MB}=\frac{b}{a}.$$ Ax. 5

Proposition 13. Bisector of an Exterior Angle

228. Theorem. *If the bisector of an exterior angle of a triangle meets the opposite side produced, it divides that side externally into segments which are proportional to the adjacent sides.*



Given the bisector of the exterior $\angle XCA$ of the $\triangle ABC$, meeting BA produced at M'.

Prove that $$\frac{AM'}{M'B} = \frac{b}{a}.$$

Proof. Construct $AP \parallel$ to $M'C$, meeting BC at P. § 107

Then $M'B : AM' = a : PC$, § 202

or $AM' : M'B = PC : a$. § 198, 5

Now $\angle XCM' = \angle CPA$, § 62

and $\angle M'CA = \angle PAC$. § 61

But $\angle XCM' = \angle M'CA$. § 11

$\therefore \angle CPA = \angle PAC$, Ax. 5

and hence $b = PC$. § 69

Substituting b for PC in the second proportion,

$$\frac{AM'}{M'B} = \frac{b}{a}.$$ Ax. 5

What follows when $CA = CB$? when $CA > CB$?

PS

Exercises. Review

1. If two circles are tangent externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circles are proportional.

2. If two circles are tangent externally, their common external tangent is the mean proportional between their diameters.

3. Two circles are tangent, either internally or externally, at P. Through P three lines are drawn, meeting one circle in X, Y, Z and the other in X', Y', Z' respectively. Prove that $\triangle XYZ$ and $X'Y'Z'$ are similar.

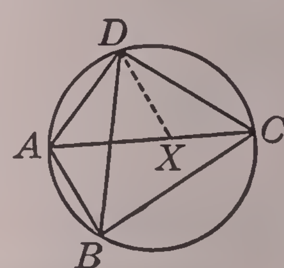
4. If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the smaller circle.

5. From a point P a secant 7.6 in. long is drawn to a circle such that the external segment is 1.9 in. Find the length of the tangent from P.

6. In a $\triangle ABC$, $AB = 16$, $BC = 14$, and $CA = 15$. Find the segments of CA made by the bisector of $\angle B$.

7. The sides of a triangle are 6, 8, and 10. Find the segments of the sides made by the bisectors of the angles.

8. In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.



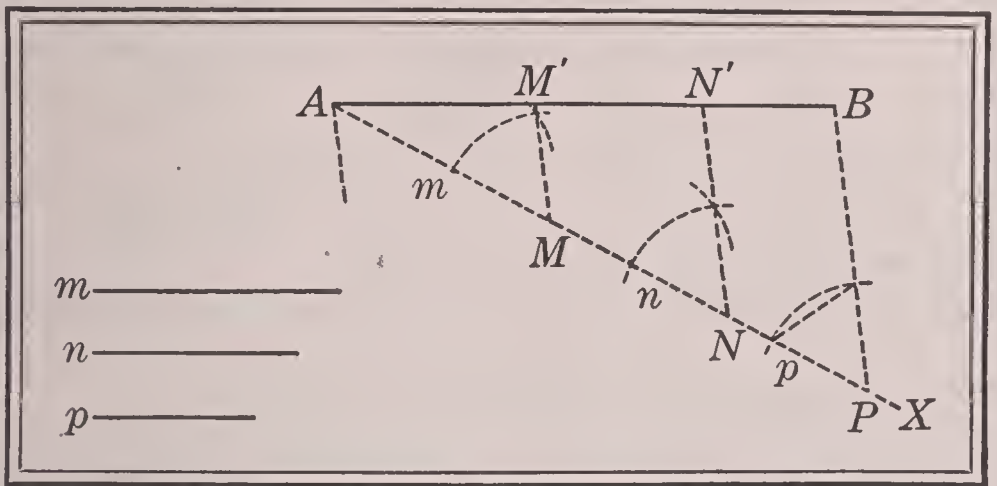
Construct DX, making $\angle XDC = \angle ADB$. Then $\triangle ABD$ and XCD are similar; and $\triangle BCD$ and AXD are also similar.

9. Given the chords AB and AC from any point A on a circle, and AD, a diameter. If the tangent at D intersects AB and AC at E and F, and if the chord BC is drawn, then the $\triangle ABC$ and AEF are similar.

II. Fundamental Constructions

Proposition 14. Dividing a Line

229. Problem. *Divide a given line segment into parts proportional to any number of given line segments.*



Given the line segments AB, m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. From A draw AX, making any convenient $\angle$ with AB. Post. 1

On AX, using dividers, take $AM = m$, $MN = n$, and $NP = p$.

Draw BP. Post. 1

At N construct $NN' \parallel$ to PB,

and at M construct $MM' \parallel$ to PB. § 107

Then M' and N' are the required points of division.

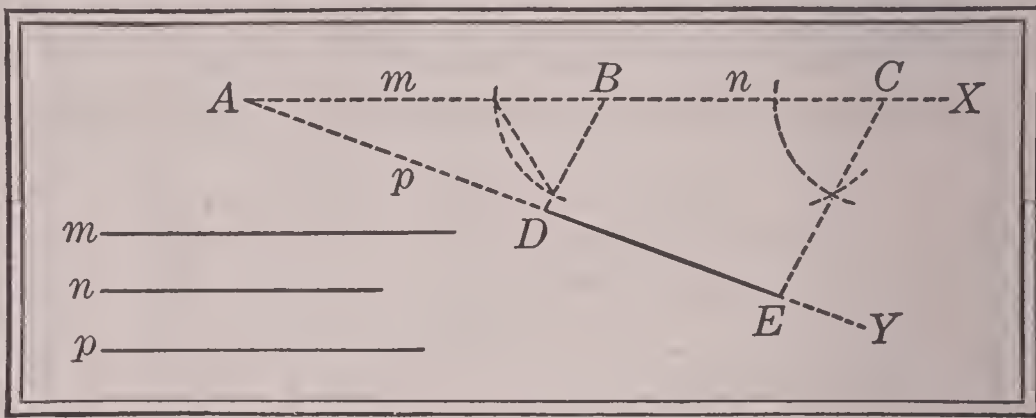
The proof is left for the student. It should be observed that § 113 is a special case of this problem.

230. Fourth Proportional. The fourth term of a proportion is called the *fourth proportional* to the terms taken in order.

Thus, in the proportion $a : b = c : d$, the term d is the fourth proportional to a, b, and c.

Proposition 15. Fourth Proportional

231. Problem. *Construct the fourth proportional to three given line segments.*



Given the three line segments m, n, and p.

Required to find the fourth proportional to m, n, *and* p.

Construction. Draw two lines AX and AY forming any convenient $\angle YAX$. Post. 1

Any acute $\angle$ will be convenient, although an obtuse $\angle$ may be used.

On AX, using dividers, take $AB = m$ and $BC = n$.
Similarly, on AY take $AD = p$.

Draw BD. Post. 1

At C construct a line $\parallel$ to BD, § 107

and designate the point where it meets AY as E.

Then DE is the required fourth proportional.

Proof. $$\frac{AB}{BC} = \frac{AD}{DE}.$$ § 201

If through two sides of a $\triangle$ a line is constructed $\parallel$ to the third side, it divides the two sides proportionally.

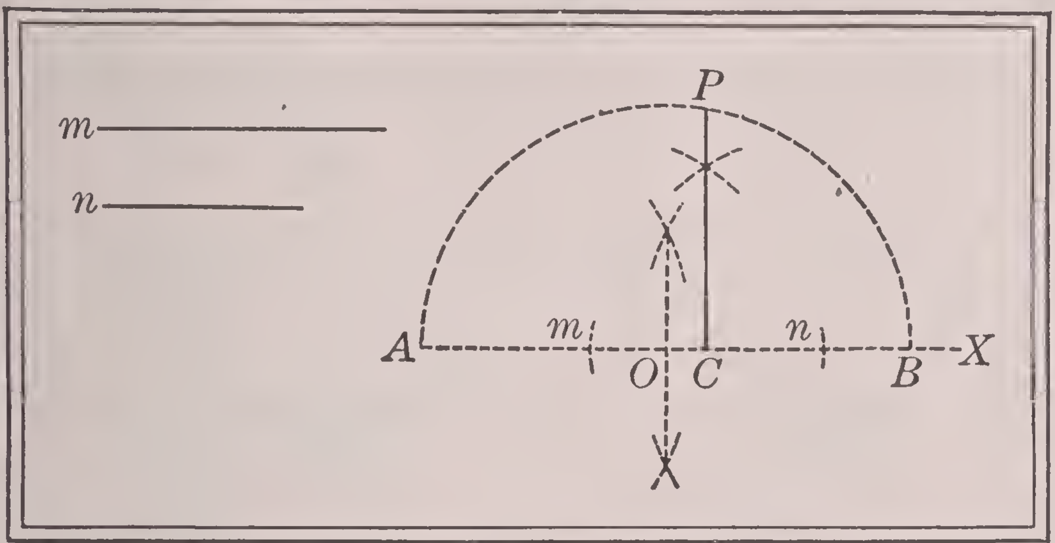
Substituting m, n, p for their equals, AB, BC, AD, we have

$$\frac{m}{n} = \frac{p}{DE}.$$ Ax. 5

$\therefore$ DE is the fourth proportional to m, n, p. § 230

Proposition 16. Mean Proportional

232. Problem. *Construct the mean proportional between two given line segments.*



Given the two line segments *m* and *n*.

Required to construct the mean proportional between m and n.

Construction. Draw any convenient line AX. Post. 1

On AX, using dividers, take

$$AC = m \text{ and } CB = n.$$

Bisect AB as at O. § 102

With O as center and OA as radius, construct a semicircle as shown. Post. 4

At C construct the $\perp$ CP, meeting the $\odot$ at P. § 104

Then CP is the mean proportional between m and n.

Proof. $$\frac{AC}{CP} = \frac{CP}{CB}.$$ § 217

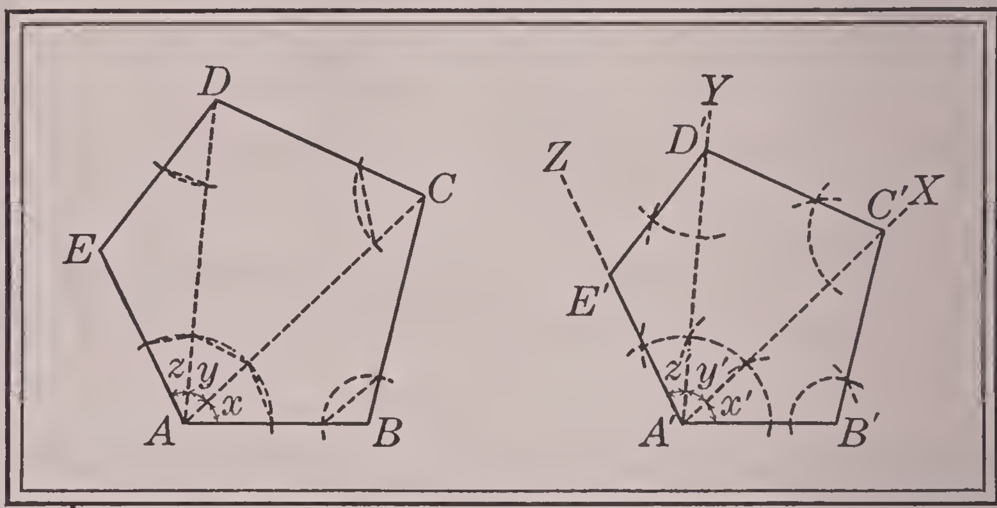
Substituting m, n for their equals, AC, CB, we have

$$\frac{m}{CP} = \frac{CP}{n}.$$ Ax. 5

$\therefore$ CP is the mean proportional between m and n. § 197

Proposition 17. Similar Polygons

233. Problem. *Upon a given line segment corresponding to a given side of a given polygon construct a polygon similar to the given polygon.*



Given the line segment *A'B'* and the polygon *ABCDE*.

Required to construct on A'B', corresponding to AB, a polygon similar to the polygon ABCDE.

Construction. From A draw the diagonals AC, AD. Post. 1

From A' construct $A'X$, $A'Y$, and $A'Z$, making

$$x' = x,$$

$$y' = y,$$

and $$z' = z.$$ § 106

Similarly, from B' construct $B'C'$, making $\angle B' = \angle B$; from C' construct $C'D'$, making $\angle D'C'A' = \angle DCA$; and from D' construct $D'E'$, making $\angle E'D'A' = \angle EDA$.

Then $A'B'C'D'E'$ is the required polygon.

Proof. $\triangle A'B'C'$ is similar to $\triangle ABC$,

$\triangle A'C'D'$ is similar to $\triangle ACD$,

and $\triangle A'D'E'$ is similar to $\triangle ADE$. § 209

$\therefore$ the two polygons are similar. § 226

Exercises. Constructions

1. If a and b are two given lines, construct a line equal to x where $x=\sqrt{ab}$. Consider the special case of $a=8$, $b=2$.

2. Construct the third proportional to two given line segments.

This means, given two line segments a and b, find x such that $a:b=b:x$; that is, find a fourth proportional to a, b, and b.

3. In Ex. 2 find x both by geometric construction and arithmetically when $a=8$ in. and $b=6$ in.

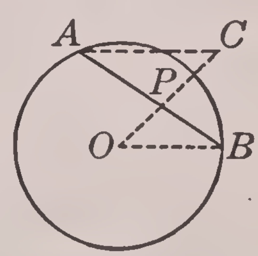
4. Determine both by geometric construction and arithmetically the fourth proportional to lines which are $2\frac{1}{2}$ in., 4 in., and $4\frac{1}{2}$ in. long respectively.

5. Determine both by geometric construction and arithmetically the mean proportional between lines which are 2.4 in. and 3.4 in. long respectively.

6. Find $\sqrt{7}$ by geometric construction. Measure the line and thus determine the approximate arithmetic value.

7. A map is drawn to the scale of 1 in. to 100 mi. How far apart are two places that are $3\frac{1}{4}$ in. apart on the map?

8. Through a given point P within a given circle construct a chord AB such that the ratio $AP:BP$ shall equal a given ratio $m:n$.



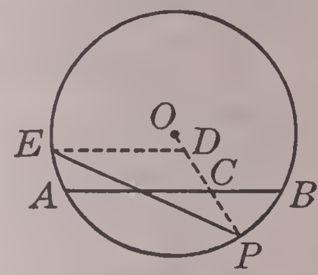
Construct OPC so that $OP:PC=n:m$. Then construct CA equal to the fourth proportional to n, m, and the radius of the circle.

9. Given the perimeter, construct a triangle similar to a given triangle.

10. Construct two circles of radii $\frac{1}{4}$ in. and $\frac{1}{2}$ in. respectively which shall be tangent externally, and construct a third circle of radius 1 in. which shall be tangent to each of these two circles and inclose both of them.

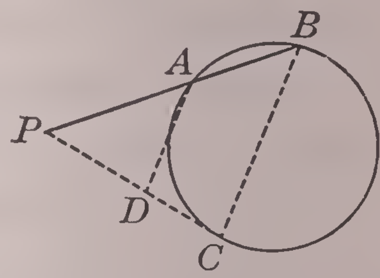
11. Given a line segment 3.5 in. long, divide it both internally and externally in the ratio $3:4$.

If AB is the given segment and P' the external point of division (§ 200), then $AP':P'B=3:4$. But $AP'=P'B-AB$, and hence it is possible to compute the length of $P'B$.



12. Through a given point P in the arc of the chord AB construct a chord which shall be bisected by AB.

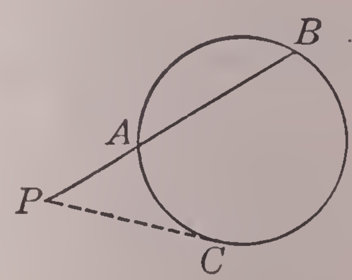
In the figure $CD=CP$ and DE is $\parallel$ to BA.



13. Through a given external point P construct a secant PAB to a given circle so that the ratio $PA:AB$ shall equal a given ratio $m:n$.

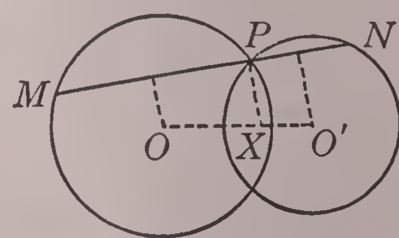
On the tangent PC construct D such that $PD:DC=m:n$. Then construct PA such that $PA:PC=PC:PB$. Consider any impossible case.

14. Through a given external point P construct a secant PAB to a given circle so that $\overline{AB}^2=PA\cdot PB$.



If PC is the tangent from P, then $PB:PC=PC:PA$, or $\overline{PC}^2=PA\cdot PB$. But it is required that $\overline{AB}^2=PA\cdot PB$. What is the relation of AB to PC? What is the locus of the midpoints of equal chords of a circle? By constructing a tangent, how can you construct the secant PAB so that $AB=PC$?

15. Through one of the points of intersection of two circles construct a secant such that the two chords that are formed shall be in a given ratio $m:n$.

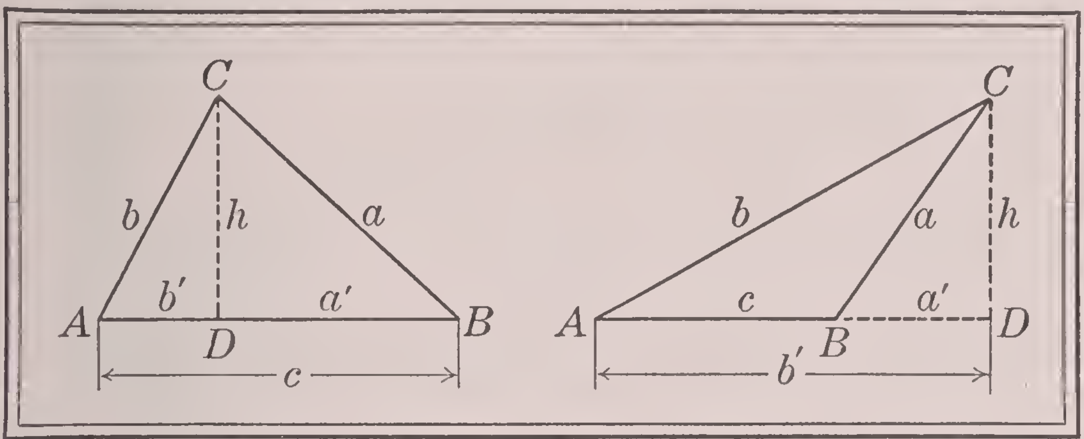


If X is constructed on the line of centers so that $OX:XO'=m:n$, if MPN is $\perp$ to PX, and if perpendiculars are drawn from O and O' to MP and PN, what follows as to the relation of MP to PN?

III. Numerical Relations

Proposition 18. Side opposite an Acute Angle

234. Theorem. *The square of the side opposite an acute angle of any triangle is equal to the sum of the squares of the other two sides diminished by twice the product of one of those sides and the projection of the other side upon it.*



Given the $\triangle ABC$ with an acute $\angle A$, and a' and b', the projections of a and b respectively upon c.

Prove that $\qquad a^2 = b^2 + c^2 - 2\,b'c.$

Proof. Depending on whether D is between A and B or not, we have $\qquad a' = c - b', \quad \text{or} \quad a' = b' - c.$ § 5

Squaring, $\qquad a'^2 = b'^2 + c^2 - 2\,b'c.$ Ax. 6

Adding h^2 to each side of this equation, we have

$$h^2 + a'^2 = h^2 + b'^2 + c^2 - 2\,b'c.$$ Ax. 1

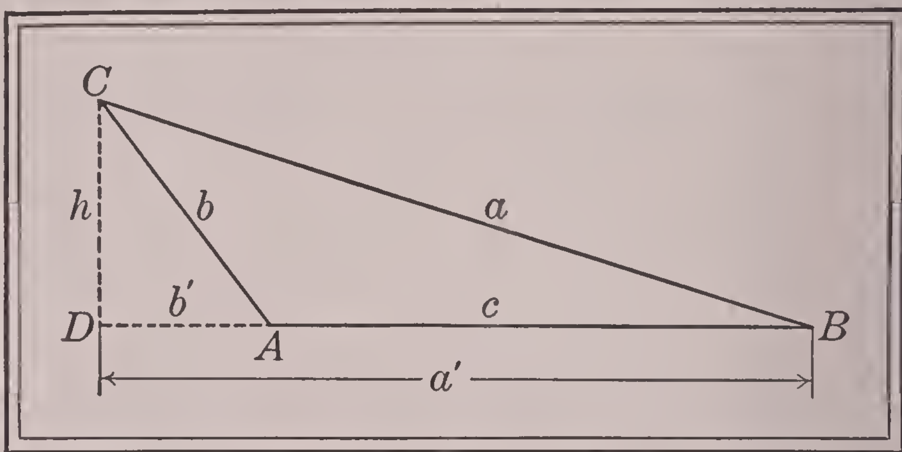
But $\qquad h^2 + a'^2 = a^2$, and $h^2 + b'^2 = b^2$. § 218

Substituting a^2 and b^2 for their equals in the above equation, we have $\qquad a^2 = b^2 + c^2 - 2\,b'c.$ Ax. 5

Pages 191–198, which illustrate the application of algebra to geometry, may be omitted without destroying the sequence.

Proposition 19. Side opposite the Obtuse Angle

235. Theorem. *The square of the side opposite the obtuse angle of any obtuse triangle is equal to the sum of the squares of the other two sides increased by twice the product of one of those sides and the projection of the other side upon it.*



Given the obtuse $\triangle ABC$ with the obtuse $\angle A$, and a' and b', the projections of a and b respectively upon c.

Prove that $a^2 = b^2 + c^2 + 2b'c.$

Proof. $a' = b' + c.$ Ax. 10

Squaring, $a'^2 = b'^2 + c^2 + 2b'c.$ Ax. 6

Adding h^2 to each side of this equation, we have

$$h^2 + a'^2 = h^2 + b'^2 + c^2 + 2b'c. \quad \text{Ax. 1}$$

But $h^2 + a'^2 = a^2$

and $h^2 + b'^2 = b^2.$ § 218

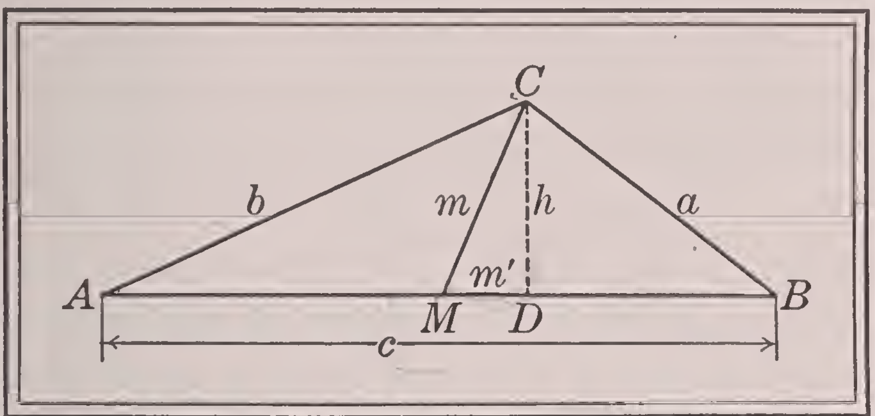
Substituting a^2 and b^2 for their equals in the above equation, we have $a^2 = b^2 + c^2 + 2b'c.$ Ax. 5

The student should notice that if b swings about A so that $\angle A$ becomes a rt. $\angle$, then b' becomes 0, and hence $a^2 = b^2 + c^2$; in other words, we have § 218. If $\angle A$ becomes acute, then b' passes through 0 and becomes negative, and hence we have § 234.

Proposition 20. Squares of Two Sides

236. Theorem. *The sum of the squares of two sides of a triangle is equal to twice the square of half the third side, increased by twice the square of the median upon it.*

The difference between the squares of two sides of a triangle is equal to twice the product of the third side and the projection of the median upon it.



Given the $\triangle ABC$ with $b > a$, the median m (or CM), and the projection m' of m upon the side c.

Prove that $\quad b^2 + a^2 = 2\,\overline{AM}^2 + 2\,m^2,$

and that $\quad b^2 - a^2 = 2\,cm'.$

Proof. $\angle CMA$ is obtuse, and $\angle CMB$ is acute. §§ 124, 18

Since it is given that $b > a$, M lies between A and D. § 118

Then $\quad b^2 = \overline{AM}^2 + m^2 + 2\,AM \cdot m',$ § 235

and $\quad a^2 = \overline{MB}^2 + m^2 - 2\,MB \cdot m'.$ § 234

Since $MB = AM$ (§ 132), if we add these equals, we have

$$b^2 + a^2 = 2\,\overline{AM}^2 + 2\,m^2. \qquad \text{Ax. 1}$$

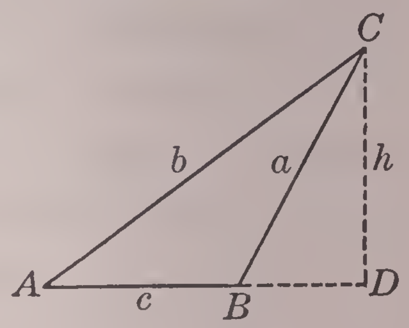
Subtracting the second equation from the first, we have

$$b^2 - a^2 = 2\,m'(AM + MB) = 2\,cm'. \qquad \text{Ax. 2}$$

The student should also consider the proposition when $a = b$. This theorem enables us to compute the medians when the three sides are known.

Exercises. Numerical Relations

1. Assuming that the area of a triangle is half the product of its base and height, as will be proved later, find the area of a triangle in terms of its sides.



At least one of the $\angle s\ A$ and B of the $\triangle ABC$ is acute. Suppose that $\angle A$ is acute.

In the $\triangle ADC$, $h^2 = b^2 - \overline{AD}^2$. § 218, Ax. 2

In the $\triangle ABC$, $a^2 = b^2 + c^2 - 2c \cdot AD$. § 234

Then $$AD = \frac{b^2 + c^2 - a^2}{2c}.$$

Hence
$$\begin{aligned} h^2 &= b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} \\ &= \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2} \\ &= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2} \\ &= \frac{[(b+c)^2 - a^2][a^2 - (b-c)^2]}{4c^2} \\ &= \frac{(a+b+c)(b+c-a)(a+b-c)(a-b+c)}{4c^2}. \end{aligned}$$

Let $a + b + c = 2s$, where s stands for semiperimeter.

Then $b + c - a = a + b + c - 2a = 2s - 2a = 2(s-a)$.

Similarly, $a + b - c = 2(s - c)$,

and $a - b + c = 2(s - b)$.

Hence $$h^2 = \frac{2s \cdot 2(s-a) \cdot 2(s-b) \cdot 2(s-c)}{4c^2}.$$

Simplifying, and finding the square root, we have
$$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

Hence area of $\triangle ABC = \frac{1}{2}ch = \sqrt{s(s-a)(s-b)(s-c)}$.

This proposition dealing with area is included here on account of its relation to the numerical theorems given in §§ 234–236. The subject of area will be treated fully in Book IV. Similarly, the propositions of §§ 234–238 are sometimes given in Book IV and stated in relation to the squares *on* the lines instead of the squares *of* the lines as here given.

2. Find the area of the triangle with sides 3 in., 4 in., 5 in.

Do this by substituting in the formula of Ex. 1, and check by the familiar rule that the area is half the product of the base and height.

3. Using Ex. 1, find to the nearest 0.01 sq. in. the area of the triangle whose sides are $2\frac{1}{2}$ in., 3 in., 4 in.

4. Find to the nearest 0.01 in. the diagonal of the square of which the side is 7 in.

5. Find to the nearest 0.01 in. the side of the square of which the diagonal is 1 ft. 8 in.

6. The minute hand and hour hand of a clock are 3 in. and $2\frac{1}{4}$ in. long respectively. How far apart are the ends of the hands at 3 o'clock?

7. From a point in the ceiling of a room 12 ft. high wires are stretched to two points on the floor 6 ft. and 10 ft. respectively from a point directly beneath the one in the ceiling. Find to the nearest 0.01 ft. the lengths of the wires.

8. The sum of the squares of the segments of two perpendicular chords of a circle is equal to the square of the diameter.

If AB, CD are the chords, draw the diameter BE, and draw AC, ED, BD. Prove that $AC = ED$.

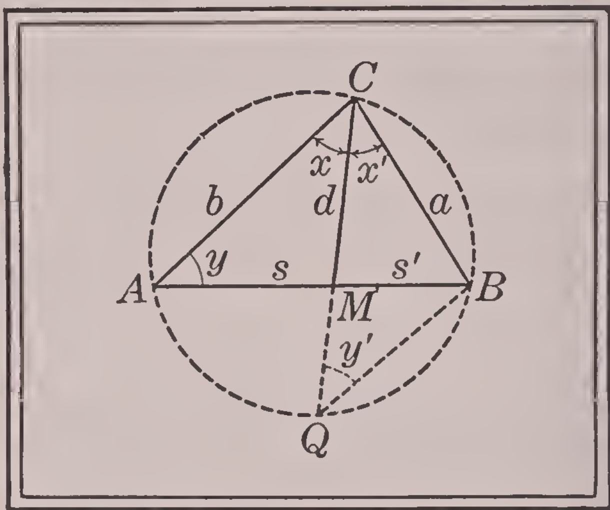
9. The difference between the squares of two sides of a triangle is equal to the difference between the squares of the segments of the third side made by the perpendicular to this side from the opposite vertex.

10. The square of one of the equal sides of an isosceles triangle is equal to the square of any line drawn from the vertex to the base, increased by the product of the segments of the base.

11. The three sides of a triangle are 3 in., 4 in., 5 in. Find to the nearest 0.01 in. the length of any median.

Proposition 21. Bisector of an Angle

237. Theorem. *The square of the bisector of an angle of a triangle is equal to the product of the sides which form this angle diminished by the product of the segments made by the bisector upon the third side of the triangle.*



Given the segment d bisecting $\angle C$ of the $\triangle ABC$ and forming the segments s and s' on AB.

Prove that $$d^2 = ab - ss'.$$

Proof. Circumscribe a $\odot$ about $\triangle ABC$ (§ 188), produce CM to cut the $\odot$ as at Q (Post. 2), and draw QB (Post. 1).

Since $x = x'$ (§ 11) and since $y = y'$ (§ 172), we see that

$\triangle BCQ$ is similar to $\triangle MCA$. § 209

Hence $CQ : b = a : d$; § 205

whence $ab = CQ \cdot d = (d + MQ)d = d^2 + MQ \cdot d$. § 198, 1

But $MQ \cdot d = ss'$, § 220

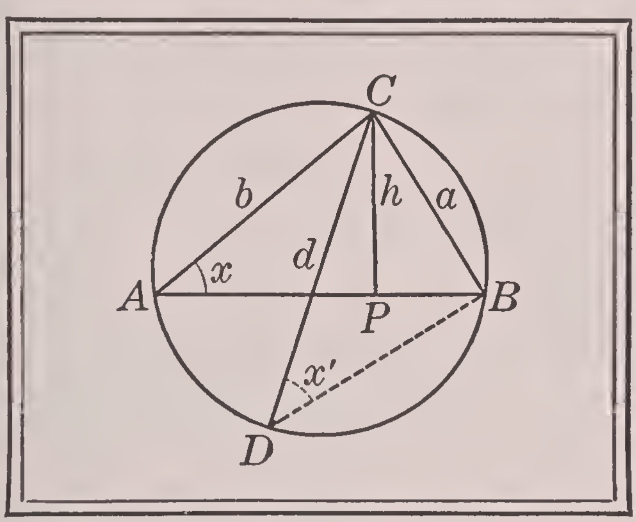
and hence $ab = d^2 + ss'$, Ax. 5

or $d^2 = ab - ss'$. Ax. 2

This theorem combined with that of § 227 enables us to compute the bisectors of the angles of a triangle terminated by the opposite sides, if the three sides are known.

Proposition 22. Product of Two Sides

238. Theorem. *The product of two sides of any triangle is equal to the product of the diameter of the circumscribed circle and the altitude upon the third side.*



Given the $\triangle ABC$ with the altitude CP (or h), and CD (or d) the diameter of the circumscribed $\odot$.

Prove that $ab = hd.$

Proof. Draw	$BD.$	Post. 1
Then	$\angle CPA$ is a rt. $\angle$,	§ 74
and	$\angle CBD$ is a rt. $\angle$.	§ 173
Further,	x is measured by $\frac{1}{2}$ arc BC,	
and	x' is measured by $\frac{1}{2}$ arc BC,	§ 172
and hence	$x = x'.$	Ax. 5
	$\therefore \triangle APC$ is similar to $\triangle DBC.$	§ 210
Hence	$\dfrac{b}{d} = \dfrac{h}{a},$	§ 205
and	$ab = hd.$	§ 198, 1

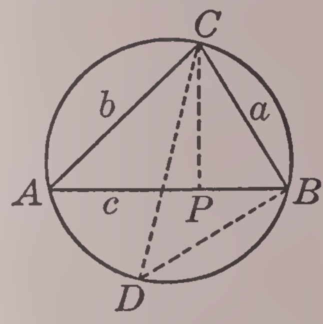
This proposition closes the list of propositions of a semialgebraic nature in Book III. As stated on page 191, they may be omitted without destroying the geometric sequence. They are needed for the exercises on page 198, but not for those on pages 199 and 200.

Exercises. Numerical Relations

If Ex. 1, page 194, has been solved, find the areas to the nearest 0.01 of triangles with sides as follows:

1. 4, 5, 6. **2.** 6, 8, 10. **3.** 7, 8, 11. **4.** 1.2, 3, 2.1.

5. In terms of the sides of a given inscribed triangle, find the radius of a circle.



Consider this exercise only in case § 238 and Ex. 1, page 194, have been studied.

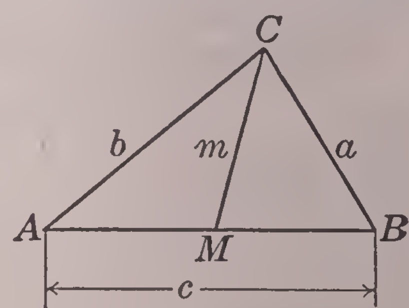
Let CD be a diameter. By § 238, what do we know about the products $CA \cdot BC$ and $CD \cdot CP$? What does this tell us of ab and $2r \cdot CP$, where r is the radius? From Ex. 1, page 194, what does CP equal in terms of the sides? From the above reasoning show that

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$

If Ex. 5 has been solved, compute the radii to the nearest 0.01 of the circles circumscribed about triangles with sides as follows:

6. 3, 4, 5. **7.** 27, 36, 45. **8.** 7, 9, 11. **9.** 10, 11, 12.

10. Find the medians of a triangle in terms of its sides.



Omit if § 236 has not been studied. What do we know about $a^2 + b^2$ as compared with $2m^2 + 2(\frac{1}{2}c)^2$?

From this relation show that for the median m in this figure,

$$m = \tfrac{1}{2}\sqrt{2(a^2+b^2) - c^2}.$$

If Ex. 10 has been solved, find to the nearest 0.01 the three medians of triangles with sides as follows:

11. 3, 4, 5. **12.** 6, 8, 10. **13.** 6, 7, 8. **14.** 7, 9, 11.

15. Find the altitude of a triangle of which the base is 4 in. and the other sides are 3 in. and 2.5 in. respectively.

Exercises. Review

1. Omitting §§ 234–238, make a list of the numbered propositions in Book III, stating under each the propositions in Books I–III upon which it depends either directly or indirectly.

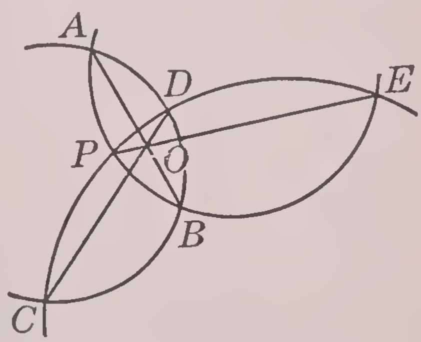
2. Omitting §§ 234–238, make another list of the numbered propositions, stating under each the propositions in Book III which depend upon it.

3. The tangents to two intersecting circles, constructed from any point in their common chord produced, are equal.

4. The common chord of two intersecting circles, if produced, bisects their common tangents.

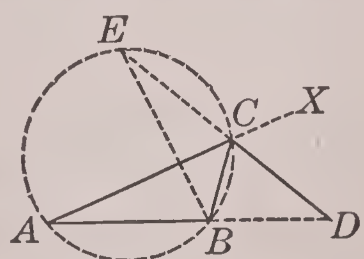
5. If two circles are tangent externally, the common internal tangent bisects the two common external tangents.

6. If three circles intersect one another, the common chords pass through the same point.



Let two of the chords, AB and CD, meet at O. Join the point of intersection E to O, and suppose that EO produced meets its two circles at two different points P and Q. Then prove that $OP = OQ$ (§ 220), and hence that the points P and Q coincide.

7. If the bisector of an exterior angle of a triangle meets the opposite side produced, the square of this segment of the bisector is equal to the product of the segments determined by it upon the opposite side, diminished by the product of the other two sides of the triangle.



In proving that $\overline{CD}^2 = AD \cdot BD - AC \cdot BC$, let CD bisect the exterior $\angle BCX$ of the $\triangle ABC$. Then prove that $\triangle$ ADC and EBC are similar (§ 209), and apply § 223.

8. If the line of centers of two circles meets the circles at the consecutive points A, B, C, D, and meets the common external tangent at P, then $PA \cdot PD = PB \cdot PC$.

9. The line of centers of two circles meets the common external tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points W, X, Y, Z. Prove that $PW \cdot PZ = PX \cdot PY$.

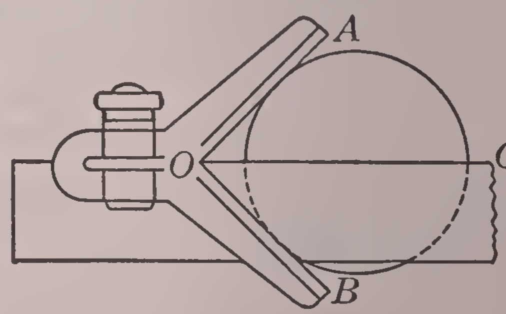
Draw radii to the points of contact, and to W, X, Y, Z. Construct perpendiculars upon PZ from the centers of the circles.

10. In a circle with a radius of 6 in., chords are drawn through a point 2 in. from the center. What is the product of the segments of each of these chords?

11. The chord AB is 6 in. long and is produced through B to the point P so that $PB = 24$ in. Find the length of the tangent to the circle from P.

12. Two line segments AB and CD intersect at O. How would you ascertain, by measuring OA, OB, OC, and OD, whether the four points A, B, C, and D lie on the same circle?

13. This figure shows a *center square*, an instrument for finding the centers of circular objects. The moveable head which has the arms OA and OB can be fixed by a set screw on the blade OC, which always bisects the $\angle BOA$. Show that, if OA and OB rest on a circle, OC passes through the center, and that by placing the square in two positions the center of the circle can be determined.



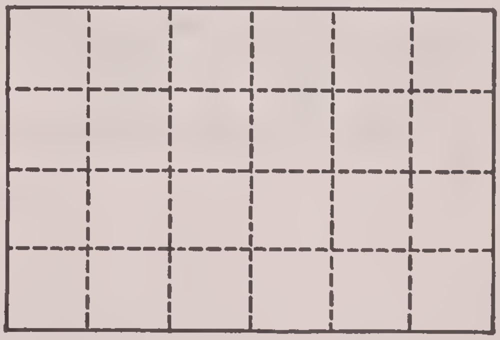
14. If three circles are tangent externally each to the other two, the tangents at their points of contact pass through the center of the circle inscribed in the triangle formed by joining the centers of the three given circles.

BOOK IV

AREAS OF POLYGONS

I. Fundamental Theorems

239. Area. If a rectangular piece of paper is 6 in. long and 4 in. wide, we may represent the rectangle by the figure here shown. We then see that there are 4 small squares in each column and that there are 6 columns; hence there are 6×4 small squares in the whole rectangle. Each of these squares is 1 in. on a side, and we define the *area* of such a square as one square inch (1 sq. in.).



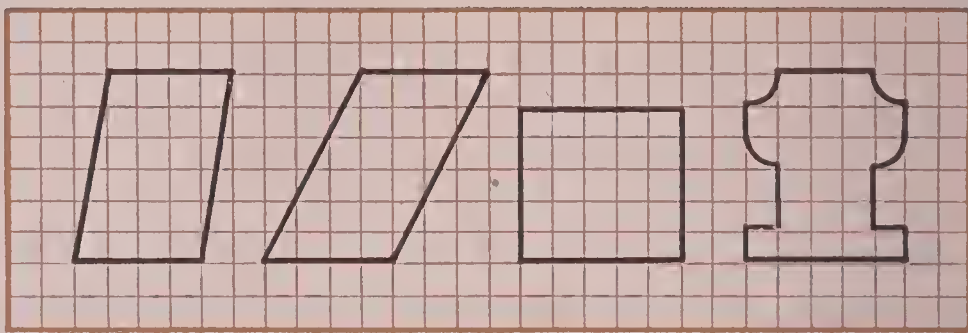
Each of the small squares is called a *unit of area*. The area of the piece of paper is 6×4 sq. in., or 24 sq. in.

As with the unit of length, a precise definition of these terms for the purposes of proof is unnecessary. Among the common units of area are 1 sq. in., 1 sq. ft., and 1 sq. mi. Sometimes a unit is taken that is not commonly in the form of a square, as in the case of the acre; but this measure contains 160 sq. rd., so that the fundamental unit in this case is 1 sq. rd.

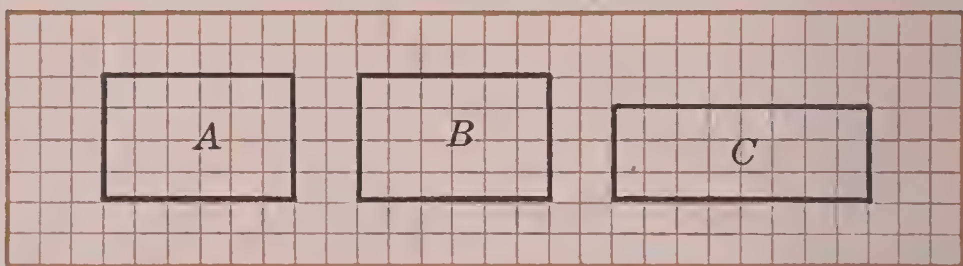
In case the sides of a rectangle are considered as incommensurable (§ 164), the subject of areas requires special treatment in a manner similar to that used in § 517. For the present we shall consider the line segments used in Book IV as commensurable, as they are for all practical purposes of measurement.

240. Equivalent Figures. Figures that have equal areas are called *equivalent figures.*

For example, the figures shown below are equivalent, since the area of each figure is 24 times the area of one of the small squares of the coordinate paper.



Since congruent figures can be made to coincide, such figures are manifestly equivalent. Equivalent figures cannot usually be made to coincide, however, and hence they are not usually congruent, as is seen above.



Of the above rectangles, A and B are both congruent and equivalent; B and C are equivalent but not congruent, and similarly for A and C.

Since the word "congruent" means identically equal, the word "equal" is commonly used to mean equivalent. Thus, since their areas are equal, equivalent figures are frequently spoken of as equal figures. The symbol $=$ may be used both for "equivalent" and for "congruent," as the conditions under which it is used will determine which meaning is to be assigned to it.

In propositions relating to areas the word "rectangle" is commonly used for area of the rectangle, and similarly for other plane figures. It is also the custom to speak of the product of two line segments when we mean the product of their numerical measures.

241. Area of a Rectangle. From the preceding discussion we may assume as true the statement that

The area of a rectangle is the product of the base and the altitude.

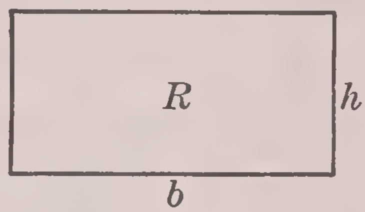
In case the sides have a common unit of measure, this is readily proved from the figure of § 239. In case they have no common measure, the proof is similar to the one given in § 517.

If the base is 3 in. and the altitude 2 in., the area is 3×2 sq. in., or 6 sq. in. This is the meaning of the expression "the product of the base and the altitude."

In industrial work 2′ is used for 2 ft. and 2″ for 2 in.

If R stands for the number of units of area of a rectangle of base b units and altitude h units, the above statement may be written

$$R = bh;$$



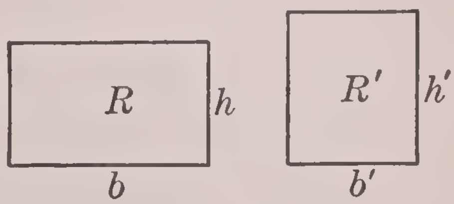
whence $$b = \frac{R}{h},$$

and $$h = \frac{R}{b},$$

formulas that we sometimes use in measuring rectangles.

242. Ratio of Two Rectangles. In considering the areas of two rectangles R and R' we see that

$$\frac{R}{R'} = \frac{bh}{b'h'}.$$



Then if $h' = h$, we have

$$\frac{R}{R'} = \frac{bh}{b'h} = \frac{b}{b'};$$

that is, *rectangles with equal altitudes are to each other as their bases.*

Similarly, *rectangles with equal bases are to each other as their altitudes.*

As stated in § 240, the word "rectangles" is here used for the areas of rectangles.

Exercises. Areas of Rectangles

1. Find the ratio of a lot 180 ft. long and 120 ft. wide to a field 80 rd. long and 40 rd. wide.

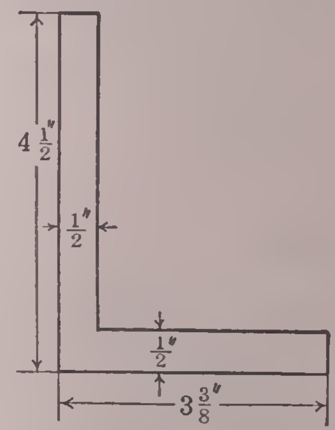
2. A square and a rectangle have equal perimeters of 576 in., and the length of the rectangle is five times the width. Which has the greater area? How much greater?

3. On a certain map the linear scale is 1 in. = 20 mi. How many acres are represented by a square $\frac{7}{8}$ in. on a side?

4. Find the area of a gravel walk 7 ft. wide which surrounds a rectangular plot of grass 80 ft. long and 50 ft. wide.

5. Find the number of rods in the perimeter of a square field that contains exactly an acre.

6. Find the number of square inches in the cross section of this L beam.



7. A machine for planing iron plates planes a surface 1 in. wide and 9 ft. long in 1 min. At the same rate per square inch, how long does it take to plane a plate 12 ft. long and 6 in. wide, allowing 28 min. for adjusting the machine during the process?

8. How many tiles, each 6 in. square, does it take to cover a floor 36 ft. 6 in. long by 18 ft. wide?

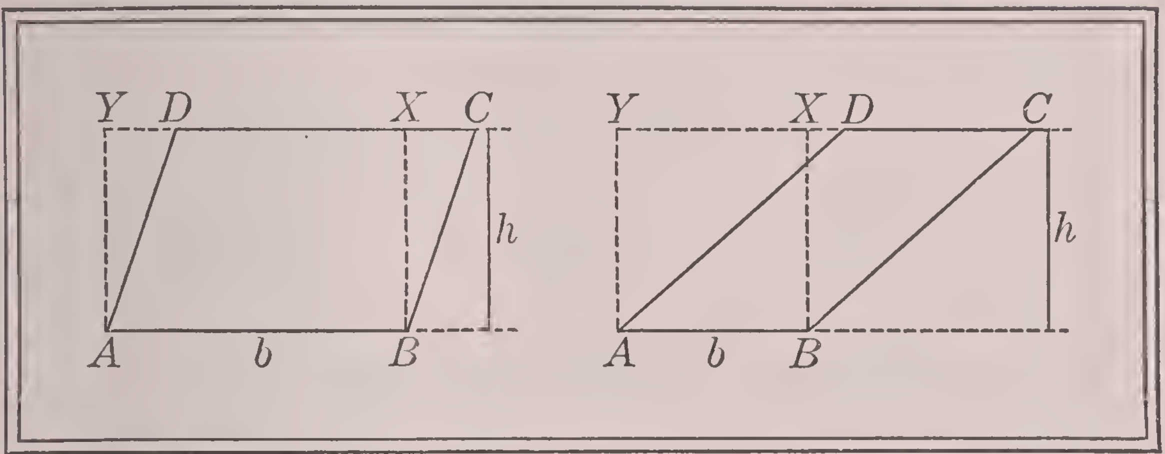
9. The length of a rectangle is four times the width. If the perimeter is 120 ft., what is the area?

10. Along two adjacent sides of a rectangular field 120 rd. long and 80 rd. wide a road 4 rd. wide is laid out inside the field. How many acres are taken for the road?

11. From one end of a rectangular sheet of iron 12 in. long a square piece is cut off such that it leaves 36 sq. in. in the rest of the sheet. How wide is the sheet?

Proposition 1. Area of a Parallelogram

243. Theorem. *The area of a parallelogram is the product of the base and the altitude.*



Given the $\square ABCD$ with base b and altitude h.

Prove that the area of $\square ABCD = bh$.

Proof. At B construct $BX \perp$ to CD, or CD produced, and at A construct $AY \perp$ to CD, or CD produced. § 105

The only cases which require special attention are shown above.

Then AY is $\parallel$ to BX. § 57

$\therefore ABXY$ is a $\square$ with base b and altitude h. § 72

Since $AY = BX$ and $AD = BC$, § 78

then rt. $\triangle ADY$ is congruent to rt. $\triangle BCX$. § 71

Now, considering the quadrilateral $ABCY$, we have

$$ABCY - \triangle BCX = \square ABXY,$$

and $ABCY - \triangle ADY = \square ABCD$. Ax. 10

But $ABCY - \triangle BCX = ABCY - \triangle ADY$, Ax. 2

and hence $\square ABXY = \square ABCD$. Ax. 5

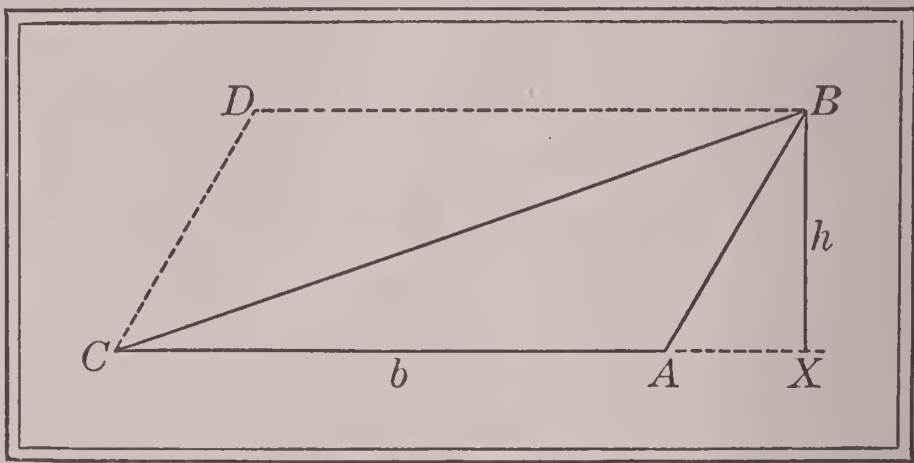
But $\square ABXY = bh$. § 241

$\therefore \square ABCD = bh$. Ax. 5

By this theorem we have proved the correctness of a formula with which the student has long been familiar.

Proposition 2. Area of a Triangle

244. Theorem. *The area of a triangle is half the product of the base and the altitude.*



Given the △*ABC* with base *b* and altitude *h*.

Prove that the area of $\triangle ABC = \frac{1}{2}bh$.

Proof. With CA and AB as adjacent sides construct the ▱$ABDC$. § 107

Then $\triangle ABC = \frac{1}{2}$ ▱$ABDC$. § 77

But ▱$ABDC = bh$. § 243

$\therefore \triangle ABC = \frac{1}{2}bh$. Ax. 4

245. Corollary. *Triangles with equal bases and equal altitudes are equivalent; and similarly for parallelograms.*

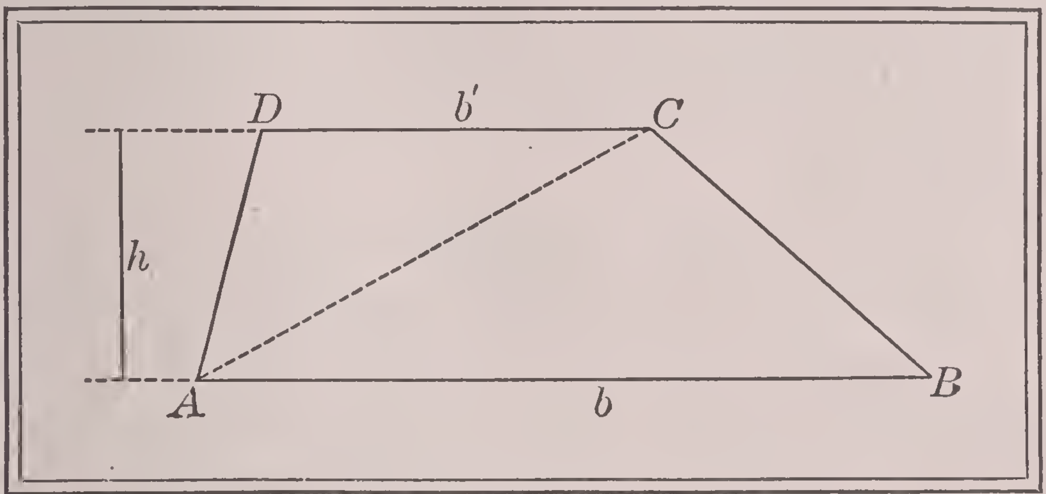
For, whatever the shape, the area of the △ is $\frac{1}{2}bh$, and the area of the ▱ is bh.

246. Corollary. *Triangles with equal bases are to each other as their altitudes; triangles with equal altitudes are to each other as their bases; any two triangles are to each other as the products of their bases and altitudes; and similarly for parallelograms.*

Has this been proved for ▭s? What is the relation of a △ to a ▭ of equal base and equal altitude? What must then be the relations of △s to one another? Can the same be proved for ▱s?

Proposition 3. Area of a Trapezoid

247. Theorem. *The area of a trapezoid is half the product of the altitude and the sum of the bases.*



Given the trapezoid *ABCD* with bases *b* and *b′* and altitude *h*.

Prove that the area of $ABCD = \frac{1}{2}h(b+b')$.

Proof. Draw the diagonal AC. Post. 1

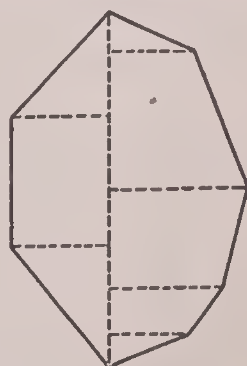
Then $\triangle ABC = \frac{1}{2}bh$,

and $\triangle ACD = \frac{1}{2}b'h$. § 244

Hence $ABCD = \frac{1}{2}bh + \frac{1}{2}b'h$; Ax. 1

that is, $ABCD = \frac{1}{2}h(b+b')$.

248. Area of an Irregular Polygon. The area of an irregular polygon may be found by dividing the polygon into triangles and trapezoids and then finding the area of each of these triangles and trapezoids separately.



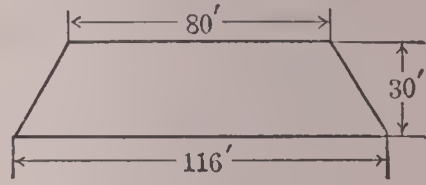
A common method used in land surveying is as follows: Draw the longest diagonal, construct perpendiculars upon this diagonal from the other vertices of the polygon, as shown in the figure, and then measure each of the dotted lines. The sum of the areas of the right triangles, rectangles, and trapezoids thus formed is the area of the polygon.

The student should see that he can now measure any rectilinear figure.

Exercises. Areas

1. Find the area of a trapezoid of which the bases are 17 in. and 13 in. respectively and the altitude is 7.5 in.

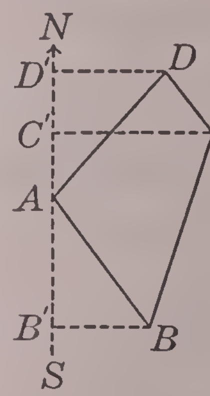
2. A railway embankment is 30 ft. high, 80 ft. wide at the top, and 116 ft. wide at the bottom. Find the area of the cross section.



3. A canal is 36 ft. deep, 240 ft. wide at the top, and 200 ft. wide at the bottom. Find the area of the cross section.

4. A polygon of six sides is made up of six congruent triangles such that the base of each triangle is 4 in. and its altitude is $2\sqrt{3}$ in. Find the area of the polygon to the nearest 0.1 sq. in.

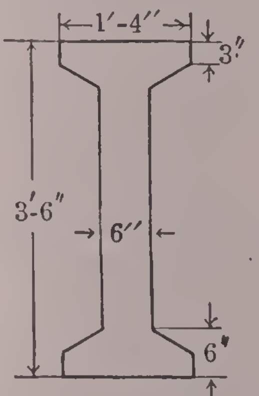
5. In surveying the field here shown a surveyor laid off a north-and-south line NS through A and then found that $BB' = 6$ rd., $CC' = 10$ rd., $DD' = 7$ rd., $B'A = 8$ rd., $B'C' = 12$ rd., $C'D' = 4$ rd. Find the area of the field.



6. In Ex. 5, what would be the area if each of the given measurements were doubled?

7. The area of a trapezoid is the product of the altitude and the line segment joining the midpoints of the nonparallel sides.

8. Find the area of the cross section of the steel girder here shown.

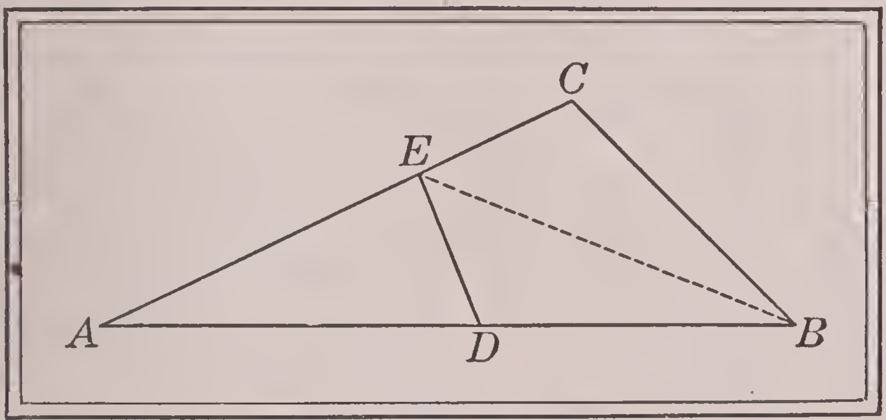


9. In Ex. 8, what would be the area if each of the given measurements were multiplied by three?

10. The product of the sides forming the right angle of a right triangle is equal to the product of the hypotenuse and the altitude upon it.

Proposition 4. Ratios of Areas of Triangles

249. Theorem. *If an angle of one triangle is equal to an angle of another, the triangles are to each other as the products of the sides forming the equal angles.*



Given the $\triangle$ ABC and ADE with the common $\angle A$.

Prove that $$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \cdot AC}{AD \cdot AE}.$$

Proof. Draw BE. Post. 1

Then $$\frac{\triangle ABC}{\triangle ABE} = \frac{AC}{AE},$$

and $$\frac{\triangle ABE}{\triangle ADE} = \frac{AB}{AD}, \qquad \text{§ 246}$$

because $\triangle$ *with equal altitudes are to each other as their bases.*

Since we are considering numerical measures, we may treat the terms of these proportions as numbers.

Taking the product of the first members and the product of the second members of these equations, we have

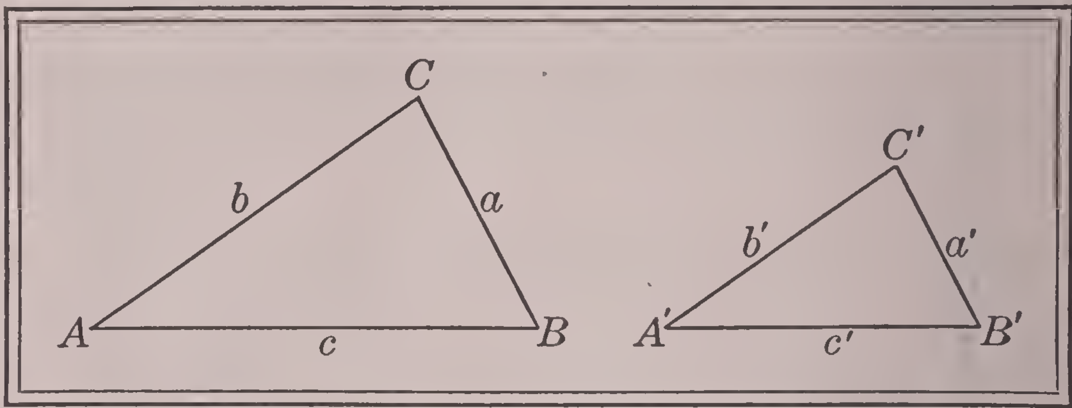
$$\frac{\triangle ABE \cdot \triangle ABC}{\triangle ADE \cdot \triangle ABE} = \frac{AB \cdot AC}{AD \cdot AE}. \qquad \text{Ax. 3}$$

Then, canceling $\triangle ABE$, we have the proportion

$$\frac{\triangle ABC}{\triangle ADE} = \frac{AB \cdot AC}{AD \cdot AE}.$$

Proposition 5. Similar Triangles

250. Theorem. *The areas of two similar triangles are to each other as the squares on any two corresponding sides.*



Given the similar $\triangle\triangle ABC$ and $A'B'C'$.

Prove that $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{c^2}{c'^2}.$$

Proof. Since $\triangle ABC$ is similar to $\triangle A'B'C'$, Given

we have $$\angle A = \angle A'.$$ § 205

Then $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{bc}{b'c'},$$ § 249

because ⋯ the $\triangle\triangle$ are to each other as the products of the sides forming the equal $\angle\angle$;

that is, $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{b}{b'} \cdot \frac{c}{c'}.$$

But $$\frac{b}{b'} = \frac{c}{c'}.$$ § 205

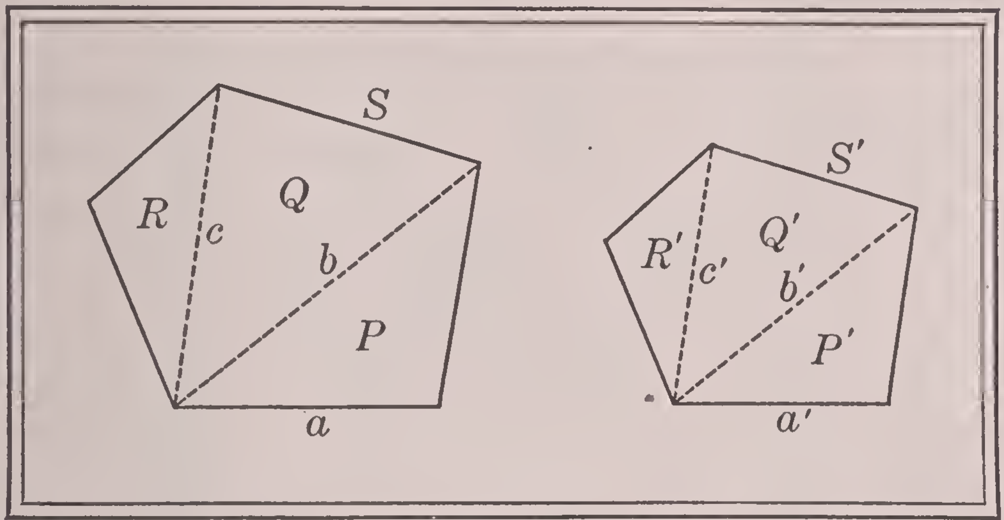
Substituting $\frac{c}{c'}$ for its equal, $\frac{b}{b'}$, we have

$$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{c}{c'} \cdot \frac{c}{c'},$$ Ax. 5

or $$\frac{\triangle ABC}{\triangle A'B'C'} = \frac{c^2}{c'^2}.$$

Proposition 6. Areas of Polygons

251. Theorem. *The areas of two similar polygons are to each other as the squares on any two corresponding sides.*



Given two similar polygons with areas S and S' respectively.

Prove that $$\frac{S}{S'}=\frac{a^2}{a'^2}.$$

Proof. By drawing all the diagonals from any two corresponding vertices the two similar polygons are separated into the similar $\triangle$s P, P'; Q, Q'; R, R'. § 225

Then $$\frac{\triangle R}{\triangle R'}=\frac{c^2}{c'^2}=\frac{\triangle Q}{\triangle Q'}=\frac{b^2}{b'^2}=\frac{\triangle P}{\triangle P'}=\frac{a^2}{a'^2}.$$ § 250

Hence $$\frac{\triangle R}{\triangle R'}=\frac{\triangle Q}{\triangle Q'}=\frac{\triangle P}{\triangle P'}.$$ Ax. 5

Then $$\frac{\triangle R+\triangle Q+\triangle P}{\triangle R'+\triangle Q'+\triangle P'}=\frac{\triangle P}{\triangle P'}.$$ § 198, 8

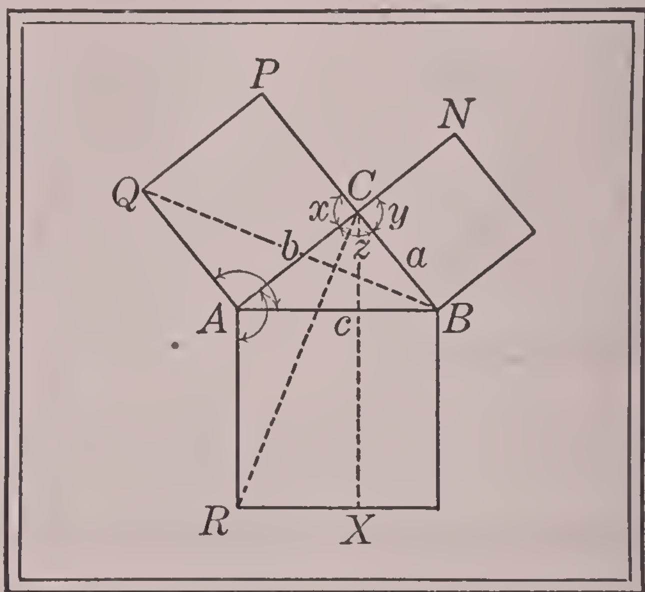
But $$\frac{\triangle P}{\triangle P'}=\frac{a^2}{a'^2}.$$ Proved

Then $$\frac{\triangle R+\triangle Q+\triangle P}{\triangle R'+\triangle Q'+\triangle P'}=\frac{a^2}{a'^2},$$ Ax. 5

and hence $$\frac{S}{S'}=\frac{a^2}{a'^2}.$$ Ax. 10

Proposition 7. Pythagorean Theorem

252. Theorem. *The square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.*



Given the rt. $\triangle\ ABC$ with the rt. $\angle C$, and the squares constructed on the sides a, b, c respectively.

Prove that $c^2 = a^2 + b^2$.

Proof. Construct $CX \parallel$ to AR (§ 107), and draw BQ and CR.

Since x and z are rt. $\angle$s, then $\angle PCB$ is a st. $\angle$. § 13

Hence PCB is a st. line; and similarly for ACN. § 18

Then $AR = AB,\ AC = AQ$, § 15

and $\angle RAC = \angle BAQ$. Ax. 1

$\therefore \triangle ARC$ is congruent to $\triangle ABQ$. § 40

But $\square AX = 2\triangle ARC$, § 244

because they have the same base AR and the same altitude RX.

Similarly, $b^2 = 2\triangle ABQ = 2\triangle ARC$.

$\therefore \square AX = b^2$. Ax. 5

Similarly, $\square BX = a^2$.

$\therefore \square AX + \square BX = b^2 + a^2$, or $c^2 = a^2 + b^2$. Ax. 1

253. Corollary. *The square on either side of a right triangle is equivalent to the difference between the square on the hypotenuse and the square on the other side.*

For, since $c^2 = a^2 + b^2$, § 252

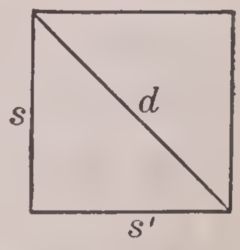
then $c^2 - a^2 = b^2$. Ax. 2

254. Pythagorean Theorem. The fact that the square on the hypotenuse is equivalent to the sum of the squares on the other two sides has, of course, long been known to the student. It is usually learned in arithmetic, and we have already given an algebraic proof in § 218. Various geometric proofs may be given, but the one in § 252 is the most satisfactory for beginners. This proof is attributed to Euclid, a famous mathematician who lived in Alexandria, in Egypt, about 300 B.C. Euclid wrote the first great textbook on geometry, and taught in the world's first great university, an institution founded by one of the Greek kings of Egypt.

It is thought, as stated in § 218, that Pythagoras gave the first proof of this theorem about 525 B.C., but it is not certain that he did so. Pythagoras founded the world's first great school of mathematics at Crotona, in the southeastern part of Italy, which was then a Greek colony.

If § 218 has been thoroughly mastered, § 252 may be omitted.

From a study of the theorem we see that the diagonal and side of a square are incommensurable.



For $d^2 = s^2 + s'^2$,

or $d^2 = 2s^2$.

Hence $d = s\sqrt{2}$.

Since $\sqrt{2}$ may be carried to as many decimal places as we please, but cannot be exactly expressed as a rational number, it has no common measure with 1. That is, $\frac{d}{s} = \sqrt{2}$, an incommensurable number, and hence the diagonal and side are incommensurable.

Exercises. Areas

Find the areas of the parallelograms whose bases and altitudes are respectively as follows:

1. 4.5 in., $2\frac{2}{3}$ in. **2.** 5.4 ft., 2.4 ft. **3.** 4 ft. 6 in., 14 in.

Find the areas of the triangles whose bases and altitudes are respectively as follows:

4. 2.8 in., 3 in. **5.** 13 ft., 6 ft. **6.** 4 ft. 6 in., 2 ft.

Find the areas of the trapezoids whose bases are the first two of the following numbers, and whose altitudes are the third numbers:

7. $2\frac{1}{2}$ ft., $1\frac{1}{4}$ ft.; 5 in. **8.** 4 ft. 7 in., 3 ft.; 16 in.

Find the altitudes of the parallelograms whose areas and bases are respectively as follows:

9. 20 sq. in., 10 in. **10.** 8 sq. ft., 3 ft. **11.** 7 sq. ft., 2 ft.

Find the altitudes of the triangles whose areas and bases are respectively as follows:

12. 9 sq. in., 4 in. **13.** 7 sq. ft., 2 ft. **14.** 11 sq. yd., 3 yd.

15. Find the altitude of the trapezoid whose area and bases are 33 sq. in., 5 in., and 6 in. respectively.

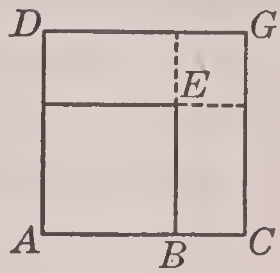
Given the sides of a right triangle as follows, find the hypotenuse to the nearest 0.01 ft.:

16. 60 ft., 80 ft. **17.** 40 ft., 60 ft. **18.** 7 ft. 6 in., 9 ft.

Given the hypotenuse and one side of a right triangle as follows, find the other side to the nearest 0.01 ft.:

19. 25 ft., 20 ft. **20.** 20 ft., 12 ft. **21.** 3 ft. 4 in., 2 ft.

22. The square constructed upon the sum of two line segments is equivalent to the sum of the squares constructed upon the two segments, increased by twice the rectangle of the segments.

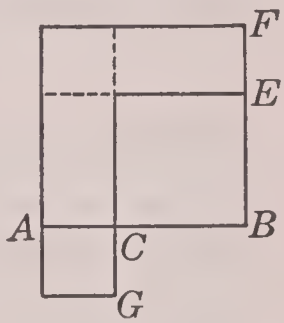


Given the two line segments AB and BC, their sum AC, and the squares AG and AE constructed upon AC and AB respectively. Complete the figure as shown. Then the square AG is the sum of the squares AE, EG and the $\boxed{\text{s}}$ DE, CE.

This proves geometrically the algebraic formula

$$(a+b)^2 = a^2 + 2\,ab + b^2.$$

23. The square constructed upon the difference between two line segments is equivalent to the sum of the squares constructed upon the two segments, diminished by twice the rectangle of the segments.

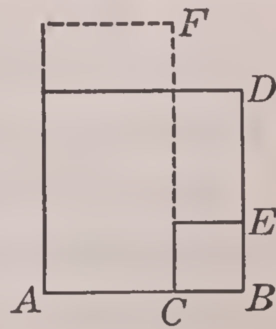


Given the two line segments AB and AC, their difference BC, the square AF constructed upon AB, the square AG upon AC, and the square CE upon BC. Complete the figure as shown. Then the square CE is the difference between the whole figure and the sum of two rectangles.

This proves geometrically the algebraic formula

$$(a-b)^2 = a^2 - 2\,ab + b^2.$$

24. The difference between the squares constructed upon two line segments is equivalent to the rectangle of the sum and difference of these lines.



Given the squares AD and CE constructed upon AB and BC respectively. Show that the difference between the squares AD and CE is equivalent to the ▭ AF, with dimensions $AB+BC$ and $AB-BC$.

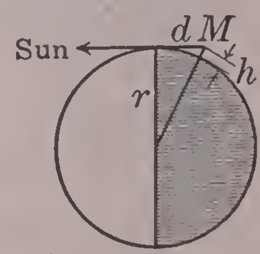
This proves geometrically the algebraic formula

$$a^2 - b^2 = (a+b)(a-b).$$

Before our present algebra was invented the algebraic laws given in Exs. 22–24 were proved as above by geometry.

25. An extension ladder 77 ft. long is placed with its top against a wall, and its foot 46.2 ft. from the base of the wall. How high, to the nearest 0.1 ft., does the ladder reach on the wall?

26. Galileo (1564–1642), who was the first to use the telescope in astronomy, found the height of a mountain on the moon by the aid of the Pythagorean Theorem. On a map of the moon he measured the distance d from the top of the mountain M when it was touched by the sun's rays to the line dividing the light half of the moon from the dark half. Representing the height by h and the radius of the moon by r, he saw that



$$(h+r)^2 = r^2 + d^2.$$

Find h, given that the radius of the moon is 1081 mi.

Students who have had quadratics should solve this equation for h, then substitute 1081 for r, and find that $h = -1081 + \sqrt{1081^2 + d^2}$, where h is in miles. An approximate solution in feet is $h = 2.44\,d^2$.

27. Find a formula for the altitude h of an equilateral triangle in terms of its side s.

28. Find a formula for the side s of an equilateral triangle in terms of its altitude h.

29. If A is the area of an equilateral triangle with side s, prove that $A = \frac{1}{4} s^2 \sqrt{3}$.

30. Find the length of the longest chord and of the shortest chord that can be drawn through a point 1 ft. from the center of a circle with a radius of 20 in.

31. If the diagonals of a quadrilateral intersect at right angles, the sum of the squares on one pair of opposite sides is equivalent to the sum of the squares on the other pair.

32. The area of a rhombus is half the product of its diagonals.

33. Two triangles are equivalent if the base of the first is equal to half the altitude of the second, and the altitude of the first is equal to twice the base of the second.

34. The area of a circumscribed polygon is half the product of the perimeter of the polygon and the radius of the inscribed circle.

35. If equilateral triangles are constructed on the sides of a right triangle, the triangle on the hypotenuse is equivalent to the sum of the triangles on the other two sides.

36. If similar polygons are constructed on the sides of a right triangle as corresponding sides, the polygon on the hypotenuse is equivalent to the sum of the polygons on the other two sides.

Ex. 36 is one of the general forms of the Pythagorean Theorem.

37. Every line drawn through the intersection of the diagonals of a parallelogram bisects the parallelogram.

38. If lines are drawn from any point within a parallelogram to the four vertices, the sum of either pair of triangles with parallel bases is equivalent to the sum of the other pair.

39. If a quadrilateral with two sides parallel is bisected by either diagonal, the quadrilateral is a parallelogram.

40. The line that bisects the bases of a trapezoid divides the trapezoid into two equivalent parts.

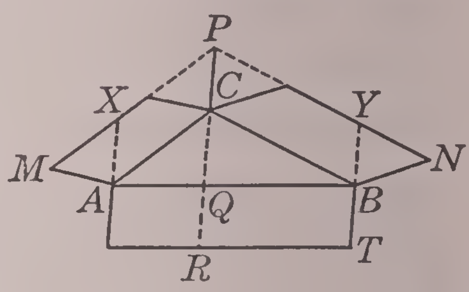
41. The triangle formed by two lines drawn from the midpoint of either of the nonparallel sides of a trapezoid to the opposite vertices is equivalent to half the trapezoid.

42. The sides of a triangle are 1.4 in., 1.2 in., and 1.4 in. respectively. Is the largest angle acute, right, or obtuse?

43. The sides of a triangle are 9.5 in., 14.1 in., and 17 in. respectively. Is the largest angle acute, right, or obtuse?

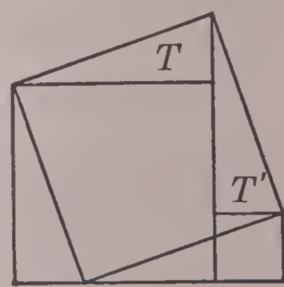
44. Find to the nearest 0.1 sq. in. the area of an isosceles triangle whose perimeter is 28 in. and whose base is 8 in.

45. Upon any two sides AC and BC of a given $\triangle ABC$ the ▱s CM and CN are constructed. Two sides of these parallelograms are produced to meet at P as here shown, the line PC is drawn and produced so that $QR = PC$, and then the ▱ AT is constructed with BT equal to and parallel to QR. Prove that $CM + CN = AT$.



This interesting generalization of the Pythagorean Theorem is due to the Greek geometer Pappus, about A.D. 300. It is not difficult to derive the Pythagorean Theorem from it by starting with a right triangle and by making CM and CN squares.

46. Prove the Pythagorean Theorem by using this figure.



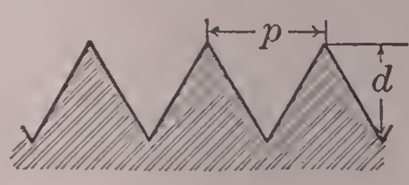
Show that the four large right triangles are congruent. If the two triangles marked T and T' are taken from the whole figure, there remains the sum of the squares on the two sides. If the other two triangles are taken from the whole figure, there remains the square on the hypotenuse.

47. Find the area of a right triangle if the hypotenuse is 3.4 in. and one of the other sides is 1.6 in.

48. Find the ratio of the altitudes of two equal triangles if the base of one is 3 in. and that of the other is 9 in.

49. The bases of a trapezoid are 68 in. and 60 in., and the altitude is 4 in. Find the side of a square with the same area.

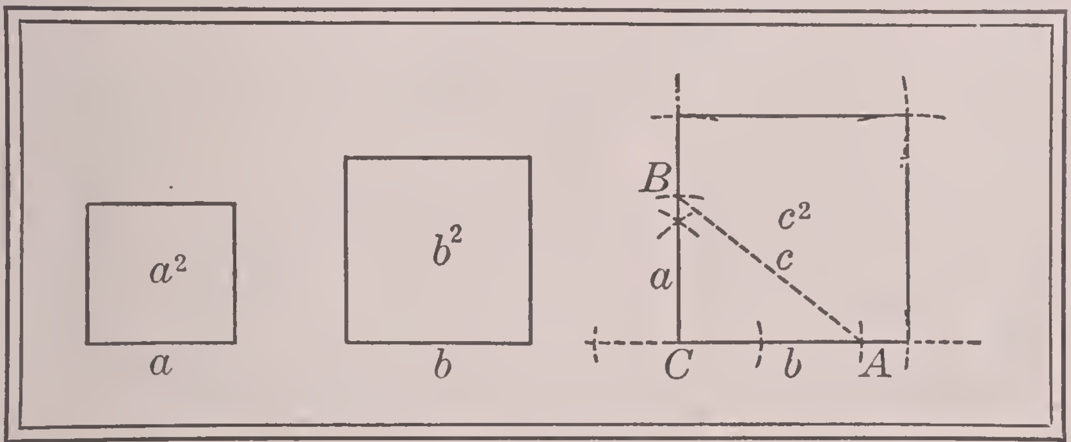
50. The cross section of a V-thread on a screw is an equilateral triangle. The distance p between successive threads is known as the *pitch* of the thread, and the distance d as the *depth* of the thread. If $p = \frac{1}{8}$ in., what is the value of d?



II. Fundamental Constructions

Proposition 8. Sum of Two Squares

255. Problem. *Construct a square equivalent to the sum of two given squares.*



Given the squares a^2 and b^2 with sides a and b respectively.

Required to construct a square equivalent to $a^2 + b^2$.

Construction. On any line construct the rt. $\angle C$ (§ 104), and on its arms take $CB = a$ and $CA = b$.

Draw AB, or c. Post. 1

With c as a radius, construct the required square c^2 by drawing arcs as shown. Post. 4

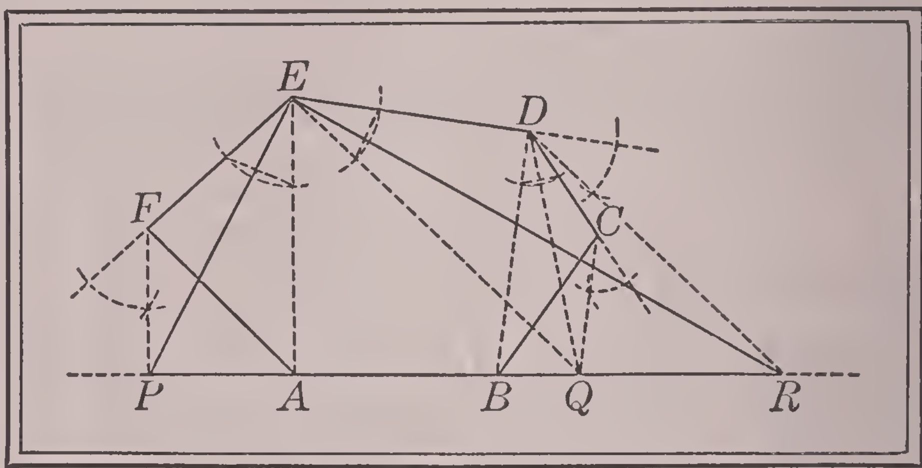
Proof. $c^2 = a^2 + b^2$. § 252

256. Purpose of These Constructions. Since the area of a square is easily found, it is often advantageous to transform a rectilinear figure into a square. It is also helpful to combine several squares into a single square, by first finding a square equivalent to two of the squares, and then combining this square with a third one, and so on.

The student may omit §§ 255–260 without interfering with the subsequent work, and should omit §§ 261 and 262 unless preparing for more advanced work in mathematics. In some courses § 257 is required.

Proposition 9. Transforming a Polygon

257. Problem. *Construct a triangle equivalent to a given polygon.*



Given the polygon *ABCDEF*.

Required to construct a △ equivalent to ABCDEF.

Construction. Let B, C, and D be any three consecutive vertices of the polygon.

Draw the diagonal DB. Post. 1

From C construct a line ∥ to DB. § 107

Produce AB to meet this line at Q, Post. 2

and draw DQ. Post. 1

Similarly, draw EQ, and from D construct a line ∥ to EQ, meeting AB produced at R, and draw ER.

Continue to reduce the number of sides of the polygon until the required $\triangle EPR$ is obtained.

Proof. Polygon $AQDEF$ has one side less than $ABCDEF$.

Now $ABDEF$ is common to both polygons,

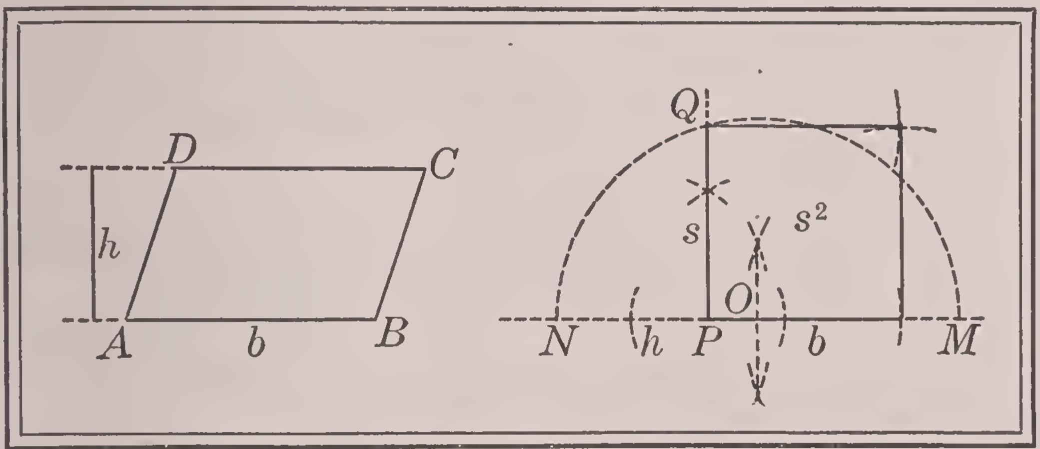
and $\triangle BQD = \triangle BCD$. § 245

$\therefore AQDEF = ABCDEF$. Ax. 1

Similarly, $AREF = AQDEF$, and $EPR = AREF$.

Proposition 10. Transforming a Parallelogram

258. Problem. *Construct a square equivalent to a given parallelogram.*



Given the $\square ABCD$ **with base** b **and altitude** h.

Required to construct a square equivalent to $\square ABCD$.

Construction. On any line take $NP = h$ and $PM = b$.

Construct the mean proportional s to h and b. § 232

With s as radius, construct the required square s^2 by drawing arcs as shown. Post. 4

Proof. Since s is $\perp$ to NM, Const.

then $h : s = s : b$. § 217

$\therefore s^2 = bh$. § 198, 1

But $\square ABCD = bh$. § 243

$\therefore s^2 = \square ABCD$. Ax. 5

259. Corollary. *Construct a square equivalent to a given triangle.*

Construct s so that $b : s = s : \frac{1}{2} h$.

260. Corollary. *Construct a square equivalent to a given polygon.*

Reduce the polygon to an equivalent $\triangle$ (§ 257), and then construct a square equivalent to this $\triangle$ (§ 259).

Exercises. Constructions

1. Construct a square which shall have twice the area of a given square.

2. Construct a triangle equivalent to the sum of any two given triangles.

3. Construct a right triangle equivalent to a given oblique triangle.

4. Construct a rectangle equivalent to a given parallelogram, and with its altitude equal to a given line.

5. Construct a triangle equivalent to a given triangle, and with one side equal to a given line.

6. Construct a right triangle equivalent to a given triangle, and with one of the sides of the right angle equal to a given line.

7. Construct a right triangle equivalent to a given triangle, and with its hypotenuse equal to a given line.

8. Divide a given triangle into two equivalent parts by a line through a given point P in the base.

9. Construct a polygon similar to two given similar polygons and equivalent to their sum.

Exs. 9–12 are often given in older geometries as fundamental constructions, but in later textbooks they are usually omitted or are given as optional problems. Since they are not needed in proving other propositions, they may be omitted except by students who are specializing in mathematics.

10. Construct a polygon similar to a given polygon and such that it has a given ratio to it.

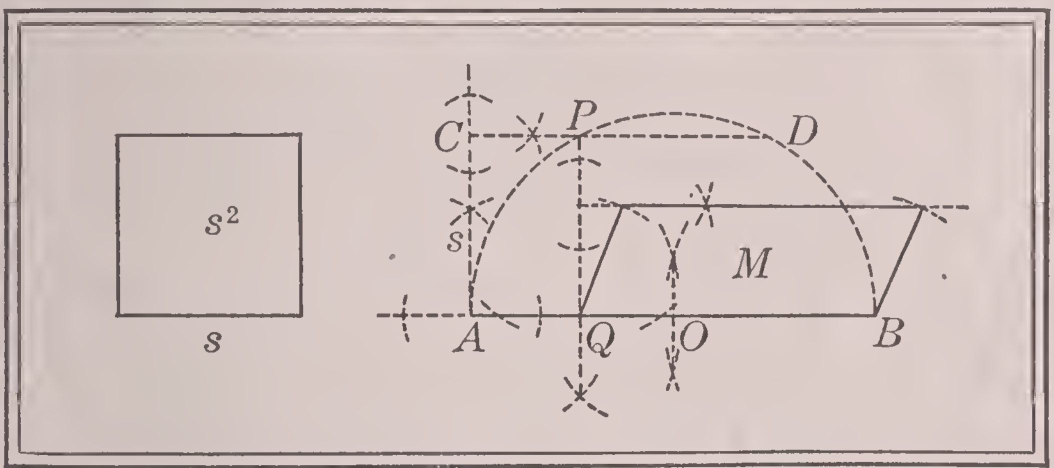
11. Construct a polygon similar to a given polygon and equivalent to another given polygon.

12. Construct a square which shall have a given ratio to a given square.

III. Supplementary Constructions

Proposition 11. Constructing a Parallelogram

261. Problem. *Construct a parallelogram equivalent to a given square, and with the sum of its base and altitude equal to a given line.*



Given the square s^2 with side s, and the line AB.

Required to construct a ▱ equivalent to s^2, and with the sum of its base and altitude equal to AB.

Construction. Bisect AB as at O (§ 102), and with O as center and OA as radius, construct a semicircle (Post. 4).

At A construct a $\perp$ to AB (§ 104), and on it take $AC = s$.

At C construct $CD \parallel$ to AB, cutting the ⊙ at P. § 107

At P construct $PQ \perp$ to AB. § 105

Then any ▱, as M, with AQ for altitude and QB for base is equivalent to s^2.

Proof. Since $AQ : PQ = PQ : QB$ (§ 217), then $\overline{PQ}^2 = AQ \cdot QB$, and since PQ is $\parallel$ to CA (§ 57), we have $PQ = CA = s$ (§ 80).

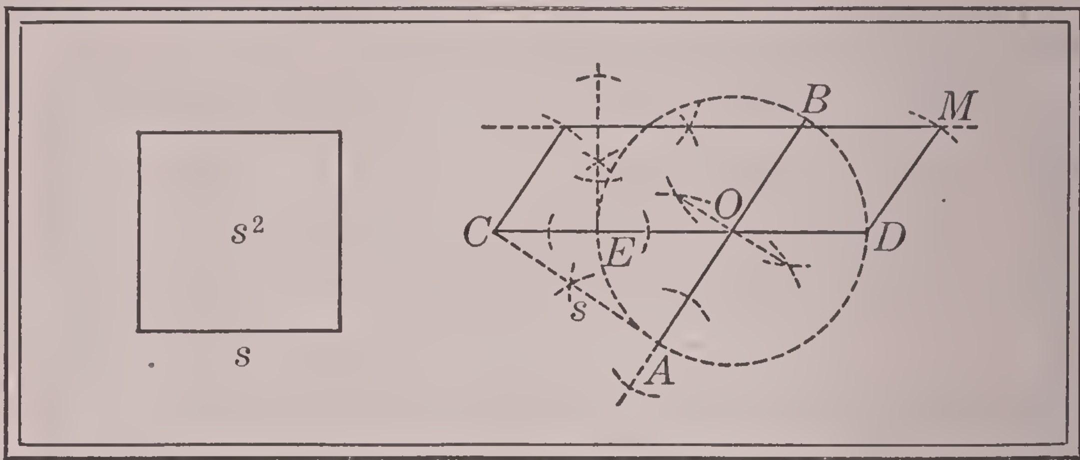
$\therefore AQ \cdot QB = s^2$. Ax. 5

Then $M = AQ \cdot QB = s^2$. § 243, Ax. 5

This theorem solves geometrically the equations $x + y = a$, $xy = b$.

Proposition 12. Constructing a Parallelogram

262. Problem. *Construct a parallelogram equivalent to a given square, and with the difference between its base and altitude equal to a given line.*



Given the square s^2 with side s, and the line AB.

Required to construct a ▱ *equivalent to* s^2, *with the difference between its base and altitude equal to AB.*

Construction. Bisect AB as at O (§ 102), and with O as center and OA as radius, construct a ⊙ (Post. 4).

At A construct a tangent to the ⊙, § 195

and on it take $AC = s$.

Through O draw CD as shown. Post. 1

Then any ▱, as CM, with CD for its base and CE for its altitude, is equivalent to s^2.

Proof. $CD : s = s : CE$. § 222

$\therefore s^2 = CD \cdot CE$. § 198, 1

But $CM = CD \cdot CE$. § 243

$\therefore CM = s^2$. Ax. 5

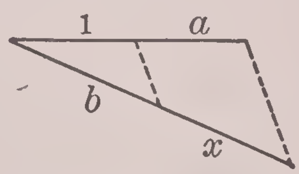
Also, $CD - CE = ED = AB$. § 134, 3

By this theorem we solve geometrically the algebraic problem of finding x and y in the equations $x - y = a$, $xy = b$.

Exercises. Review

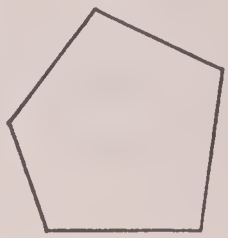
1. Omitting §§ 255–262, make a list of the numbered propositions in Book IV, stating under each the propositions in Books I–IV upon which it depends either directly or indirectly.

2. This figure shows an angle cut by parallel lines. Prove that $x = ab$, and thus show that we may construct a line segment equal to the product of two line segments.



We thus see that we may think of a line, as well as a rectangle, as representing the product of two line segments.

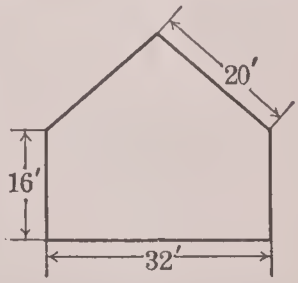
3. Draw a figure of about this shape. Then construct a triangle equivalent to this polygon. Finally, construct a square equivalent to the triangle, measure the square, and thus find the area of the original polygon.



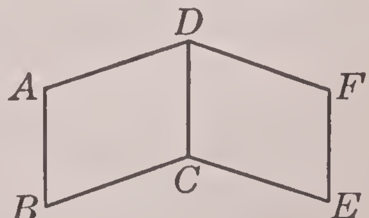
4. Construct a square equivalent to the difference between two given squares.

5. This figure represents the cross section of a barn. Find the area of the section.

In finding the number of cubic feet in the barn we multiply the area of the cross section by the length of the barn. This shows a reason for finding the areas of the cross sections of barns, pipes, canals, railway embankments, and the like.

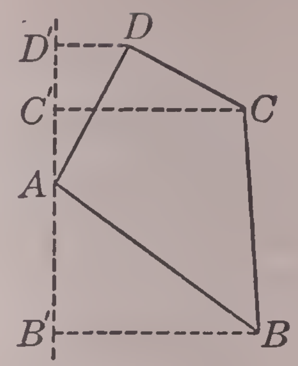


6. In this figure the $\square$ $BCDA$ and $ECDF$ are equivalent. Prove that the triangle formed by joining F to A and B is equivalent to either parallelogram.

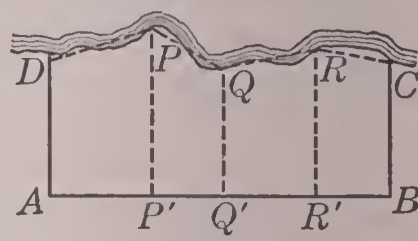


7. In the figure of Ex. 6 draw the diagonals AC and FC. Then prove the quadrilateral $ACFD$ equivalent to either parallelogram.

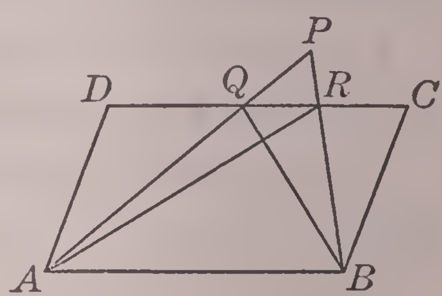
8. In surveying the field $ABCD$ a surveyor runs a north-and-south line through A, and from it lays off the ⊥s BB', CC', and DD'. By measuring he finds that $BB'=38$ rd., $CC'=35$ rd., $DD'=14$ rd., $B'A=28$ rd., $B'C'=42$ rd., and $AD'=26$ rd. Find the area of the field in square rods; in acres.



9. Wishing to find the area of a field $ABCD$ bounded on one side by a river, a surveyor made a map as here shown by constructing the ⊥s AD, $P'P$, $Q'Q$, $R'R$, BC to AB. He found that $AP'=26$ rd., $P'Q'=18$ rd., $Q'R'=23$ rd., $R'B=18$ rd., $AD=35$ rd., $PP'=42$ rd., $QQ'=32$ rd., $RR'=38$ rd., $BC=35$ rd. Find the approximate area of the field.



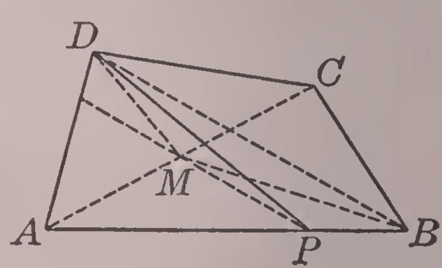
10. In this figure, $ABCD$ is a parallelogram. Prove that $\triangle PQB$ is equivalent to $\triangle PRA$.



11. Generalize Ex. 10 by first letting P move down to rest on the line DC and seeing if Ex. 10 holds true. Then let P move down below DC so as to lie within the parallelogram, and let Q lie on AP produced and R on BP produced.

12. If P is any point in the diagonal AC of ▱$ABCD$, then $\triangle ABP$ is equivalent to $\triangle APD$.

13. A surveyor wishes to divide a field $ABCD$ into two equivalent parts by a line DP drawn from the vertex D. How should he proceed to do it?



Let M bisect AC and construct $MP \parallel$ to DB. From this suggestion show how the surveyor solved the problem.

BOOK V

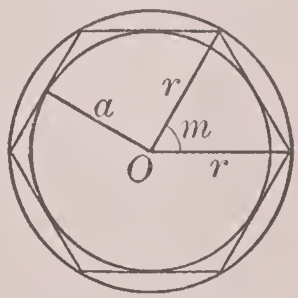
REGULAR POLYGONS AND THE CIRCLE

I. Fundamental Theorems

263. Regular Polygon. A polygon that is both equiangular and equilateral is called a *regular polygon* (§ 92).

264. Circumscribed and Inscribed Circles. It will be proved in §§ 269 and 270 that a circle can be circumscribed about, and a circle can be inscribed in, any regular polygon (§ 156), and that these circles are concentric (§ 157).

265. Radius. The radius of the circle circumscribed about a regular polygon is called the *radius* of the polygon.



In this figure, r is the radius of the polygon.

266. Apothem. The radius of the circle inscribed in a regular polygon is called the *apothem* of the polygon.

In the figure, a is the apothem of the polygon. The apothem is evidently perpendicular to the side of the regular polygon (§ 147).

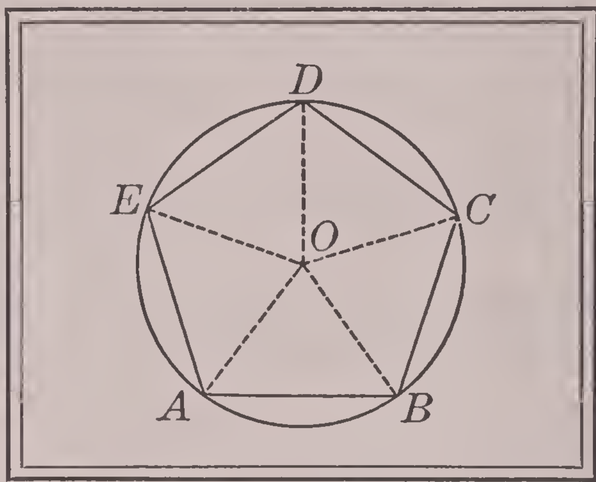
267. Center. The common center of the circles circumscribed about and inscribed in a regular polygon is called the *center* of the polygon.

268. Angle at the Center. The angle between the radii drawn to the extremities of any side of a regular polygon is called an *angle at the center* of the polygon.

In the figure, m is an angle at the center of the polygon.

Proposition 1. Circumscribed Circle

269. Theorem. *A circle can be circumscribed about any regular polygon.*



Given the regular polygon *ABCDE*.

Prove that a ⊙ can be circumscribed about ABCDE.

Proof. Let O be the center of a ⊙ constructed through three vertices A, B, C of the polygon. § 190

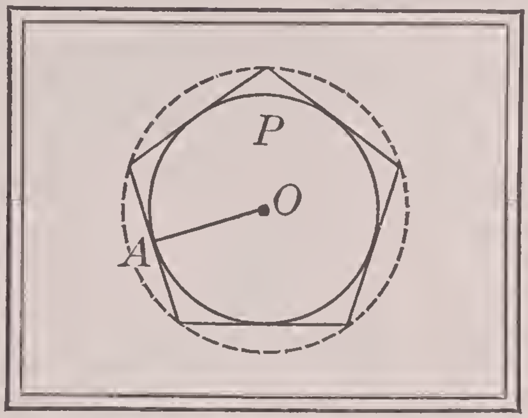
Draw	$OA, OB, OC, OD.$	Post. 1
Then	$OB = OC.$	§ 134, 1
Further,	$AB = CD.$	§ 263
Also,	$\angle CBA = \angle DCB,$	§ 263
and	$\angle CBO = \angle OCB.$	§ 42
	$\therefore \angle OBA = \angle DCO.$	Ax. 2
Then	$\triangle OAB$ is congruent to $\triangle ODC,$	§ 40
and hence	$OA = OD.$	§ 38
Then	the ⊙ through A, B, C passes through $D.$	§ 134, 6

In like manner, it can be proved that the ⊙ through $B, C,$ and D passes through E; and so on.

Hence the ⊙ constructed with O as center and OA as radius is circumscribed about the polygon. § 156

Proposition 2. Inscribed Circle

270. Theorem. *A circle can be inscribed in any regular polygon.*



Given the regular polygon P.

Prove that $\quad$ *a ⊙ can be inscribed in P.*

Proof. Let O be the center of the ⊙ circumscribed about polygon P. § 269

Since the sides of P are equal chords of the circumscribed ⊙ (§ 156), they are equidistant from O. § 150

Hence the ⊙ constructed with O as center and with the $\perp$ OA as radius (§ 146) is inscribed in the polygon. § 156

271. Corollary. *The angles at the center of any regular polygon are equal, and each is supplementary to an interior angle of the polygon.*

The ∠s at the center are corresponding ∠s of congruent △s.

Further, in the figure of § 269, $\angle AOB + \angle OBA + \angle BAO = 180°$, and $\angle BAO = \angle CBO$. Hence $\angle AOB + \angle CBA = 180°$.

272. Corollary. *An equilateral polygon inscribed in a circle is a regular polygon.*

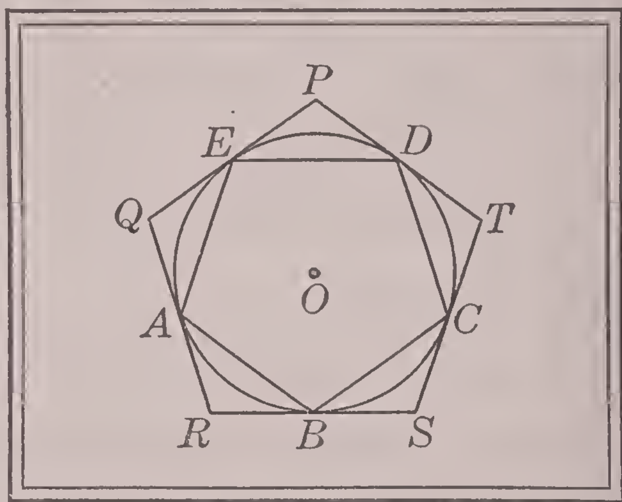
Why are the ∠s also equal?

273. Corollary. *An equiangular polygon circumscribed about a circle is a regular polygon.*

By joining consecutive points of contact of the sides show that certain isosceles △s are congruent, and thus prove the polygon equilateral.

Proposition 3. Inscribed and Circumscribed Polygons

274. Theorem. *If a circle is divided into any number of equal arcs, the chords of these arcs form a regular inscribed polygon; and the tangents at the points of division form a regular circumscribed polygon.*



Given the ⊙ O divided into equal arcs by A, B, C, D, and E, the chords AB, BC, CD, DE, EA, and the tangents PQ, QR, RS, ST, TP at E, A, B, C, D respectively.

Prove that ABCDE is a regular inscribed polygon and that PQRST is a regular circumscribed polygon.

Proof. The arcs AB, BC, CD, DE, EA are equal. Given

Hence $AB = BC = CD = DE = EA$, § 139

because if two arcs ⋯ are equal, the arcs have equal chords.

Also, $ABCDE$ is inscribed in the ⊙. § 156

∴ $ABCDE$ is a regular inscribed polygon. § 272

Since the arcs are equal, Given

$\angle P = \angle Q = \angle R = \angle S = \angle T$, § 179

because an ∠ formed by ⋯ two tangents ⋯ is measured by half the difference between its intercepted arcs.

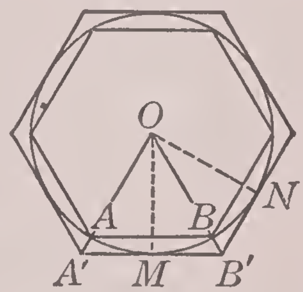
Also, $PQRST$ is circumscribed about the ⊙. § 156

∴ $PQRST$ is a regular circumscribed polygon. § 273

275. Corollary. *Tangents to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.*

For it is shown in § 274 that $PQRST$ is a regular circumscribed polygon. It has as many sides as there are vertices of $ABCDE$, and $ABCDE$ has as many vertices as it has sides.

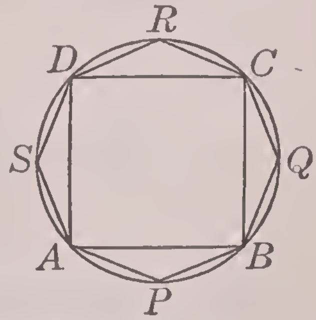
276. Corollary. *Tangents to a circle at the midpoints of the arcs of the sides of a regular inscribed polygon form a regular circumscribed polygon, whose sides are parallel to the sides of the inscribed polygon and whose vertices lie on radii produced of the inscribed polygon.*



Since M is the midpoint of arc AB (given), then $\angle A'OM = \angle B'OM$ (§ 137). Also, $OA = OB$ (§ 134, 1), and OM is a common side. Hence OM bisects AB (§§ 40, 38). Then, since two corresponding sides AB and $A'B'$ are both $\perp$ to OM (§§ 142, 147), they are $\parallel$ (§ 57).

Further, since the tangents MB' and NB' intersect at a point equidistant from OM and ON (§ 149), they intersect upon the bisector of $\angle MON$ (§ 183). But OB bisects $\angle MON$ (§ 137). Hence MB' and NB' intersect on OB produced.

277. Corollary. *Lines drawn from each vertex of a regular inscribed polygon to the midpoints of the arcs of adjacent sides of the polygon form a regular inscribed polygon of double the number of sides.*



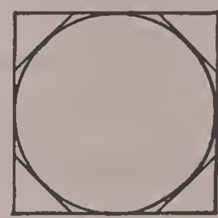
Let $ABCD$ be the given inscribed polygon, and P, Q, R, S the midpoints of the arcs of its adjacent sides. Then because arcs $AP, PB, BQ, \cdots$ are halves of equal arcs (§ 140), they are equal (Ax. 4). Hence $APBQC \cdots$ is a regular inscribed polygon (§ 274), and since each arc of polygon $ABCD$ now has two chords in place of one, the polygon $APBQC \cdots$ has double the number of sides of the polygon $ABCD$.

The work on inscribed and circumscribed polygons is essential to the understanding of the propositions in connection with the measurement of the circle, as will be shown later.

Exercises. Inscribed and Circumscribed Polygons

1. The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides; and the perimeter of a regular circumscribed polygon is greater than that of a regular circumscribed polygon of double the number of sides.

2. Tangents at the midpoints of the arcs between adjacent points of contact of the sides of a regular circumscribed polygon form a regular circumscribed polygon of double the number of sides.



3. The radius drawn to any vertex of a regular polygon bisects the angle at the vertex.

In a square of side s and radius r find the following:

4. r when $s=8$. **5.** s when $r=9$.

In an equilateral triangle of side s, radius r, apothem a, and area A find the following:

6. s when $r=4$. **8.** s when $a=\sqrt{3}$.

7. a when $s=\sqrt{3}$. **9.** A when $s=\sqrt{3}$.

Find the area of the square inscribed in a circle of radius:

10. 4 in. **11.** 6 in. **12.** 10 in. **13.** n inches.

In a regular octagon (§ 90) find the number of degrees in:

14. The angle at the center.

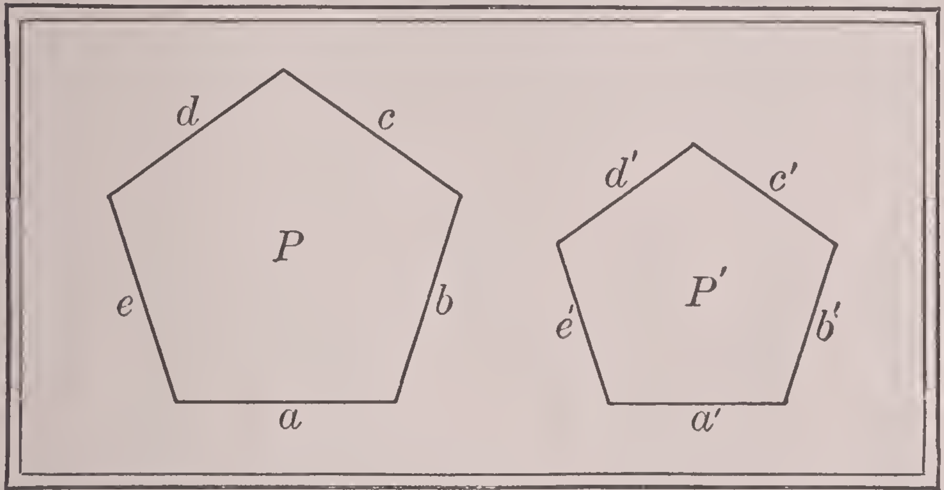
15. Each angle of the polygon.

16. The sum of one angle at the center and one angle of the polygon.

17. The radius of an equilateral triangle is how many times the apothem? what part of the side?

Proposition 4. Similar Regular Polygons

278. Theorem. *Two regular polygons of the same number of sides are similar.*



Given the regular polygons P and P', each of n sides.

Prove that $\quad$ *P and P' are similar.*

Proof. Since $\quad P$ and P' are regular, $\quad$ Given

then $\quad$ each $\angle$ of $P = (n-2)/n$ st. $\angle$s,

and $\quad$ each $\angle$ of $P' = (n-2)/n$ st. $\angle$s. $\quad$ § 96

$\therefore$ P and P' are mutually equiangular. $\quad$ Ax. 5

Furthermore, $\quad a = b = c = d = e,$

and $\quad a' = b' = c' = d' = e'.$ $\quad$ § 263

Then $\quad \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'} = \dfrac{d}{d'} = \dfrac{e}{e'};$ $\quad$ Ax. 4

that is, the corresponding sides of P and P' are proportional.

$\therefore$ P and P' are similar. $\quad$ § 205

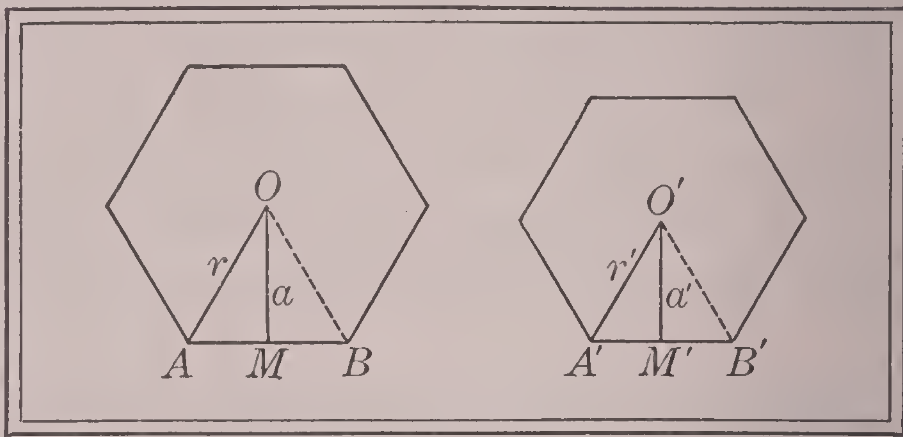
279. Corollary. *The areas of two regular polygons of the same number of sides are to each other as the squares on any two corresponding sides.*

Since the polygons are similar (§ 278), their areas are to each other as the squares on any two corresponding sides (§ 251).

PS

Proposition 5. Perimeters of Regular Polygons

280. Theorem. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, and also as their apothems.*



Given two regular polygons of n sides, with centers O and O', perimeters p and p', radii r and r' (or OA and $O'A'$), and apothems a and a' (or OM and $O'M'$) respectively.

Prove that $p:p'=r:r'=a:a'$.

Proof. Draw the radii OB, $O'B'$. Post. 1

Now $p:p'=AB:A'B'$. §§ 278, 224

Furthermore, $\angle AOB=\angle A'O'B'$, § 271, Post. 9

and $OA:OB=1=O'A':O'B'$. § 134, 1

Hence $\triangle OAB$ and $O'A'B'$ are similar, § 213

and $AB:A'B'=r:r'$. § 205

Also, $\triangle AMO$ and $A'M'O'$ are similar. § 210

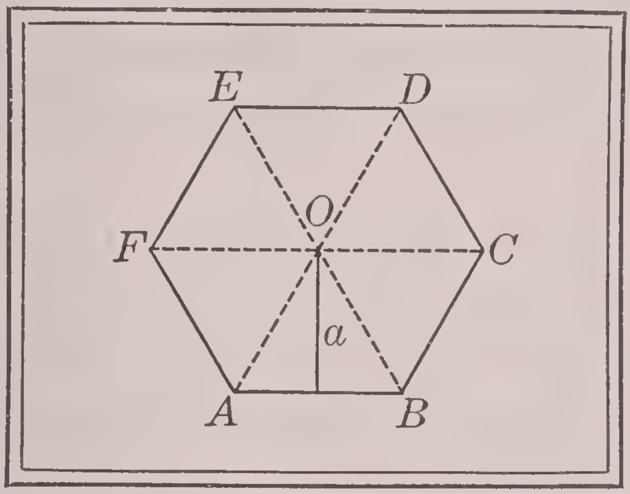
Hence $r:r'=a:a'$. § 205

$\therefore p:p'=r:r'=a:a'$. Ax. 5

281. Corollary. *The areas of two regular polygons of the same number of sides are to each other as the squares on the radii of the circumscribed circles, and also as the squares on the radii of the inscribed circles.*

Proposition 6. Area of a Regular Polygon

282. Theorem. *The area of a regular polygon is half the product of its apothem and its perimeter.*



Given the regular polygon *ABCDEF* with apothem *a*, perimeter *p*, and area *S*.

Prove that $S = \frac{1}{2}ap.$

Proof. Draw the radii OA, OB, OC, $\cdots$ to the successive vertices of the polygon, thus dividing the polygon into as many congruent ⨺ (§ 47) as it has sides.

The apothem is the common altitude of these ⨺, and the area of each $\triangle$ is $\frac{1}{2}a \times$ the base. § 244

Hence the sum of the areas of all the congruent ⨺ is $\frac{1}{2}a \times$ the sum of all the bases. Ax. 1

But the sum of the areas of the ⨺ is the area of the polygon, and the sum of the bases is its perimeter. Ax. 10

$\therefore S = \frac{1}{2}ap.$ Ax. 5

283. Similar Parts. In different circles *similar arcs, similar sectors,* and *similar segments* are such arcs, sectors, and segments as correspond to equal angles at the center.

For example, two arcs of 30° in different circles are similar arcs, and the sectors formed by drawing radii to the ends of the arcs are similar sectors.

Exercises. Regular Polygons

1. Find the ratio of the perimeters and the ratio of the areas of two regular hexagons whose sides are 4 in. and 8 in. respectively.

2. Find the ratio of the perimeters and the ratio of the areas of two regular octagons whose sides are in the ratio 4 : 2.

3. Find the ratio of the perimeters of two squares whose areas are 484 sq. in. and 121 sq. in. respectively.

4. Find the ratio of the perimeters and the ratio of the areas of two equilateral triangles whose altitudes are 9 in. and 36 in. respectively.

5. The area of one equiangular triangle is 16 times that of another. Find the ratio of their altitudes.

6. The area of the cross section of a steel beam 2 in. thick is 24 sq. in. What is the area of the cross section of a beam of the same proportions and $1\frac{1}{2}$ in. thick?

7. Squares are inscribed in two circles of radii 6 in. and 18 in. respectively. Find the ratio of the areas of the squares, and also the ratio of the perimeters.

8. Squares are inscribed in two circles of radii 6 in. and 24 in. respectively, and on the sides of these squares equilateral triangles are constructed. What is the ratio of the areas of these triangles?

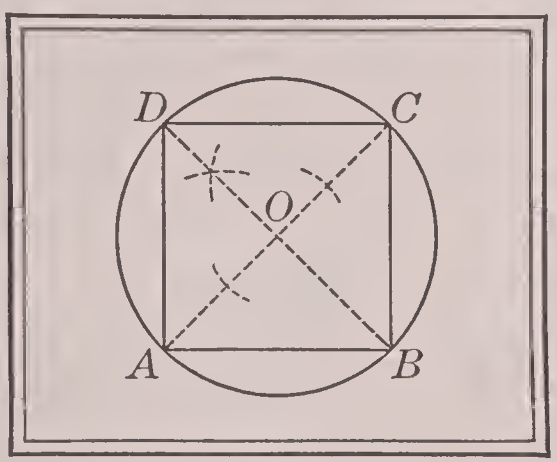
9. A square piece of timber is sawed from a round log 2 ft. in diameter so as to have the cross section of the timber the largest possible. What is the area of this cross section? What is the area of the cross section of the largest square beam that can be cut from a log of half this diameter?

10. Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides.

II. Fundamental Constructions

Proposition 7. Inscribed Square

284. Problem. *Inscribe a square in a given circle.*



Given a ⊙ with center *O*.

Required to inscribe a square in the ⊙.

Construction. Draw any diameter *AOC*. Post. 1

At *O* construct the diameter $DB \perp$ to *AC*. § 104

Draw *AB*, *BC*, *CD*, and *DA*. Post. 1

Then *ABCD* is the required square.

Proof. The ∠s *CBA*, *DCB*, *ADC*, *BAD* are rt. ∠s. § 173

Since the ∠s at the center are rt. ∠s, Const.

then the arcs *AB*, *BC*, *CD*, and *DA* are equal. § 136

$\therefore AB = BC = CD = DA$. § 139

Hence the quadrilateral *ABCD* is a square. § 15

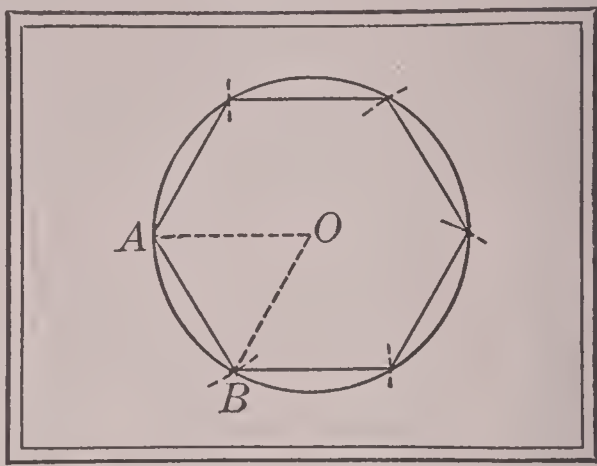
285. Corollary. *Inscribe regular polygons of 8, 16, 32, 64, ⋯ sides in a given circle.*

By bisecting the successive arcs in the figure of § 284, a regular polygon of eight sides may be inscribed in the ⊙. By continuing the process regular polygons of how many sides may be inscribed?

In general we may say that this corollary allows us to inscribe a regular polygon of 2^n sides, where n is any positive integer.

Proposition 8. Regular Inscribed Hexagon

286. Problem. *Inscribe a regular hexagon in a given circle.*



Given a ⊙ with center O.

Required to inscribe a regular hexagon in the ⊙.

Construction. Draw any radius, as OA. Post. 1

With A as center and a radius equal to OA, construct an arc intersecting the ⊙ at B. Post. 4

Draw AB. Post. 1

Then AB is a side of a regular hexagon.

Hence the required hexagon is inscribed by applying AB six times as a chord.

Proof. Draw OB. Post. 1

Then $\triangle OAB$ is equiangular. § 43

$\therefore \angle AOB$ is $\frac{1}{3}$ of a st. $\angle$, or $\frac{1}{6}$ of 2 st. $\angle$s. § 65

Hence arc AB is $\frac{1}{6}$ of the ⊙, § 171

and chord AB is a side of a regular inscribed hexagon. § 272

287. Corollary. *Inscribe an equilateral triangle in a given circle.*

Join the alternate vertices of a regular inscribed hexagon.

288. Corollary. *Inscribe regular polygons of 12, 24, 48, ⋯ sides in a given circle.*

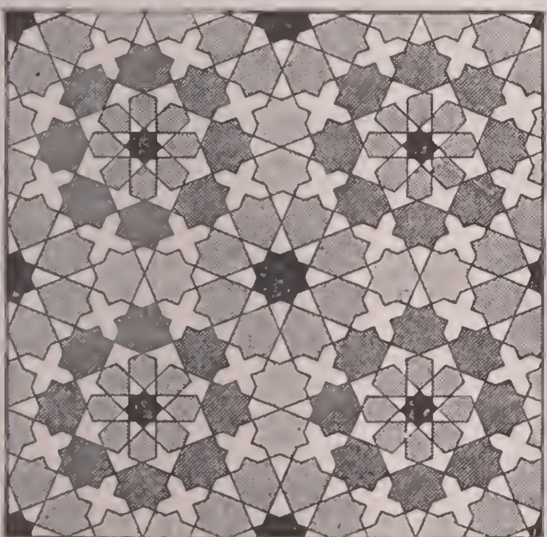
289. Extreme and Mean Ratio. If a line segment is divided into two segments such that one is the mean proportional between the whole line and the other, the line segment is said to be divided in *extreme and mean ratio.*

The name comes from the fact that one part is a mean and the whole line segment and the other part are extremes.

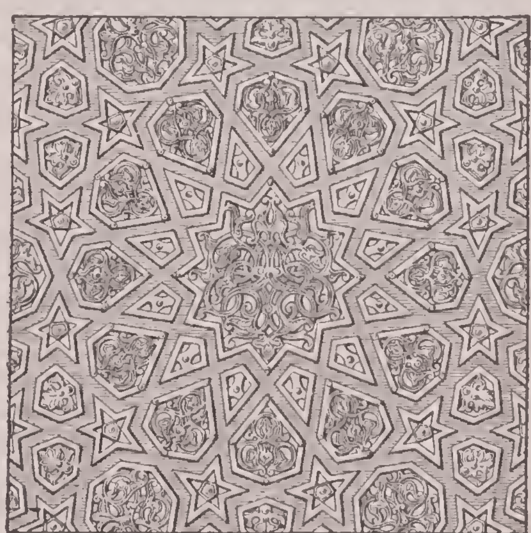
For example, the line segment a is divided in extreme and mean ratio if a segment x is found such that $a:x=x:a-x$. From this equation it can be shown that $x=0.618\,a$. That is, $x=0.6\,a$ and $a-x=0.4\,a$, approximately, so that the division is about 2:3.

This division of a line segment is often called the *Golden Section,* a relatively modern term. At one time, about 1500, it was commonly called the *Divine Proportion.*

290. Geometric Forms in Art. Since the division of a line in the ratio 2 : 3 is especially pleasing to the eye, the Golden Section is often seen in architecture and in the general plans of paintings. It is also seen in leaves and flowers.



Mosaic from Damascus



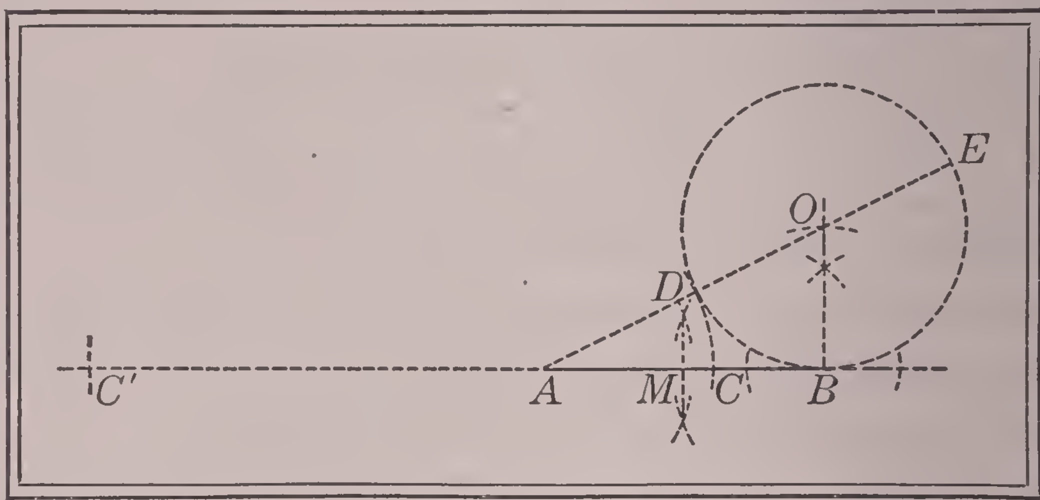
Arabic Pattern

The use of geometric forms in art is so familiar as to require only brief mention. The figures here shown illustrate combinations of regular and semiregular polygons.

Except for students specializing in mathematics, §§ 289–296 may be omitted. They are not generally required in standard courses.

Proposition 9. Golden Section

291. Problem. *Divide a given line segment in extreme and mean ratio.*



Given the line segment AB.

Required to divide AB in extreme and mean ratio.

Construction. At B construct a $\perp$ to AB, § 104
and on it take $BO = \frac{1}{2} AB = BM$. § 102

With O as center and BO as radius, construct a $\odot$. Post. 4

Draw AO, meeting the $\odot$ at D and E. Post. 1

On AB take $AC = AD$, and on BA produced take $AC' = AE$.

Then C, C' are the required points of division; that is,

$$AB : AC = AC : CB, \text{ and } AB : AC' = AC' : C'B.$$

Proof. $AE : AB = AB : AD$. § 222

Then, by the laws given in § 198, we have

$AE - AB : AB = AB - AD : AD.$	$AB + AE : AE = AD + AB : AB.$
$\therefore AE - DE : AB = AB - AC : AC.$	$\therefore AB + AC' : AC' = AD + DE : AB.$
$\therefore AC : AB = CB : AC.$	$\therefore C'B : AC' = AC' : AB.$
$\therefore AB : AC = AC : CB.$	$\therefore AB : AC' = AC' : C'B.$

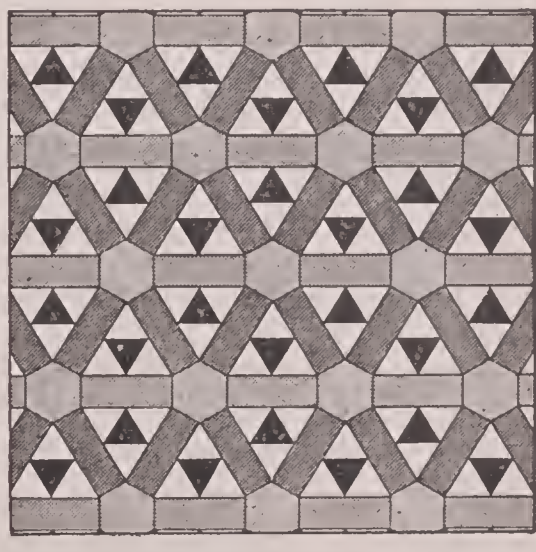
Exercises. Review

1. Given an equilateral triangle inscribed in a circle, circumscribe an equilateral triangle about the circle.

2. Given an equilateral triangle inscribed in a circle, inscribe a regular hexagon in the circle and circumscribe a regular hexagon about the circle.

3. Divide a line 2 in. long in extreme and mean ratio. Measure to the nearest $\frac{1}{16}$ in. the lengths of the two segments of both the internal and the external division and compare the results with the ratio given in § 289.

4. Consider Ex. 3 for a line $2\frac{1}{2}$ in. long; a line 3 in. long.



5. In this illustration from a mosaic in an ancient church at Constantinople it looks as if the broad bands which connect the regular hexagons formed equilateral triangles. It also looks as if the midpoints of the sides of these triangles were the vertices of other equilateral triangles. Investigate these two possibilities geometrically.

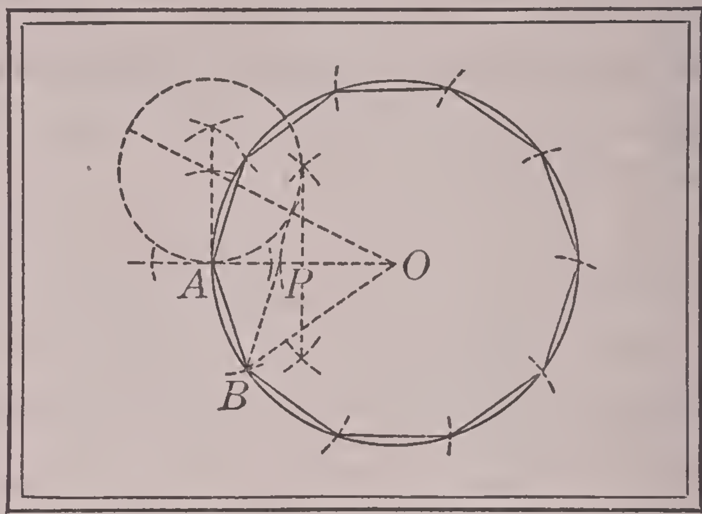
6. Find the ratio of the side of an inscribed equilateral triangle to the side of a similar circumscribed triangle.

7. In the internal division of the given line segment in § 291, which part is the mean proportional, the long part or the short one? How is it in the case of the external division? Write a statement of these two facts.

8. Find a point within a given triangle such that lines from this point to the vertices divide the triangle into three equivalent parts.

Proposition 10. Regular Inscribed Decagon

292. Problem. *Inscribe a regular decagon in a given circle.*



Given a ⊙ with center O.

Required to inscribe a regular decagon in the ⊙.

Construction. Draw any radius OA. Post. 1

Divide OA in extreme and mean ratio; § 291

that is, so that $OA:OP=OP:AP$.

With A as center and OP as radius, construct an arc intersecting the ⊙ at B. Post. 4

Draw AB. Post. 1

Then AB is a side of a regular decagon.

Hence the required regular decagon is inscribed by applying AB ten times as a chord.

Proof. Draw PB and OB. Post. 1

Now $OA:OP=OP:AP$,

and $AB=OP$. Const.

$\therefore OA:AB=AB:AP$. Ax. 5

Moreover, $\angle BAO=\angle BAP$. Iden.

Then △ OAB and BAP are similar, § 213

and hence $OA:BA=OB:BP$. § 205

But	$OA = OB.$	§ 134, 1
Then	$BA = BP,$	§ 198, 2
and hence	$BA = BP = OP.$	Ax. 5
	$\therefore \angle APB = \angle BAP$ and $\angle POB = \angle OBP.$	§ 42
But	$\angle APB = \angle POB + \angle OBP,$	§ 66
and hence	$\angle BAP = 2\angle POB.$	Ax. 5
Now	$\angle BAP = \angle BAO = \angle OBA,$	Iden., § 42
and	$\angle POB = \angle AOB.$	Iden.
Hence	$\angle BAO = \angle OBA = 2\angle AOB,$	Ax. 5
and	the sum of the $\angle$s of $\triangle OAB = 5\angle AOB.$	Ax. 1
But	the sum of these $\angle$s = a st. $\angle$.	§ 65
Hence	$5\angle AOB =$ a st. $\angle$,	Ax. 5
and	$10\angle AOB = 2$ st. $\angle$s;	Ax. 3
whence	$\angle AOB = \frac{1}{10}$ of 2 st. $\angle$s.	Ax. 4
Hence	arc AB is $\frac{1}{10}$ of the $\odot$,	§ 171
and chord AB is a side of a regular inscribed decagon.		§ 272

293. Corollary. *Inscribe a regular pentagon in a given circle.*

Join the alternate vertices of a regular inscribed decagon.

From the regular pentagon it is possible to construct the regular five-pointed star here shown.

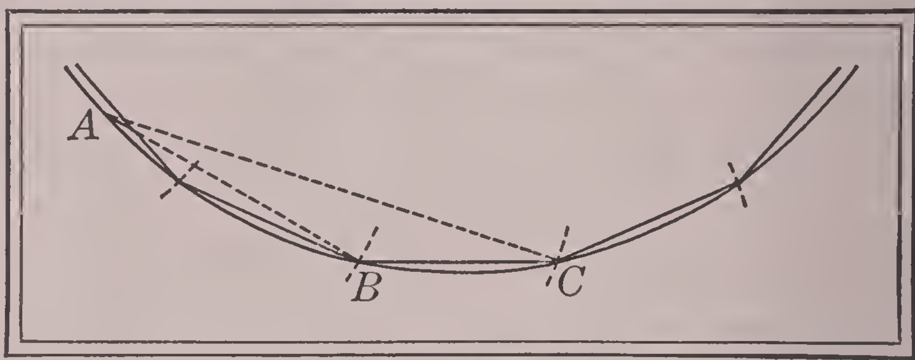
The Pythagoreans (§ 254), about 525 B.C., are supposed to have been the first to solve the problem of constructing a regular pentagon. Because of this fact they chose the regular five-pointed star as the badge of a brotherhood made up of members of their famous school.

294. Corollary. *Inscribe regular polygons of 20, 40, 80, ⋯ sides in a given circle.*

By bisecting the arcs of the sides of a regular inscribed decagon, a regular polygon of how many sides may be inscribed in the $\odot$? By continuing the process, regular polygons of how many sides may be inscribed in the $\odot$?

Proposition 11. Regular Polygon of 15 Sides

295. Problem. *Inscribe a regular polygon of fifteen sides in a given circle.*



Given a ⊙.

Required to inscribe a regular polygon of 15 sides in the ⊙.

Construction. From any point A on the ⊙ construct a chord AC equal to the radius of the ⊙ (§ 286), and a chord AB equal to a side of a regular inscribed decagon (§ 292).

In order to obtain a distinct figure only a portion of the ⊙ is shown, and the detailed construction of the chord AB is assumed from § 292.

Draw BC. Post. 1

Then BC is a side of a regular polygon of 15 sides.

Hence the required polygon is inscribed by applying BC fifteen times as chord.

Proof. Since arc AC is $\frac{1}{6}$ of the ⊙ § 286

and arc AB is $\frac{1}{10}$ of the ⊙, § 292

then arc BC is $\frac{1}{6}-\frac{1}{10}$, or $\frac{1}{15}$ of the ⊙. Ax. 2

Hence chord BC is a side of a regular inscribed polygon of 15 sides. § 272

A polygon of 15 sides is called a *pentadecagon*, but the term is rarely used.

296. Corollary. *Inscribe regular polygons of 30, 60, 120, ⋯ sides in a given circle.*

Exercises. Regular Polygons

1. A five-cent piece is placed on the table. How many five-cent pieces can be placed around it, each tangent to it and tangent to two of the others? Prove it.

Circumscribe about a given circle the following regular polygons:

2. Triangle. **4.** Hexagon. **6.** Pentagon.

3. Square. **5.** Octagon. **7.** Decagon.

Construct an angle of:

8. 36°. **9.** 18°. **10.** 9°. **11.** 24°. **12.** 12°.

With a side of given length construct:

13. An equilateral triangle. **16.** A regular octagon.

14. A square. **17.** A regular pentagon.

15. A regular hexagon. **18.** A regular decagon.

19. A regular polygon of fifteen sides.

20. Prove that the diagonals AC, BD, CE, DF, EA, FB of the regular hexagon $ABCDEF$ form another regular hexagon.

21. Prove that the diagonals AC, BD, CE, DA, EB of the regular pentagon $ABCDE$ form another regular pentagon.

In a regular inscribed polygon in which n is the number of sides, a the apothem, r the radius, A an angle, and C an angle at the center, prove the following:

22. If $n=3$, then $A=60°$, $a=\frac{1}{2}r$, and $C=120°$.

23. If $n=4$, then $A=90°$, $a=\frac{1}{2}r\sqrt{2}$, and $C=90°$.

24. If $n=6$, then $A=120°$, $a=\frac{1}{2}r\sqrt{3}$, and $C=60°$.

25. If $n=10$, then $A=144°$, $a=\frac{1}{4}r\sqrt{10+2\sqrt{5}}$, and $C=36°$.

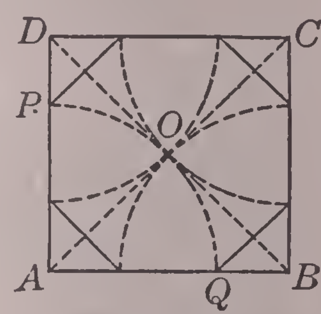
26. Find the perimeter of an equilateral triangle inscribed in a circle of radius 3 in.

27. Find the perimeter of an equilateral triangle circumscribed about a circle of radius 2 in.

28. Find the perimeter of a regular hexagon circumscribed about a circle of radius 4 in.

29. From a circular log with a diameter of 18 in. a builder wishes to cut a column with its cross section as large a regular octagon as possible. Find the length of a side of the cross section.

30. In the figure here shown $ABCD$ is a square. Arc POQ is constructed as part of a circle with center A and radius AO, and the other arcs are constructed in a similar manner. Prove that the octagon seen in the figure is regular.



31. The area of a regular inscribed hexagon is what part of the area of a regular hexagon circumscribed about the same circle?

32. Construct a regular pentagon, given one of the diagonals.

33. In a given equilateral triangle inscribe three equal circles, tangent each to the other two and to two sides of the triangle.

34. The points $A, B, C, D, \cdots$ are consecutive vertices of a regular inscribed octagon, and $A, B', C', D', \cdots$ are consecutive vertices of a regular polygon of twelve sides inscribed in the same circle. Find the angle formed by each pair of the following lines, produced if necessary:

(1) AB and AB'. (3) AB and AC. (5) AB' and AD.

(2) AB and AC'. (4) AB and AD. (6) $B'C'$ and AC.

III. Circle Measurement

297. Plan of Measurement. For practical purposes we can find the circumference of a circle very easily. If we wind a piece of paper about a cylinder, prick through the paper with a needle where the paper overlaps, and then flatten the paper out on a table, we can measure with a fair degree of accuracy the distance between the two points thus made. Evidently, however, this is not as accurate as the measurement of a straight line by means of a pair of dividers or compasses, because paper tends to stretch or to contract.

For scientific purposes we therefore resort to mathematics. One reason for showing how to inscribe and circumscribe regular polygons, and then to double the number of sides, is to construct polygons that approach nearer and nearer to the circle. Since we can measure these polygons, both as to perimeter and as to area, we can thus approximate the circumference, and can also approximate the area which the circle incloses. We may carry this approximation to any degree of accuracy that we wish.

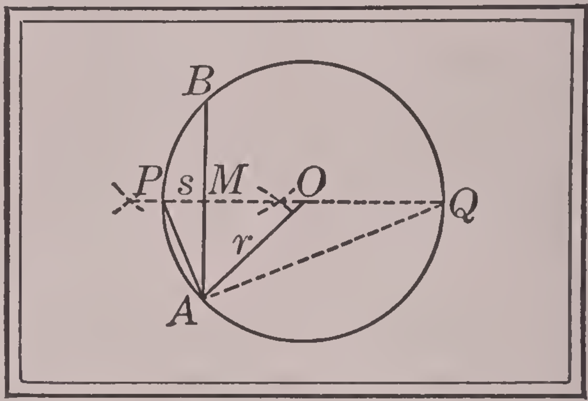
For example, if we find the perimeter of an inscribed square, then find the perimeter of an inscribed regular octagon, and continue this process for polygons of 16, 32, 64, $\cdots$ sides, we can find a perimeter which approaches as near the circumference as we choose, and similarly for the area inclosed by the circle.

In this way we can find the approximate ratio of the circumference of a circle to its diameter. The student who takes up the calculus in college will there find a simpler method of solving this problem.

We shall, therefore, first consider the problem of finding the perimeter of a regular polygon of double the number of sides of a given regular polygon; or, what is more simple, of finding one side of such a polygon.

Proposition 12. Doubling the Sides

298. Problem. *Given the side and the radius of a regular inscribed polygon, find the side of a regular inscribed polygon of double the number of sides.*



Given s (or AB), the side, and r, the radius, of a regular polygon inscribed in the ⊙ with center O.

Required to find a side of a regular inscribed polygon of double the number of sides.

Solution. Construct PQ, the $\perp$ bisector of s. §§ 102, 104

Draw AP and AQ. Post. 1

Then PQ is a diameter and bisects arc AB. § 143

$\therefore AP$ is the required side. § 277

Since $AM = \frac{1}{2}s$, $\overline{OM}^2 = r^2 - \frac{1}{4}s^2$; § 253

whence $OM = \sqrt{r^2 - \frac{1}{4}s^2}$. Ax. 6

$\therefore PM = r - OM = r - \sqrt{r^2 - \frac{1}{4}s^2}$. Ax. 5

Further, $\triangle$ AMP and QAP are similar, § 210

and hence $PM : AP = AP : PQ$. § 205

Then $\overline{AP}^2 = PQ \cdot PM$; § 198, 1

whence $\overline{AP}^2 = 2r\left(r - \sqrt{r^2 - \frac{1}{4}s^2}\right)$. Ax. 5

Hence $AP = \sqrt{2r\left(r - \sqrt{r^2 - \frac{1}{4}s^2}\right)}$, Ax. 6

or $AP = \sqrt{r\left(2r - \sqrt{4r^2 - s^2}\right)}$.

299. Constant and Variable. If we inscribe a regular polygon in a given circle, and then continue to double the number of sides of this polygon, the perimeter continues to vary in size, approaching nearer and nearer the circle, which remains constantly the same in size. A quantity considered as having a fixed value throughout a given discussion is called a *constant*, and a quantity considered as having different successive values is called a *variable*.

In the above case, the perimeter of the polygon, as we increase the number of sides, is a variable, but the circle is a constant.

300. Limit. When a variable so approaches a constant that the difference between the two may become and remain less than any assigned positive quantity, however small, the constant is called the *limit* of the variable.

Sometimes variables can reach their limits and sometimes they cannot. For example, a chord may increase in length up to a certain limit, the diameter, and it can reach this limit and still be a chord; it may decrease, approaching the limit 0, but it cannot reach this limit and still be a chord as we define it in elementary work.

If p is the perimeter of a regular inscribed or of a regular circumscribed polygon and c is the circle, we say that "p tends to c," or "p approaches c as its limit," indicating this by the symbol $p \longrightarrow c$.

301. Principles of Limits. From the above definition we may assume as postulates the following principles:

1. *If a variable x approaches a finite limit l, and if c is a constant, then cx approaches the limit cl, and $\frac{x}{c}$ approaches the limit $\frac{l}{c}$.*

That is, if $x \longrightarrow l$, then $cx \longrightarrow cl$ and $\frac{x}{c} \longrightarrow \frac{l}{c}$.

2. *If, while approaching their respective limits, two variables are always equal, their limits are equal.*

For if the limits were unequal, the two variables would be unequal when they were very near their limits.

302. Area of a Circle. The area of the space inclosed by a circle is called the *area* of the circle.

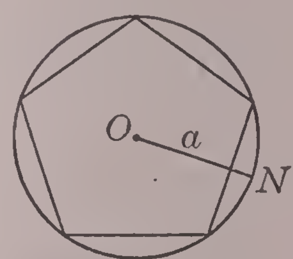
With the modern definition of a circle as a line, the expression "area of a circle" has no meaning unless it is specifically defined. We therefore define it as a brief form of the longer expression "area inclosed by a circle."

303. Limits related to the Circle. From what has been said concerning the circle and the regular inscribed polygon we may assume as true the following statements:

1. *The circumference of a circle is the limit of the perimeter of a regular inscribed or of a regular circumscribed polygon as the number of sides is indefinitely increased.*

2. *The area of a circle is the limit of the area of a regular inscribed or of a regular circumscribed polygon as the number of sides is indefinitely increased.*

3. *If the number of sides of a regular inscribed polygon is indefinitely increased, the apothem of the polygon approaches the radius of the circle as its limit.*



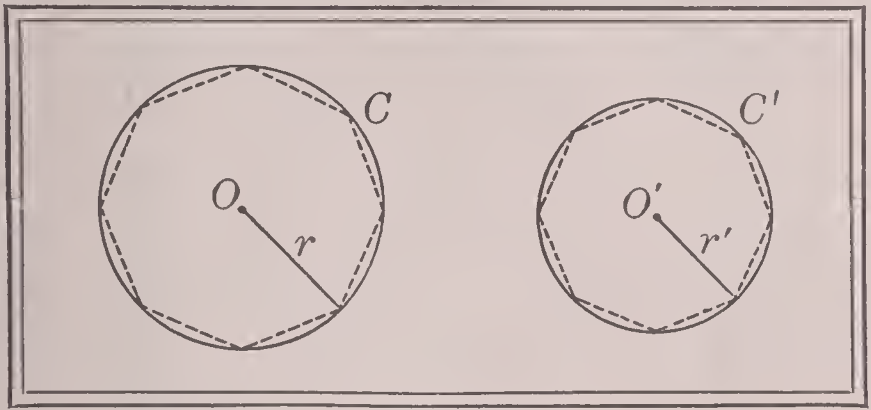
In this figure, if n is the number of sides of the polygon, then $a \longrightarrow ON$ as $n \longrightarrow \infty$; that is, a approaches ON as its limit as the number of sides increases without limit. We are not justified in saying that the expression $n \longrightarrow \infty$ means that n approaches infinity as a limit, because the word "infinity" means without limit. We may, however, say that "n tends to infinity" or that "n approaches infinity."

In higher mathematics the statements given above are proved with the same care with which we prove a proposition in the geometry of rectilinear figures, but in an elementary treatment of measurement it is impossible to give satisfactory proofs; indeed, the truth of the statements would be no more evident if the proofs were given. By informal discussion their truth is as apparent as that of any postulate.

In the case of a regular circumscribed polygon the apothem is always the same as the radius of the circle, and hence, with this fact understood, we may say that all three assumptions apply to either inscribed or circumscribed regular polygons.

Proposition 13. Ratio of Circumferences

304. Theorem. *Two circumferences have the same ratio as the radii.*



Given the ⓢ O and O' with circumferences C and C' and radii r and r' respectively.

Prove that $C : C' = r : r'$.

Proof. Let p, p' be the perimeters of two similar regular inscribed polygons. § 269

Then $p : p' = r : r'$. § 280

$\therefore pr' = p'r$. § 198, 1

Let the number of sides be increased uniformly.

Then $p \to C$, and $p' \to C'$, § 303, 1

and hence $Cr' = C'r$. § 301

$\therefore C : C' = r : r'$. § 198, 3

305. Corollary. *The ratio of any circle to its diameter is constant.*

Since $C : C' = 2r : 2r'$, then $C : 2r = C' : 2r'$.

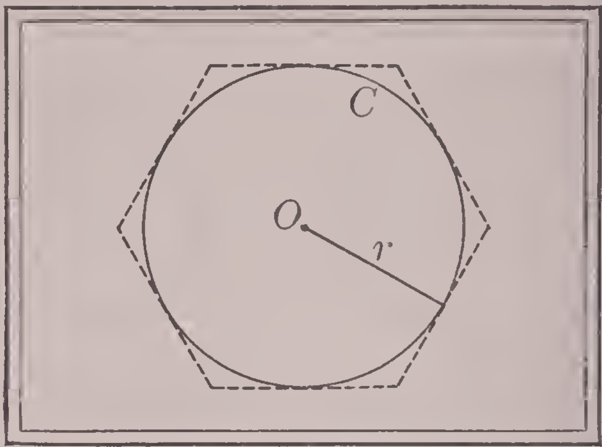
306. Symbol π. The constant ratio of a circle to its diameter is represented by the Greek letter π (pī).

307. Corollary. *In any circle, $C = 2\pi r = \pi d$.*

By definition (§ 306), $\pi = \frac{C}{2r} = \frac{C}{d}$; whence $C = 2\pi r$ and $C = \pi d$.

Proposition 14. Area of a Circle

308. Theorem. *The area of a circle is half the product of the radius and the circumference.*



Given the $\odot O$ with radius r, circumference C, and area A.

Prove that $A = \frac{1}{2} rC$.

Proof. Circumscribe about the $\odot$ a regular polygon of n sides, and let p be its perimeter and A' its area. § 270

Then $A' = \frac{1}{2} rp$. § 282

Let n be increased indefinitely.

Since $p \to C$ § 303, 1

and r is constant,

then $\frac{1}{2} rp \to \frac{1}{2} rC$. § 301, 1

Also, $A' \to A$. § 303, 2

But, always, $A' = \frac{1}{2} rp$. § 282

$\therefore A = \frac{1}{2} rC$. § 301, 2

309. Corollary. *The area of a circle is π times the square on the radius.*

For $A = \frac{1}{2} rC = \frac{1}{2} r \times 2 \pi r = \pi r^2$.

310. Corollary. *The areas of two circles are to each other as the squares on the radii.*

Exercises. Circumference and Area

1. *The area of a sector is half the product of the radius and the arc.*

2. If the circumference of one circle is twice that of another, the square on the radius of the first is how many times the square on the radius of the second?

3. If the circumference of one circle is four times that of another, an equilateral triangle constructed on the diameter of the first as side has how many times the area of an equilateral triangle constructed on the diameter of the second as side?

4. A water pipe with a diameter of 3 in. has a circumference of 9.425 in. Find the circumference of a water pipe which has a diameter of 4 in.

5. A wheel with a circumference of 8 ft. has a diameter, expressed to the nearest 0.01 ft., of 2.55 ft. Find the circumference of a wheel with a diameter of 3.175 ft.

6. A regular hexagon is 4 in. on a side. Find both its apothem and its area to the nearest 0.01.

7. If the radius of one circle is four times that of another, and if the area of the smaller circle is 31.4 sq. in., what is the area of the larger circle?

8. If the radius of one circle is five times that of another, and if the area of the smaller circle is 9.6 sq. in., what is the area of the larger circle?

9. The circumferences of two cylindric steel shafts are 7 in. and $3\frac{1}{2}$ in. respectively. The area of the cross section of the first shaft is how many times that of the second?

10. If the arc of a sector of a circle $3\frac{1}{2}$ in. in diameter is 2 in. long, what is the area of the sector?

Use Ex. 1, above, in finding the required area.

Proposition 15. The Value of π

311. Problem. *Find the approximate value of the ratio of the circumference of a circle to its diameter.*

Given a ⊙ with circumference *C* and diameter *d*.

Required to find the approximate value of π.

Solution. Let s_6 be the length of a side of a regular inscribed polygon of 6 sides, s_{12} of 12 sides, and so on.

The student need not perform the computations or recall the following steps, but he should understand the general nature of the work.

Then $s_{12} = \sqrt{r(2r - \sqrt{4r^2 - s_6^2})}$. § 298

But, when $r = 1$, $s_6 = 1$. § 286

Hence, using the successive values of s, we have

Form of Computation	Length of Side	Perimeter
$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}}$	0.51763809	6.21165708
$s_{24} = \sqrt{2 - \sqrt{4 - 0.51763809^2}}$	0.26105238	6.26525722
$s_{48} = \sqrt{2 - \sqrt{4 - 0.26105238^2}}$	0.13080626	6.27870041
$s_{96} = \sqrt{2 - \sqrt{4 - 0.13080626^2}}$	0.06543817	6.28206396
$s_{192} = \sqrt{2 - \sqrt{4 - 0.06543817^2}}$	0.03272346	6.28290510
$s_{384} = \sqrt{2 - \sqrt{4 - 0.03272346^2}}$	0.01636228	6.28311544
$s_{768} = \sqrt{2 - \sqrt{4 - 0.01636228^2}}$	0.00818121	6.28316941

Since $C = 2\pi r$, § 307

when $r = 1$, $\pi = \frac{1}{2}C$.

But, when $n = 768$, $C = 6.28317$, approximately,

and hence $\pi = 3.14159$, approximately.

For thousands of years the world tried to find the value of the incommensurable number π. The ancients generally considered the value as 3 or as $3\frac{1}{7}$. We generally use the following values: $\pi = 3.1416$, $\frac{22}{7}$, or $3\frac{1}{7}$, and $1/\pi = 0.31831$.

Exercises. Circle Measurement

Find the circumferences of circles with radii as follows:

1. 2 in.	3. 3.2 in.	5. $6\frac{1}{4}$ in.	7. 3 ft. 6 in.
2. 3 in.	4. 4.3 in.	6. $7\frac{3}{8}$ in.	8. 4 ft. 2 in.

In all the work on this page use the value 3.1416 for π.

Find the circumferences of circles with diameters as follows:

9. 4 in.	11. 6.2 in.	13. $3\frac{1}{2}$ ft.	15. 30 cm.
10. 22 in.	12. 8.3 in.	14. $2\frac{1}{8}$ in.	16. 42 mm.

Find the radii of circles with circumferences as follows:

17. 3π.	19. 15.708 in.	21. 18.8496.	23. 345.576.
18. 4π.	20. 21.9912 ft.	22. 125.664.	24. 3487.176.

Find the diameters of circles with circumferences as follows:

25. 8π.	27. $2\pi r$.	29. 188.496 in.	31. 3361.512 in.
26. π^3.	28. $3\pi a^2$.	30. 219.912 in.	32. 3173.016 in.

Find the areas of circles with radii as follows:

33. $2x$.	35. 16 ft.	37. $4\frac{1}{2}$ in.	39. 3 ft. 4 in.
34. 3π.	36. 5.8 ft.	38. $3\frac{5}{8}$ in.	40. 5 ft. 8 in.

Find the areas of circles with diameters as follows:

41. $10ab$.	43. 3.5 ft.	45. $2\frac{2}{3}$ yd.	47. 2 ft. 4 in.
42. $12\pi^2$.	44. 4.3 in.	46. $3\frac{1}{4}$ yd.	48. 3 ft. 6 in.

Find the areas of circles with circumferences as follows:

49. 3π.	50. πk.	51. 18.8496 in.	52. 333.0096 in.

Find the radii of circles with areas as follows:

53. πa^2.	54. π.	55. 12.5664.	56. 78.54.

Exercises. Applications

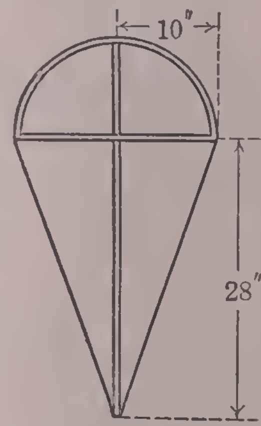
1. The diameter of a bicycle wheel is 28 in. How many revolutions does the wheel make in 8 mi.?

2. Find the diameter of an automobile wheel which makes r revolutions in half a mile.

3. A circular pond 200 yd. in diameter is surrounded by a walk 8 ft. wide. Find the area of the walk.

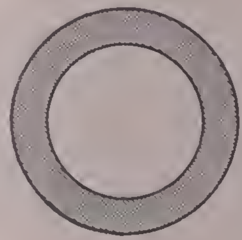
4. The span (chord) of a bridge in the form of a circular arc is 60 ft., and the highest point of the arch is 7 ft. 6 in. above the piers. Find the radius of the arch.

5. Two branch drain pipes lead into a main drain pipe. It is necessary that the cross-section area of the main pipe shall equal the sum of the cross-section areas of the two branch pipes, which are respectively 6 in. and 8 in. in diameter. Find the diameter of the main pipe.



6. The top part of the kite here shown is a semicircle and the lower part is a triangle. Find the area of the kite.

7. In making a drawing for an arch it is necessary to mark off on a circle drawn with a radius of $10\frac{1}{2}$ in. an arc that shall be 11 in. long. This is best done by finding the angle at the center. How many degrees are there in this angle?



8. In the iron washer here shown, the diameter of the hole is $2\frac{3}{4}$ in. and the width of the metal ring is $\frac{3}{4}$ in. Find the area of one face of the washer.

9. Find the area of a fan which opens out into a sector of 120° with a radius of 10 in.

10. Consider Ex. 9 for a radius of 5 in.

IV. General Review

Exercises. Review

Write a classification of the different kinds of:

1. Lines.
2. Angles.
3. Triangles.
4. Quadrilaterals.
5. Polygons.
6. Parallelograms.

State the conditions under which:

7. Two triangles are congruent; are equal in area; are similar.

8. Two straight lines are parallel.

9. Two parallelograms are equal in area.

10. Two polygons are similar.

Complete the following statements in general terms:

11. In a right triangle the square on the $\cdots$.

12. If two parallel lines are cut by a transversal, $\cdots$.

13. An angle formed by two secants drawn to a circle is measured by $\cdots$.

14. The perimeters of two similar polygons are to each other as $\cdots$, and their areas are to each other as $\cdots$.

15. Equal chords of the same circle or of equal circles $\cdots$.

16. Two central angles of the same circle or of equal circles have $\cdots$.

17. If two secants intersect within, on, or outside a circle, the product of $\cdots$.

18. The sum of the interior angles of $\cdots$.

19. The area of a polygon is $\cdots$.

20. One formula for the $\cdots$ of a circle is $\frac{1}{4}\pi d^2$.

21. One formula for a $\cdots$ is πr.

Exercises. Loci

1. Find the locus of the center of the circle inscribed in a triangle which has a given base and a given angle at the vertex.

2. Given a line segment, find the locus of the end of a tangent to a given circle such that the length of the tangent is equal to the length of the given segment.

3. Find the locus of a point from which tangents drawn to a given circle form a given angle.

4. Find the locus of the intersection of the perpendiculars from the three vertices to the opposite sides of a triangle which has a given base and a given angle at the vertex.

5. Find the locus of the midpoint of a line segment drawn from a given point to a given line.

6. Find the locus of the vertex of a triangle which has a given base and a given altitude.

7. Find the locus of a point such that the sum of its distances from two given parallel lines is constant.

8. Find the locus of a point such that the difference between its distances from two given parallel lines is constant.

9. Find the locus of a point such that the sum of its distances from two given intersecting lines is constant.

10. Find the locus of a point such that the difference between its distances from two given intersecting lines is constant.

11. Find the locus of a point such that its distances from two given points are in the ratio 3 : 4.

12. Find the locus of a point such that its distances from two given parallel lines are in the ratio $m:n$.

Exercises. Constructions

1. In a given circle inscribe a regular polygon similar to a given regular polygon.

2. Divide the area of a given circle into two equivalent parts by a circle which has the same center as the given circle.

3. Construct a circle with its circumference equal to the sum of the circumferences of two circles of given radii.

4. Construct a circle with its circumference equal to the difference between two circumferences of given radii.

5. Construct a circle with its area equal to the sum of the areas of two circles of given radii.

6. Construct a circle such that its area is three times the area of a given circle.

7. Construct a circle such that the ratio of its area to that of a given circle is $m:n$.

8. In a given square inscribe four equal circles such that each circle is tangent to two of the others and to two sides of the square.

9. In a given square inscribe four equal circles such that each circle is tangent to two of the others and to one side and only one side of the square.

10. Construct a common secant to two given circles, which are exterior to each other, such that the intercepted chords shall have the given lengths a and b.

11. Through a point of intersection of two given intersecting circles construct a common secant of a given length.

12. Construct a tangent to a given circle such that the segment intercepted between the point of contact and a given line has a given length.

Exercises. Formulas

If r is the radius of a circle, s one side of a regular inscribed polygon, and n the number of sides, prove the following, and find s to the nearest 0.01 when $r=1$:

1. If $n=3$, $s=r\sqrt{3}$.
2. If $n=4$, $s=r\sqrt{2}$.
3. If $n=6$, $s=r$.
4. If $n=5$, $s=\frac{1}{2}r\sqrt{10-2\sqrt{5}}$.
5. If $n=8$, $s=r\sqrt{2-\sqrt{2}}$.
6. If $n=10$, $s=\frac{1}{2}r(\sqrt{5}-1)$.

7. If a regular pentagon of side s is inscribed in a circle of radius r, find the apothem.

8. If a regular polygon of side s and apothem a is inscribed in a circle of radius r, prove that

$$a=\frac{1}{2}\sqrt{4r^2-s^2}.$$

9. A regular polygon of side s is inscribed in a circle of radius r. If a side of the similar circumscribed regular polygon is s', prove that

$$s'=\frac{2sr}{\sqrt{4r^2-s^2}}.$$

10. Three equal circles are constructed, each tangent to the other two. If the common radius is r, find the area inclosed by the arcs between the points of tangency.

11. Given p and P, the perimeters of regular polygons of n sides respectively inscribed in and circumscribed about a given circle of radius r, find p' and P', the perimeters of regular polygons of $2n$ sides respectively inscribed in and circumscribed about the given circle.

12. A circular plot of land a feet in diameter is surrounded by a walk b feet wide. Find the area of the circular plot and the area of the walk.

13. In Ex. 12 find the circumference at the outer edge of the walk.

Exercises. Review

1. The segment which joins the midpoints of the diagonals of a trapezoid is equal to half the difference between the bases.

2. If from any point on a circle a chord and a tangent are drawn, the perpendiculars drawn to them from the midpoint of the minor arc are equal.

3. Consider Ex. 2 with respect to the midpoint of the major arc.

4. If two equal chords are produced to meet outside a circle, the secants thus formed are equal.

5. If squares are constructed outwardly on the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular polygon of twelve sides.

6. The sum of the perpendiculars drawn to any tangent to a circle from the ends of a diameter is equal to the diameter.

7. No oblique parallelogram can be inscribed in a circle.

An oblique parallelogram has oblique angles (§ 16).

8. Two points C and D are taken on a semicircle of diameter AB. If AD and BC meet in E, and AC and BD meet in F, then EF is $\perp$ to AB.

9. If the tangents from a given point P to three given circles which do not intersect are all equal, the circle drawn with center P and passing through the points of contact of these tangents cuts the given circles at right angles.

Two circles are said to intersect at right angles if their tangents at a point of intersection are perpendicular to each other.

10. State and prove the converse of the proposition that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.

Exercises. Applications

1. On a railway curve which is the arc of a circle two points P and Q are taken and the chord PQ is found to be 400 ft. The distance from the midpoint of the arc to the midpoint of the chord is 28 ft. Find the radius of the circle.

2. Two rectangular city lots have the same depth, the frontage of the first is twice that of the second, and their combined frontage is equal to their common depth. Find the ratio of their areas and the ratio of their perimeters.

3. A ladder 50 ft. long reaches a window 40 ft. from the ground on one side of a street, and when tipped backward to rest against the building on the opposite side it reaches a window 30 ft. from the ground. How wide is the street?

4. Two wheat bins of the same height are respectively 8 ft. and 10 ft. square on the bottom. Find the dimensions of the square bottom of a third bin which has the same height as each of the other two and the same volume as the other two combined.

5. Two forces of 180 lb. and 240 lb. make an angle of 90° with each other. Compute the resultant.

The resultant is represented graphically by the diagonal of a rectangle of sides 180 and 240. See Ex. 6, p. 106.

6. In laying out a park it is desired to plant eight trees equidistant from one another and each 200 ft. from a fountain. Construct a figure with all construction lines to show how the trees should be placed.

7. A water main is to be laid to two branch pipes which have diameters of 12 in. and 18 in. respectively. The diameter of the main must be such that the area of its cross section is equal to the sum of the cross-section areas of the branches. Find the diameter of the main to the nearest $\frac{1}{4}$ in.

Exercises. Review

1. If the three points of tangency of a circle inscribed in a triangle are joined, the angles of the resulting triangle are all acute.

2. If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles of the quadrilateral form a right angle.

3. The two line segments which join the midpoints of the opposite sides of a quadrilateral bisect each other.

4. If two triangles have equal bases and equal angles at the vertex, the areas of the circumscribed circles are equal.

5. If two circles are concentric, the segments intercepted between them on any line are equal.

6. If any two consecutive sides of an inscribed hexagon are respectively parallel to their opposite sides, the remaining two sides are parallel.

7. The lines which bisect any angle of an inscribed quadrilateral and the exterior angle at the opposite vertex intersect on the circle.

8. In order that a parallelogram can be circumscribed about a circle, the parallelogram must have equal sides.

9. The area of a triangle is half the product of its perimeter and the radius of the inscribed circle.

10. The perimeter of a triangle is to any side as the altitude from the opposite vertex of the triangle is to the radius of the inscribed circle.

11. If two equivalent triangles have the same base and lie on the same side of this base, any line which cuts the triangles and is parallel to the base cuts off equal areas from the triangles.

12. In the triangle whose sides are 10, 36, and 40 compute the length of the projection of the longest side upon the shortest side.

13. Within a rhombus $ABCD$, in which A and C are opposite vertices, the point P is chosen so that $PB = PD$. Prove that A, P, and C are in the same straight line, and that $AP \cdot PC = \overline{AB}^2 - \overline{PB}^2$.

14. An isosceles $\triangle ABC$ is inscribed in a circle, and from the vertex A a chord AD is drawn to cut the base BC in the point E. Prove that $\overline{AB}^2 - \overline{AE}^2 = BE \cdot CE$.

15. In an acute $\triangle ABC$ the altitudes BD and CE intersect in the point O. Prove that $OB : OC = OE : OD$.

16. From an external point P two secants are drawn, one cutting the circle at the points A and B, and the other at the points C and D, so that $PA = 5$ in., $AB = 35$ in., and $PC = CD$. Find the length of PD.

17. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the product of the apothem and the number of sides.

18. Find the perimeter and the area of a regular octagon inscribed in a circle with a diameter of 32 in.

19. On the sides of a square $ABCD$ of side a, the points P, Q, R, S are taken such that $AP = BQ = CR = DS = \frac{2}{5}a$. Prove that $PQRS$ is a square and then find its area.

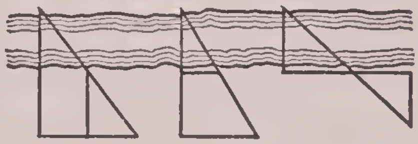
20. Each side of a triangle is $2a$ inches, and about each vertex as a center a circle is constructed with a radius of a inches. Find the area bounded by the three arcs which lie outside the triangle, and the area bounded by the three arcs which lie inside the triangle.

21. Every equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.

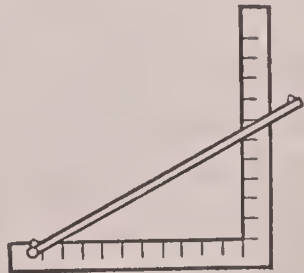
Exercises. Miscellaneous Applications

1. Extend your arm toward a distant object, and, closing your left eye, sight across a finger tip with your right eye. Now keep the finger in the same position and sight with your left eye. The finger then seems to point to an object some distance to the right of the one at which you were pointing. If you can estimate the distance between these two objects, which can often be done with a fair degree of accuracy when there are houses between them, then your distance from the objects is approximately ten times the estimated distance between them. Draw a plan which shows that the lines of sight are sides of triangles, and explain the geometric principle involved.

2. The distance across a stream can be found by the principle involved in any one of these three diagrams. Explain the method in each case and state the geometric principles involved.

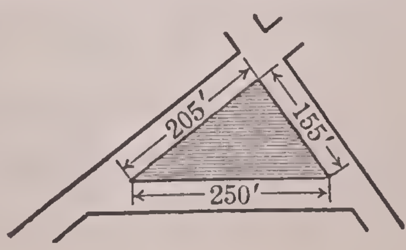


3. An instrument like the one here shown is used in measuring heights. The base is graduated in equal divisions, say 50, and the upright arm is similarly divided. At each end of the hinged bar is a sight. If an observer lying 50 ft. from a tree sights at the top, and finds that the hinged bar cuts the upright arm at 27, he knows that the tree is 27 ft. high. Explain the geometric principle involved.



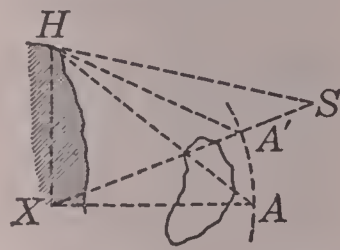
4. If three streets intersect as here shown, find the area of the shaded triangle.

Use the formula in Ex. 1, page 194.

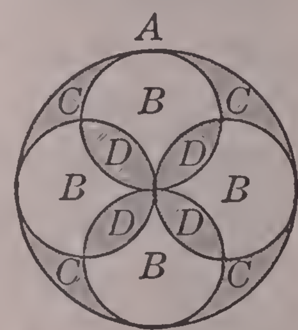


5. Can the triangle of Ex. 4 be a right triangle? Prove your answer.

6. If a dangerous shoal lies near a headland, the *vertical danger angle* is the angle ($\angle HAX$) between the level of the water and the line of sight to the headland H from any point, as A, on a circle of sufficient radius to inclose the dangerous area. In order to avoid the shoal, ships coming near the headland should be careful to keep far enough away, say at S, so that the $\angle HSX$ is less than the known danger angle. Explain the geometric principle involved.



7. On his voyage to Egypt, Napoleon is said to have suggested to his staff the problem of dividing a circle into four equal parts by the use of circles alone. It is also said that the problem was solved by using the figure here shown. How was it done?



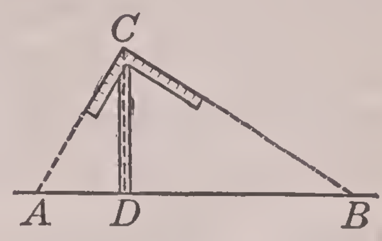
Prove that the area of $\odot B$ is one fourth that of $\odot A$. Then prove that the sum of the four areas marked D is equal to the sum of the four areas marked C. Then prove that one of the D's, the white part of one of the B's, and one of the C's together make one fourth of $\odot A$.

8. In locating the site for a union-school building for three villages A, B, and C, it is desired to place the school so that it shall be equidistant from the three villages. If A is $4\frac{1}{2}$ mi. from B and 6 mi. from C, and B is $5\frac{3}{4}$ mi. from C, draw a map to the scale of 1 in. = 1 mi. and show the location for the school.

While in practice the established roads between the villages would have to be considered, it may be assumed here that all distances are measured in a straight line.

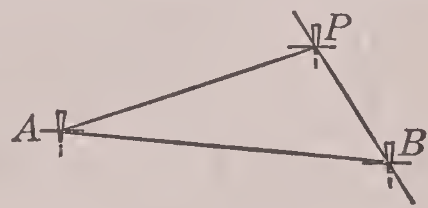
9. By measuring the map in Ex. 8, find how far it will be from each village to the school, and check your answer by the formula given in Ex. 5, page 198.

10. If a carpenter's square is placed on top of an upright stick, as here shown, and an observer sights along the arms to a distant point B and to a point A near the stick, then if AD and DC are measured, the length of DB can be found. Show how this can be done, explaining the geometric principle involved.

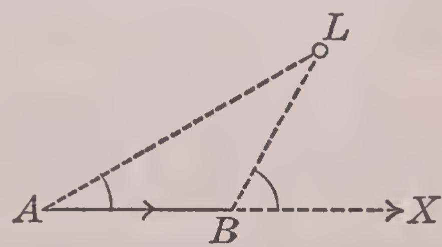


Roman surveyors knew this method two thousand years ago.

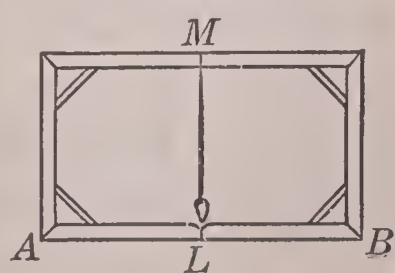
11. Surveyors sometimes lay off a right angle in a field by setting two stakes P and B on a line 3 ft. apart. They then hold the end of a tape at B and the 9-foot mark at P, stretch the tape taut toward A, and set a stake at A on the 5-foot mark. Prove that $\angle P$ is a right angle.



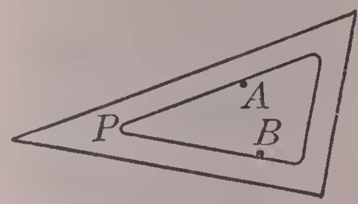
12. The captain of a ship which is sailing on the course ABX observes a lighthouse L when the ship is at A, and measures $\angle A$. He then observes the lighthouse until the angle at B is just twice that at A. He determines the distance AB from his log, an instrument which tells how far a ship has gone. He then knows that BL, the distance from the lighthouse, is the same as AB, the distance sailed. State the geometric principle involved in this method, which is known as "doubling the angle on the bow."



13. The rectangular frame here shown has a plumb line ML hung from M, the midpoint of the upper strip of wood. Prove that when the point of the plumb bob is at the midpoint of AB, the base of the frame is level.

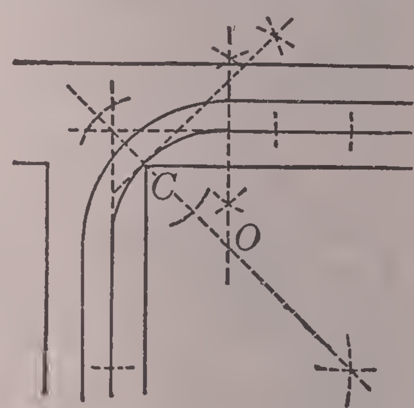


14. A draftsman's triangle is placed over two nails driven into a board at A and B. If a pencil point is placed at P, it will mark an arc of a circle as the triangle is moved about so that the arms of $\angle P$ always touch the two nails. State the geometric principle involved.

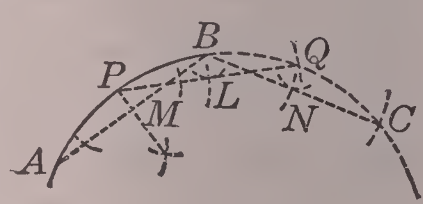


15. In Ex. 14, if P is taken as the vertex of the inside right angle of the triangle and its arms always touch A and B, what kind of arc is formed upon AB as chord?

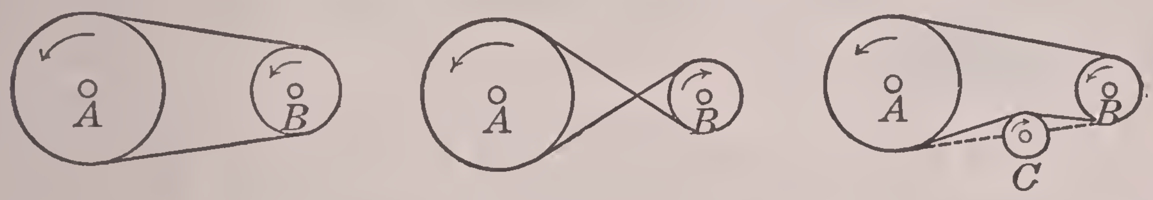
16. In laying out the tracks for a street railway, which is to turn a right-angled corner as shown in this plan, the curve is to be tangent to both the vertical tracks and to the horizontal tracks. The curve is also to be as large as possible without running the inside track beyond the corner C. Show how to find the *center of curvature*; that is, the center O from which the arcs for the curve are drawn.



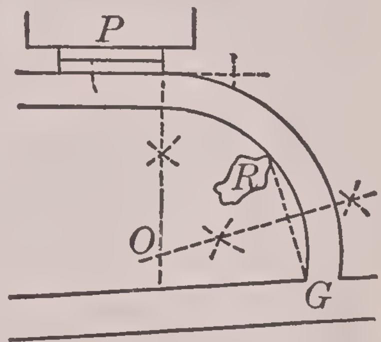
17. If an engineer has to extend a curve which he knows is an arc of a circle, but which is too large to be drawn with a tapeline, or which cannot be easily reached from the center, the following method is sometimes used: Take P as the midpoint of the known part APB of the curve. Then stretch the tape from A to B and construct $PM \perp$ to AB. Then swing the length AM about P, and the length PM about B, until they meet at L, and stretch the length AB along PL to Q, thus fixing the point Q. The point C is fixed in the same way, and so on for as many points as are necessary. Explain the geometric principle involved.



18. In shops where two pulleys are driven by belting, we have a case of two tangents to two given circles. If the belt runs straight between the pulleys, we have the case of two exterior tangents. If the belt is crossed so that

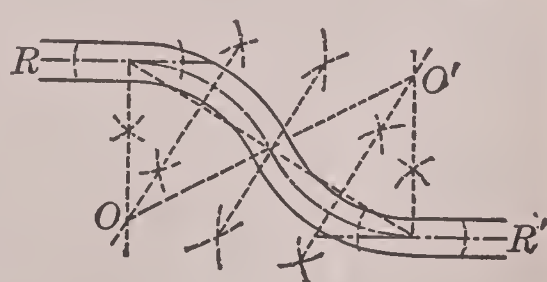


the pulleys turn in opposite directions, we have the case of two interior tangents. In case the belt is liable to change its length, on account of stretching or variation in heat or moisture, a third pulley C is often used. We then have the case of tangents to three pairs of circles. Construct the figure for each of the three cases.



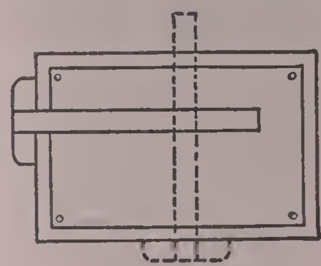
19. This figure shows how a circular driveway was laid out from a gate G to a porch P so as to avoid a group of rocks R. Explain how the plan was constructed and state the geometric principles involved.

20. In making the plans for a park a landscape architect wished to connect two parallel roads R and R' by the curve here shown, which consists of two arcs and is known as a *reversed curve*. From the figure explain how the architect proceeded to construct the plan, and state the geometric principles involved at each step.

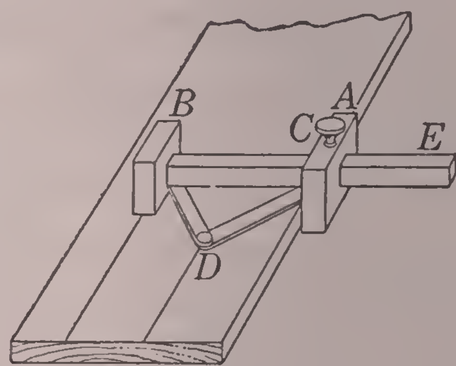


The architect located the center line of the curve, the dot-and-dash line, before drawing the lines which represent the sides of the road. Considering the center line, notice that each arc is tangent to a road and that the arcs are tangent to each other.

21. A draftsman who wished to draw one long line perpendicular to another used his T-square in the two positions shown in the figure, instead of using a triangle. State the geometric principles involved in drawing the lines in this way.

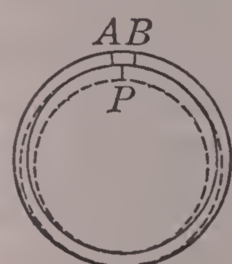


22. This instrument is used for drawing a line parallel to the edge of a board. Block B is fastened to the end of bar E and has a sharp marking point on its underside. Block A can be clamped in any position on bar E by the set screw C. If block A is moved along one edge of the board, will the point on B trace a line parallel to the edge? Why?

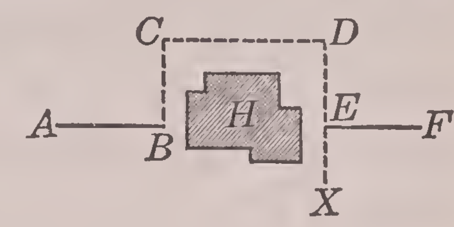


23. The gauge in Ex. 22 is also used for dividing a board into two equal parts. The equal brass arms AD and BD are pivoted at D by a marking point, and are also pivoted at A and B. Blocks A and B are set to the width of the board to be divided, and then block A is moved along one edge of the board while point D traces the dividing line. State the geometric principle involved.

24. In turning a piston ring for an engine a larger ring is made than is needed in the cylinder. Usually the outside diameter of the ring is made 1.5% longer than the diameter of the cylinder. The piece AB is then cut out, the ring is drawn together at P, as shown by the dotted lines, and is fitted in place. If the diameter of the cylinder is 4 in., what diameter should be used in turning the ring and what length should be cut off (AB) to make the ring fit the cylinder?



25. In surveying it is often necessary to run a straight line beyond an obstacle through which it is impossible to sight and over which it is impossible to pass. One of the methods, which is illustrated by the adjoining figure, is as follows: Suppose that the surveyor desires to run the line AB beyond the house H; he first runs a line BC at right angles to AB; at C he runs a line CD at right angles to BC; at D he runs a line DX at right angles to CD; on DX he lays off $DE = CB$, and at E he runs a line EF at right angles to DE. Prove that EF is part of the straight line AB prolonged.



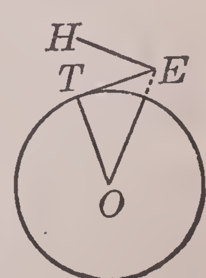
26. An 8-inch pipe can carry how many times as much water as a 1-inch pipe? as a 2-inch pipe? as a 4-inch pipe?

In an 8-inch pipe the internal diameter is 8 in.

27. The diameter of the safety valve of a boiler is $2\frac{7}{8}$ in. Find the total pressure of the steam upon the face of the valve when the steam gauge indicates that the pressure is 140 lb. per square inch.

28. The drive wheel of a locomotive is 6 ft. in diameter and makes 1722 revolutions while the locomotive is going 6 mi. Find the distance lost through the slipping of the wheel on the track.

29. The "dip of the horizon" is the $\angle TEH$ in this figure. It is the angle formed at the eye E of an observer by the line EH which is $\perp$ to OE, the earth's radius produced, and ET, the tangent from E to the sea horizon. Prove that the dip of the horizon is equal to the $\angle O$ at the center of the earth.



The proportions of such a figure are necessarily exaggerated in drawing. Those who have studied physics will also observe that in practice the question of the bending of the light rays must be considered.

Exercises. College Entrance Examinations

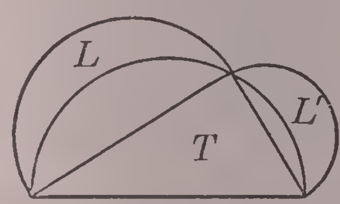
1. The sum of the angles of a triangle is 180°, and the sum of the angles of polygon P is 180°. What do you infer as to the number of sides of P? The sum of the sides of a certain triangle is 180 in., and the sum of the sides of polygon P is 180 in. What do you infer as to the number of sides of P?

State your reasons in both cases, and similarly in Ex. 2.

2. If three parallels cut off equal segments on one transversal, they cut off equal segments on every other transversal. Three given lines do cut off equal segments on one transversal and also cut off equal segments on another transversal. What do you infer as to whether or not these three lines are parallel?

3. The sum of the four sides of any quadrilateral is greater than the sum of the two diagonals.

4. In the fifth century B.C., Hippocrates, a Greek mathematician, proved a theorem which asserts that if three semicircles are constructed on the sides of a right triangle as diameters, as here shown, the crescents L and L' are together equivalent to the $\triangle T$. Prove the statement.



This statement is in a somewhat more general form than the one given by Hippocrates.

5. If two altitudes of a triangle are equal, the triangle is isosceles; if three altitudes are equal, it is equilateral.

6. From an external point P a tangent PA is drawn to a circle. If the diameter AB and the secant PB, cutting the circle at Q, are also drawn, then $\triangle PAB$ is similar to $\triangle AQB$.

The exercises on pages 272 and 273 have been adapted from various examination questions, and represent cases of average difficulty.

7. Through a point P inside a circle with center O chords whose midpoints are M_1, M_2, M_3, $\cdots$ are drawn. Find the locus of M_1, M_2, M_3, $\cdots$.

8. Given a line segment a, construct an equilateral triangle with altitude a.

9. If the side BA of a $\triangle ABC$ is produced through A to D, and if the bisector of $\angle B$ meets the bisector of $\angle CAD$ at E, then $\angle AEB = \frac{1}{2} \angle C$.

10. The bisectors of the base $\measuredangle$s A and B of the equilateral $\triangle ABC$ meet in the point P. From P lines are constructed $\parallel$ to AC and BC and meeting the base in X and Y respectively. Prove that X and Y trisect the base.

11. Construct a circle which shall have half the area of a given circle of radius r.

12. A circular arch of masonry of radius r feet rests on two piers which are d feet apart. Find the height of the center of the arch above the level of the top of the piers. Discuss the result when $r = 25$, $d = 40$; when $r = 25$, $d = 50$.

13. Without performing the actual construction, show how to construct an equilateral triangle equivalent to a given square of side s.

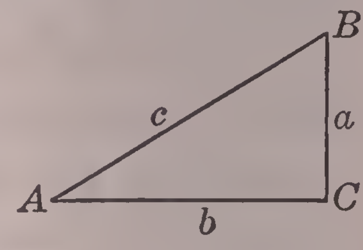
14. A circle of radius 2 in. rolls around the outside of a square of side 4 in. Find the length of the path made by the center of the circle.

15. Construct the locus of the center of a circle of radius 0.5 in. which rolls around an equilateral triangle of altitude 2 in. Find the length of this locus to the nearest 0.1 in.

16. While the wind is blowing directly from the north at the rate of 10 mi. per hour, a steamer is sailing directly east at the same rate. In what direction is a weather vane on the ship pointing? State the reason.

Exercises. Optional Trigonometry

1. Using the right triangle here shown, define $\sin A$, $\cos A$, and $\tan A$ in terms of a, b, and c.



This page is intended only for those who have studied trigonometry and are preparing for an examination that includes the trigonometry of the right triangle. The following table of natural functions is sufficient for the exercises given below:

Angle	sin	cos	tan	Angle	sin	cos	tan
30°	0.500	0.866	0.577	60°	0.866	0.500	1.732
40°	.643	.766	.839	70°	.940	.342	2.747
50°	.766	.643	1.192	80°	.985	.174	5.671

Given the following, find the other parts and the area of the right triangle shown above:

2. $A = 30°$, $a = 20$ ft.
3. $B = 40°$, $b = 70$ ft.
4. $A = 50°$, $b = 9.5$ in.
5. $B = 60°$, $a = 8.2$ in.
6. $A = 70°$, $c = 83$ yd.
7. $b = 20$ ft., $a = 23.84$ ft.

8. The angle of elevation of a balloon from a point P is 60°, and the distance from P to a point directly beneath the balloon is 375 yd. Find the height of the balloon.

9. When a pole 59.6 ft. high casts a horizontal shadow 50 ft. long, what is the angle of elevation of the sun?

10. A flagpole is broken by the wind, and the upper part falls over so as to form a right triangle with the lower part and the ground. If the upper part makes an angle of 70° with the ground and the top of the pole is 15 ft. from its foot, find the original height of the pole.

11. Two sides of a parallelogram are 7 ft. and 9 ft. 6 in. respectively, and the included angle is 80°. Find the area of the parallelogram.

SOLID GEOMETRY

BOOK VI

LINES AND PLANES IN SPACE

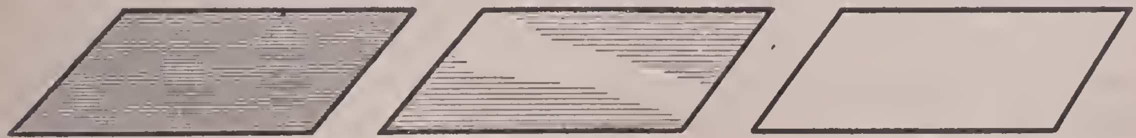
I. Lines and Planes

312. Nature of Solid Geometry. In plane geometry we considered figures lying in a plane, studied their properties and relations, and measured their dimensions and areas. Such figures are, in general, two-dimensional.

In solid geometry we shall consider not only figures of one dimension and two dimensions, but also three-dimensional figures, such as cubes and spheres.

We shall not need to construct the solid figures by means of the straightedge and compasses, and hence we shall not discuss any problems of construction.

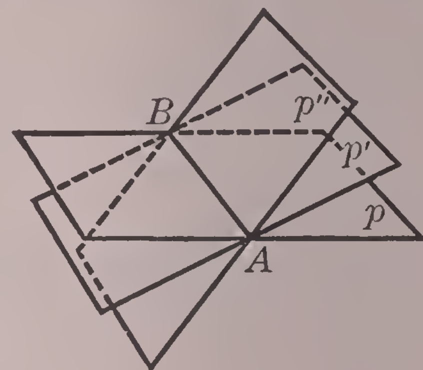
313. Plane. A surface such that a straight line joining any two of its points lies wholly in the surface is called a *plane surface*, or simply a *plane*.



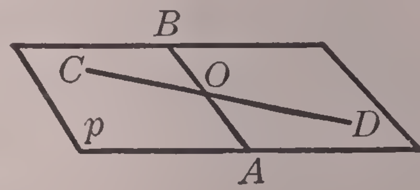
A plane has no thickness, and is understood to be indefinite in extent. A plane may be conveniently represented by a thin rectangular solid seen obliquely, as in any of the three ways shown above.

314. Postulates of Planes. Just as in plane geometry we assumed certain postulates upon which to build the proofs of the propositions, we now assume certain postulates respecting planes. The following are the ones needed in the elementary part of solid geometry:

1. *Two intersecting straight lines determine a plane.*



Although the plane p may turn about one of its lines AB, as shown in the upper figure, and occupy any number of positions, as p', p'', $\cdots$, it cannot turn if it must also pass through an intersecting line CD, as shown in the lower figure. In other words, the intersecting lines AB and CD determine the plane p.



This postulate may be taken to include the following statements:

A straight line and a point not on the line determine a plane.

For example, the line AB and the point C in the second figure are sufficient to determine the plane p.

Three points not in a straight line determine a plane.

If two of them are connected with the third, we have the case of the postulate as first stated.

Two parallel straight lines determine a plane.

By definition (§ 51) they must lie in a plane, and one of the parallels and any point on the other determine the plane.

Any one of the four statements given above may be referred to as § 314, 1.

2. *If two planes have one point in common, they have at least one other point in common.*

It is evident that they must then coincide or else that they must intersect in a straight line.

Exercises. Planes

1. We commonly say that we live in a space of three dimensions, these dimensions being length, width, and thickness. We may, then, consider a plane as a space of how many and what dimensions? Similarly, a line is a space of how many and what dimensions?

2. Explain the meaning of the statement that a plane passes through a line; that it cuts or intersects the line. Draw a figure to illustrate each case.

3. Two lines in a plane may have no point in common, in which case they are parallel; they may have one point in common, in which case they intersect; or they may have an infinite number of points in common. Write a similar statement respecting two planes in three-dimensional space.

4. Write a statement mentioning three points in the room and describing the position of the plane determined by them. Illustrate the statement by a drawing.

5. Write a statement explaining why a three-legged stool stands firmly on the floor while a four-legged chair may not do so.

6. Write a statement describing the position of two lines in the room which are so situated that they do not determine a plane and do not meet however far produced.

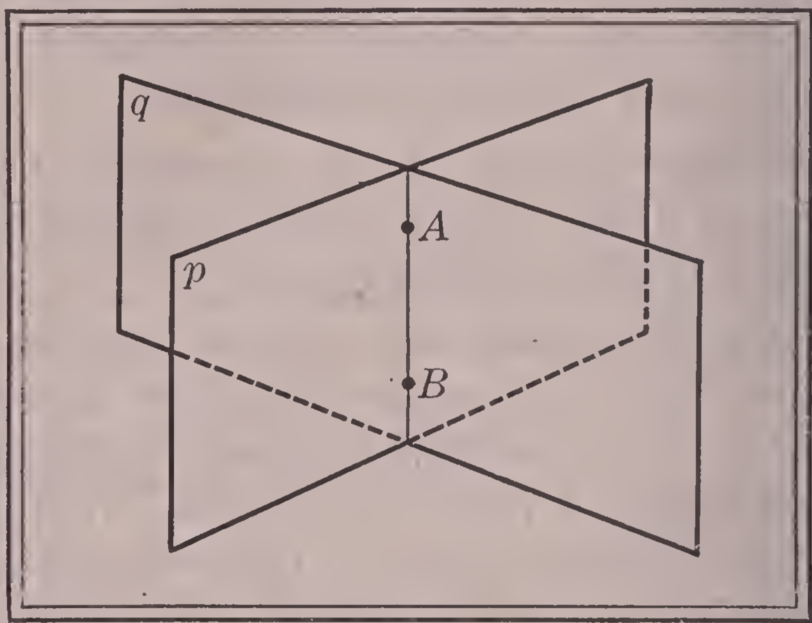
7. State the geometric reason why a triangle is necessarily a plane figure while a quadrilateral in three-dimensional space need not be.

8. In three-dimensional space how many different planes are determined by four points? by five points?

9. If n lines, no two of which are parallel, meet a given line l, how many planes are determined, and upon what postulate does your answer depend?

Proposition 1. Intersection of Planes

315. Theorem. *If two planes meet, they intersect in a straight line.*



Given two planes p and q which meet.

Prove that p and q intersect in a st. line.

Proof. Since p and q meet, they must have at least one point, as A, in common. Hence they must have at least one other point, as B, in common. § 314, 2

Draw AB. Post. 1

Then AB lies in both p and q, § 313
because otherwise p and q would not be planes.

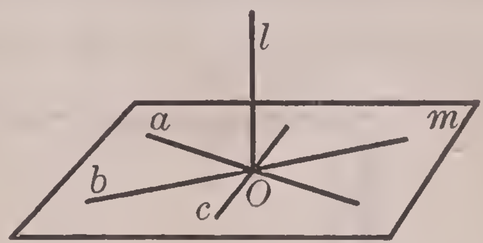
Also, no point not on AB can be in both p and q, § 314, 1
because p and q would then coincide instead of meeting.

Hence the st. line determined by A and B contains all points common to p and q.

$\therefore$ AB is the intersection of p and q,
because this is the meaning of intersection.

Hence p and q intersect in a st. line.

316. Perpendicular to a Plane. If a straight line which meets a plane is perpendicular to every straight line which lies in the plane and passes through the point of meeting, the line is said to be *perpendicular to the plane,* and the plane is said to be *perpendicular to the line.*



In this figure we may have any number of planes containing l, and in each plane we may have a perpendicular to l at O. Hence we may have any number of perpendiculars, as $a, b, c, \cdots,$ to l at O.

If, as will be shown (§ 321) to be the case, $a, b, c, \cdots$ all lie in one plane m, then l is $\perp$ to m, and m is $\perp$ to l.

If we invert the above definition (§ 9), we see that if a straight line is perpendicular to a plane, the line is perpendicular to every line in the plane that passes through the point of meeting.

317. Foot of a Perpendicular. The point at which a perpendicular meets a plane is called the *foot of the perpendicular.*

318. Oblique to a Plane. If a straight line which meets a plane is not perpendicular to the plane, the line is said to be *oblique to the plane,* and the plane is said to be *oblique to the line.*

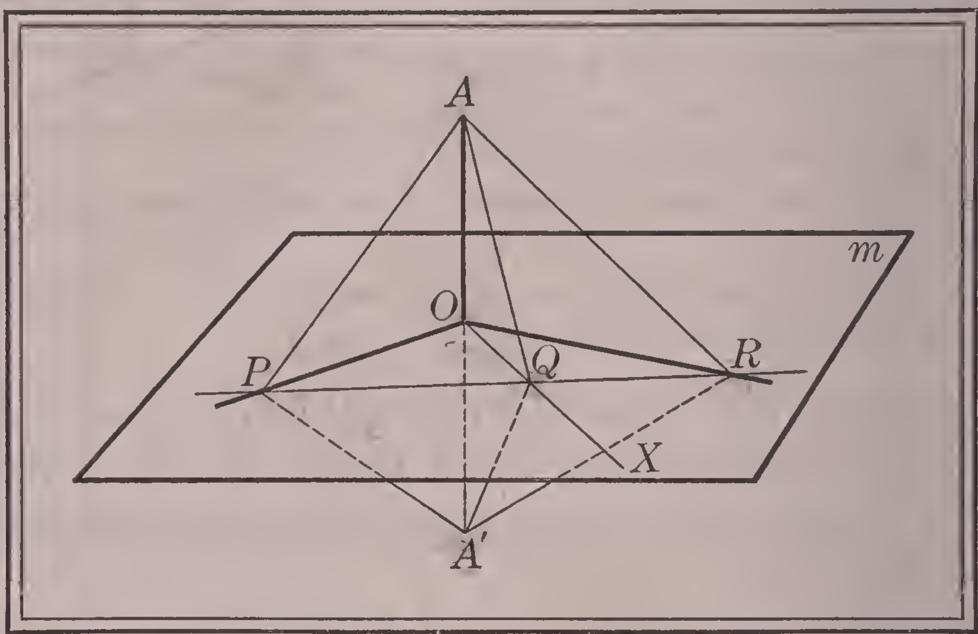
Lines which are perpendicular or oblique to a plane are called *perpendiculars* or *obliques* respectively.

When we speak of a perpendicular or an oblique from a point to a plane, we mean the line segment from the point to the plane.

319. Parallel to a Plane. If a straight line cannot meet a plane, however far each is produced, the line is said to be *parallel to the plane,* and the plane is said to be *parallel to the line.* Similarly, if one plane cannot meet another plane, however far each is produced, the planes are said to be *parallel.*

Proposition 2. Perpendicular to a Plane

320. Theorem. *If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.*



Given $AO \perp$ **to** OP **and** OR **at** O**, and** m**, the plane of** OP **and** OR.

Prove that AO *is* $\perp$ *to* m.

Proof. Through O draw any other line OX in the plane m, and draw PR, cutting OP, OX, OR in P, Q, and R respectively. On AO produced take $OA' = OA$.

Join A and A' to P, Q, and R respectively.

Then $AP = A'P$ and $AR = A'R$, § 117

because OP and OR are each $\perp$ to AA' at its midpoint.

$\therefore \triangle APR$ is congruent to $\triangle A'PR$. § 47

Then $\angle RPA = \angle RPA'$. § 38

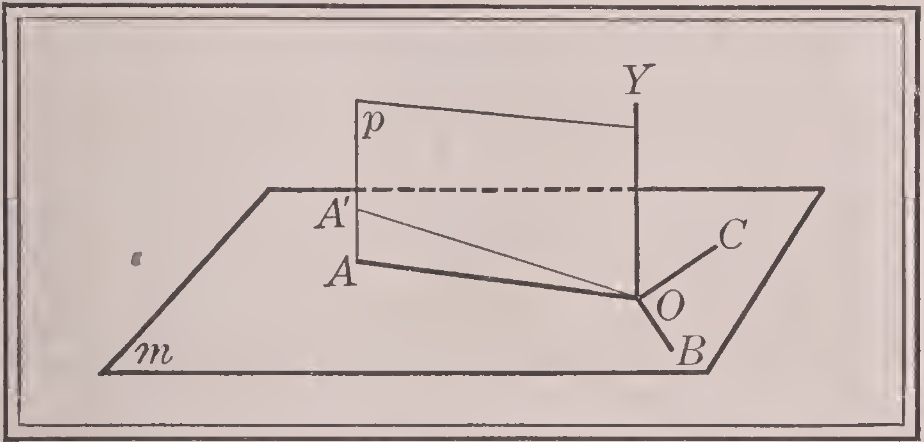
$\therefore \triangle PQA$ is congruent to $\triangle PQA'$. § 40

Then, since $AQ = A'Q$ (§ 38), OQ is $\perp$ to AA' at O. § 182

Hence AO is $\perp$ to any line in the plane m through O, and thus is $\perp$ to m. § 316

Proposition 3. Perpendiculars to a Line

321. Theorem. *Every line perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point.*



Given OA, OB, OC, $\cdots$ and the plane m, all $\perp$ to OY at O.

Prove that OA, OB, OC, $\cdots$ *lie in* m.

Proof. Suppose that the plane p, determined by OA and OY, does not intersect m in OA, but intersects it in OA'.

Then OA' is $\perp$ to OY. § 316

But OA is $\perp$ to OY. Given

Hence the supposition is false.

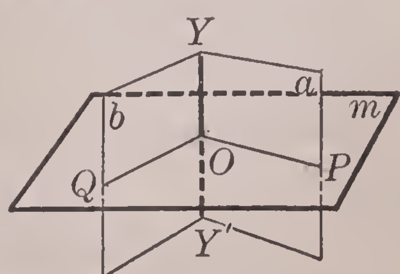
$\therefore$ p intersects m in OA. Post. 10

Hence OA lies in m, and similarly for OB, OC, $\cdots$. § 313

322. Corollary. *Through a given internal point there can be one and only one plane perpendicular to a given line.*

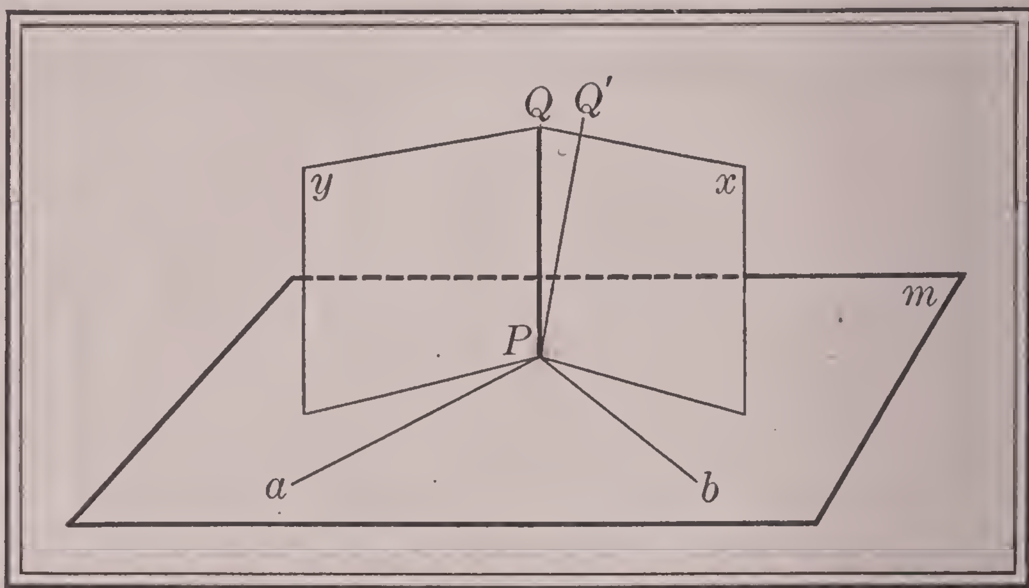
323. Corollary. *Through a given external point there can be one and only one plane perpendicular to a given line.*

In a, the plane of YY' and the given point P, let PO be $\perp$ to YY'. In b, any other plane containing YY', let OQ be $\perp$ to YY'. Then m, the plane of OP and OQ, is $\perp$ to YY' at O (§ 321). Now prove that m is the only $\perp$ plane by using Post. 10.



Proposition 4. Perpendicular through Internal Point

324. Theorem. *Through a given internal point there can pass one and only one line perpendicular to a plane.*



Given the plane *m* and the internal point *P*.

Prove that through P there can pass one and only one line which is ⊥ to m.

Proof. If a and b are any two lines in the plane m passing through P, then through P there is a plane x which is ⊥ to a and a plane y which is ⊥ to b. § 322

Since a and b are not identical but meet at P, x and y are not identical and must intersect in a st. line PQ. § 315

Since a is ⊥ to x, it is ⊥ to PQ. § 316

Similarly, b is ⊥ to PQ,

and hence PQ is ⊥ to m. § 320

Now if another line PQ' could pass through P and be ⊥ to m, it would be ⊥ to a and to b. § 316

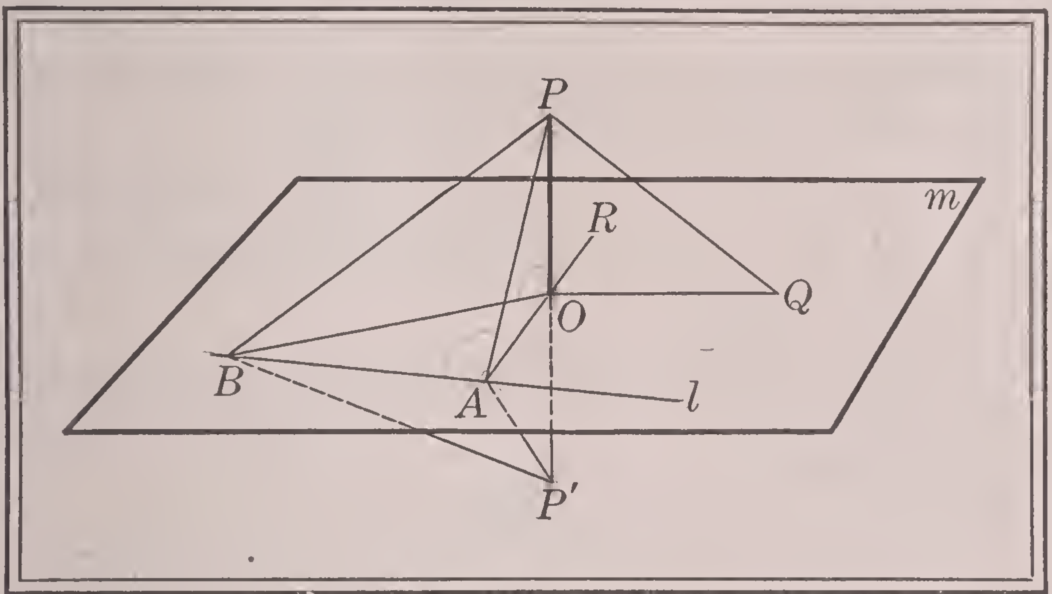
Hence PQ' would lie in both x and y, § 316

and PQ' would coincide with PQ.

∴ PQ is the one and only ⊥ to m through P.

Proposition 5. Perpendicular through External Point

325. Theorem. *Through a given external point there can pass one and only one line perpendicular to a plane.*



Given the plane m and the external point P.

Prove that through P there can pass one and only one line which is $\perp$ to m.

Proof. Let l be any line in m, and let PA be $\perp$ to l.

In m construct $AR \perp$ to l, and in the plane of AR and P construct $PO \perp$ to AR. §§ 104, 105

Produce PO to P', making $OP' = OP$. Post. 2

Let OB be any other line from O to l, and draw PB, $P'A$, and $P'B$.

Then l is $\perp$ to plane $AP'P$. § 316

Now prove that rt. $\triangle APB$ is congruent to rt. $\triangle AP'B$, hence that $\triangle OPB$ is congruent to $\triangle OP'B$, and hence that PO is $\perp$ to OB.

Then PO is $\perp$ to m. § 316

Further, if PQ is any other line from P to m and QO is drawn, then $\angle QOP$ is a rt. $\angle$. § 316

Hence PQ is not $\perp$ to m, and PO is the only $\perp$.

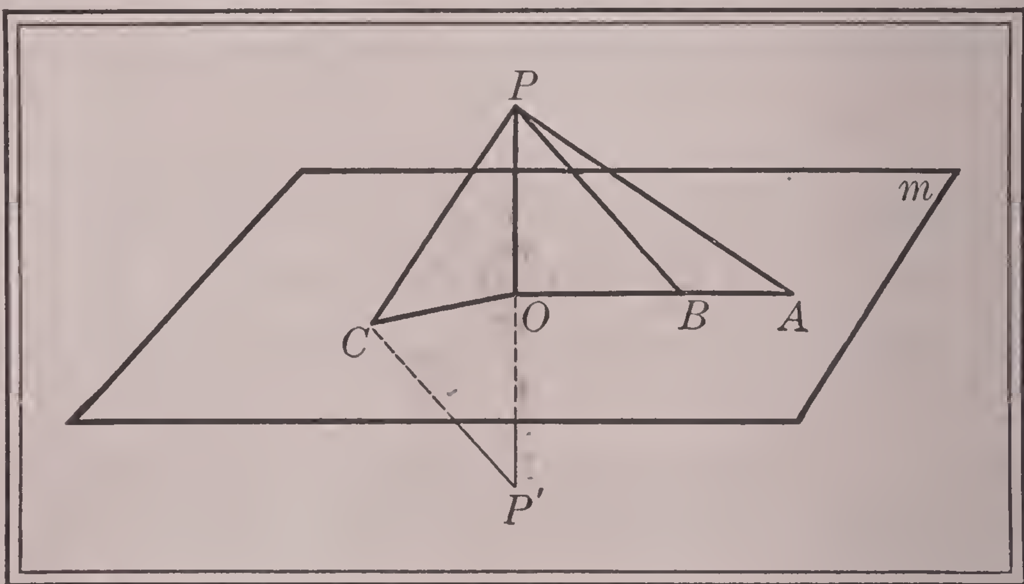
Proposition 6. A Perpendicular and Obliques

326. Theorem. *If from an external point a perpendicular and obliques are drawn to a plane,*

1. *The perpendicular is shorter than any oblique;*

2. *Obliques meeting the plane at equal distances from the foot of the perpendicular are equal;*

3. *Of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.*



Given the plane m, the external point P, $PO \perp$ to m, and the obliques PA, PB, PC drawn to m so that $OA > OB = OC$.

Prove that $PO < PC$, $PB = PC$, *and* $PA > PC$.

Proof. Produce PO to P', making $OP' = OP$. Post. 2

Draw $P'C$. Post. 1

Now prove that $PP' < PC + CP'$, and hence that $PO < PC$.

Then prove that $\triangle OBP$ is congruent to $\triangle OCP$,

and hence that $PB = PC$.

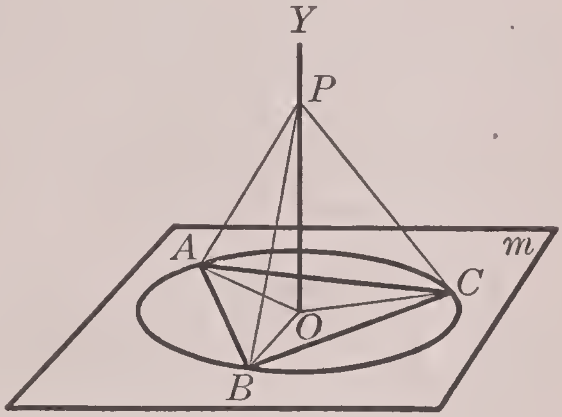
Finally, prove that $PA > PB$,

and hence that $PA > PC$.

327. Distance. The length of the perpendicular from a point to a plane is called the *distance* from the point to the plane.

The following corollaries (§§ 328, 329) extend the idea of a locus with which the student is familiar from plane geometry.

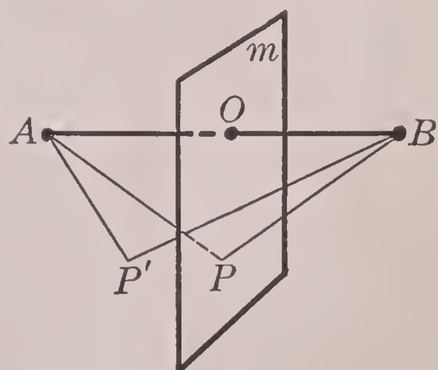
328. Corollary. *The locus of points equidistant from the vertices of a triangle is a line through the center of the circumscribed circle, and perpendicular to the plane of the triangle.*



We have first to prove (§ 181) that any point P on the $\perp OY$ satisfies the conditions; that is, that $AP = BP = CP$. But this follows from §§ 134, 1 and 326, 2.

We have then to prove that any point P which satisfies the conditions that $AP = BP = CP$ is on the line OY. Now $AO = BO = CO$ (§ 134, 1). Then since OP is a common side, $\triangle AOP$, BOP, and COP are congruent (§ 47). Then the $\angle$s made by OP with any lines in m are equal, and hence they are rt. $\angle$s. Hence OP is $\perp$ to m at O (§ 316), and since OP is part of OY (§ 324), P is on OY.

329. Corollary. *The locus of points equidistant from two given points is the plane perpendicular at the midpoint to the line segment joining the points.*



We have first to prove (§ 181) that any point P in m satisfies the condition that $PA = PB$. We have then to prove that any point P' such that $P'A = P'B$ lies in m, which is best done by an indirect proof.

The proof of each of these steps is left for the student.

We here meet a case in which the locus is a plane instead of a line. In plane geometry, as the student has already found, a locus is usually a line; in solid geometry a locus may be a line or it may be a surface.

Exercises. Lines and Planes

1. Equal oblique lines drawn from a point to a plane meet the plane at equal distances from the foot of the perpendicular from the point to the plane; and of two unequal oblique lines the greater meets the plane at the greater distance from the foot of the perpendicular.

2. The locus of points equidistant from all points on a circle is a line through the center, perpendicular to the plane of the circle.

3. Find the locus of points at a given distance from each of two given points.

4. Explain how a carpenter might proceed to set a joist so that it shall be perpendicular to a horizontal floor. Draw a figure to illustrate any method which seems practical to you for the carpenter to use.

5. What geometric principle is involved in the statement that if the spoke of a wheel is perpendicular to the axle, the spoke determines a plane as the wheel revolves?

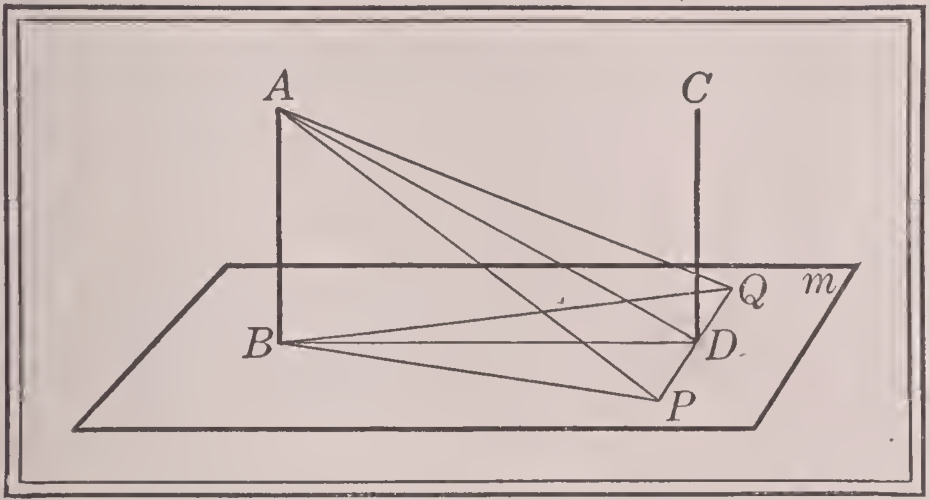
6. A steel smokestack 80 ft. high is braced by four steel wires each 100 ft. long and reaching from the top of the stack to the ground. If the wires are straight, at what distance from the foot of the stack does each reach the ground? Would three wires serve as well? Would two serve as well? Would one serve as well? State the geometric principle involved in each case.

7. Through a point P there pass four lines such that no three are in the same plane. Find the number of planes determined by the four lines.

8. In a plane P there lie four lines such that no three pass through the same point. Find the greatest number of points that can be determined by the four lines.

Proposition 7. Perpendiculars to a Plane

330. Theorem. *Two lines perpendicular to the same plane are parallel.*



Given AB and CD, each $\perp$ to the plane m.

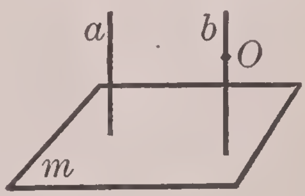
Prove that AB *is* $\parallel$ *to* CD.

Proof. Draw AD and BD, and in m draw $PQ \perp$ to BD at D, making $DP = DQ$. Draw AP, AQ, BP, BQ.

By congruent △ (§ 47) prove that $\angle ADP = \angle ADQ = 90°$, and then that BD, CD, and AD lie in the same plane (§ 321). Then prove that AB also lies in this plane (§ 313), and then that AB and CD are each $\perp$ to BD (§ 316).

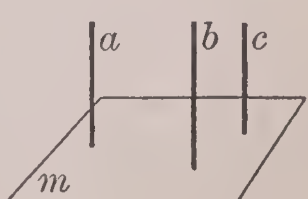
$\therefore$ AB is $\parallel$ to CD. § 57

331. Corollary. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*



For if through any point O of b a line is drawn $\perp$ to m, how is it related to a (§ 330)? Now apply § 52.

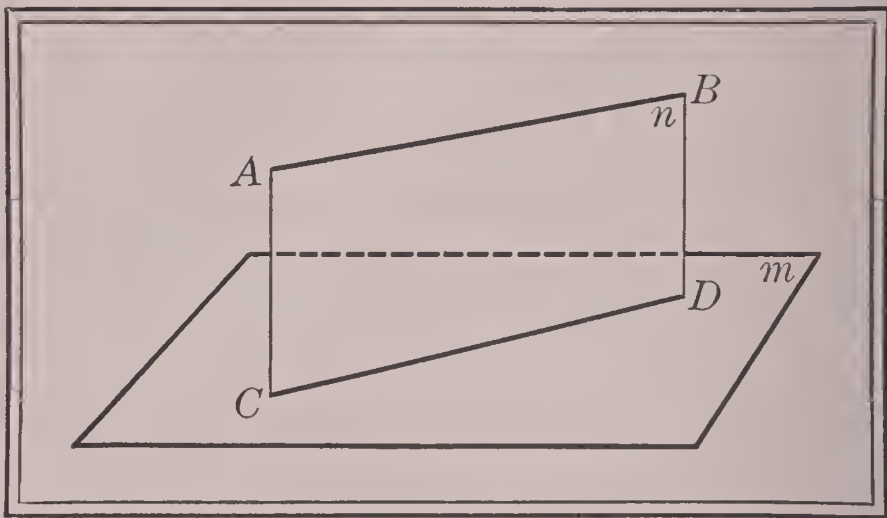
332. Corollary. *If two lines are parallel to a third line, they are parallel to each other.*



For if b is $\perp$ to m, so are a and c (§ 331).

Proposition 8. Parallel Lines

333. Theorem. *If two lines are parallel, every plane containing one and only one of the lines is parallel to the other.*

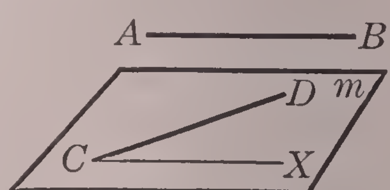


Given the ∥ lines *AB* and *CD*, and the plane *m* containing *CD* but not *AB*.

Prove that *m is ∥ to AB.*

Proof. *AB* and *CD* determine a plane *n*, § 314, 1
and *AB* lies in *n*, however far each is produced. § 313
Hence, if *AB* meets *m*, it meets *CD*. § 313
Since *AB* cannot meet *CD*, § 51
AB cannot meet *m*;
that is, *m* is ∥ to *AB*. § 319

334. Corollary. *Through either of two lines not in the same plane one and only one plane can pass parallel to the other line.*

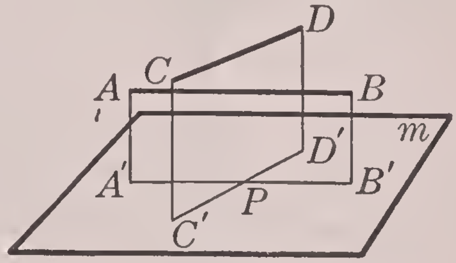


For if *AB* and *CD* are the given lines, and if *CX* is ∥ to *AB*, what can be said of the plane *m*, which is determined by *CD* and *CX*, with respect to the line *AB*? Why can there be only one such plane?

Lines placed like *AB* and *CD* in this figure are called *skew lines.*

335. Corollary. *Through a given point one and only one plane can pass parallel to each of two given lines not in the same plane.*

Let P be the given point and AB and CD the given lines. If, now, we construct through P the line $A'B' \parallel$ to AB, and the line $C'D' \parallel$ to CD, these lines determine the plane m. Then prove that m is $\parallel$ to AB and CD, and that no other such plane is possible through P.



In the above figure, the lines AB and CD are said to form an angle, although they do not meet. This angle is defined as the $\angle C'PB'$, but the concept is rarely used in elementary geometry.

Exercises. Lines and Planes

1. State the geometric principle by which we know that a straight edge results from folding a piece of paper.

2. In a given plane what is the locus of points equidistant from two parallel lines in the plane? Given two parallel planes instead of two parallel lines, what is the corresponding locus in a space of three dimensions? Draw the figures but give no proofs.

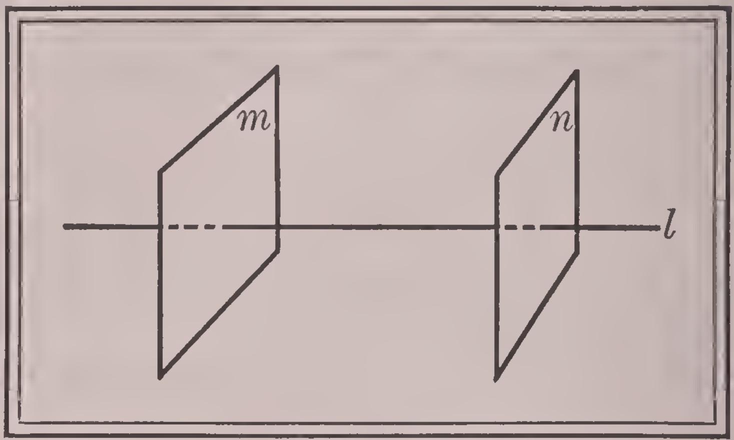
3. If a given line is parallel to a given plane, the intersection of the plane with any plane passed through the given line is parallel to that line.

4. If a given line is parallel to a given plane, a line parallel to the given line drawn through any point of the plane lies in the plane.

5. If equal oblique lines are drawn from a given external point to a plane, they make equal angles with lines drawn from the points where the oblique lines meet the plane to the foot of the perpendicular drawn from the given point to the plane.

Proposition 9. Parallel Planes

336. Theorem. *Two planes perpendicular to the same line are parallel.*



Given the planes m and n, each $\perp$ to the line l.

Prove that m *is* $\parallel$ *to* n.

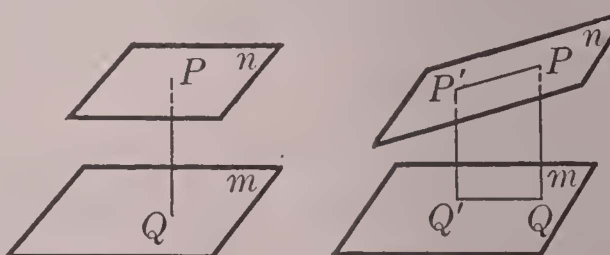
Proof. If m is not $\parallel$ to n, it must meet n, in which case we should have two planes through a point in their intersection both $\perp$ to l.

Since this is impossible (§ 322), m is $\parallel$ to n.

The following corollary, which is analogous to the Postulate of Parallels (§ 52), may be assumed, if desired, without proof.

337. Corollary. *Through a given external point one and only one plane can pass parallel to a given plane.*

If P is the point and m is the plane, as shown in the left-hand figure, there is only one line PQ that is $\perp$ to m (§ 325). Through P there is one and only one plane n that is $\perp$ to PQ (§ 322), and this is $\parallel$ to m (§ 336).



If through P there were two planes $\parallel$ to m, one would be oblique to PQ, as shown in the right-hand figure, and would contain some line PP' that would meet its projection QQ' in m. Then n would not be $\parallel$ to m (§ 319). Hence only one plane through P is $\parallel$ to m.

Exercises. Review

1. If from the foot of a perpendicular to a plane a line is constructed at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point on the perpendicular is perpendicular to the line in the plane.

2. If two perpendiculars extend from a given external point to a plane and to a line in that plane respectively, the line joining the feet of the two perpendiculars is perpendicular to the given line.

3. From two vertices of a triangle perpendiculars are constructed upon the opposite sides. From the intersection of these perpendiculars there is a perpendicular to the plane of the triangle. Prove that a line drawn to any vertex of the triangle from any point on this perpendicular is perpendicular to the line drawn through that vertex parallel to the opposite side.

4. Find the point in a plane to which lines may be drawn from two given external points on the same side of the plane so that their sum shall be the least possible.

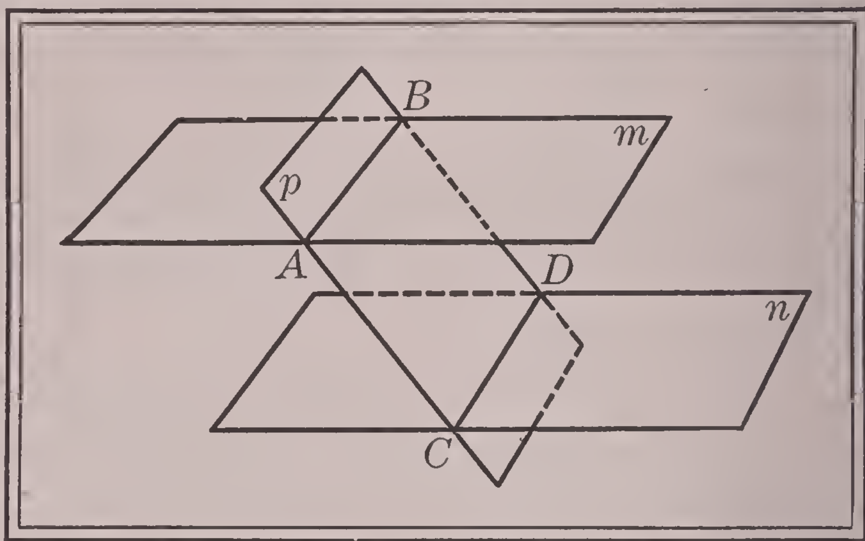
From one point A suppose that a line AO is $\perp$ to the plane and that it is produced to A', making $OA' = OA$. Connect A' and the other point B by a line cutting the plane at P. Then $AP + PB$ is the least sum.

5. If three equal oblique lines are drawn from an external point to a plane, the perpendicular from the point to the plane meets the plane at the center of the circle circumscribed about the triangle which has for its vertices the points where the oblique lines meet the plane.

6. State and prove the propositions of plane geometry corresponding to §§ 330, 331, and 332. Why do not the proofs of those propositions apply to the corresponding propositions of solid geometry?

Proposition 10. Parallel Planes Intersected

338. Theorem. *If two parallel planes are cut by a third plane, the lines of intersection are parallel.*



Given the ∥ planes *m* and *n*, intersected by a third plane *p* in *AB* and *CD* respectively.

Prove that *AB is ∥ to CD.*

Proof. *AB* and *CD* are in the same plane *p*. Given

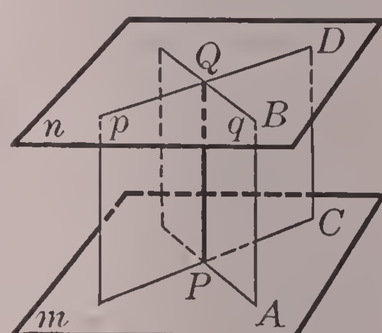
Since *AB* is always in *m* and *CD* is always in *n*, § 313
m and *n* must meet if *AB* and *CD* meet.

But *m* is ∥ to *n*, Given

and hence *m* and *n* cannot meet. § 319

∴ *AB* is ∥ to *CD*. § 51

339. Corollary. *A line perpendicular to one of two parallel planes is perpendicular to the other also.*



Let *PQ* be ⊥ to *m*, and let *p* and *q* be two planes containing *PQ*. Now prove (§ 338) that *QB* is ∥ to *PA* and that *QD* is ∥ to *PC*. Then prove that *PQ* is ⊥ to *QB* and *QD*, and hence that *PQ* is ⊥ to *n*.

340. Distance between Parallel Planes. The length of a perpendicular line segment between two parallel planes is called the *distance* between the planes.

It has been shown (§ 339) that if this segment is perpendicular to one of two parallel planes it is perpendicular to the other. It will now be shown (§ 341) that the length is the same whatever perpendicular between the planes is taken.

341. Corollary. *Two parallel planes are everywhere equidistant from each other.*

In the figure of § 339, if AB is constructed $\parallel$ to PQ, then $ABQP$ is a ▱ (§ 72). Since it is given that PQ is $\perp$ to m, then AB is also $\perp$ to m (§ 331). Both AB and PQ are then $\perp$ to n (§ 339) and represent distances measured on any two ⊥s. But $AB = PQ$ (§ 76), and hence m and n are everywhere equidistant from each other.

342. Parallel in the Same Sense. If two parallel rays lie on the same side of the line segment which joins their end points, they are said to be *parallel in the same sense.*

Exercises. Parallel Planes

1. Parallel lines included between parallel planes are equal.

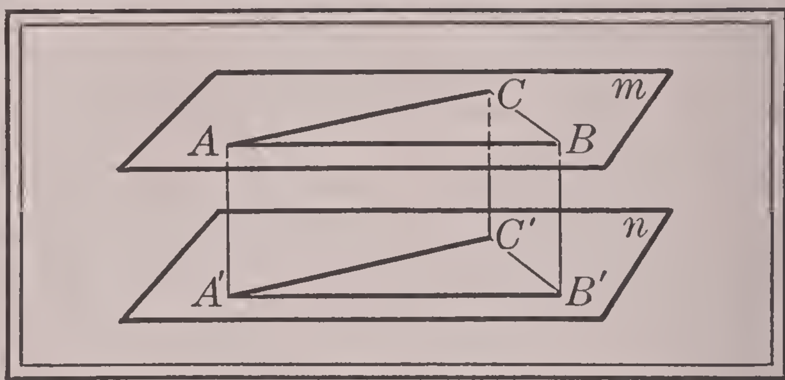
2. The locus of points equidistant from two parallel planes is a plane which is perpendicular to a line perpendicular to the planes and which bisects the segment cut off by them.

3. The locus of points equidistant from two parallel lines is a plane which is perpendicular to a line perpendicular to the given lines and which bisects the segment cut off by them.

4. The locus of points at a given distance from a plane is a pair of parallel planes, each at the given distance from the given plane.

Proposition 11. Arms of Angles Parallel

343. Theorem. *If two angles not in the same plane have their arms respectively parallel in the same sense, the angles are equal and their planes are parallel.*



Given the $\angle$s A and A' in the planes m and n respectively, with their arms respectively $\parallel$ in the same sense.

Prove that $\angle A = \angle A'$ *and that* m *is* $\parallel$ *to* n.

Proof. Take $AB = A'B'$, and $AC = A'C'$, and draw BC, $B'C'$, AA', BB', CC'.

Then AA' is equal and $\parallel$ to BB' and to CC'. § 81

Hence $BB' = CC'$ (Ax. 5), and BB' is $\parallel$ to CC' (§ 332).

Since $BC = B'C'$ (§ 76), $\triangle ABC$ is congruent to $\triangle A'B'C'$. § 47

$\therefore \angle A = \angle A'$. § 38

Now if m is not $\parallel$ to n, they will meet in a line l. § 315

Since AB and AC are $\parallel$ to n (§ 333), neither can meet l.

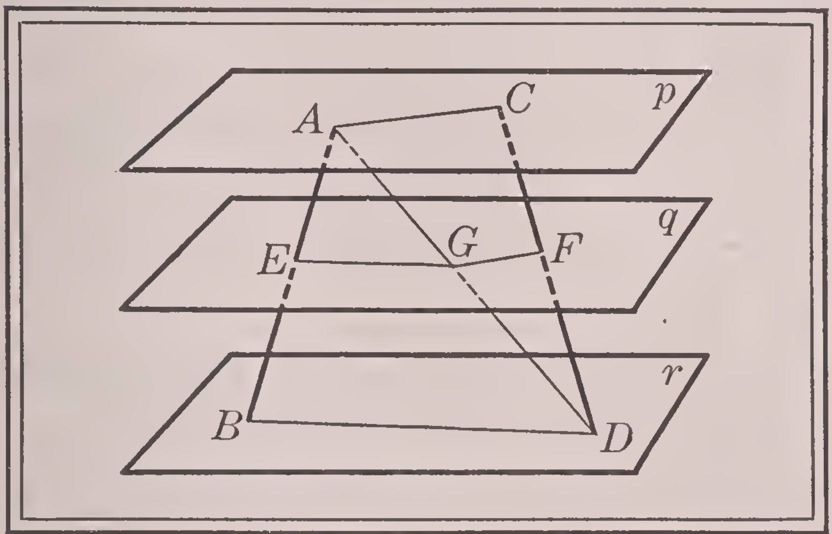
But since both AB and AC cannot be $\parallel$ to l (§ 332), m cannot meet n, and hence m is $\parallel$ to n.

344. Corollary. *If two intersecting lines are each parallel to a plane, the plane of these lines is parallel to that plane.*

In the figure of § 343, if AB and AC are both $\parallel$ to n, show that AB cannot meet a line $A'B'$, which is the intersection of n and the plane of AA' and AB. Similarly, AC is $\parallel$ to $A'C'$. Hence m is $\parallel$ to n (§ 343).

Proposition 12. Transversals in Space

345. Theorem. *If two lines are cut by three parallel planes, their corresponding segments are proportional.*



Given *AB* and *CD*, cut by the ∥ planes *p*, *q*, *r*, in the points *A*, *E*, *B*, and *C*, *F*, *D* respectively.

Prove that
$$\frac{AE}{EB} = \frac{CF}{FD}.$$

Proof. Let q intersect the plane of A, B, D in EG and the plane of A, D, C in GF.

Then EG is ∥ to BD and GF is ∥ to AC. § 338

Hence
$$\frac{AE}{EB} = \frac{AG}{GD},$$

and
$$\frac{CF}{FD} = \frac{AG}{GD}. \quad \text{§ 201}$$

Hence
$$\frac{AE}{EB} = \frac{CF}{FD}. \quad \text{Ax. 5}$$

It should be observed that this proposition is a generalization of § 203, which applies only to a figure lying in one plane. It may be stated still more generally as follows: *If two lines are cut by any number of parallel planes, their corresponding segments are proportional.*

Consider also the case of AB intersecting CD between the planes.

Exercises. Review

1. In a given plane find the locus of points equidistant from two given points not in the plane.

2. Find the locus of points equidistant from three given points not in a straight line.

3. Find the locus of points equidistant from two given parallel planes and also equidistant from two given points.

4. What is the locus of points at a given distance from each of two planes?

5. The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AE = 3$ in., $EB = 4$ in., and $CD = 6$ in., what are the lengths of CF and FD?

6. In Ex. 5, if $AB = 16$ in., $CF = 10$ in., and $CD = 18$ in., what are the lengths of AE and EB?

7. It is proved in plane geometry that if three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal. State and prove a corresponding proposition in solid geometry.

8. It is proved in plane geometry that the line which joins the midpoints of two sides of a triangle is parallel to the third side. State and prove a proposition in solid geometry which shall refer to a plane passing through the midpoints of two sides of a triangle.

9. A cylindric water tank which is 16 ft. deep and 12 ft. in diameter is filled with water to a depth of 9 ft. A pole standing obliquely in the tank just reaches from a point on the circumference of the base to a point exactly opposite on the upper rim. Find the length of that part of the pole which is under water.

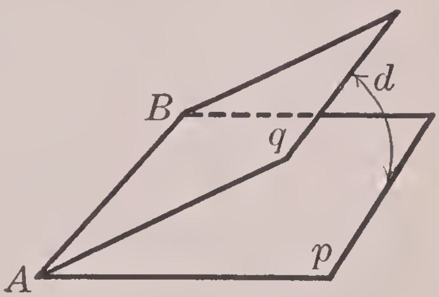
10. Consider Ex. 9 when the water level rises 3 ft.

II. Dihedral Angles

346. Half-Planes. Any straight line in a plane is said to divide the plane into two *half-planes.*

This term corresponds to the term *rays* in plane geometry.

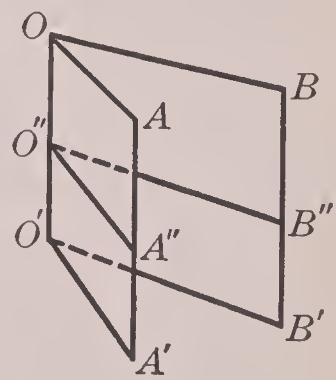
347. Dihedral Angle. If two half-planes proceed from the same line, they form a *dihedral angle.*



In this figure the half-planes p and q are the *faces* of the dihedral angle, and AB is the *edge.*

This dihedral angle may be designated by pq, d, p-AB-q, or AB, of which the first two forms are the most convenient.

348. Plane Angle of a Dihedral Angle. A plane angle whose arms are perpendicular to the edge of a dihedral angle and lie respectively in the faces is called the *plane angle of the dihedral angle.*



For example, if OA, OB; $O'A'$, $O'B'$; $O''A''$, $O''B''$ are each perpendicular to OO' in this figure, each of the $\angle$s O, O', O'' may be taken as the plane angle of the dihedral angle.

349. Corollary. *Two dihedral angles have the same ratio as their plane angles.*

For it is evident that the amount of turning necessary to generate a dihedral angle is the same as that which is necessary to generate its plane angle. Hence their numerical measures are always identical.

If a proof were necessary, it would be substantially the same as the one in § 136.

350. Kinds of Dihedral Angles. A dihedral angle is *right, acute,* or *obtuse* according as its plane angle is right, acute, or obtuse.

Similarly we may use the terms *straight, vertical, adjacent, complementary, supplementary,* and *oblique* in connection with dihedral angles.

351. Relation of Dihedral Angles to Plane Angles. It is evident that the proofs of many properties of dihedral angles are identical with those of analogous properties of plane angles. A few of the more important propositions will be proved, but the following may be assumed without proof or may be taken as exercises:

1. *If a plane meets another plane, it forms with it two adjacent dihedral angles whose sum is equal to two right dihedral angles.*

2. *If the sum of two adjacent dihedral angles is equal to two right dihedral angles, their exterior faces are in the same plane.*

3. *If two planes intersect each other, the vertical dihedral angles are equal.*

4. *If a plane intersects two parallel planes, the alternate dihedral angles are equal; the corresponding dihedral angles are equal; and the two interior dihedral angles on the same side of the transverse plane are supplementary.*

5. *When two planes are cut by a third plane, if the alternate dihedral angles are equal, or the corresponding dihedral angles are equal, and the edges of the dihedral angles thus formed are parallel, the two planes are parallel.*

6. *Two dihedral angles whose faces are parallel each to each are either equal or supplementary.*

7. *Two planes parallel to the same plane are parallel to each other.*

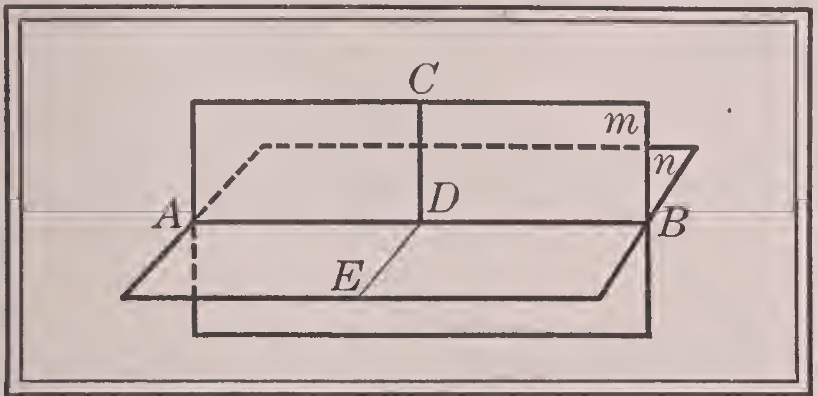
While the proofs of the above propositions are valuable as exercises in logic, they are not essential.

352. Perpendicular Planes. If two intersecting planes form a right dihedral angle, the planes are said to be *perpendicular to each other.*

As in all definitions we may invert the statement and say that, if the planes are perpendicular, they form a right dihedral angle.

Proposition 13. Perpendicular Planes

353. Theorem. *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Given the planes m and n $\perp$ to each other, and the line CD in m $\perp$ to AB, the intersection of m and n.

Prove that CD *is* $\perp$ *to* n.

Proof. In n construct $DE \perp$ to AB at D. § 104

Then $\angle EDC$ is a rt. $\angle$. § 352

Also, $\angle CDA$ is a rt. $\angle$. Given

$\therefore CD$ is $\perp$ to n. § 320

354. Corollary. *If two planes are perpendicular to each other, a line perpendicular to one of them at any point of their intersection lies in the other.*

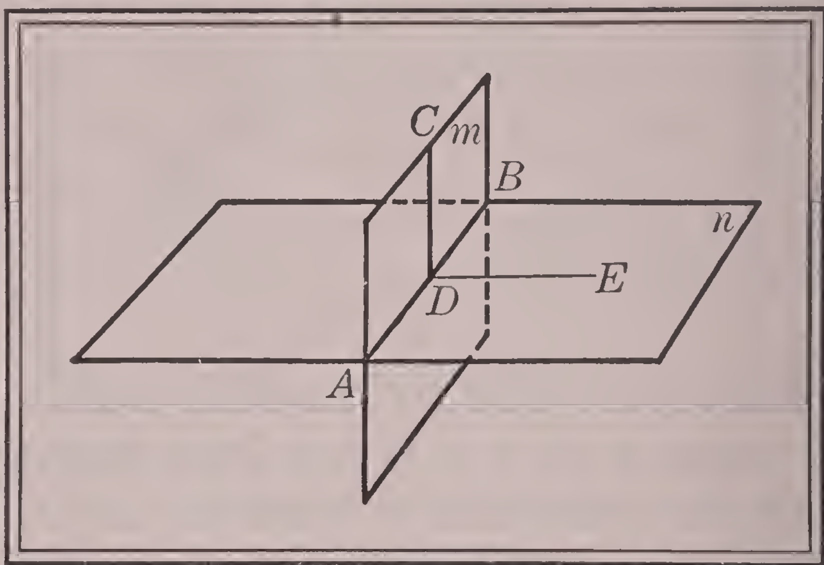
In the figure of § 353, let DC be constructed in the plane $m \perp$ to AB at any point D. Then DC is $\perp$ to n (§ 353). Now through D there can be only one $\perp$ to n (§ 324). Hence a $\perp$ to n through D must coincide with DC and lie in the plane m.

355. Corollary. *If two planes are perpendicular to each other, a line perpendicular to one plane through any point in the other lies in the second plane.*

In the figure of § 353, if CD is constructed in the plane $m \perp$ to AB through C, it is $\perp$ to n (§ 353). But only one such $\perp$ is possible (§ 325).

Proposition 14. Plane through a Perpendicular

356. Theorem. *If a line is perpendicular to a plane, every plane passed through this line is perpendicular to the plane.*



Given the line **CD** $\perp$ to the plane **n** at the point **D**, and any plane **m** through **CD**.

Prove that m *is* $\perp$ *to* n.

Proof. Let AB be the intersection of the planes m and n.

In n construct $DE \perp$ to AB at D. § 104

Since CD is $\perp$ to n, Given

then CD is $\perp$ to AB. § 316

If a st. line is $\perp$ to a plane, the line is $\perp$ to every line in the plane that passes through the point of meeting.

Now $\angle EDC$ measures the dihedral $\angle nm$, § 349

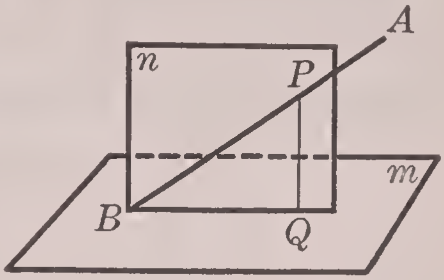
because the plane $\angle$ of a dihedral $\angle$ may be taken as the measure of the dihedral $\angle$.

But $\angle EDC$ is a rt. $\angle$, § 316

because CD is given $\perp$ to n.

$\therefore m$ is $\perp$ to n. § 352

357. Corollary. *Through a line not perpendicular to a given plane, one and only one plane can pass perpendicular to the given plane.*



Let AB be any line oblique to plane m, and from any point P on AB let a $\perp PQ$ be drawn to m. Then the plane n, determined by AB and PQ, is $\perp$ to m (§ 356), so that one such plane is possible.

If another such plane, say x, were possible, PQ would lie in x (§ 355), so that x would be determined by AB and PQ, and hence would coincide with n.

Exercises. Planes and Perpendiculars

1. A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.

2. If two intersecting planes are respectively perpendicular to two intersecting lines, the line which is determined by the planes is perpendicular to the plane which is determined by the lines.

3. Consider Ex. 2 after interchanging the words "planes" and "lines" in every instance.

4. The plane passing through a given point P and perpendicular to the edge of a given dihedral angle contains the perpendiculars from P to the faces of the angle.

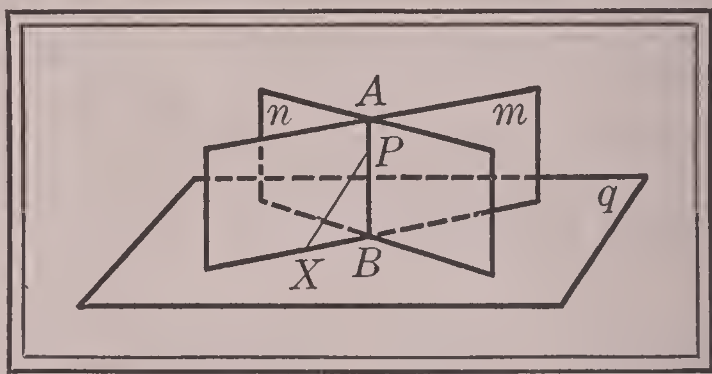
5. A workman with a 12-foot pole, a piece of chalk, and a string stands in a room that is 10 ft. high. By the use of these instruments how can he find a point on the floor that is directly below a given point on the ceiling?

6. If two planes are perpendicular to each other, a line perpendicular to either plane through an internal point not on their intersection is parallel to the other plane.

7. Consider Ex. 6 after substituting the word "plane" for "line" and "line" for "plane."

Proposition 15. Two Perpendicular Planes

358. Theorem. *If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.*



Given the planes m and n intersecting in AB and each $\perp$ to the plane q.

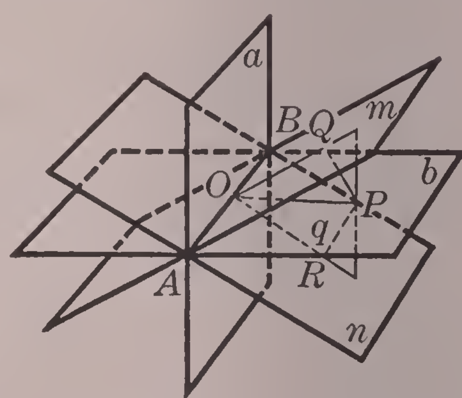
Prove that AB *is* $\perp$ *to* q.

Proof. Through P, any point on AB, there can pass but one line, as PX, which is $\perp$ to q. § 325

Now PX must lie in both m and n. § 355

Hence PX coincides with AB (§ 315); that is, AB is $\perp$ to q.

359. Corollary. *The locus of points equidistant from two intersecting planes is the pair of planes which bisect the dihedral angles formed by the given planes.*



Let plane b bisect the dihedral $\angle nm$. We must first show that, if the $\perp$s PQ, PR from any point P to m and n are equal, P lies in b. Now q, the plane of PQ and PR, is $\perp$ to m and n (§ 356). Then OA is $\perp$ to q (§ 358). Since $\triangle OPQ$ is congruent to $\triangle OPR$ (§ 71), $\angle POQ = \angle POR$ (§ 38). Then P lies in b. Now prove by congruent $\triangle$s (§ 68) that for any point P lying in b, $PQ = PR$.

A similar proof evidently holds for the plane a.

Exercises. Review

1. If three equal lines are drawn to a plane from an external point, the perpendicular from the point to the plane determines the center of the circle circumscribed about the triangle determined by the intersections of the planes of the three lines with the given plane.

2. If three lines not in the same plane meet in a point, how shall a line be drawn so as to make equal angles with all three of these lines?

After proving Exs. 3–5, interchange the words "line" and "plane" and prove the resulting statements:

3. If a line is parallel to one of two parallel planes, it is also parallel to the other plane.

Exs. 3–5 illustrate what are known as *dual propositions*, which are propositions that are also true if we interchange such terms as "line" and "plane." Another kind of dual propositions is seen in Exs. 6 and 7.

4. If a plane and a line not in the plane are perpendicular to the same line, they are parallel.

"In the plane" will become "through the line" in the dual.

5. Parallel planes make equal angles with a given line.

If P is an external point and PA is a segment from P to a plane, let PQ be perpendicular to the plane. Then the *inclination* of PA to the plane, or the angle which it makes with the plane, is the $\angle QAP$.

In Exs. 6 and 7 interchange "point" and "plane" and draw figures to illustrate each statement and its dual:

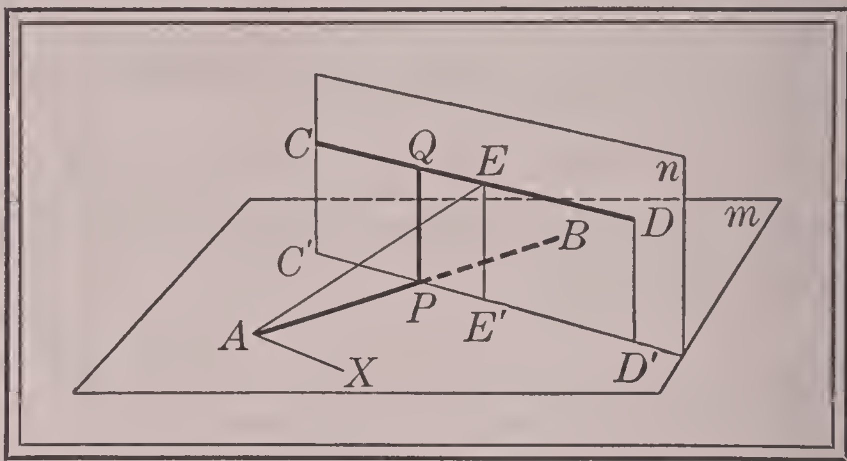
6. Three points, in general, determine a plane.

7. A point and a line determine a plane.

8. If a plane is perpendicular to one of two parallel lines, it is perpendicular to the other line also. Write the dual, as in Exs. 3–5, and prove it.

Proposition 16. Common Perpendicular

360. Theorem. *Between two lines not in the same plane there is one and only one common perpendicular.*



Given *AB* and *CD*, two lines not in the same plane.

Prove that there is one and only one common ⊥ between AB and CD.

Proof. In the plane of A and CD let AX be ∥ to CD. § 107

Then m, the plane of AX and AB, is ∥ to CD. § 333

Let DD' be the ⊥ from D to m. § 325

Then n, the plane of DD' and CD, is ⊥ to m. § 356

Let n intersect m in $C'D'$.

Then CD cannot meet $C'D'$ without meeting m. § 313

But m is ∥ to CD. Proved

∴ CD is ∥ to $C'D'$. § 51

If AB is ∥ to $C'D'$,

AB is also ∥ to CD. § 332

Since AB and CD are not in the same plane, Given

AB is not ∥ to CD. § 51

∴ AB must intersect $C'D'$.

Designate the point of intersection as P.

Let	PQ be $\perp$ to m.	§ 324
Then	PQ is $\perp$ to AB and to $C'D'$,	§ 316
and hence	PQ is $\perp$ to CD.	§ 63
Hence	there is one common $\perp$.	

If there is another common $\perp$, suppose it to be EA.

Then	EA is $\perp$ to AX,	§ 63
and hence	EA is $\perp$ to m.	§ 320
Let	EE' be $\perp$ to $C'D'$.	§ 105
Then	EE' is $\perp$ to m.	§ 353
But if	EA is $\perp$ to m,	
	EE' cannot be $\perp$ to m.	§ 325

That is, the supposition that there is a second common $\perp$, as EA, leads to an impossible result.

Hence there is one and only one common $\perp$ between the lines AB and CD.

Exercises. Review

1. The common perpendicular between two lines not in the same plane is the shortest line joining them.

2. If three lines passing through a given point P are cut by a fourth line that does not pass through P, the four lines all lie in the same plane.

3. If seven lines, no three of which lie in the same plane, pass through the same point, how many planes are determined by these lines?

4. A cubic tank 10 in. deep is filled with water to a depth of 7 in. A foot rule resting on and oblique to the bottom just reaches the top edge of the tank. Make a sketch of the tank, and compute the length of the rule covered by water.

5. A plane perpendicular to one of two parallel planes is also perpendicular to the other plane.

6. If the walls of a room are perpendicular to both the ceiling and the floor, what is the geometric reason for asserting that the ceiling is parallel to the floor?

7. If four points lie in a straight line, can their projections on a plane lie in a straight line? Must they lie in a straight line? Draw the figures to illustrate your answers and state the geometric principles involved.

8. In plane geometry we find that three lines, in general, determine three points; namely, the vertices of the triangle formed by the lines. Draw a figure illustrating the corresponding case for planes in solid geometry and state what is determined.

9. The equal sides of an isosceles triangle make equal angles with any plane that contains the base.

10. The base BC of the isosceles $\triangle ABC$ in the plane m is 3 in., and the perimeter of the triangle is 10 in. If the triangle revolves about its base as an axis, what is the greatest distance, to the nearest 0.001 in., from the plane that is reached by A?

In finding roots or powers, the student should make use of the table given on page 462.

11. A point P moves so as to be constantly 10 in. from each of the points A and B, which are 8 in. apart. Find to the nearest 0.001 in. the length of the locus of P.

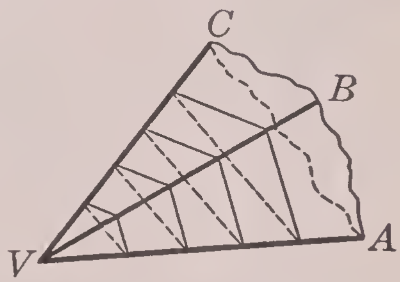
12. Two parallel planes m and n are cut by a third plane P so that one of the dihedral angles contains $32° 45'$. Find the sizes of the other dihedral angles.

13. From a point P which is 12 in. above a plane m, PO is drawn perpendicular to m. With O as center and a radius of 9 in. a circle is drawn in m. At Q, any point on this circle, a tangent QR, 20 in. in length, is drawn in m. Find the length of PR to the nearest 0.01 in.

III. Polyhedral Angles

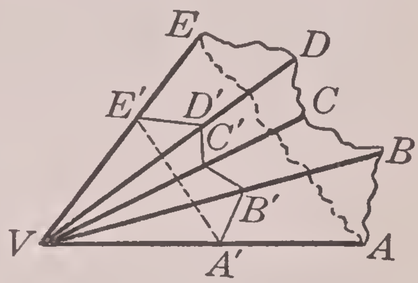
361. Polyhedral Angle. If three or more planes meet in a point, they form a *polyhedral angle.*

If we consider angles greater than 360°, two lines which meet form an infinite number of plane angles; but this fact never confuses us in plane geometry. Similarly, in solid geometry, if three planes meet in a point, they form an infinite number of polyhedral angles; but since we shall always make clear the angle to be considered, this fact will cause no difficulty.



The three planes *AVB, BVC,* and *CVA,* in the above figure, form a polyhedral angle which we designate by *V,* or by *V-ABC.* The letters are given in the order in which they occur around the figure.

If the figure formed on a plane which cuts all the planes forming a polyhedral angle is convex, the angle is said to be *convex*; if the figure is concave, as here shown, the angle is said to be *concave.* Since we shall consider only convex polyhedral angles, this distinction need not be memorized.



362. Parts of a Polyhedral Angle. The common point at which the planes meet to form a polyhedral angle is called the *vertex* of the angle. The intersections of the planes are called the *edges* of the angle. The portions of the planes lying between the edges are called the *faces* of the angle. The angles formed by adjacent edges are called the *face angles* of the polyhedral angle. The vertex, edges, faces, face angles, and the dihedral angles formed by the faces are the *parts* of a polyhedral angle.

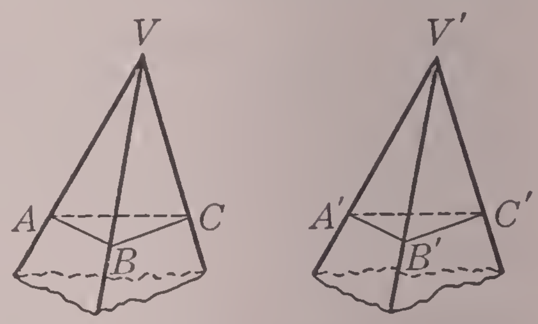
In a polyhedral angle of n faces there are n edges, n face angles, and n dihedral angles.

In the first of the above figures, the vertex is *V*; the edges are *VA, VB, VC*; the faces are *AVB, BVC, CVA*; the face angles are $\angle AVB$, $\angle BVC$, $\angle AVC$; and the dihedral angles are *VA, VB, VC.*

363. Classes of Polyhedral Angles. A polyhedral angle of three faces is called a *trihedral angle.*

Polyhedral angles of four, five, six, seven, ⋯ faces take the prefixes tetra-, penta-, hexa-, hepta-, ⋯. Since, however, we rarely refer to polyhedral angles of more than three faces, these names need not be memorized.

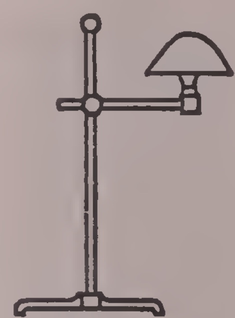
364. Equal Polyhedral Angles. If two polyhedral angles can be placed so that their vertices and edges coincide, the angles are said to be *equal.*



Conversely, if polyhedral angles are equal, they can be made to coincide by superposition, because, if the vertices and edges coincide, all the corresponding parts coincide. In this figure the trihedral $\measuredangle V$ and V' are equal.

Exercises. Review

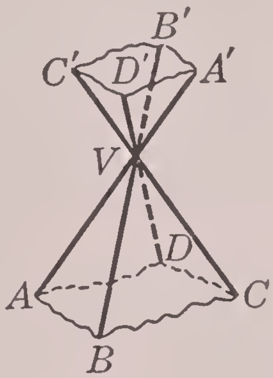
1. A reading lamp is attached to an upright rod which is fastened to two iron pieces, or feet, resting on the floor as here shown. If the rod is perpendicular to the two pieces, is it perpendicular to the floor? Would three pieces be better? Would four be better? Would five be still better? State the geometric principles involved in your answers.

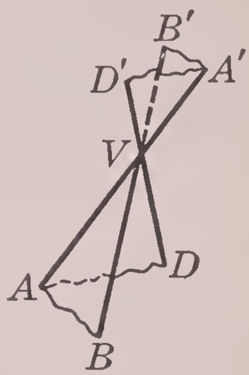


2. Two adjacent walls and the ceiling of a rectangular room form a trihedral angle. Write a statement of the relations of the parts of the angle; for example, that each dihedral angle has a certain size.

3. It is known that the projections of four points upon a plane (that is, the feet of the perpendiculars from the points to the plane) lie in a straight line. Write your conclusion as to whether or not these points lie in a straight line; lie in a plane; are scattered at random in space.

365. Symmetric Polyhedral Angles. If the faces of the polyhedral $\angle V\text{-}ABCD$ are produced through the vertex V, another polyhedral angle, the $\angle V\text{-}A'B'C'D'$, is formed. The $\angle V\text{-}A'B'C'D'$ is said to be *symmetric* with respect to $\angle V\text{-}ABCD$.





The face $\measuredangle AVB$, BVC, $\cdots$, in the figure at the left, are equal respectively to the face $\measuredangle A'VB'$, $B'VC'$, $\cdots$ of the polyhedral $\angle V\text{-}A'B'C'D'$ (§ 35).

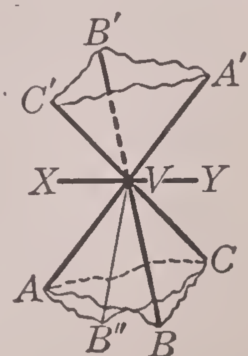
Also, the dihedral $\measuredangle VA$, VB, $\cdots$ are equal respectively to the dihedral $\measuredangle VA'$, VB', $\cdots$ (§ 351, 3). The figure at the right shows a pair of these vertical dihedral angles.

Looked at from the point V, the edges of $\angle V\text{-}ABCD$ are arranged counterclockwise (from left to right) in the order VA, VB, VC, VD, but the edges of $\angle V\text{-}A'B'C'D'$ are arranged clockwise (from right to left) in the order VA', VB', VC', VD'; that is, in an order which is the reverse of the order of the edges of $\angle V\text{-}ABCD$. Therefore,

Two symmetric polyhedral angles have all their parts equal, each to each, but arranged in reverse order.

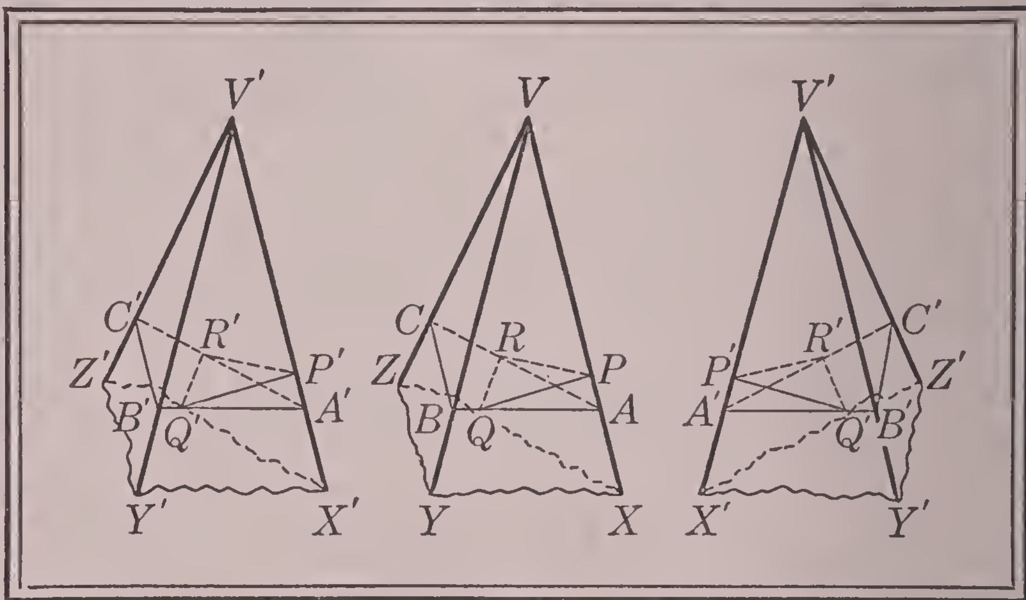
366. Symmetric Polyhedral Angles not Superposable. In general, two symmetric polyhedral angles cannot be superposed.

If the trihedral $\angle V\text{-}A'B'C'$ here shown is made to turn 180° about XY, the bisector of $\angle CVA'$, then VA' coincides with VC, VC' with VA, and the face $A'VC'$ with the face AVC. But since the dihedral $\angle VA$, and hence the dihedral $\angle VA'$, is not equal to the dihedral $\angle VC$, the face $A'VB'$ does not coincide with the face BVC, nor does $C'VB'$ with AVB. Hence VB' takes some position as VB''; that is, symmetric trihedral angles cannot, in general, be superposed.



Proposition 17. Trihedral Angles

367. Theorem. *If the three face angles of one trihedral angle are equal respectively to the three face angles of another, the trihedral angles are either equal or symmetric.*



Given the trihedral ∡s $V\text{-}XYZ$ and $V'\text{-}X'Y'Z'$ with face ∡s YVX, ZVX, ZVY equal respectively to face ∡s $Y'V'X'$, $Z'V'X'$, $Z'V'Y'$.

Prove that the trihedral ∡s $V\text{-}XYZ$ and $V'\text{-}X'Y'Z'$ are either equal or symmetric.

Proof. On the edges of the trihedral ∡s take the six equal segments VA, VB, VC, $V'A'$, $V'B'$, $V'C'$.

Draw AB, BC, CA, $A'B'$, $B'C'$, $C'A'$.

Since the face ∡s are respectively equal, Given

then $\triangle BAV$ is congruent to $\triangle B'A'V'$,

$\triangle CAV$ is congruent to $\triangle C'A'V'$,

and $\triangle CBV$ is congruent to $\triangle C'B'V'$. § 40

Hence $AB = A'B'$, $BC = B'C'$, $CA = C'A'$. § 38

$\therefore \triangle BAC$ is congruent to $\triangle B'A'C'$. § 47

From any point P on VA construct PQ in the face XVY and PR in the face XVZ, each $\perp$ to VA. § 104

Since $\angle$s VAB and VAC are equal $\angle$s of isosceles $\triangle$s,

$\angle$s VAB and VAC are acute. § 65

Hence the $\perp$s PQ and PR meet AB and AC respectively.

Draw QR.

On $A'V'$ take $A'P'=AP$.

In the faces $X'V'Y'$ and $X'V'Z'$ construct $P'Q'$ and $P'R'$ respectively, each $\perp$ to $V'A'$.

Draw $Q'R'$.

Since $AP=A'P'$, Const.

since $\angle QPA=\angle Q'P'A'$, Post. 6

and since $\angle PAQ=\angle P'A'Q'$, § 38

then $\triangle APQ$ is congruent to $\triangle A'P'Q'$. § 44

Hence $AQ=A'Q'$

and $PQ=P'Q'$. § 38

Similarly, $AR=A'R'$ and $PR=P'R'$.

Now, since $\triangle BAC$ is congruent to $\triangle B'A'C'$, Proved

we have $\angle CAB=\angle C'A'B'$. § 38

Then $\triangle ARQ$ is congruent to $\triangle A'R'Q'$, § 40

and hence $RQ=R'Q'$. § 38

Then $\triangle QPR$ is congruent to $\triangle Q'P'R'$, § 47

and hence $\angle RPQ=\angle R'P'Q'$. § 38

$\therefore$ dihedral $\angle VA=$ dihedral $\angle V'A'$. § 349

Similarly, dihedral $\angle VB=$ dihedral $\angle V'B'$,

and dihedral $\angle VC=$ dihedral $\angle V'C'$.

Hence $\angle V\text{-}XYZ=\angle V'\text{-}X'Y'Z'$, § 364

or these trihedral $\angle$s are symmetric. § 365

The symmetric angles are shown in the two figures at the right.

Exercises. Review

1. Make a list of the numbered propositions in Book VI, stating under each the previous propositions employed directly in its proof.

2. Make another list of the numbered propositions, stating under each the subsequent propositions or corollaries in Book VI in which it is used as an authority.

3. Mention three trihedral angles which may be found in a room, and state whether they are equal, symmetric, or both equal and symmetric.

4. By consulting a dictionary find the derivation of the words "dihedral," "trihedral," and "polyhedral," and explain how these derivations apply to the figures.

5. If each of two trihedral angles have right angles for their face angles, they are both equal and symmetric.

6. Consider whether or not two trihedral angles are equal if two face angles and the included dihedral angle of one are equal respectively to two face angles and the included dihedral angle of the other, and give the proof.

7. State a condition under which two polyhedral angles of four faces are equal, and prove the equality.

8. In the trihedral $\angle V\text{-}ABC$, what is the locus of points equidistant from the faces VAB and VBC? from the faces VAB and VCA? from the faces VAB and VBC, and also from the faces VAB and VCA? To what proposition in plane geometry does this correspond?

9. If two intersecting planes pass through two parallel lines a and b respectively, their line of intersection is also parallel to a and b.

10. A line parallel to each of two intersecting planes is parallel to their line of intersection.

BOOK VII

POLYHEDRONS, CYLINDERS, AND CONES

I. Prisms

368. Polyhedron. A solid bounded by planes is called a *polyhedron.*

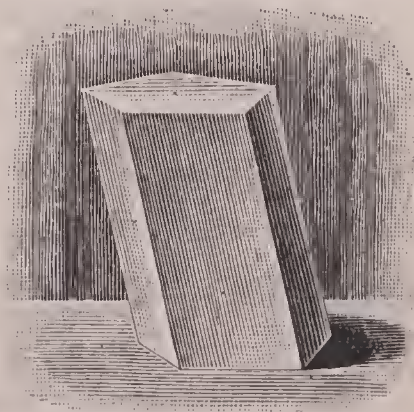
For example, in the second figure of § 367, if we consider that part of the $\angle V\text{-}XYZ$ cut off by the plane ABC, we have a polyhedron.

The bounding planes are called the *faces* of the polyhedron, the intersections of the faces are called the *edges* of the polyhedron, and the intersections of the edges are called the *vertices* of the polyhedron.

A line joining any two vertices not in the same face is called a *diagonal* of the polyhedron.

If every plane which cuts a given polyhedron forms a convex polygon, the polyhedron is said to be *convex.* We shall consider only convex polyhedrons in this course.

369. Prism. A polyhedron of which two faces are congruent polygons in parallel planes, and the other faces are parallelograms, is called a *prism.*



The figure at the right shows a prism. The parallel polygons are called the *bases* of the prism, the parallelograms are called the *lateral faces,* and the intersections of the lateral faces are called the *lateral edges.* The lateral edges of a prism are equal (§ 76).

The sum of the areas of the lateral faces is called the *lateral area* of the prism.

The perpendicular distance between the planes of the bases is called the *height* or *altitude* of the prism.

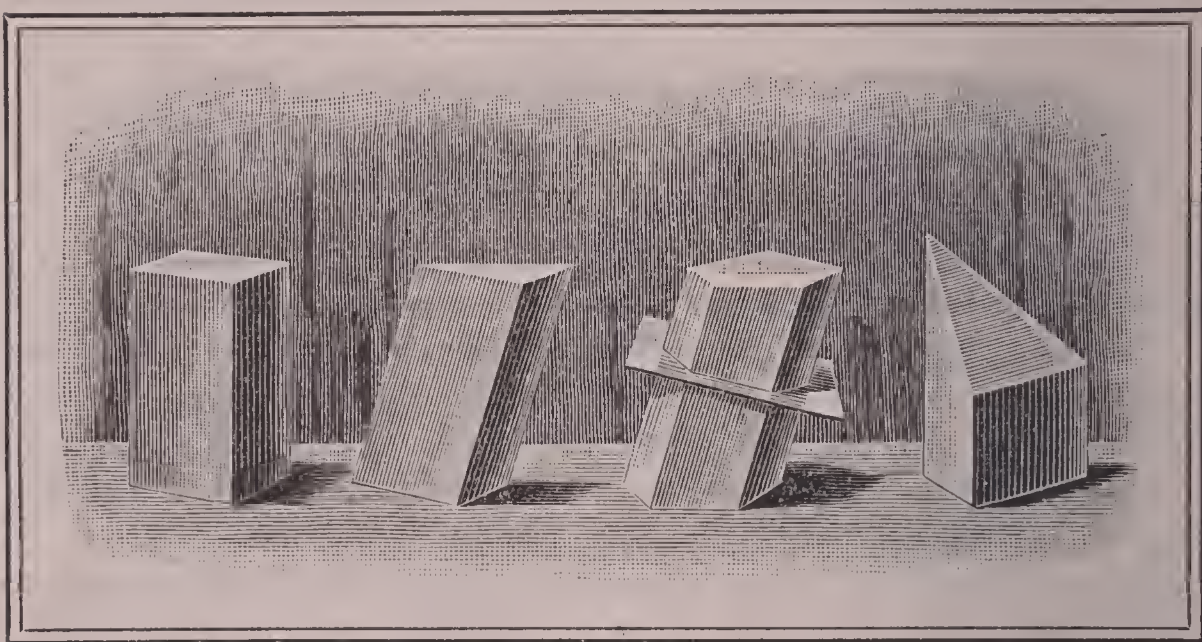
The meaning of the terms *congruent* and *equivalent* as applied to polyhedrons is evident from plane geometry.

370. Right Prism. A prism whose lateral edges are perpendicular to its bases is called a *right prism.*

The first of the figures below shows a right prism. The lateral edges of a right prism are equal to the altitude (§ 341).

371. Oblique Prism. A prism whose lateral edges are oblique to its bases is called an *oblique prism.*

The second of the figures below shows an oblique prism.



372. Prisms classified as to Bases. Prisms are said to be *triangular*, *quadrangular*, and so on, according as their bases are triangles, quadrilaterals, and so on.

Thus, the first figure above shows a quadrangular prism, the second shows a triangular prism, and so on.

373. Right Section. The polygon formed by the intersections of the lateral faces of a prism with a plane which cuts all the lateral edges, produced if necessary, and is perpendicular to them, is called a *right section.*

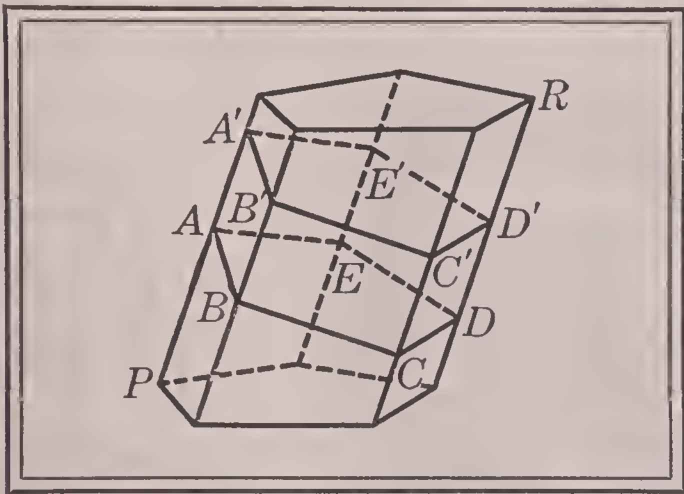
The third figure above shows how a right section is formed.

374. Truncated Prism. The part of a prism included between the base and a section made by a plane oblique to the base is called a *truncated prism.*

The fourth of the above figures shows a truncated prism.

Proposition 1. Sections of a Prism

375. Theorem. *The sections of a prism made by parallel planes cutting all the lateral edges are congruent polygons.*



Given the prism PR and the sections AD and $A'D'$ made by ∥ planes cutting all the lateral edges.

Prove that AD *is congruent to* $A'D'$.

Proof. Lettering the figure as shown above, we see that

AB is ∥ to $A'B'$, BC is ∥ to $B'C'$, CD is ∥ to $C'D'$,

and so on for all the corresponding sides. § 338

If two ∥ planes are cut by a third plane, the lines of intersection are ∥.

Then $AB = A'B'$, $BC = B'C'$, $CD = C'D'$,

and so on for all the corresponding sides. § 76

The opposite sides of a ▱ are equal ⋯.

Also, $\angle CBA = \angle C'B'A'$, $\angle DCB = \angle D'C'B'$,

and so on for all the corresponding ∠s. § 343

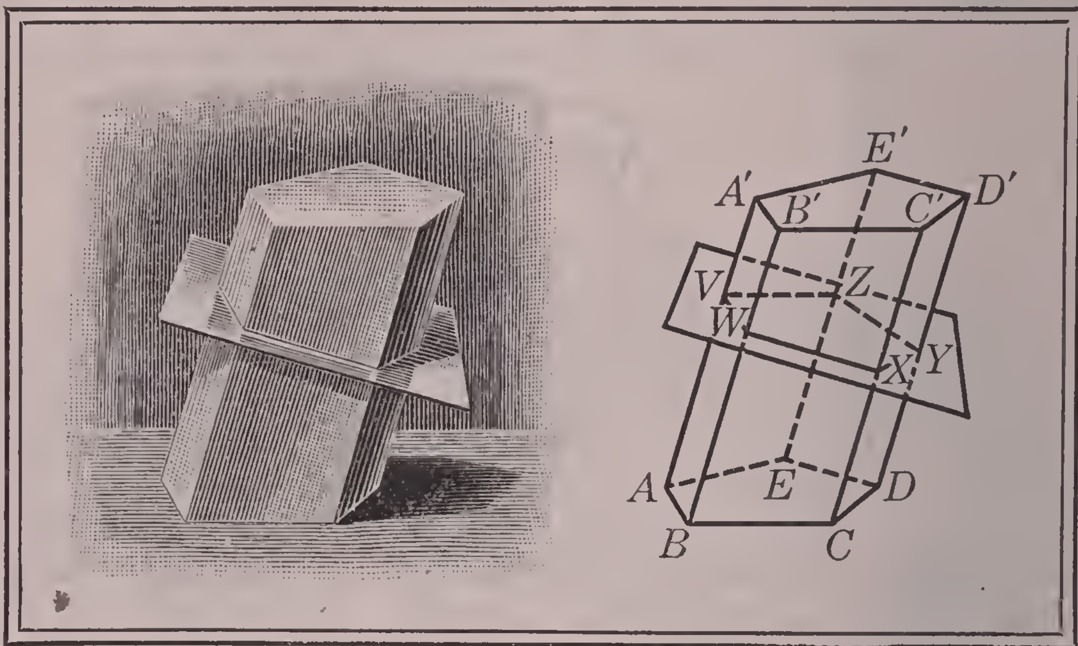
$\therefore$ AD is congruent to $A'D'$. § 37

Since all their corresponding parts are equal, the sections can be made to coincide by superposition.

As a special case of this theorem, *all right sections of a prism are congruent.*

Proposition 2. Lateral Area

376. Theorem. *The lateral area of a prism is the product of a lateral edge and the perimeter of a right section.*



Given VY, a rt. section of the prism AD'; L, the lateral area; e, a lateral edge; and p, the perimeter of the rt. section.

Prove that $L = ep.$

Proof. Lettering the figure as shown, we see that

$AA' = BB' = CC' = DD' = EE' = e.$ § 369

Also, VW is $\perp$ to BB'. § 316

Similarly, WX is $\perp$ to CC', and so on.

$\therefore \square AB' = BB' \cdot VW = e \cdot VW.$ § 243

Similarly, $\square BC' = CC' \cdot WX = e \cdot WX,$

and so on for all the lateral faces.

Now L is the sum of these areas. § 369

$\therefore L = e(VW + WX + XY + YZ + ZV).$ Ax. 1

But $VW + WX + XY + YZ + ZV = p.$ § 7

$\therefore L = ep.$ Ax. 5

Exercises. Practical Measurements

1. *The lateral area of a right prism is the product of the altitude and the perimeter of the base.*

2. The right section of a steel rod 10 ft. long is a square whose area is 4.41 sq. in. If the lateral surface is to be nickeled, how many square inches are to be covered?

3. The right section of an iron rod 8 ft. long is an equilateral triangle whose area is $\sqrt{3}$ sq. in. If the lateral surface is enameled, how many square inches are covered?

4. The lateral surface of an iron bar 6 ft. long is to be gilded. If the right section is a square whose area is 1.69 sq. in., how many square inches are covered?

5. A right prism is $2\frac{3}{4}$ in. long and its base is an equilateral triangle whose altitude is 0.866 in. (or $\frac{1}{2}\sqrt{3}$ in.). Find the lateral area.

6. Find the total area of a right prism twice as long as it is thick, and whose base is a square 5.76 sq. in. in area.

7. What is the total area of a right prism whose altitude is 28 in., and whose base is a right triangle of which the hypotenuse is 70 in. and one side is 42 in.?

Find the lateral areas of the right prisms whose altitudes (h) and perimeters (p) of bases are as follows:

8. $h = 16$ in., $p = 27$ in.
9. $h = 2$ ft. 9 in., $p = 3$ ft. 8 in.

Find the lateral areas of the prisms whose lateral edges (e) and perimeters (p) of right sections are as follows:

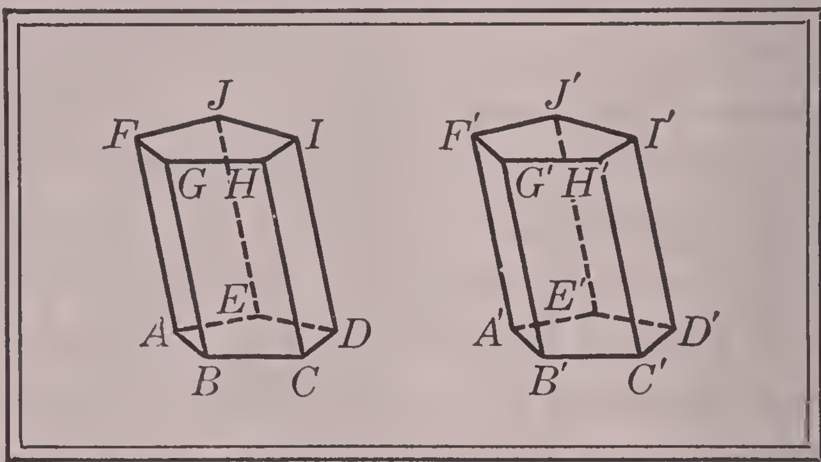
10. $e = 15$ in., $p = 28$ in.
11. $e = 1$ ft. 7 in., $p = 2$ ft. 8 in.

Find the lateral edges of the prisms whose lateral areas (L) and perimeters (p) of right sections are as follows:

12. $L = 169$, $p = 2$.
13. $L = 225$ sq. ft., $p = 9$ ft.

Proposition 3. Congruent Prisms

377. Theorem. *If the three faces which include a trihedral angle of one prism are congruent respectively to three faces which include a trihedral angle of another, and are similarly placed, the prisms are congruent.*



Given the prisms **AI**, **$A'I'$** with faces **AD**, **AG**, **AJ** congruent respectively to faces **$A'D'$**, **$A'G'$**, **$A'J'$**, and similarly placed.

Prove that AI *is congruent to* $A'I'$.

Proof. Lettering as shown, we see that $\angle BAE = \angle B'A'E'$, $\angle BAF = \angle B'A'F'$, $\angle EAF = \angle E'A'F'$ (§ 38), and that they are arranged in the same order (given).

$\therefore$ trihedral $\angle A =$ trihedral $\angle A'$. § 367

Now show by superposition that face AD can be made to coincide with face $A'D'$, face AG with face $A'G'$, and face AJ with face $A'J'$; C lying on C', and D on D'.

Then show by §§ 52, 314 that the planes of the upper bases coincide.

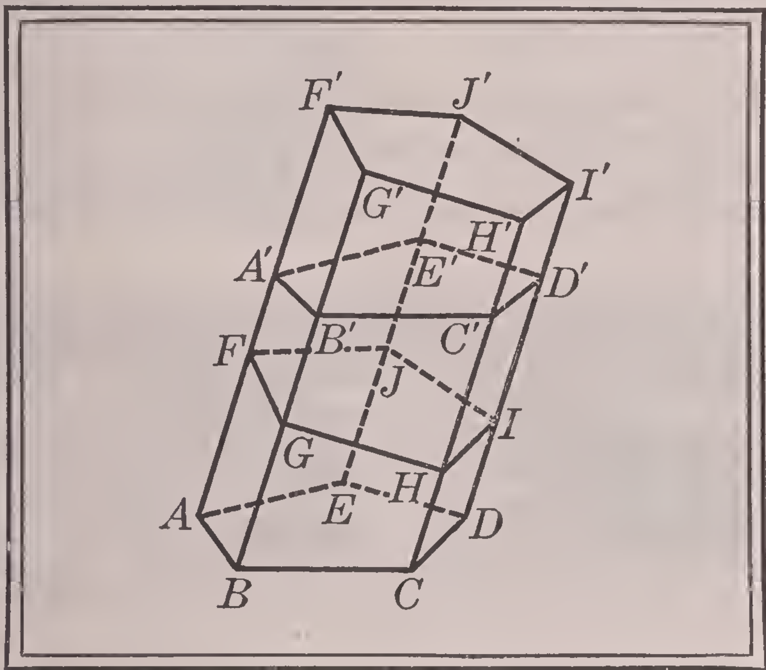
Finally show that the prisms coincide and hence are congruent.

378. Corollary. *Two truncated prisms are congruent under the conditions given in § 377.*

379. Corollary. *Two right prisms which have congruent bases and equal altitudes are congruent.*

Proposition 4. Equivalent Prisms

380. Theorem. *An oblique prism is equivalent to a right prism whose base is equal to a right section of the oblique prism, and whose altitude is equal to a lateral edge of the oblique prism.*



Given FI, a rt. section of the oblique prism AD', and FI', a rt. prism whose altitude is equal to a lateral edge of AD'.

Prove that AD' *is equivalent to* FI'.

Proof. Lettering the figure as shown, we see that if we take from the equal lateral edges of AD' and FI' (§ 370) the lateral edges of FD', which are common to both, we have

$$AF = A'F',\ BG = B'G',\ CH = C'H',\ \cdots. \qquad \text{Ax. 2}$$

Also, base FI is congruent to base $F'I'$. § 369

Now prove the truncated prisms AI and $A'I'$ congruent (§ 378).

Then $AI + FD' = A'I' + FD'$. Ax. 1

Now $AI + FD' = AD'$,

and $A'I' + FD' = FI'$. Ax. 10

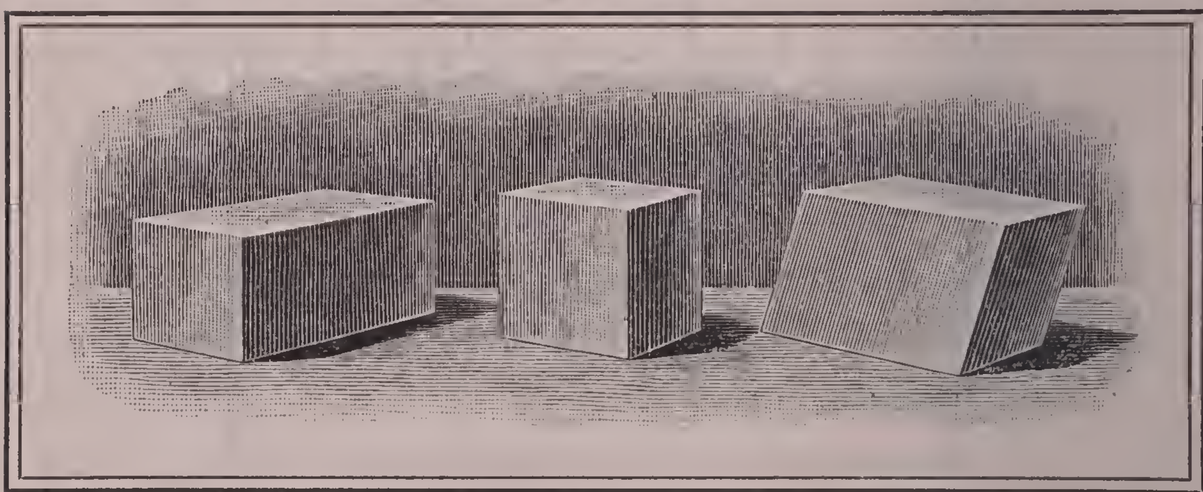
$\therefore$ AD' is equivalent to FI'. Ax. 5

381. Parallelepiped. A prism whose bases are parallelograms is called a *parallelepiped.*

382. Right Parallelepiped. A parallelepiped with edges perpendicular to the bases is called a *right parallelepiped.*

383. Rectangular Parallelepiped. A right parallelepiped whose bases are rectangles is called a *rectangular parallelepiped.*

The first figure below shows a rectangular parallelepiped. The four lateral faces of this solid are also rectangles.



The terms *parallelepiped, right parallelepiped,* and *rectangular parallelepiped* are not commonly used outside the school, the last being generally called a *rectangular solid* or a *square solid.* Some teachers employ the term *cuboid,* but it has not come into practical use.

Among the common illustrations of rectangular solids are boxes, bricks, rooms, and the like.

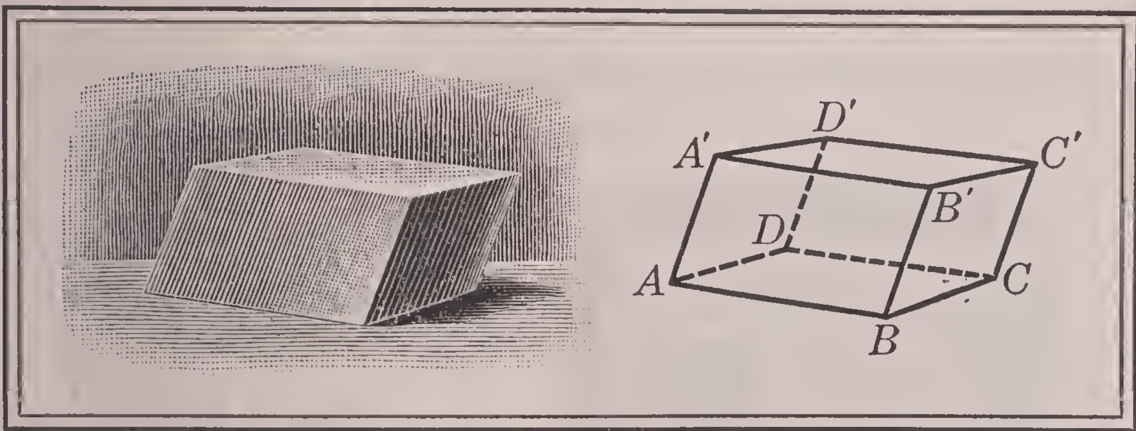
The oblique parallelepiped illustrated in the third figure above is rarely found in practice, and no special definition is necessary.

384. Cube. A parallelepiped whose six faces are all squares is called a *cube.*

The second figure above shows a cube. It might properly be asked how we know that such a figure is possible. We may, for example, speak of a seven-edged polyhedron, but such a figure does not exist. In elementary work, however, it is not expected that attention will be given to such details.

Proposition 5. Opposite Faces

385. Theorem. *The opposite faces of a parallelepiped are congruent and parallel.*



Given the parallelepiped AC'.

Prove that the faces AB', DC' and the faces AD', BC' are respectively congruent and ∥.

What previous proposition can be used to prove that AB' is ∥ to DC'? Can it be proved by superposition that AB' is congruent to DC'? The proof is left for the student.

Exercises. Parallelepipeds

1. The diagonals of a parallelepiped bisect each other.

2. The lateral faces of a right parallelepiped are rectangles, and its four diagonals are equal.

3. Compute the lengths of the diagonals of a rectangular parallelepiped whose edges meeting at any vertex are a, b, c.

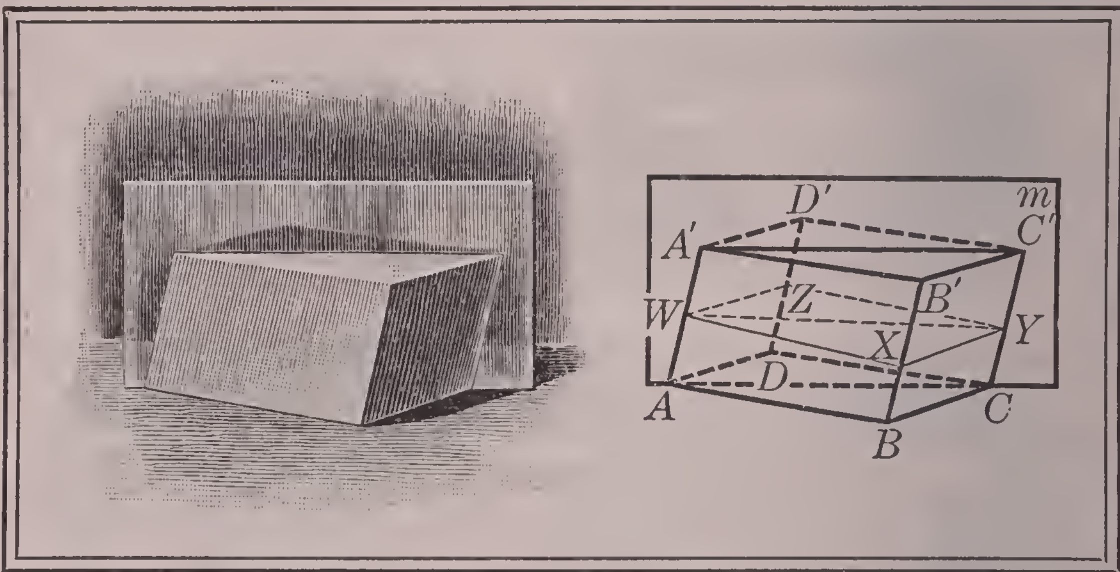
4. Every section of a parallelepiped made by a plane parallel to the lateral edges is a parallelogram.

5. Investigate Ex. 4 for any prism whatever.

6. Given a rectangular parallelepiped, lettered as in the figure of § 385, with $AB = 8$, $BC = 6$, and $CC' = 7\frac{1}{2}$, find the length of the diagonal AC'.

Proposition 6. Diagonal Plane of a Parallelepiped

386. Theorem. *The plane passed through diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.*



Given the plane m passed through the opposite edges AA' and CC' of the parallelepiped AC'.

Prove that AC' is divided into two equivalent triangular prisms ABC-B' and CDA-D'.

Proof. Let $WXYZ$ be a right section of AC'. § 373

Then faces AB' and DC' are congruent and $\parallel$,

and faces AD' and BC' are congruent and $\parallel$. § 385

Hence in the section $WXYZ$,

WX is $\parallel$ to ZY, and XY is $\parallel$ to WZ. § 338

Hence $WXYZ$ is a $\square$, § 72

and $\triangle WXY$ is congruent to $\triangle YZW$. § 77

Now prism ABC-B' is equivalent to a rt. prism with base WXY and altitude AA', and prism CDA-D' is equivalent to a rt. prism with base YZW and altitude AA'. § 380

The rest of the proof is left for the student.

Exercises. Practical Measurements

1. Given that the three plane angles at one of the vertices of a parallelepiped are 80°, 70°, and 75° respectively, find all the other angles in all the faces.

2. The three edges of the trihedral angle at one of the vertices of a rectangular parallelepiped are 10 in., 12 in., and 14 in. respectively. Find the area of the total surface of the parallelepiped.

3. The three face angles at one vertex of a parallelepiped are each 60°, and the three edges of the trihedral angle at that vertex are 6 in., 4 in., and 2 in. respectively. Find the area of the total surface to the nearest 0.01 sq. in.

4. In a rectangular parallelepiped the square of any diagonal is equivalent to the sum of the squares of any three edges which meet at one vertex.

5. In a box 6 in. deep and 12 in. wide, a wire 2 ft. long reaches from one corner to the diagonally opposite corner. Find the length of the box to the nearest 0.01 in.

6. The height of a rectangular parallelepiped is 22 in. and the length of the diagonal of the base is 30 in. Find the length of the diagonal of the parallelepiped.

7. The total area of the six faces of a cube is 108 sq. in. Find the length of the diagonal of the cube.

8. The diagonal of the face of a cube is $\sqrt{6}$. Find the diagonal of the cube.

9. The diagonal of a cube is $5\frac{1}{2}\sqrt{3}$. Find the diagonal of a face of the cube.

10. A water tank is 4 ft. long, 3 ft. wide, and 2 ft. deep. How many square feet of zinc will be required to line the four sides and the base, allowing 2 sq. ft. for overlapping and for turning the top edge?

11. A square sheet of galvanized iron 8 ft. on a side leans against a wall and is inclined at an angle of 60° to the horizontal. What area of ground does the sheet protect from rain falling vertically?

12. Through a point P on the sheet of iron in Ex. 11, what line in the inclined plane will make the largest angle with the horizontal? Give the reason for your answer, and state the number of degrees in this angle.

13. A rectangular solid has for its lower base the $\square ABCD$ and for its upper base the $\square A'B'C'D'$, lettered in the corresponding way. What plane passing through the diagonal DB' is $\parallel$ to AB, and what angle does the plane make with the lower base?

14. The length of the diagonal of a rectangular solid is 17 in. and the area of the total surface is 552 sq. in. Find the sum of the three dimensions.

15. If the length, width, and height of a room are a, b, and c respectively, what is the total area of the four walls? of the walls, floor, and ceiling?

16. The outside dimensions of a closed wooden box are 8 in., 10 in., and 12 in., and the area of the total inside surface is 376 sq. in. Find the thickness of the wood used in making the box.

17. The area of the total surface of a rectangular block is 1332 sq. in. and the dimensions are proportional to 4, 5, and 6. Find the dimensions.

18. The lower base of a cube which is $\sqrt{2}$ on an edge is $ABCD$, and the upper base is $A'B'C'D'$, lettered in the corresponding way. If a plane passes through A', C', and B, what is the area of the $\triangle A'BC'$?

19. The area of $\triangle A'BC'$ in Ex. 18 is what part of the area of the diagonal plane $AB'C'D$?

387. Unit of Volume. In measuring volumes, a cube whose edges are all equal to the unit of length is taken as the *unit of volume.*

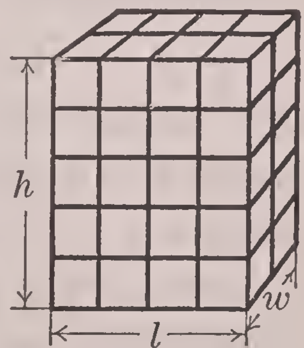
Thus, if we are measuring the contents of a box of which the dimensions are given in feet, we take 1 cu. ft. as the unit of volume.

388. Volume. The number of units of volume contained by a solid is called its *volume.*

389. Equivalent Solids. Two solids which have equal volumes are said to be *equivalent.*

390. Dimensions. The lengths of the three edges of a rectangular parallelepiped which meet at a common vertex are called its *dimensions.*

391. Volume of a Rectangular Parallelepiped. Assuming that the three edges are commensurable, suppose that l, the length, contains 4 units; that w, the width, contains 2 units; and that h, the height, contains 5 units. Then V, the volume, contains $4 \times 2 \times 5$, or 40, cubic units.



In general, if there are l units of length, w units of width, and h units of height, then

$$V = lwh;$$

that is, *the volume of a rectangular parallelepiped is the product of its three dimensions.*

For on each square unit of base there is one cubic unit for every unit of height. Then, since there are lw square units of base (§ 241), there are lw cubic units for every unit of height. Hence for h units of height, there are lwh cubic units of volume.

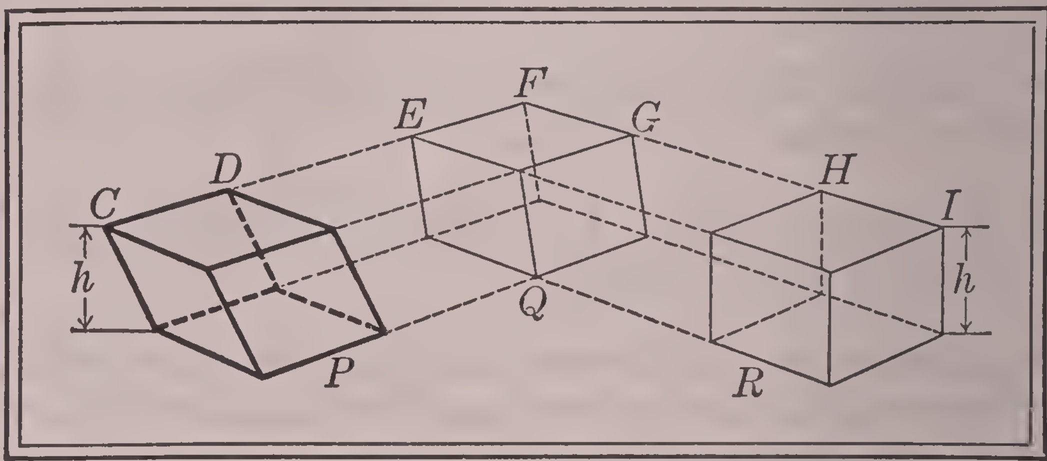
The incommensurable case is considered in § 517.

392. Corollary. *The volume of a rectangular parallelepiped is the product of its base and altitude.*

For, if B is the area of the base, $B = lw$, and hence $V = Bh$.

Proposition 7. Volume of a Parallelepiped

393. Theorem. *The volume of any parallelepiped is the product of the base and the altitude.*



Given P, a parallelepiped with no two faces $\perp$ to each other; V, the volume; B, the area of the base; and h, the altitude.

Prove that $V = Bh$.

Proof. Produce the edge CD and the edges $\parallel$ to CD, and cut them perpendicularly by two $\parallel$ planes whose distance apart, EF, is equal to CD. We then have the oblique parallelepiped Q whose base is a $\square$.

Produce FG and the edges $\parallel$ to FG, and cut them perpendicularly by two $\parallel$ planes whose distance apart, HI, is equal to FG. We then have the rectangular parallelepiped R.

Since $P = Q$, and $Q = R$, § 380

then $P = R$. Ax. 5

Also, P, Q, and R have the common altitude h. § 341

Let B' be the area of the base of Q, and B'' that of R.

Since $B = B'$, and $B' = B''$, § 245

then $B = B''$. Ax. 5

Now the volume of R is $B''h$. § 392

$\therefore V = Bh$. Ax. 5

Exercises. Parallelepipeds

1. *Two rectangular parallelepipeds with congruent bases are to each other as their altitudes.*

In all such cases the words "the volumes of" are understood.

2. *Two rectangular parallelepipeds with equal altitudes are to each other as their bases.*

3. *Two rectangular parallelepipeds with one dimension in common are to each other as the products of their other two dimensions.*

4. *Two rectangular parallelepipeds with two dimensions in common are to each other as their third dimensions.*

5. *Two rectangular parallelepipeds are to each other as the products of their three dimensions.*

6. *The volume of any parallelepiped is equal to that of a rectangular parallelepiped of equivalent base and equal altitude.*

7. Find the ratio of two rectangular parallelepipeds which have the dimensions a, b, c, and $2a$, $3b$, and $4c$ respectively.

8. Find the ratio of two rectangular parallelepipeds both of whose altitudes are h inches, and whose bases are a inches by $2b$ inches, and $2b$ inches by $3a$ inches respectively.

9. Find the volume of a rectangular parallelepiped l feet long, w inches wide, and h yards high, expressing the result as cubic inches; as cubic feet; as cubic yards.

10. Find the volume of a rectangular parallelepiped whose base is B square feet and whose altitude is h inches, expressing the result as cubic inches; as cubic feet.

11. The volume of a parallelepiped is a^2 cubic inches and the area of the base is a square feet. Express the altitude in inches; in feet.

Exercises. Practical Measurements

1. The volume of a rectangular parallelepiped with a square base is 84 cu. in. and the altitude is 6 in. Find the dimensions.

2. A rectangular tank full of water is 9 ft. long and 5 ft. 6 in. wide. How many cubic feet of water must be drawn off in order that the surface may be lowered 1 ft.?

In all such cases the measurements are supposed to be made on the inside unless the contrary is stated.

3. What dimensions should be allowed for a rectangular container which shall hold just 1 gal. (231 cu. in.) if each dimension must be a whole number of inches?

4. The inside dimensions of a covered box, made of steel $\frac{1}{8}$ in. thick and weighing 490 lb. per cubic foot, are 16 in., 9 in., and 4 in. Find the total weight of the box.

5. A steel bar 6 ft. long is 2 in. wide and $1\frac{3}{4}$ in. thick. At 490 lb. per cubic foot, how much does it weigh?

6. If 3 cu. in. of gold beaten into gold leaf will cover 75,000 sq. in. of surface, what is the thickness of the leaf?

7. The sum of the squares of the four diagonals of a parallelepiped is equal to the sum of the squares of the twelve edges.

8. The volume of a cube is 216 cu. in. Find to the nearest 0.01 in. the length of the diagonal.

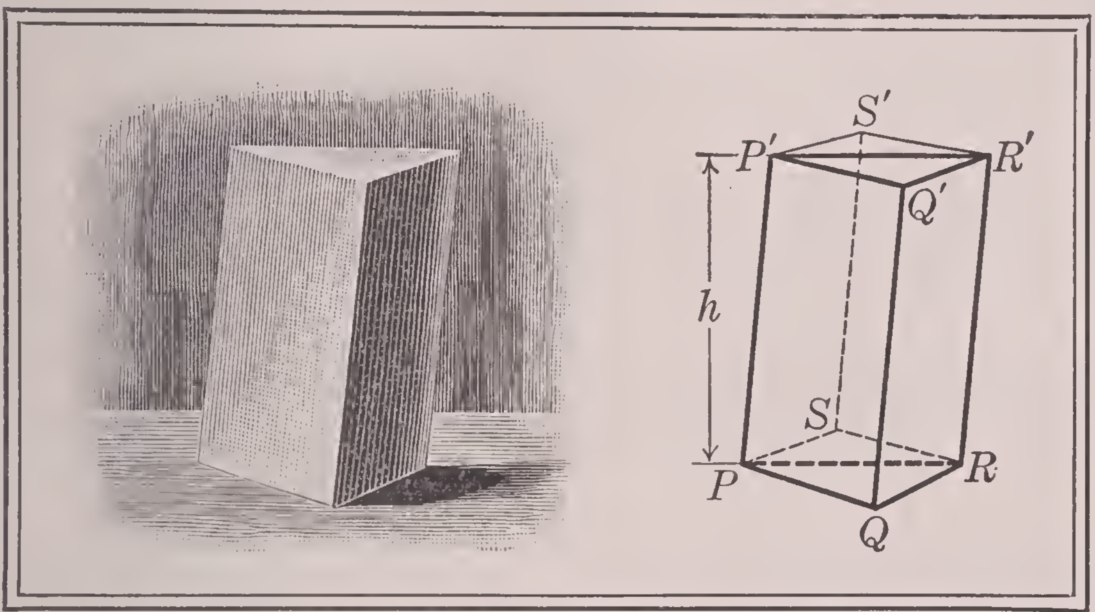
9. Find a formula for the total surface of a cube in terms of the diagonal of the cube.

10. If the total surface of one cube is n times that of another cube, the volume of the first is how many times the volume of the second?

11. Find the weight of a wooden beam 8 in. by 10 in., 18 ft. long, and weighing 54 lb. per cubic foot.

Proposition 8. Triangular Prism

394. Theorem. *The volume of a triangular prism is the product of the base and the altitude.*



Given $PQR\text{-}Q'$, a triangular prism; V, the volume; B, the area of the base; and h, the altitude.

Prove that $V = Bh$.

Proof. Since, in any plane, we can construct a line ∥ to a given line, we can construct PS ∥ to QR, RS ∥ to QP, SS' ∥ to PP', $P'S'$ ∥ to PS, and $R'S'$ ∥ to RS, thus forming the parallelepiped $PQRS\text{-}Q'$.

Then $PQR\text{-}Q' = \frac{1}{2}\, PQRS\text{-}Q'$. § 386

But $PQRS\text{-}Q' = PQRS \cdot h$, § 393

and $PQRS = 2\,B$. § 77

Hence $PQRS\text{-}Q' = 2\,B \cdot h$,

and $PQR\text{-}Q' = \frac{1}{2}(2\,B \cdot h)$, Ax. 5

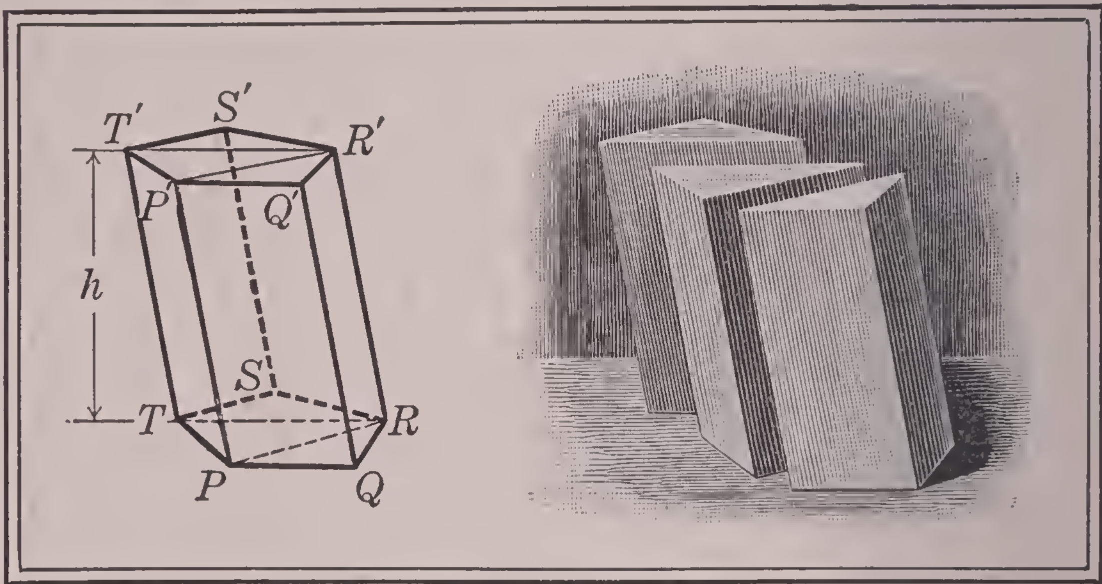
or $PQR\text{-}Q' = Bh$.

Substituting V for $PQR\text{-}Q'$, we have

$$V = Bh.$$

Proposition 9. Volume of a Prism

395. Theorem. *The volume of any prism is the product of the base and the altitude.*



Given PR', **a prism;** V, **the volume;** B, **the area of the base; and** h, **the altitude.**

Prove that $V = Bh$.

Proof. From the vertex R in the lower base $PQRST$ draw the diagonals RP and RT.

From the vertex R' in the upper base $P'Q'R'S'T'$ draw the diagonals $R'P'$ and $R'T'$.

Taken together, the triangular prisms thus determined form the given prism.

Similarly, taken together, the respective bases of the triangular prisms form the bases of the given prism.

Now the volume of each triangular prism is the product of the base and the altitude h. § 394

Hence the total volume of the prism PR' is the sum of the bases of the triangular prisms multiplied by the common altitude h. Ax. 1

That is, $V = Bh$.

Exercises. Review

1. *Prisms with equivalent bases and equal altitudes are equivalent.*

2. *Prisms with equivalent bases are to each other as their altitudes; prisms with equal altitudes are to each other as their bases.*

3. The volume of a triangular prism is equal to half the product of a lateral face and the distance of this face from the opposite edge.

4. The edge of a cube is e. Write a formula for the sum of the edges; for the area of the total surface; for the diagonal of the cube.

5. The edge of a cube is e. Find the volume of a cube that has an edge twice as long. Find the edge of a cube of twice the volume.

6. The altitude of a prism is 7 in., and the base is an equilateral triangle which is 7 in. on a side. Find the volume of the prism.

7. The number of square millimeters in the area of the total surface of a certain cube is equal to the number of cubic millimeters in the volume of the cube. Find the length of each edge.

8. In Ex. 7 find the length of the diagonal of the cube.

9. The number of square millimeters in the area of the total surface of a right triangular prism with an equilateral base is equal to the number of cubic millimeters in the volume, and the altitude of the prism is equal to the side of the base. Find the altitude.

10. If the base of a right prism is a regular polygon of apothem a, and the area of the lateral surface is L, the volume V is given by the formula $V = \frac{1}{2} aL$.

Exercises. Practical Measurements

1. If the length of a rectangular parallelepiped is 22 in., the width 8 in., and the height 6 in., what is the area of the total surface?

2. Find the volume of a triangular prism whose height is 14 in. and whose base has the sides 10 in., 8 in., and 6 in.

3. Find the volume of a prism whose height is 12 ft. and whose base is an equilateral triangle 10 in. on a side.

4. The base of a right prism is a rhombus of which one side is 10 in., and the shorter diagonal 12 in. The height of the prism is 15 in. Find the area of the total surface and the volume of the prism.

5. An open tank 8 ft. long and $5\frac{1}{2}$ ft. wide holds 264 cu. ft. of water. How many square feet of sheet lead will it take to line the sides and bottom?

6. How much sheet lead will be required to line an open tank which is 5 ft. long, 3 ft. 6 in. wide, and contains 105 cu. ft.?

7. The diagonal of one of the faces of a cube is $\sqrt{7}$ in. Find the volume of the cube.

8. The three dimensions of a rectangular parallelepiped are a, b, c. Find in terms of a, b, and c the volume, the area of the total surface, and the length of the diagonal.

9. If the height of a prism is 5 in., and the base is a regular hexagon 1 in. on a side, what is the volume?

10. An open cistern is made of iron $\frac{1}{4}$ in. thick, and the inside dimensions are as follows: length, 6 ft.; width, 4 ft.; depth, 3 ft. What will the cistern weigh when empty? when full of water?

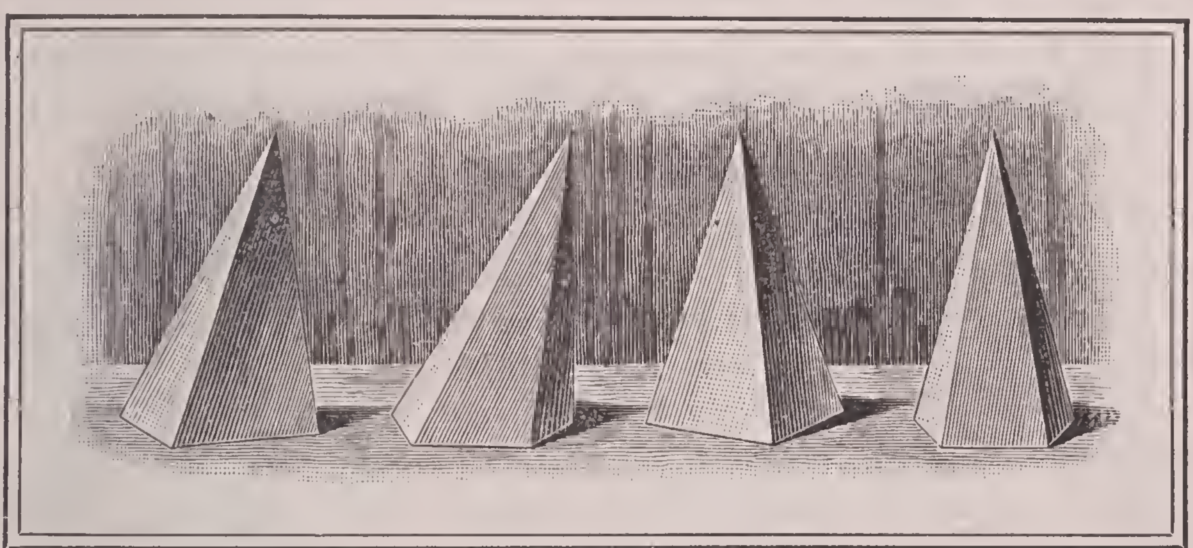
A cubic foot of water weighs $62\frac{1}{2}$ lb. The specific gravity of iron is 7.2; that is, iron is 7.2 times as heavy as water.

II. PYRAMIDS

396. Pyramid. A polyhedron of which one face, called the *base*, is any polygon and the other faces are triangles with a common vertex is called a *pyramid.*

The triangular faces with the common vertex are called the *lateral faces*, their intersections are called the *lateral edges*, and their common vertex is called *the vertex* of the pyramid. In our work we shall consider only pyramids whose bases are convex polygons.

The sum of the areas of the lateral faces is called the *lateral area* of the pyramid. The perpendicular distance from the vertex to the plane of the base is called the *height* or *altitude* of the pyramid.



397. Pyramids classified as to Bases. Pyramids are said to be *triangular*, *quadrangular*, and so on, according as their bases are triangles, quadrilaterals, and so on.

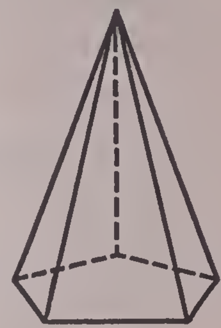
A triangular pyramid is also called a *tetrahedron.*

398. Regular Pyramid. If the base of a pyramid is a regular polygon whose center coincides with the foot of the perpendicular from the vertex to the base, the pyramid is called a *regular pyramid.*

The altitude of the triangle which forms one of the lateral faces of a regular pyramid is called the *slant height* of the pyramid.

The figures above show different types of pyramids, the two at the right being regular pyramids.

399. Properties of Regular Pyramids. The proofs of the following obvious properties of regular pyramids depend upon the theorems indicated:



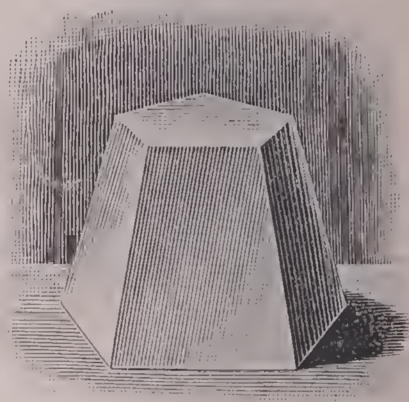
1. *The lateral edges of a regular pyramid are equal* (§ 326).

For they meet the base at equal distances from the center (§ 398).

2. *The lateral faces of a regular pyramid are congruent isosceles triangles* (§ 47).

3. *The slant height of a regular pyramid is the same for all the lateral faces* (§ 326).

400. Frustum of a Pyramid. The portion of a pyramid included between the base and a section parallel to the base is called a *frustum of a pyramid.*

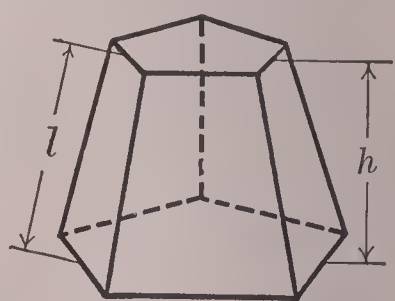


The figure at the right shows a frustum of a regular pyramid.

A more general term, including a frustum as a special case, is *truncated pyramid,* which is applied to the portion of a pyramid included between the base and any section whatever made by a plane that cuts all the lateral edges. This term, however, is little used at the present time.

The base of the pyramid and the parallel section are called the *bases* of the frustum.

The perpendicular distance between the bases is called the *height* or *altitude* of the frustum. The altitude is represented by h in the figure here shown.

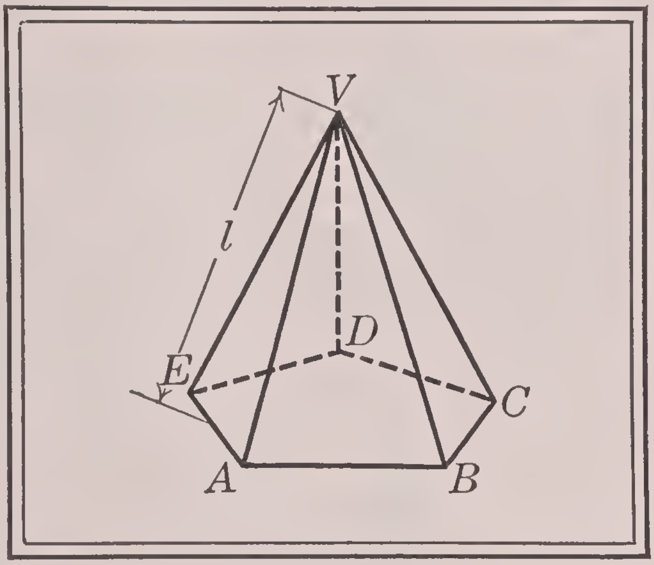


The portions of the lateral faces of a pyramid that lie between the bases of a frustum are called the *lateral faces* of the frustum, and the sum of their areas is called the *lateral area* of the frustum.

The altitude of one of the lateral faces of a frustum of a regular pyramid is called the *slant height* of the frustum. The slant height is represented by l in the above figure.

Proposition 10. Lateral Area of a Pyramid

401. Theorem. *The lateral area of a regular pyramid is half the product of the slant height and the perimeter of the base.*



Given *V-ABCDE*, a regular pyramid; *L*, the lateral area; *l*, the slant height; and *p*, the perimeter of the base.

Prove that $L = \frac{1}{2}lp.$

Proof. The lateral faces are congruent △. § 399, 2

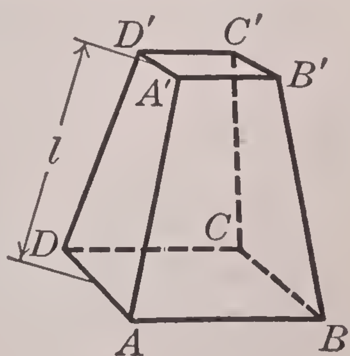
Now the area of each face is $\frac{1}{2}l$ times the base, § 244

and the sum of the bases of the △ is p. § 7

Then the sum of the areas of the △ is $\frac{1}{2}lp$. Ax. 1

$\therefore L = \frac{1}{2}lp.$ Ax. 5

402. Corollary. *The lateral area of a frustum of a regular pyramid is half the product of the slant height of the frustum and the sum of the perimeters of the bases.*

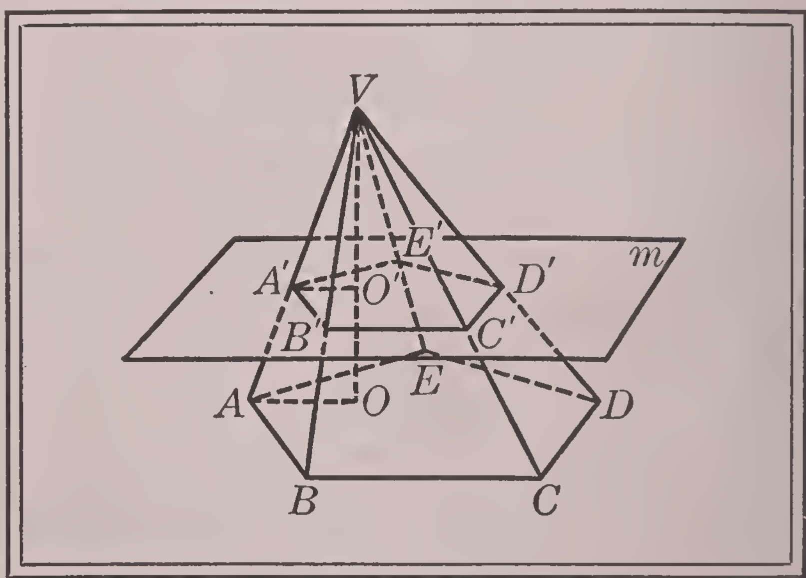


How is the area of a trapezoid found (§ 247)? Are the faces congruent trapezoids? What is the sum of their lower bases? of their upper bases? What is the sum of their areas? Write the formula and give the proof in full.

Proposition 11. Section Parallel to Base

403. Theorem. *If a pyramid is cut by a plane parallel to the base,*

1. *The lateral edges and the altitude are divided proportionally.*
2. *The section is a polygon similar to the base.*
3. *The area of the section is to the area of the base as the square of the distance of the plane from the vertex is to the square of the altitude of the pyramid.*



Given the pyramid $V\text{-}ABCDE$ cut by m, a plane $\parallel$ to the base and intersecting the altitude VO in O', and $A'B'C'D'E'$, the section thus formed.

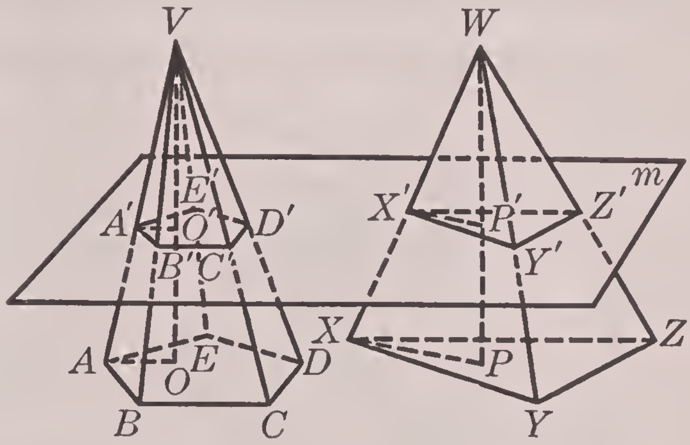
Prove that $\dfrac{VA'}{VA} = \dfrac{VB'}{VB} = \cdots = \dfrac{VO'}{VO}$; *that* $A'B'C'D'E'$ *is similar to* $ABCDE$; *and that* $\dfrac{A'B'C'D'E'}{ABCDE} = \dfrac{\overline{VO'}^2}{\overline{VO}^2}$.

1. Use §§ 338 and 201 to prove the first proportion.

2. Prove $\triangle VA'B'$ similar to $\triangle VAB$, and so on. Then prove the necessary conditions (§ 205) under which two polygons are similar.

3. Prove that $A'B'C'D'E' : ABCDE = \overline{A'B'}^2 : \overline{AB}^2 = \overline{VO'}^2 : \overline{VO}^2$.

404. Corollary. *If two pyramids have equal altitudes and equivalent bases, sections made by planes parallel to the bases at equal distances from the vertices are equivalent.*



$$\frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2}. \quad \text{§ 403, 3}$$

$$\frac{X'Y'Z'}{XYZ} = \frac{\overline{WP'}^2}{\overline{WP}^2}. \quad \text{§ 403, 3}$$

$$\frac{VO'}{VO} = \frac{WP'}{WP}. \quad \text{Ax. 4}$$

$$\therefore \frac{\overline{VO'}^2}{\overline{VO}^2} = \frac{\overline{WP'}^2}{\overline{WP}^2}. \quad \text{Ax. 6}$$

Then $$\frac{A'B'C'D'E'}{ABCDE} = \frac{X'Y'Z'}{XYZ}, \quad \text{Ax. 5}$$

and hence $$A'B'C'D'E' = X'Y'Z'. \quad \text{§ 198, 2}$$

Exercises. Review

1. The base of a pyramid is an equilateral triangle 2 in. on a side. Find the area of a section parallel to the base and halfway between the vertex and the base.

2. A section of a pyramid parallel to the base is equal to half the base. If the altitude of the pyramid is 10 in., how far is the section from the base?

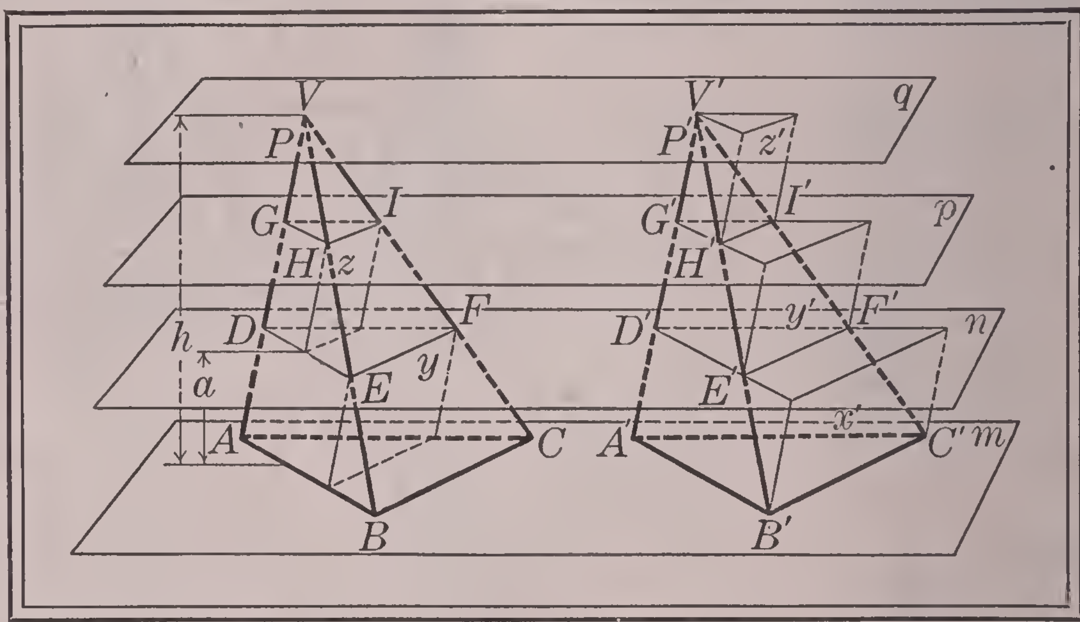
3. Solve Ex. 2 when the section is one nth of the base and the altitude of the pyramid is h feet.

4. The perimeter of the base of a regular pyramid is 40 mm. and the lateral area is 320 sq. mm. Find the slant height of the pyramid.

5. The top of a certain obelisk is a regular pyramid with a square base 225 sq. in. in area and an altitude of 30 in. Find to the nearest 0.1 in. the slant height of the pyramid, and then find the lateral area.

Proposition 12. Triangular Pyramids

405. Theorem. *Two triangular pyramids with equivalent bases and equal altitudes are equivalent.*



Given the triangular pyramids P and P' with the equivalent bases ABC and $A'B'C'$ and the equal altitudes h.

Prove that $P = P'$.

Proof. Place the bases in the plane m, divide h into any number of equal parts, as a, and through the points of division of h let the planes $n, p, q, \cdots$ pass ∥ to m, as shown.

Let the prisms $y, z, \cdots$ with lateral edges ∥ to VA have DEF, GHI, $\cdots$ for their upper bases, and let the prisms $x', y', z', \cdots$ with lateral edges ∥ to $V'A'$ have $A'B'C'$, $D'E'F'$, $G'H'I'$, $\cdots$ for their lower bases.

In the above figure there are two prisms in one case and three in the other, but the proof may be applied to any number.

Since	$DEF = D'E'F'$	§ 404
and	the altitudes are equal,	Const.
we have	$y = y'$.	§ 395
Similarly,	$z = z'$.	

Hence $x' + y' + z' - (y + z) = x'$. Ax. 2

Then by substituting P', which is less than $x' + y' + z'$, for $x' + y' + z'$, and by substituting P, which is greater than $y + z$, for $y + z$, we have

$$P' - P < x'.$$

That is, the difference between the pyramids is less than x', which is the difference between the sets of prisms.

Now by increasing indefinitely the number of parts into which h is divided, and consequently decreasing a indefinitely, x' can be made as small as we please.

Hence whatever difference we assume to exist between the pyramids, x' can be made smaller than that difference.

But this is impossible, since we have shown that x' is greater than the difference, if any exists.

Hence it leads to an impossibility to suppose that

$$P' > P, \text{ or that } P' < P.$$

$$\therefore P = P'.$$

Exercises. Review

1. The slant height of a regular pyramid is 12 in., and the base is an equilateral triangle whose altitude is $4\sqrt{3}$ in. Find the lateral area.

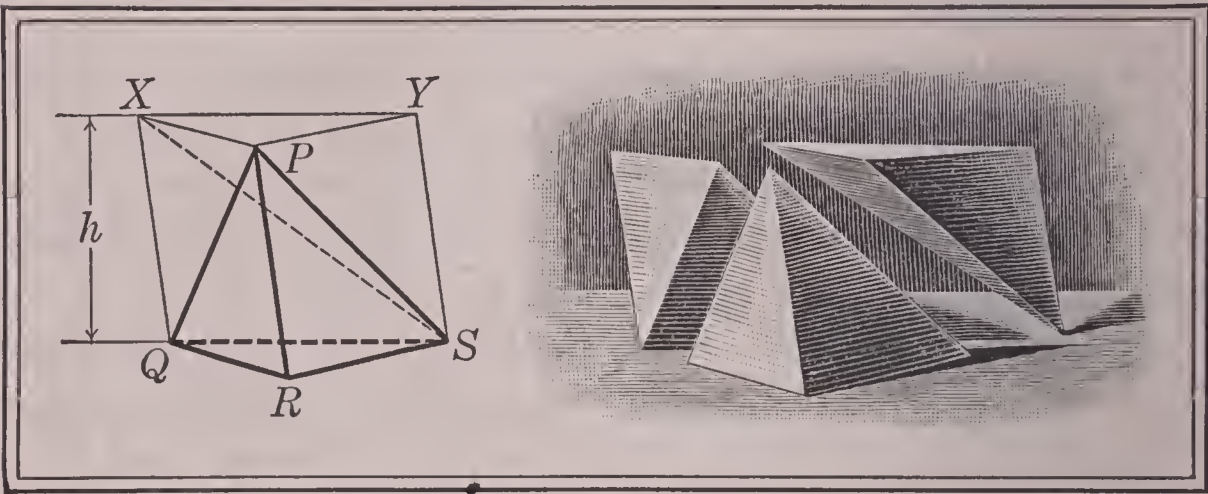
2. The slant height of a regular triangular pyramid is equal to the altitude of the base, and the area of the base is $\sqrt{5}$ sq. ft. Find the area of the total surface.

3. If one pyramid has for its base a right triangle with hypotenuse 10 and shortest side 6, and another pyramid of equal altitude has for its base an equilateral triangle which is $4\sqrt{2\sqrt{3}}$ on a side, the pyramids are equivalent.

4. The base of one of two equivalent pyramids 6 in. high is 8 sq. in. in area, and that of the other is an equilateral triangle. Find the lateral area of the second pyramid.

Proposition 13. Volume of a Triangular Pyramid

406. Theorem. *The volume of a triangular pyramid is one third the product of the base and the altitude.*



Given ***P-QRS***, a triangular pyramid; ***V***, the volume; ***B***, the area of the base; and ***h***, the altitude.

Prove that $V = \frac{1}{3} Bh$.

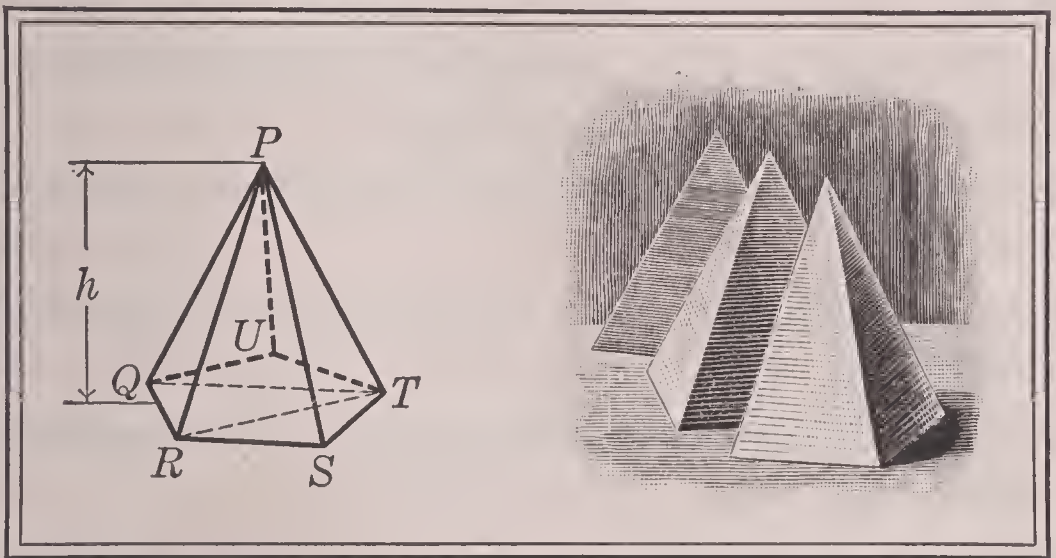
Proof. On QRS as base let there stand the prism $QRS\text{-}XPY$, and let the plane XPS pass through XP and S.

Then the prism is composed of the three triangular pyramids $P\text{-}QRS$, $P\text{-}SYX$, and $P\text{-}QSX$.

Now $P\text{-}SYX$ and $P\text{-}QSX$ have the same altitude, § 396
and base $SYX =$ base QSX. § 77
$\therefore P\text{-}SYX = P\text{-}QSX$. § 405
But $P\text{-}SYX$ is the same as $S\text{-}XPY$.
Now $S\text{-}XPY$ has the same altitude as $P\text{-}QRS$, § 341
and base $XPY =$ base QRS. § 369
$\therefore P\text{-}SYX = P\text{-}QRS$. § 405
Then $P\text{-}QRS = P\text{-}SYX = P\text{-}QSX$, Ax. 5
and hence $P\text{-}QRS = \frac{1}{3}$ of $QRS\text{-}XPY$.
$\therefore V = \frac{1}{3} Bh$. § 394

Proposition 14. Volume of any Pyramid

407. Theorem. *The volume of any pyramid is one third the product of the base and the altitude.*



Given $P\text{-}QRSTU$, a pyramid; V, the volume; B, the area of the base; and h, the altitude.

Prove that $V = \frac{1}{3} Bh.$

Proof. From any vertex of the base draw diagonals to the other vertices. In the above figure, let these diagonals be TQ and TR.

Then the planes determined by PT and the diagonals TQ, TR divide the given pyramid into three triangular pyramids, each of which has the altitude h.

The volume of each of these triangular pyramids is $\frac{1}{3} h$ times the area of the base. § 406

Hence the volume of $P\text{-}QRSTU$, the sum of the triangular pyramids, is $\frac{1}{3} h$ times the sum of the bases. Ax. 1

But the sum of the bases is B. Ax. 10

$\therefore V = \frac{1}{3} Bh.$ Ax. 5

The proof is evidently the same whatever the number of the triangular pyramids into which the given pyramid is divided.

Exercises. Properties of Pyramids

1. *The volume of a triangular pyramid is equal to one third the volume of a triangular prism of the same base and altitude.*

2. *The volumes of two pyramids are to each other as the products of their bases and altitudes.*

3. *Pyramids with equivalent bases are to each other as their altitudes.*

4. *Pyramids with equal altitudes are to each other as their bases.*

5. *Pyramids with equivalent bases and equal altitudes are equivalent.*

6. In the tetrahedron (§ 397) $V\text{-}ABC$, the midpoints of VA, VC, BA, BC are vertices of a parallelogram.

7. The lines joining the midpoints of the opposite edges of a tetrahedron meet in a point.

In the figure described in Ex. 6 the opposite edges are VA and BC; VB and AC; VC and AB.

8. The plane which passes through an edge of a tetrahedron and the midpoint of the opposite edge divides the tetrahedron into two equivalent tetrahedrons.

9. Given that a regular triangular pyramid has all its four faces congruent, and that its volume is known, show how to find the area of the total surface.

10. Show how to find the volume of any polyhedron by dividing the polyhedron into pyramids.

11. Given the edge a of the base and the area T of the total surface of a regular pyramid with a square base, find the height h in terms of a and T.

12. In Ex. 11 find the volume V in terms of a and T.

Exercises. Measuring the Pyramid

1. What is the lateral area of a regular pyramid whose slant height is 34 in., and the perimeter of the base 57 in.?

2. Find the volume of a pyramid with an altitude of 7 in. and a base 9 sq. in. in area.

3. The base of a regular pyramid is an octagon 3 m. on a side and the slant height is 5 m. Find the lateral area of the pyramid.

4. Find the volume of a pyramid with an altitude of 6.75 m. and a square base whose diagonal is $3\sqrt{2}$ m.

5. The volume of a regular pyramid with a square base is 912 cu. ft. and the altitude is 19 ft. Find the lateral area.

6. The volume of a regular pyramid with a hexagonal base is 249.4 cu. m., and the altitude is 8 m. Find the length of each side of the base.

7. The base of a pyramid is a triangle with sides which are 6 in., 8 in., and 10 in., and the volume is 240 cu. in. Find the height of the pyramid.

8. A pyramid 12 in. high has a base which is an equilateral triangle 10 in. on a side. Find the volume.

9. Find the volume of a regular pyramid with a lateral edge of 100 ft. and a square base whose side is 40 ft.

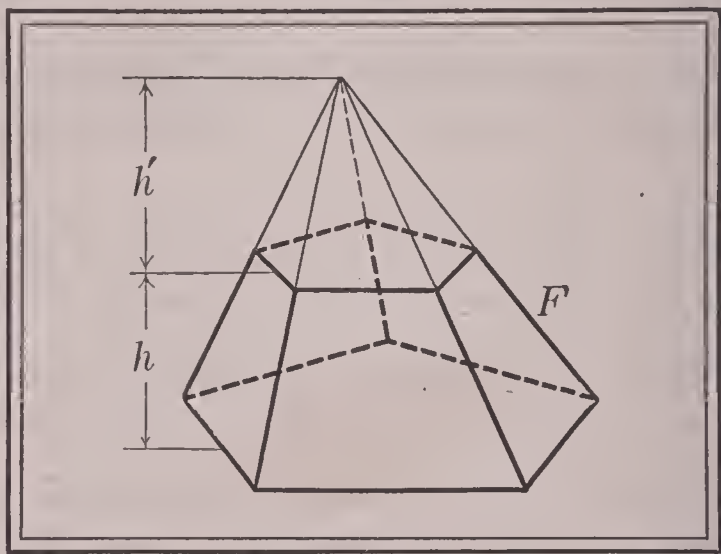
10. Find the volume of a regular pyramid whose slant height is 12 ft. and whose base is an equilateral triangle inscribed in a circle 10 ft. in radius.

11. The eight edges of a regular pyramid with a square base are equal and the area of the total surface is T. Find the edge.

12. Given the height h and the area T of the total surface, find the base edge a of a regular quadrangular pyramid.

Proposition 15. Volume of a Frustum

408. Theorem. *The volume of a frustum of a pyramid is one third the product of the altitude by the sum of the two bases and the mean proportional between them.*



Given F, a frustum; V, the volume; B, B', the areas of the bases; and h, the altitude.

Prove that $\quad V=\frac{1}{3}h(B+B'+\sqrt{BB'})$.

Proof. Let P be the volume of the pyramid from which F is cut, and let P' and h' be the volume and altitude respectively of the small pyramid remaining after F is removed.

Then $\quad V=P-P'=\frac{1}{3}B(h+h')-\frac{1}{3}B'h'.$ § 407

Now $\quad \sqrt{B}:\sqrt{B'}=h+h':h'.$ § 403, 3; Ax. 6

Solving, $\quad h'=\dfrac{h\sqrt{B'}}{\sqrt{B}-\sqrt{B'}}=\dfrac{h\sqrt{B'}(\sqrt{B}+\sqrt{B'})}{B-B'}.$

Then $\quad V=\frac{1}{3}\left[Bh+(B-B')\dfrac{h\sqrt{B'}(\sqrt{B}+\sqrt{B'})}{B-B'}\right].$ Ax. 5

Simplifying, $\quad V=\frac{1}{3}h(B+B'+\sqrt{BB'}).$

This is a subsidiary proposition in mensuration and may be omitted, together with pages 345 and 346, without disturbing the sequence.

Exercises. Frustum of a Pyramid

1. A piece of marble is in the form of a frustum of a regular pyramid with a square base. The frustum is 8 ft. high and the sides of the bases are 3 ft. and 2 ft. respectively. Taking the weight of 1 cu. ft. of marble as 165 lb., find the weight of the piece.

2. The slant height of a frustum of a regular pyramid is 10 ft. and the sides of the square bases are 3 ft. and 2 ft. respectively. Find the area of the total surface.

3. How much earth was removed in an excavation which is 6 ft. deep, 40 ft. square at the top, and 36 ft. square at the bottom?

4. A pile of broken stone is in the form of a frustum of a pyramid. The lower base is a rectangle 75 ft. long and 9 ft. wide, the upper base is 50 ft. by 6 ft., and the height of the frustum is 6 ft. If the broken stone is spread over a road 30 ft. wide to a depth of 3 in., what length of road will it cover?

5. A pyramid 4 in. high with a base whose area is 16 sq. in. is cut by a plane parallel to the base and 2 in. from the vertex. Find the volume of the frustum.

6. A pyramid 6 in. high with a base whose area is 324 sq. in. is cut by a plane parallel to the base and 2 in. from the vertex. Find the volume of the frustum.

7. The lower base of a frustum of a pyramid is a square 8 in. on a side. The side of the upper base is half that of the lower base, and the altitude of the frustum is the same as the side of the upper base. Find the volume of the frustum.

8. Consider the formula $V=\frac{1}{3}h(B+B'+\sqrt{BB'})$ of § 408 when $B'=0$. Discuss the meaning of the result. Also discuss the case in which $B=B'$.

Exercises. Review

1. The lower base of a frustum of a pyramid is a square 6 in. on a side. The area of the upper base is half that of the lower base, and the altitude of the frustum is 4 in. Find to the nearest 0.01 cu. in. the volume of the frustum.

2. A pyramid has six edges, each 2 in. long. Find to the nearest 0.01 cu. in. the volume of the pyramid.

3. A regular pyramid 8 in. high has a triangular base, and the volume of the pyramid is $16\sqrt{2}$ cu. in. Find to the nearest 0.01 in. the length of a side of the base.

4. In Ex. 3 find the area of the total surface.

5. The base of a regular pyramid is a square l feet on a side, and the slant height is s feet. Find the area of the total surface.

6. The lower base of a frustum of a pyramid is a quadrilateral whose sides are 5 in., 6 in., 7 in., and 9 in., respectively, and the corresponding sides of the upper base are 3 in., x inches, y inches, and z inches. Find x, y, and z.

7. A schoolroom 30 ft. long, 24 ft. wide, and 14 ft. high is ventilated by an electric fan which discharges every 20 min. a volume of air equal to the volume of the room. Find the amount of air discharged per minute.

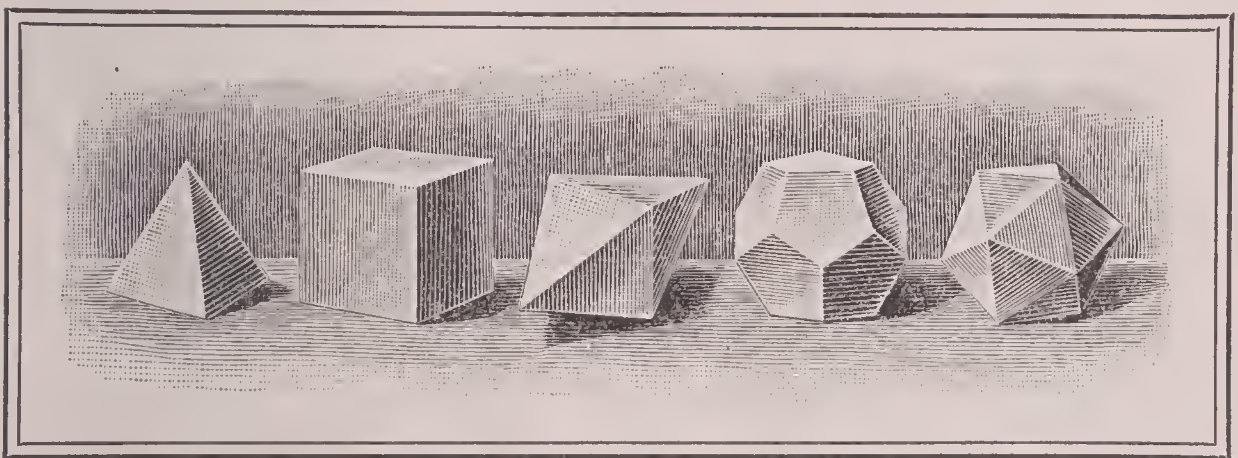
8. An oblique pyramid with a square base 10 in. on a side is cut by a plane parallel to the base so that the altitude, measured from the vertex, is divided in the ratio 3 : 2. Find the area of the section.

9. A frustum of a square pyramid is 18 ft. high and the sides of the bases are 1 ft. and 3 ft. respectively. If the frustum is divided into three parts by planes passed parallel to the bases and dividing the altitude equally, what is the volume of each part?

III. General Polyhedrons

409. Polyhedrons classified as to Faces. A polyhedron of four faces is called a *tetrahedron*; one of six faces, a *hexahedron*; one of eight faces, an *octahedron*; one of twelve faces, a *dodecahedron*; one of twenty faces, an *icosahedron.*

410. Regular Polyhedron. A polyhedron whose faces are congruent regular polygons, and whose polyhedral angles are equal, is called a *regular polyhedron.*



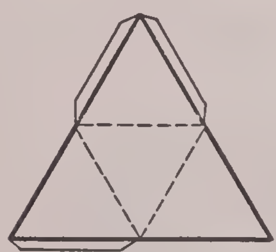
It is proved in § 413 that there cannot be more than five regular polyhedrons, and as a matter of fact there are just five. The five regular polyhedrons are shown in the above illustration in the order in which the polyhedrons are mentioned in § 409.

These regular solids occupied the attention of Pythagoras and his followers (about 550 B.C.). They were also studied so extensively in the school of Plato (about 375 B.C.) that they are often known as the Platonic Bodies. The early Greek writers connected them in a fanciful way with various phenomena of nature. For example, they assigned the tetrahedron to fire, the hexahedron to earth, the octahedron to air, the icosahedron to water, and the dodecahedron, apparently the last one discovered, to the universe.

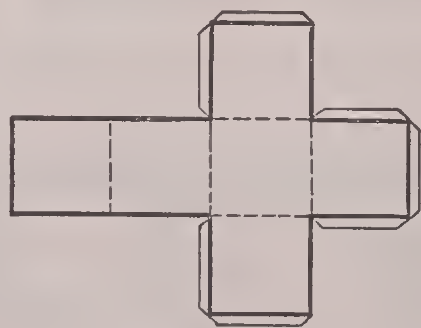
Some of the regular polyhedrons are met in the study of crystals. Thus, the cube is found in salt crystals and the regular octahedron in certain compounds of copper.

Pages 347–350, while important in the study of crystals, may be omitted without affecting the sequence of propositions.

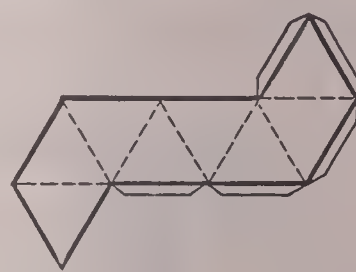
411. Models of Regular Polyhedrons. In a work printed in 1525 a great artist, Albrecht Dürer, showed how to make paper models of the five regular polyhedrons. His description suggested drawing on stiff paper the diagrams shown below, and then cutting along the full lines and pasting strips of thin paper on the edges as indicated. By folding on the dotted lines and keeping the edges together by the pasted strips of paper, the models can be easily made.



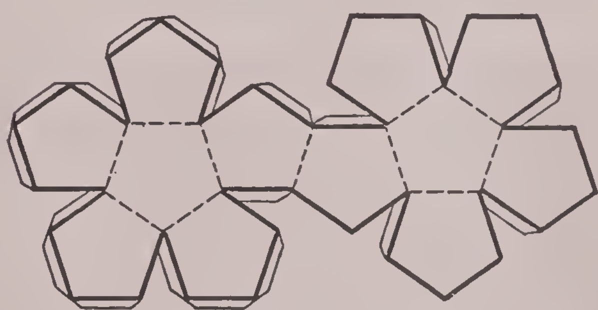
Tetrahedron



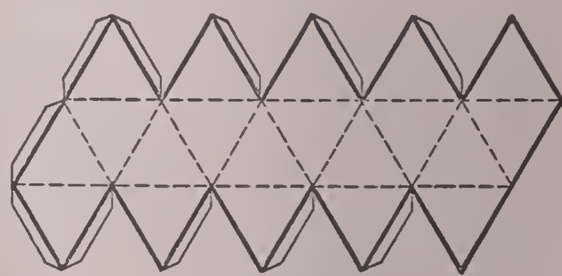
Hexahedron



Octahedron



Dodecahedron



Icosahedron

412. Relation of Parts of a Polyhedron. If the number of edges of a polyhedron is represented by e, the number of vertices by v, and the number of faces by f, then $e + 2 = v + f$. This remarkable relation was possibly known to the Greek mathematician Archimedes (about 250 B.C.), but was first clearly stated by the French writer Descartes (about 1635). It was also discovered independently by Euler (1752), and is often known as Euler's Theorem.

The proof of this law is too difficult to be given at this time, but the student will be asked to verify it for certain special cases on page 350.

Proposition 16. Regular Polyhedrons

413. Theorem. *There cannot be more than five regular polyhedrons.*

Proof. A polyhedral ∠ has at least three faces. § 361

Also, the sum of its face ∠s is less than 360°, § 13

because the polyhedral ∠ would flatten out into a plane at 360°.

Since each ∠ of an equilateral △ is 60°, § 65

polyhedral ∠s may be formed with three, four, or five equilateral △s as faces.

Now the sum of six face ∠s of 60° is 360°. Ax. 1

Hence not more than three regular polyhedrons with equilateral △s as faces are possible.

Since each ∠ of a square is 90°, § 15

a polyhedral ∠ may be formed with three squares as faces.

Now the sum of four face ∠s of 90° is 360°. Ax. 1

Hence not more than one regular polyhedron with squares as faces is possible.

Since each ∠ of a regular pentagon is 108°, § 96

a polyhedral ∠ may be formed with three regular pentagons as faces.

Now the sum of four face ∠s of 108° is 432°. Ax. 1

Hence not more than one regular polyhedron with regular pentagons as faces is possible.

Now the sum of three ∠s of a regular hexagon is 360°; of a regular heptagon, more than 360°; and so on.

Hence there cannot be more than five regular polyhedrons.

It is not of enough importance to prove that there are actually five regular polyhedrons as stated in § 410. In elementary work the fact may be safely assumed.

Exercises. Polyhedrons

1. Count the number of edges, vertices, and faces on each of the five regular polyhedrons and then fill in the blank spaces in the following table:

NAME	EDGES	VERTICES	FACES
Tetrahedron . . .	6	4	4
Hexahedron (cube) .			
Octahedron			
Dodecahedron . . .			
Icosahedron			

2. From the above table show that in each case the law $e+2=v+f$ holds true.

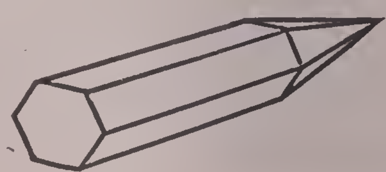
3. Assuming that the law in Ex. 2 is true for all polyhedrons, prove that a seven-edged polyhedron is impossible.

4. If the centers of the six faces of a cube are joined, what kind of polyhedron is constructed? Draw the figure and prove that any two edges are equal.

5. If the centers of the four faces of a regular tetrahedron are joined, what kind of polyhedron is constructed? Draw the figure and prove any two edges equal.

6. As in Ex. 4, consider the figure which results from joining the centers of the faces of a regular octahedron.

7. A quartz crystal is in the form of a hexagonal prism with a pyramid on one of its bases, as here shown. Show that the relation stated in Ex. 2 holds true.



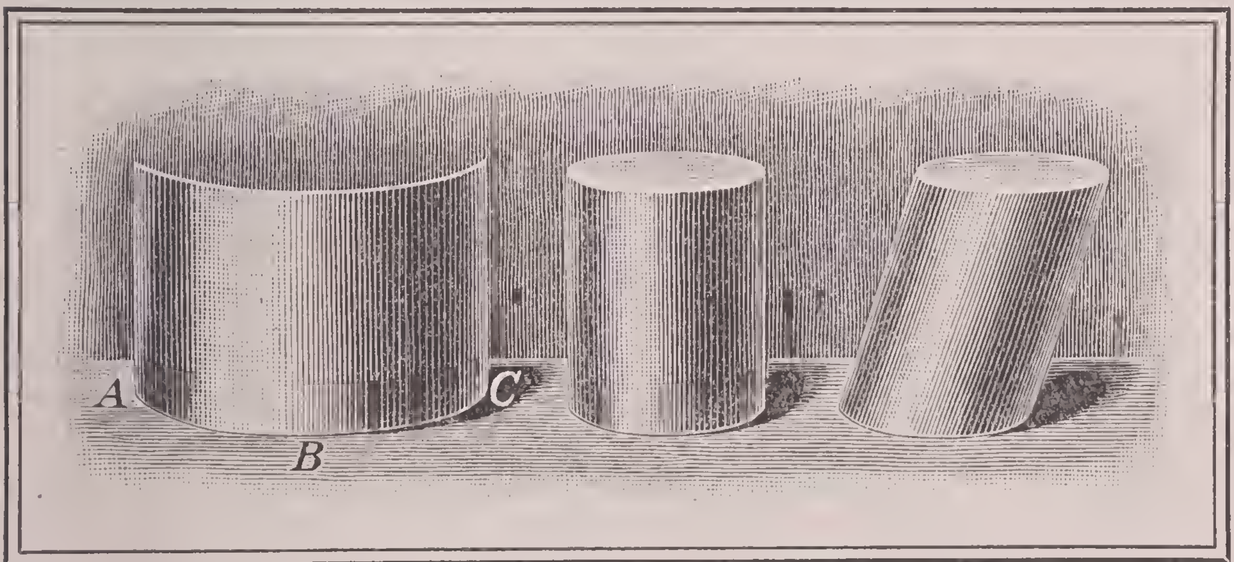
8. A given polyhedron has six vertices and five faces. How many edges are there?

IV. Cylinders

414. Cylindric Surface. A surface generated by a moving straight line which is always parallel to a fixed straight line, and touches a fixed curve not in the plane of the fixed line, is called a *cylindric surface*, or a cylindrical surface.

The moving line is called the *generatrix* and the fixed curve is called the *directrix*. In the first figure below, ABC is the directrix of the cylindric surface shown. The generatrix in any position is called an *element*.

415. Cylinder. A solid bounded by a cylindric surface and two parallel plane surfaces cutting all the elements is called a *cylinder*.



It is evident that *all elements of a cylinder are equal.* The terms *bases, lateral area,* and *altitude* are used as with prisms.

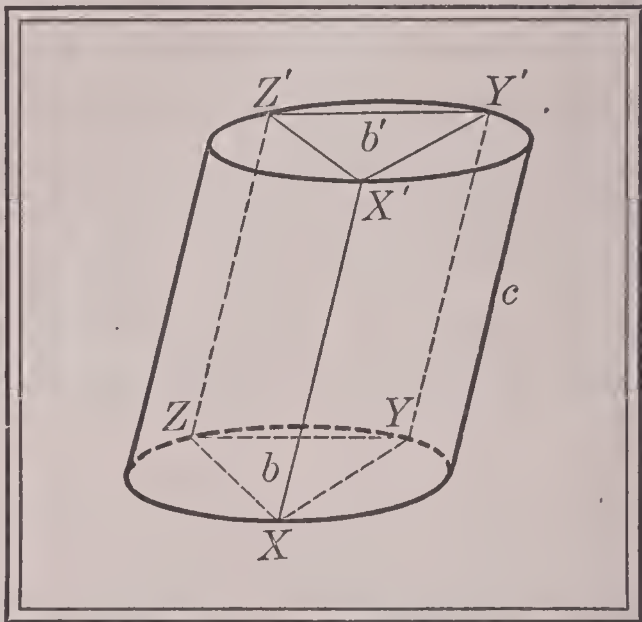
416. Cylinders classified. If the elements of a cylinder are perpendicular to the bases, the cylinder is called a *right cylinder*; if oblique, it is called an *oblique cylinder.* A cylinder whose bases are circles is called a *circular cylinder.*

The second figure above shows a right circular cylinder, and the third an oblique cylinder. The straight line through the centers of the bases of a circular cylinder is called the *axis* of the cylinder.

Since a right circular cylinder can be generated by revolving a rectangle about one side as an axis, it is also called a *cylinder of revolution.*

Proposition 17. Bases of a Cylinder

417. Theorem. *The bases of a cylinder are congruent.*



Given the cylinder c with bases b and b'.

Prove that b *is congruent to* b'.

Proof. Let X, Y, Z be any three points on the perimeter of b, and let XX', YY', ZZ' be elements.

Draw $XY, YZ, ZX, X'Y', Y'Z', Z'X'$. Post. 1

Then XY is $\parallel$ to $X'Y'$, and XX' is $\parallel$ to YY'. §§ 338, 414

Hence $XYY'X'$ is a $\square$, § 72

and $XY = X'Y'$. § 76

Similarly, $YZ = Y'Z'$ and $ZX = Z'X'$,

and hence $\triangle XYZ$ is congruent to $\triangle X'Y'Z'$. § 47

Place b on b' so that X lies on X', and Y on Y'; then Z, which is any third point on the perimeter of b, has one and only one corresponding point on the perimeter of b'.

Hence the same is true of every point on the perimeter of b; that is, b can be made to coincide with b'.

$\therefore$ b is congruent to b'. § 37

Exercises. Cylinders

1. *Every section of a cylinder made by a plane passing through two elements is a parallelogram.*

2. *Every section of a right cylinder made by a plane passing through two elements is a rectangle.*

3. *Any two sections of a cylinder made by parallel planes which cut all the elements are congruent.*

4. *Any section of a cylinder parallel to the base is congruent to the base.*

5. *The straight line joining the centers of the bases of a circular cylinder passes through the centers of all sections of the cylinder parallel to the bases.*

6. If a rectangle revolves about one of its sides, it forms a right circular cylinder.

7. If two right circular cylinders have equal bases and equal altitudes, they are congruent.

8. The locus of points equidistant from a given line is a cylindric surface.

The given line is called the *axis* of the surface.

9. The center of any section of a circular cylinder parallel to the base is on the axis.

10. If parallel planes cut all the elements of a cylindric surface, the sections thus formed are congruent.

11. If a section of an oblique cylinder is made by a plane parallel to an element, is the resulting figure a parallelogram? Can it be a rectangle? Give the proofs.

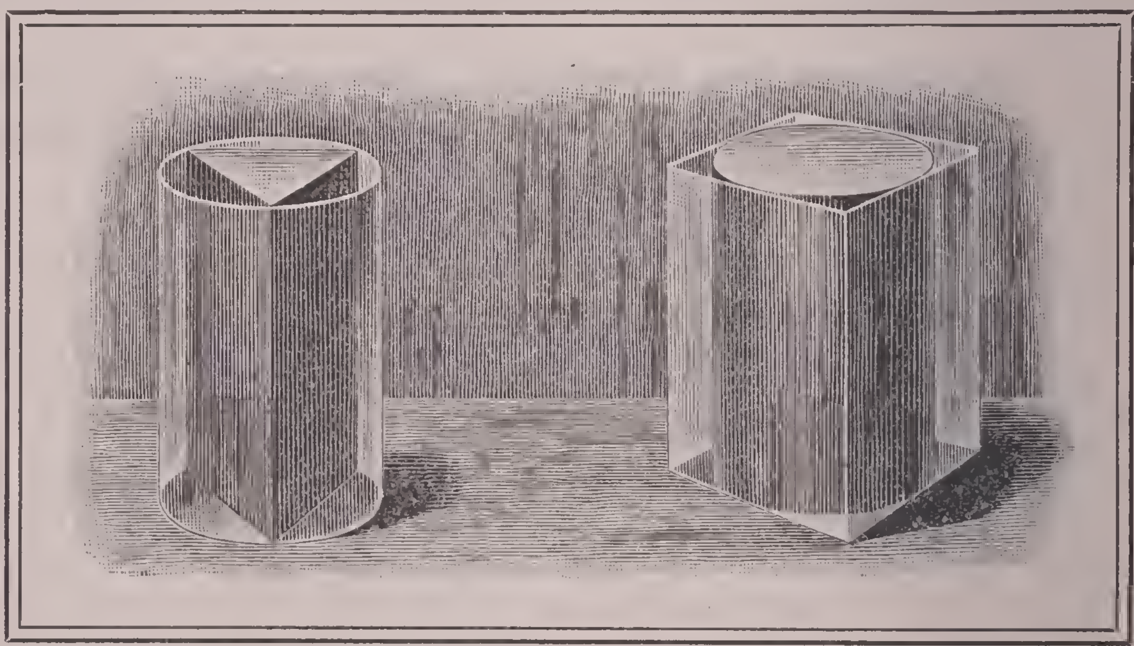
12. From the center of the upper base of a right circular cylinder 4 in. high lines are drawn to the perimeter of the lower base. If the diameters of the bases are 6 in., what is the length of each line?

418. Tangent Plane. A plane which contains an element of a cylinder, but does not cut the surface, is called a *tangent plane.*

From this definition it is evident that

A plane passing through a tangent to the base of a circular cylinder and the element through the point of contact is tangent to the cylinder.

If a plane is tangent to a circular cylinder, its intersection with the plane of the base is tangent to the base.



419. Inscribed Prism. A prism whose lateral edges are elements of a cylinder and whose bases are inscribed in the bases of the cylinder is called an *inscribed prism.*

The first figure above shows an inscribed prism. The cylinder is said to be *circumscribed about the prism.*

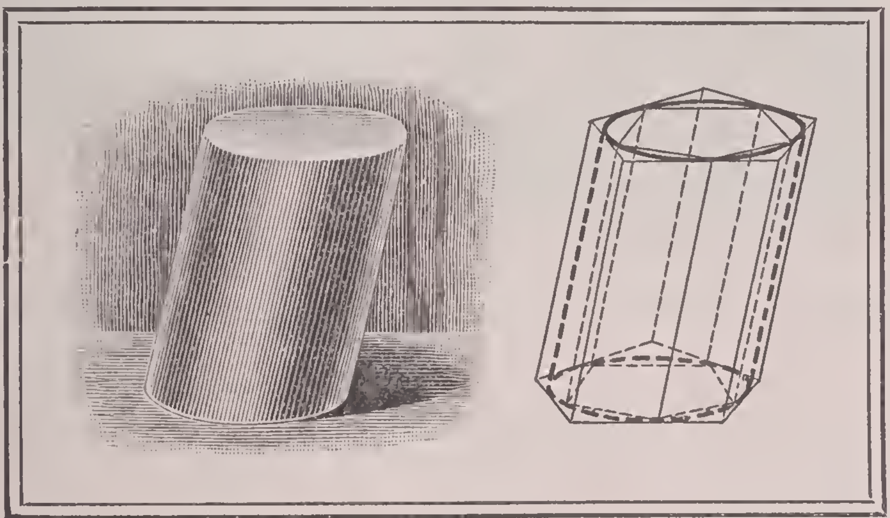
420. Circumscribed Prism. A prism whose lateral faces are tangent to the lateral surface of a cylinder and whose bases are circumscribed about the bases of the cylinder is called a *circumscribed prism.*

The second figure above shows a circumscribed prism. The cylinder is said to be *inscribed in the prism.*

421. Transverse Section. A section of a cylinder made by a plane that cuts all the elements is called a *transverse section* of the cylinder.

If the plane is perpendicular to the elements, the section is called a *right section.*

422. Cylinder as a Limit. From the principles of limits studied in plane geometry, and from the nature of inscribed and circumscribed prisms, the properties of the cylinder stated below may be assumed without proof.



If a prism whose base is a regular polygon is inscribed in or circumscribed about a circular cylinder, and if the number of sides of the prism is indefinitely increased,

1. *The volume of the cylinder is the limit of the volume of the prism.*

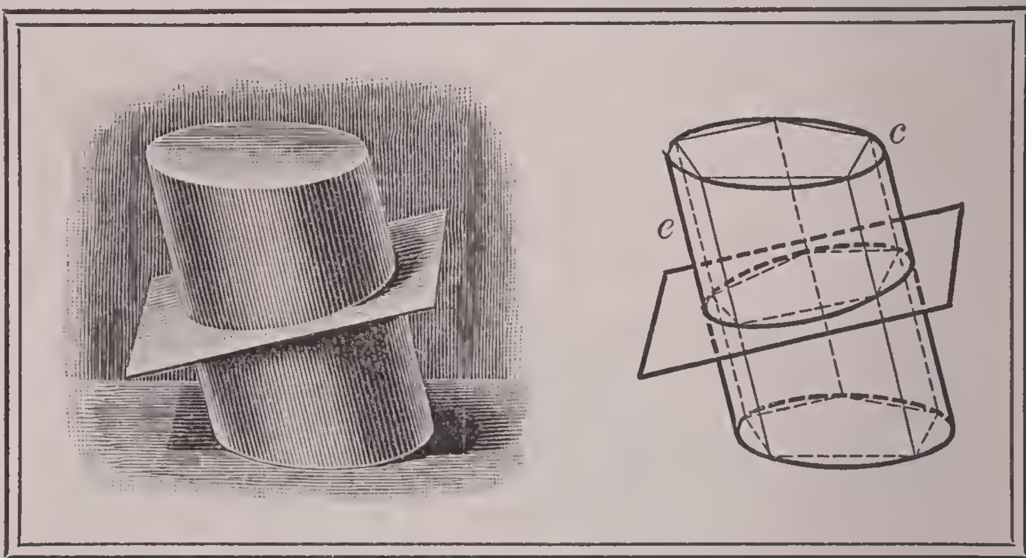
2. *The lateral area of the cylinder is the limit of the lateral area of the prism.*

3. *The perimeter of any transverse section of the cylinder is the limit of the perimeter of the corresponding section of the prism.*

As we increase the number of sides of the inscribed or circumscribed prism whose base is a regular polygon, the perimeter of the base approaches the circle as its limit (§ 303, 1). This brings the lateral surface of each prism nearer and nearer the cylindric surface.

Proposition 18. Lateral Area of a Cylinder

423. Theorem. *The lateral area of a circular cylinder is the product of an element and the perimeter of a right section.*



Given *c*, a circular cylinder; *S*, the lateral area; *e*, an element; and *p*, the perimeter of a rt. section.

Prove that $S = ep.$

Proof. Let a prism whose base is a regular polygon be inscribed in c, and let L be the lateral area and p' the perimeter of a rt. section.

Then $L = ep'.$ § 376

If the number of lateral faces is indefinitely increased,

$L \rightarrow S,$ § 422, 2

$p' \rightarrow p,$ § 422, 3

and $ep' \rightarrow ep.$ § 301, 1

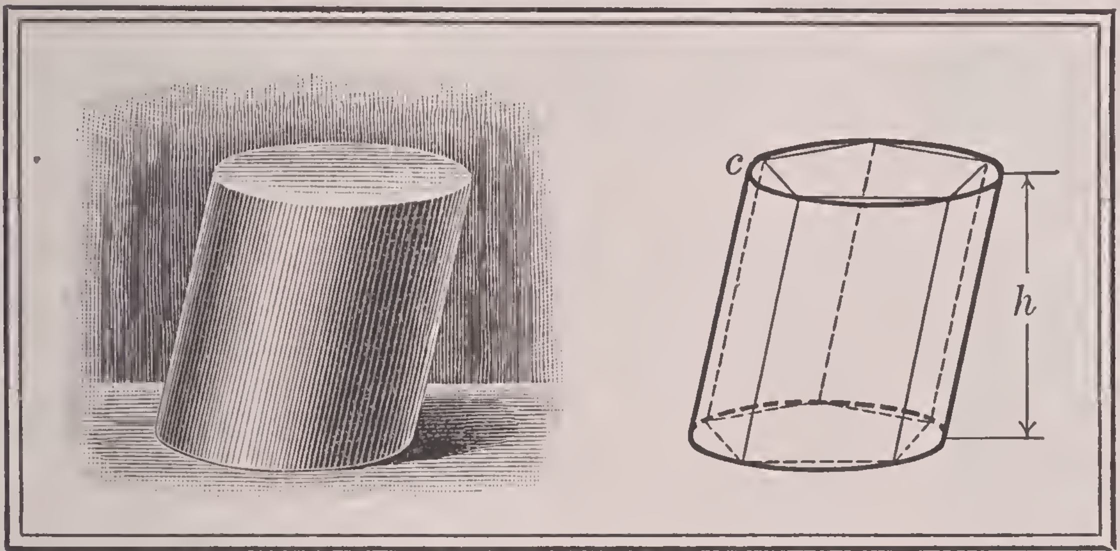
$\therefore S = ep.$ § 301, 2

424. Corollary. *In a cylinder of revolution of lateral area S, total area T, altitude h, and radius r,*

$$S = 2\pi rh, \text{ and } T = 2\pi r(h + r).$$

Proposition 19. Volume of a Cylinder

425. Theorem. *The volume of a circular cylinder is the product of the base and altitude.*



Given c, a circular cylinder; V, the volume; B, the area of the base; and h, the altitude.

Prove that $V = Bh.$

Proof. Let a prism whose base is a regular polygon be inscribed in c, and let V' be the volume and B' the area of the base.

Then $V' = B'h.$ § 395

If the number of lateral faces is indefinitely increased,

$V' \rightarrow V,$ § 422, 1

$B' \rightarrow B,$ § 303, 2

and $B'h \rightarrow Bh.$ § 301, 1

$\therefore V = Bh.$ § 301, 2

426. Corollary. *In a cylinder of revolution of volume V, radius r, and altitude h,* $V = \pi r^2 h.$

Since $B = \pi r^2$ (§ 309), then $V = \pi r^2 h$. This formula and the first one of § 424 are those most used in practical work.

Exercises. Cylinders

1. The diameter of a well is 5 ft. and the water is 8 ft. deep. Reckoning $7\frac{1}{2}$ gal. to the cubic foot, how many gallons of water are there in the well?

2. When an irregular solid body is placed under water in a right circular cylinder 50 cm. in diameter, the level of the water rises 30 cm. Find the volume of the body.

3. How many cubic yards of earth were removed in constructing a tunnel which is 140 yd. long and whose cross section is a semicircle 18 ft. in radius?

4. Find to the nearest 0.01 in. the radius of a hollow cylinder 16 in. high and containing 3 cu. ft.

5. Given that the height of a cylindric container which holds 20 qt. is equal to the diameter, find the altitude and the diameter.

6. Given that the area of the lateral surface of a right circular cylinder is S, the volume is V, and the altitude is h, find two formulas for the radius r.

7. Given that the circumference of the base of a right circular cylinder is C and the altitude is h, find the volume V.

8. From the formula $T = 2\pi r(h + r)$ in § 424 find the value of r.

Ex. 8 should be omitted unless quadratics have been studied.

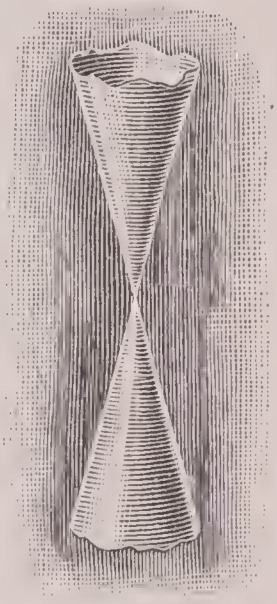
9. Defining *similar cylinders of revolution* as cylinders formed by the revolution of similar rectangles about corresponding sides, prove that the lateral areas of two such cylinders are to each other as the squares of the altitudes or as the squares of the radii.

10. Is Ex. 9 true for total areas? Prove your answer.

11. Consider Ex. 9 for volumes instead of lateral areas, changing the statement as may be necessary.

V. Cones

427. Conic Surface. A surface generated by a moving straight line which always touches a fixed plane curve and passes through a fixed point not in the plane of the curve is called a *conic surface*.



If a pencil is held by the point and the other end is allowed to swing round a circle, the pencil will generate a conic surface.

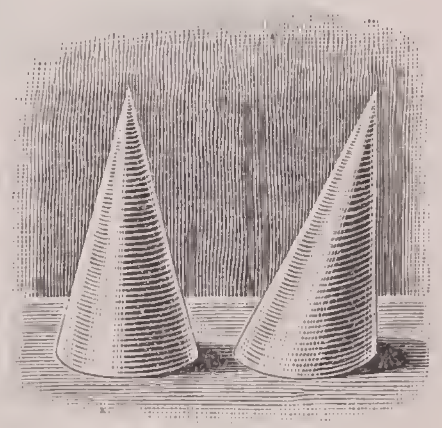
We may also swing a blackboard pointer about any point near the middle in such a way that either end shall touch a fixed plane curve, and thus generate a conic surface. Such a surface is represented in the figure here shown.

The moving line is called the *generatrix*, the fixed curve the *directrix*, and the fixed point the *vertex*.

The generatrix in any position is called an *element* of the conic surface.

Since the generatrix is of indefinite length, the conic surface consists of two portions,— one above and the other below the vertex, as shown in this figure. These portions are called the *upper nappe* and *lower nappe* respectively.

428. Cone. A solid bounded by a conic surface and a plane cutting all the elements is called a *cone*.



Since the terms *base*, *lateral area*, *vertex*, and *altitude* are used as with pyramids, their further definition is unnecessary.

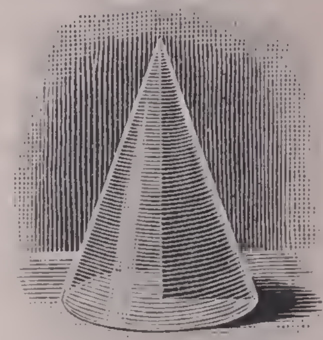
429. Circular Cones. A cone whose base is a circle is called a *circular cone*.

The straight line joining the vertex of a circular cone and the center of the base is called the *axis* of the cone.

Each of these figures shows a circular cone. In the one at the left, the axis is perpendicular to the base; in the other it is oblique.

Many machine parts, such as roller bearings, bevel gears, taper spindles, and the like, are made in the form of circular cones or parts of such cones.

430. Further Classification of Cones. If the axis of a circular cone is perpendicular to the base, the cone is called a *right circular cone*; if oblique, it is called an *oblique circular cone.*



Since a right circular cone may be generated by the revolution of a right triangle about one of the sides of the right angle, as shown in this figure, it is also called a *cone of revolution.*

The hypotenuse of the triangle corresponds to an element of the surface and is called the *slant height* of the cone.

We seldom have occasion to measure any except right circular cones.

431. Conic Section. A section formed by the intersection of a plane and the conic surface of a cone of revolution, as in the figures below, is called a *conic section.*

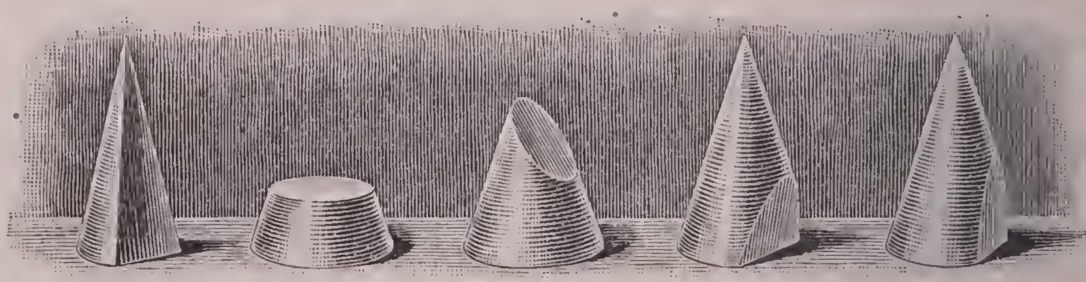


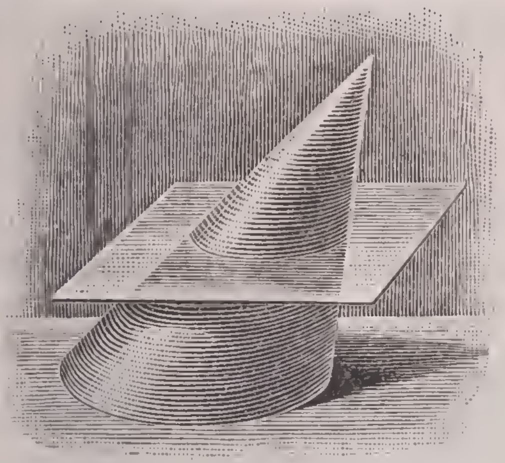
Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5

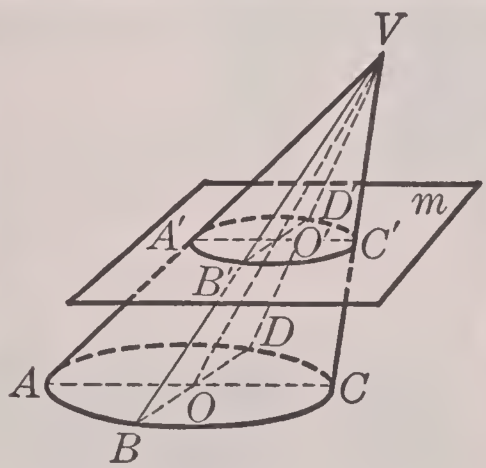
In Fig. 1 the conic section is two intersecting straight lines. In Fig. 2 the conic section is a circle, which is discussed in § 432. In Fig. 3 the conic section is an *ellipse*, the form a circle seems to take when looked at obliquely. The orbit of a planet is an ellipse. In Fig. 4 the cutting plane is parallel to an element, and the conic section is a *parabola*, the path of a projectile in a vacuum. In Fig. 5 the cutting plane is parallel to the axis, and the conic section is a *hyperbola.*

The general study of conic sections is not a part of elementary geometry, but the names of the sections are so frequently met in general reading that they should be known.

Proposition 20. Section Parallel to Base

432. Theorem. *The section of a circular cone made by a plane parallel to the base is a circle.*





Given a circular cone with the section $A'B'C'D'$ made by the plane $m \parallel$ to the base.

Prove that $A'B'C'D'$ *is a* $\odot$.

Proof. Let O be the center of the base, and let O' be the point in which the axis VO pierces the plane m.

Let the planes of VO and any elements VA, VB cut the base in the radii OA, OB and the plane m in $O'A'$, $O'B'$.

Then $O'A'$ is $\parallel$ to OA, and $O'B'$ is $\parallel$ to OB. § 338

Since $O'A'$, $O'B'$ divide VA, VO, VB proportionally, § 201

then $\triangle AOV$ is similar to $\triangle A'O'V$,

and $\triangle BOV$ is similar to $\triangle B'O'V$. § 213

Hence $$\frac{OA}{O'A'} = \frac{VO}{VO'} = \frac{OB}{O'B'}.$$ § 205

Since $OA = OB$ (§ 134, 1), then $O'A' = O'B'$. § 198, 2

Since A and B are any points on $ABCD$, then A' and B' are any points on the intersection of m and the cone.

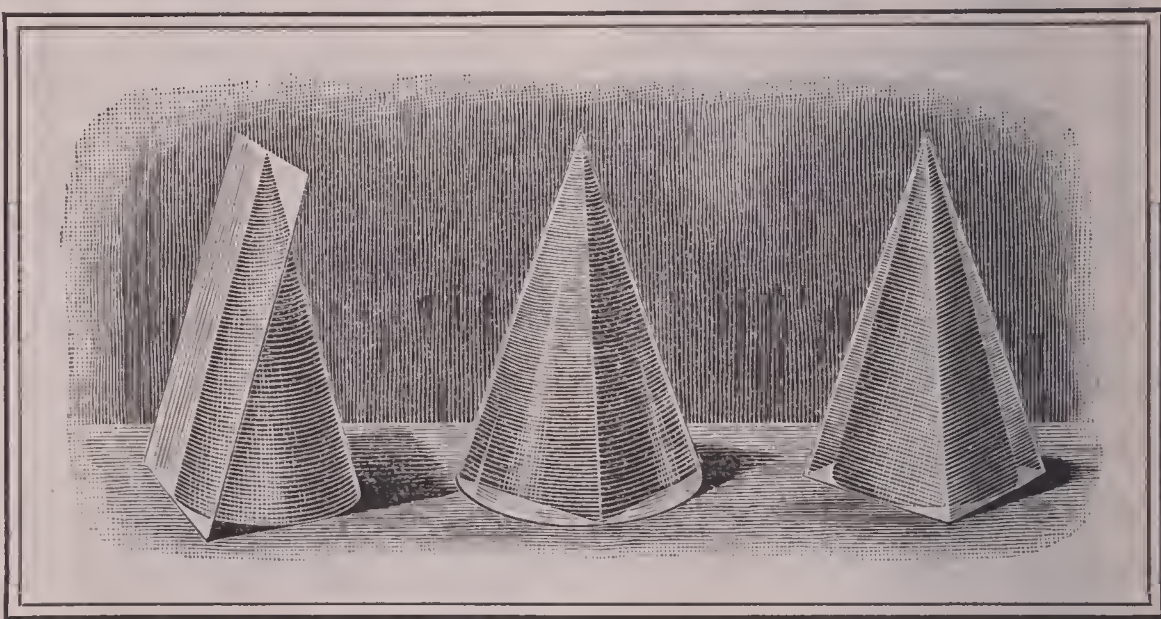
$\therefore A'B'C'D'$ is a $\odot$. § 134, 6

433. Tangent Plane. A plane which contains an element of a cone, but does not cut the surface (as in the first figure below), is called a *tangent plane.*

From this definition, it is evident that

A plane passing through a tangent to the base of a circular cone and the element through the point of contact is tangent to the cone.

If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.



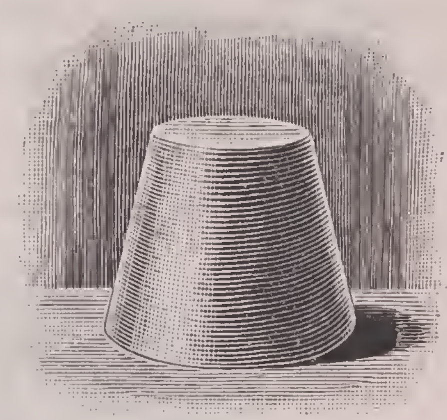
434. Inscribed Pyramid. A pyramid whose lateral edges are elements of a cone and whose base is inscribed in the base of the cone is called an *inscribed pyramid.*

The second figure above shows an inscribed pyramid. The cone is said to be *circumscribed about the pyramid.*

435. Circumscribed Pyramid. A pyramid whose lateral faces are tangent to the lateral surface of a cone and whose base is circumscribed about the base of the cone is called a *circumscribed pyramid.*

The third figure above shows a circumscribed pyramid. The cone is said to be *inscribed in the pyramid.*

436. Frustum of a Cone. The portion of a cone included between the base and a section parallel to the base is called a *frustum of a cone.*



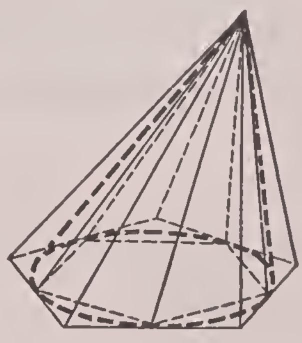
This figure shows a frustum of a cone of revolution.

The base of the cone and the parallel section are together called the *bases* of the frustum.

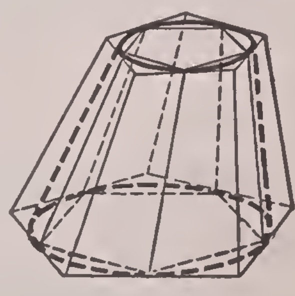
The terms *altitude* and *lateral area* as applied to a frustum of a cone, and *slant height* as applied to a frustum of a right circular cone, have the same meaning as they do when applied to a frustum of a pyramid. As with frustums of regular pyramids, only frustums of cones of revolution have a slant height.

437. Cones and Frustums as Limits. The following properties of cones and frustums, similar to those given in § 422, may be assumed without proof from a study of the figures accompanying the statements:

1. *If a pyramid whose base is a regular polygon is inscribed in or circumscribed about a circular cone, and if the number of lateral faces of the pyramid is indefinitely increased, the volume of the cone is the limit of the volume of the pyramid, and the lateral area of the cone is the limit of the lateral area of the pyramid.*

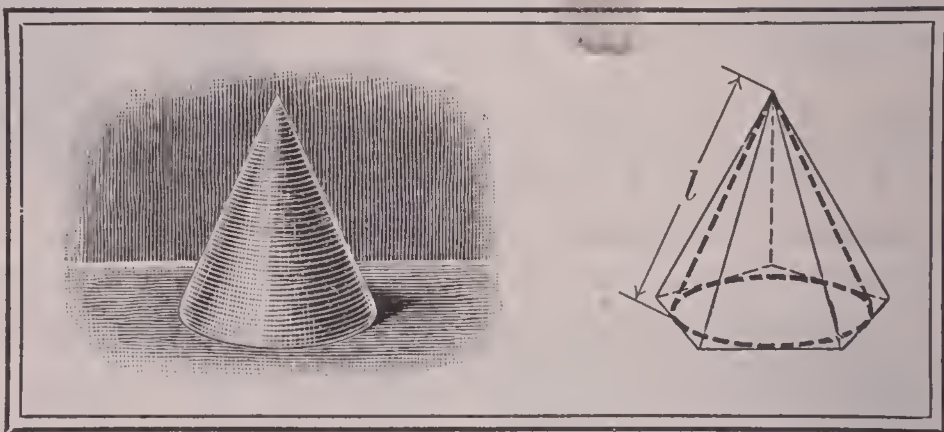


2. *The volume of a frustum of a cone is the limit of the volumes of the corresponding frustums of the inscribed and circumscribed pyramids whose bases are regular polygons as the number of lateral faces is indefinitely increased, and the lateral area of the frustum of a cone is the limit of the lateral areas of the corresponding frustums of the inscribed and circumscribed pyramids.*



Proposition 21. Lateral Area of a Cone

438. Theorem. *The lateral area of a cone of revolution is half the product of the slant height and the circumference of the base.*



Given a cone of revolution; S, the lateral area; C, the circumference of the base; and l, the slant height.

Prove that $S = \frac{1}{2} lC$.

Proof. In a regular circumscribed pyramid whose lateral area is L and whose perimeter of base is p, $L = \frac{1}{2} lp$. § 401

If the number of lateral faces is indefinitely increased,

$L \rightarrow S$, § 437, 1

and $p \rightarrow C$. § 303, 1

Then $\frac{1}{2} lp \rightarrow \frac{1}{2} lC$. § 301, 1

$\therefore S = \frac{1}{2} lC$. § 301, 2

439. Corollary. *In a cone of revolution of lateral area S, total area T, slant height l, and radius r,*

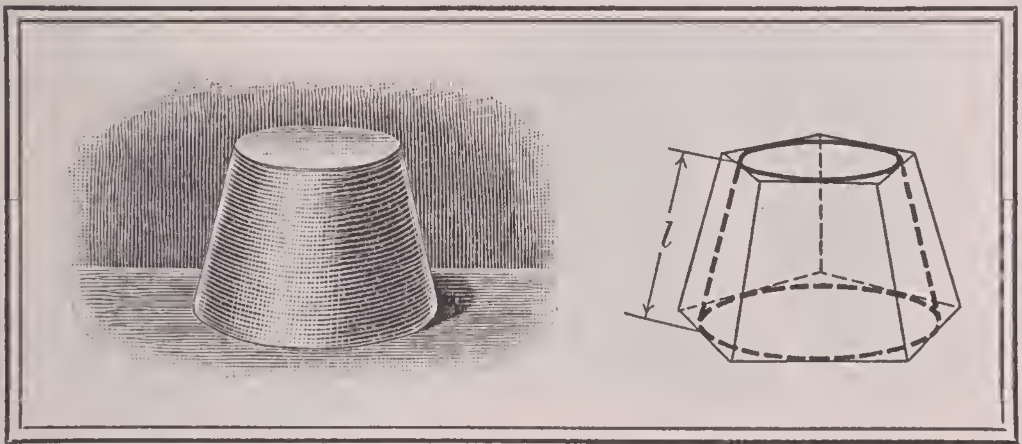
$$S = \pi rl, \text{ and } T = \pi r(l + r).$$

440. Corollary. *The theorem of § 403 applies to a circular cone.*

The necessary changes in wording are obvious.

Proposition 22. Lateral Area of a Frustum

441. Theorem. *The lateral area of a frustum of a cone of revolution is half the product of the slant height and the sum of the circumferences of its bases.*



Given a frustum of a cone of revolution; S, the lateral area; C and C', the circumferences of the bases; and l, the slant height.

Prove that $S = \frac{1}{2} l(C + C')$.

Proof. Let L be the lateral area of the corresponding frustum of a regular circumscribed pyramid, and let p, p' be the perimeters of the bases corresponding to C, C' respectively.

Then $L = \frac{1}{2} l(p + p')$. § 402

If the number of lateral faces is indefinitely increased,

$L \rightarrow S$. § 437, 2

From § 303, 1, we may assume that

$$p + p' \rightarrow C + C'.$$

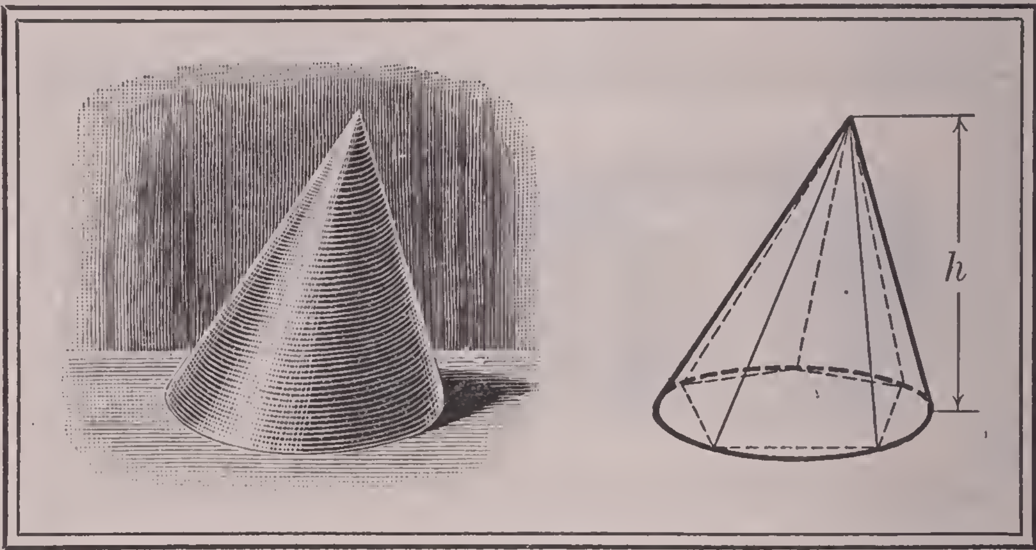
Then $\frac{1}{2} l(p + p') \rightarrow \frac{1}{2} l(C + C')$. § 301, 1

$\therefore S = \frac{1}{2} l(C + C')$. § 301, 2

442. Corollary. *The lateral area of a frustum of a cone of revolution is the product of the slant height and the circumference of a section equidistant from the bases.*

Proposition 23. Volume of a Cone

443. **Theorem.** *The volume of a circular cone is one third the product of the base and the altitude.*



Given a circular cone; V, the volume; B, the area of the base; and h, the altitude.

Prove that $V = \frac{1}{3}Bh.$

Proof. Let a pyramid whose base is a regular polygon be inscribed in the cone.

Let V' be the volume and B' the area of the base of the inscribed pyramid.

Then $V' = \frac{1}{3}B'h.$ § 407

If the number of lateral faces is indefinitely increased,

$V' \rightarrow V,$ § 437, 1

and $B' \rightarrow B.$ § 303, 2

Then $\frac{1}{3}B'h \rightarrow \frac{1}{3}Bh.$ § 301, 1

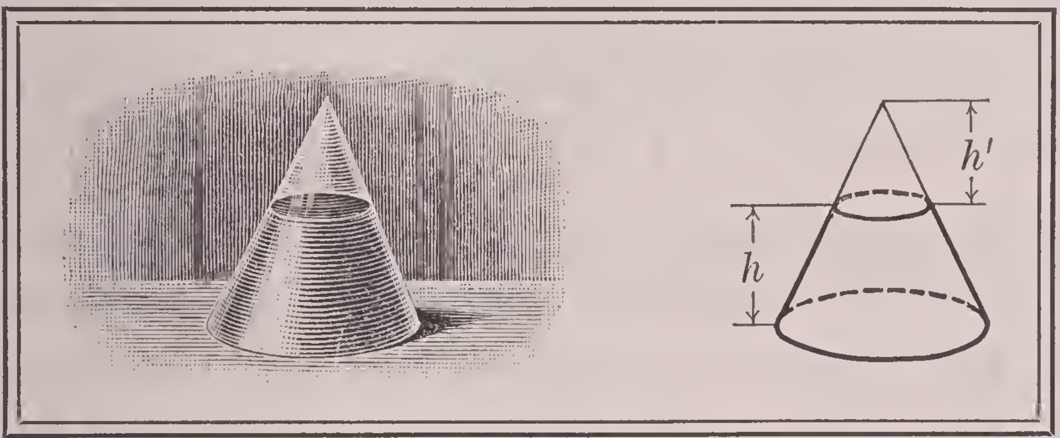
$\therefore V = \frac{1}{3}Bh.$ § 301, 2

444. **Corollary.** *In a circular cone of volume V, radius r, and altitude h,*

$$V = \frac{1}{3}\pi r^2 h.$$

Proposition 24. Volume of a Frustum

445. Theorem. *The volume of a frustum of a circular cone is one third the product of the altitude of the frustum by the sum of the areas of the bases and the mean proportional between them.*



Given a frustum of a circular cone; V, the volume; B, B', the areas of the bases; and h, the altitude.

Prove that $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.

Proof. Let C be the volume of the cone from which the frustum is cut; and let C' and h' be the volume and altitude respectively of the small cone remaining after the frustum is removed.

Then $V = C - C' = \frac{1}{3}B(h + h') - \frac{1}{3}B'h'$. § 443

Now $\sqrt{B} : \sqrt{B'} = h + h' : h$. § 440, Ax. 6

Solving, $h' = \dfrac{h\sqrt{B'}}{\sqrt{B} - \sqrt{B'}} = \dfrac{h\sqrt{B'}(\sqrt{B} + \sqrt{B'})}{B - B'}$.

Then $V = \frac{1}{3}\left[Bh + (B - B')\dfrac{h\sqrt{B'}(\sqrt{B} + \sqrt{B'})}{B - B'}\right]$. Ax. 5

Simplifying, $V = \frac{1}{3}h(B + B' + \sqrt{BB'})$.

Since $B = \pi r^2$ and $B' = \pi r'^2$, the above formula may be written

$$V = \tfrac{1}{3}\pi h(r^2 + r'^2 + rr').$$

This subsidiary proposition may be omitted, if desired.

Exercises. Numerical Computations

Find the lateral areas of cones of revolution, given the slant heights and the circumferences of the bases respectively as follows:

1. $4\frac{1}{2}$ in., $5\frac{1}{2}$ in. **2.** 4.7 in., 6.2 in. **3.** 3 ft. 6 in., 5 ft.

Find the lateral areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

4. 2.5 in., 2.1 in. **5.** 5.8 in., 5.6 in. **6.** 3 ft. 3 in., 7 in.

Find the total areas of cones of revolution, given the slant heights and the radii of the bases respectively as follows:

7. 4 in., $3\frac{1}{2}$ in. **9.** 16 in., 7 in. **11.** 8 ft., $3\frac{1}{2}$ ft.

8. 6 in., $3\frac{1}{2}$ in. **10.** 18 in., 7 in. **12.** 14 ft., 7 ft.

Find the volumes of circular cones, given the altitudes and the areas of the bases respectively as follows:

13. $4\frac{1}{2}$ in., $8\frac{3}{8}$ sq. in. **15.** 6 in., 5.8 sq. in.

14. $6\frac{1}{2}$ in., 10 sq. in. **16.** 8.1 in., 18.8 sq. in.

Find the volumes of circular cones, given the altitudes and the radii of the bases respectively as follows:

17. 8 in., 4.2 in. **19.** 18 in., 7 in. **21.** 4.9 in., 2.1 in.

18. 3 in., 2.1 in. **20.** 21 in., 7 in. **22.** 10.5 in., 6.2 in.

23. How many cubic feet are there in a conical tent which is 14 ft. in diameter and 9 ft. high?

24. How many cubic feet are there in a conical pile of earth which is 18 ft. in diameter and 10 ft. high?

25. An isosceles triangle whose altitude is 4 in. and whose equal sides are each 5 in. revolves about the base. Find the volume of the double cone thus formed.

Exercises. Formulas

Deduce, if possible, formulas for the following, stating the impossible cases, if any:

1. The lateral area of a cone of revolution in terms of the radius of the base and the altitude.

2. The slant height of a cone of revolution in terms of the lateral area and the circumference of the base.

3. The slant height of a cone of revolution in terms of the lateral area and the radius of the base.

4. The radius of the base of a cone of revolution in terms of the lateral area and the slant height.

5. The slant height of a cone of revolution in terms of the total area and the radius of the base.

6. The circumference of the base of a circular cylinder in terms of the lateral area and the slant height.

7. The volume of a cylinder in terms of the altitude and the lateral area.

8. The altitude of a circular cone in terms of the volume and the area of the base.

9. The area of the base of a circular cone in terms of the volume and the altitude.

10. The altitude of a circular cylinder in terms of the volume and the radius of the base.

11. The radius of the base of a circular cylinder in terms of the volume and the altitude.

12. The volume of a cone of revolution in terms of the slant height and the radius of the base.

13. The slant height and the altitude of a cone of revolution in terms of the volume and the circumference of the base.

Exercises. Theory of the Cone

1. Every section of a cone made by a plane passing through the vertex is a triangle.

2. The axis of a circular cone passes through the center of every section which is parallel to the base.

3. Defining *similar cones of revolution* as cones formed by the revolution of similar right triangles about corresponding sides, prove that the lateral areas of two similar cones of revolution are to each other as the squares of their altitudes.

4. In Ex. 3, consider the case of the total areas.

5. Consider Ex. 3 with "slant heights" substituted for "altitudes."

6. Consider Ex. 3 with "radii" substituted for "altitudes."

7. Consider Ex. 3 with respect to volumes instead of lateral areas, changing the statement as may be necessary.

8. If the lateral surface of a cone of revolution is cut along one of the elements and unrolled on a plane, show that it becomes a sector of a circle. Show also that there can be deduced a formula for the area of a sector of a circle that shall be the same as the formula for the lateral area of a cone.

9. Deduce a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the areas of the bases.

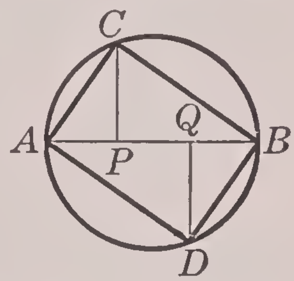
If the subsidiary proposition of § 445 was not taken, omit Exs. 9 and 10.

10. Deduce a formula for finding the altitude of a frustum of a cone of revolution in terms of the volume and the radii of the bases.

Exercises. Industrial Problems

1. A steamer's funnel 4 ft. 8 in. in diameter is built up of four equal plates in girth, with a lap at each joint of $1\frac{3}{8}$ in. Find one dimension of each plate.

2. There is a rule for calculating the strongest beam that can be cut from a cylindric log, as follows: Trisect the diameter AB, and at the points of division P, Q erect the $\perp$s PC, QD on opposite sides of AB, and meeting the circle in C and D. Then $ADBC$ is a section of the strongest beam. Given that the diameter AB of a log is 18 in., calculate the dimensions AD and DB of the strongest beam that can be cut from the log.



3. A tubular boiler has 128 tubes each $3\frac{3}{4}$ in. in diameter and 16 ft. long. Find the area of the total surface of the tubes to the nearest square foot.

4. In a room of a factory heated by steam pipes, there are 280 ft. of 2-inch pipe, 36 ft. of 3-inch pipe, and 6 ft. of $4\frac{1}{4}$-inch feed pipe. Find the total heating surface to the nearest square foot.

5. A triangular plate of wrought iron $\frac{3}{4}$ in. thick is 3 ft. 4 in. on each side. If the weight of a plate 1 ft. square and $\frac{1}{8}$ in. thick is 5 lb., what is the weight, to the nearest pound, of the triangular plate?

6. A cylinder 18 in. in diameter is required to hold 60 gal. of water. Allowing 231 cu. in. to the gallon, what must be the height of the cylinder to the nearest 0.1 in.?

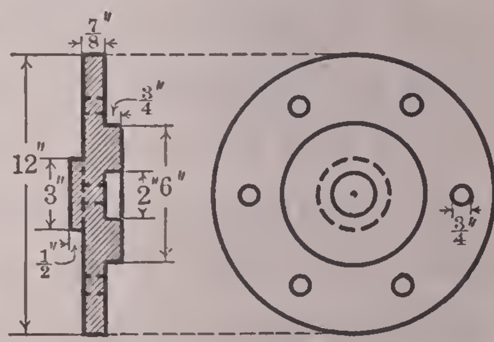
7. The water level of an upright cylindric boiler is 1 ft. 4 in. below the top of the boiler. If the cross-section area of the boiler is 14 sq. ft., what is the volume of the steam space above the water?

8. Allowing 490 lb. per cubic foot, find to the nearest 0.01 lb. the weight of a steel plate 5 ft. by 4 ft. 6 in. by $1\frac{5}{8}$ in.

9. Through a steel plate 5 ft. long, 3 ft. 8 in. wide, and $\frac{5}{8}$ in. thick, a porthole 12 in. in diameter is cut. Allowing 0.29 lb. per cubic inch, find the weight of the finished plate.

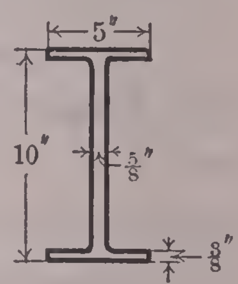
10. A cast-iron base for a column is in the form of a frustum of a regular pyramid. The lower base is a square 26 in. on a side, the upper base has one fourth the area of the lower base, and the altitude of the frustum is 10 in. If the total surface is to be painted, what area must be covered?

11. A cylinder head for a steam engine has the shape shown in this figure. Allowing 41 lb. for the weight of a steel plate 1 ft. square and 1 in. thick, find the weight of the cylinder head to the nearest 0.1 lb.



12. A hollow steel shaft 14 ft. long has an exterior diameter of 16 in. and an interior diameter of 9 in. Allowing 0.29 lb. per cubic inch, find the weight to the nearest pound.

13. A steel beam in the form here shown is 16 ft. long. Allowing 0.29 lb. per cubic inch, find the weight of the beam to the nearest pound.



14. How many square feet of tin are required to make a funnel, if the diameters at the top and bottom are to be 32 in. and 16 in. respectively, the height is to be 24 in., and 4 sq. in. are allowed for waste?

15. Find the cost, at $1.35 per square foot, of finishing the curve surface of a frustum of a right circular cone whose slant height is 12 ft. and the radii of whose bases are 3 ft. 6 in. and 2 ft. 4 in. respectively.

Exercises. Equivalent Solids

1. When a cube of iron 6 in. on an edge is melted it just fills a mold in the form of a right prism whose base is a rectangle 8 in. long and 6 in. wide. Find the height of the prism and the difference between its total area and the total area of the cube.

2. A pile of bricks in the form of a regular pyramid 10 ft. high is repiled in the form of a regular prism with an equivalent base. Assuming no loss due to piling the bricks a different way, find the height of the prism.

3. The diameter of a cylinder is 12 ft. and its height is 6 ft. Find the height of an equivalent right prism, the base of which is a square 5 ft. on a side.

4. If one edge of a cube is e, what is the height h of an equivalent right circular cylinder whose radius is r?

5. The dimensions of a rectangular parallelepiped are a, b, c. Find the height of an equivalent right circular cylinder which has a for the radius of its base. Also find the height of an equivalent right circular cone which has a for the radius of its base.

6. A right circular cylinder 5 ft. in diameter is equivalent to a right circular cone 5 ft. in diameter. If the height of the cone is 6 ft., what is the height of the cylinder?

7. The heights of two equivalent right circular cylinders are in the ratio $4:9$. If the diameter of the first is 5 ft., what is the diameter of the second?

8. A frustum of a cone of revolution is 5 ft. high and the diameters of its bases are 2 ft. and 4 ft. respectively. Find the height of an equivalent right circular cylinder whose radius is 5 ft.

Omit Ex. 8 if § 445 was not taken.

Exercises. Miscellaneous Problems

1. The slant height of a frustum of a regular pyramid is 24 ft., and the bases are squares whose sides are 50 ft. and 20 ft. respectively. Find the volume.

Exs. 1–6 should be omitted if §§ 408 and 445 were not taken.

2. Given that the bases of a frustum of a pyramid are regular hexagons whose sides are 2 ft. and 3 ft. respectively, and that the volume is 16 cu. ft., find the altitude.

3. From a right circular cone whose slant height is 24 ft. and the circumference of whose base is 8 ft., there is cut off by a plane parallel to the base a cone whose slant height is 5 ft. Find the lateral area and the volume of the frustum.

4. Find the difference between the volume of a frustum of a pyramid, whose altitude is 8 ft. and whose bases are squares 16 ft. and 12 ft. respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.

5. A stone windmill is in the shape of a frustum of a right circular cone. The mill is 14 m. high, the outer diameters at the bottom and the top are 18 m. and 14 m., and the inner diameters are 14 m. and 8 m. respectively. How many cubic meters of stone were required to build it?

6. A brick chimney has the shape of a frustum of a regular pyramid. The chimney is 160 ft. high, its upper and lower bases are squares 9 ft. and 14 ft. on a side respectively, and a square flue 6 ft. on a side runs from top to bottom. How much brickwork does the chimney contain?

7. Two right triangles whose bases are 5 in. and 7 in., and whose hypotenuses are $8\frac{1}{3}$ in. and $11\frac{2}{3}$ in. respectively, revolve about their third sides. Find the ratio of the total areas of the solids generated and find their volumes.

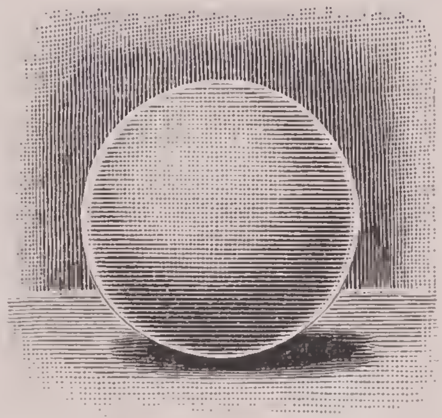
BOOK VIII

THE SPHERE

I. Fundamental Theorems

446. Sphere. The locus of points in space at a given distance from a given point is called a *sphere.*

The terms *center, radius, chord,* and *diameter* are used as in the case of the circle.



This is the modern definition, analogous to the modern definition of a circle, but it will be found that no confusion will arise if the student considers a sphere as the solid inclosed by a *spherical surface* or *spheric surface.* In modern mathematics the volume of a sphere is considered to mean the volume inclosed by the surface which is called a sphere.

447. A Point and a Sphere. A point may be *on a sphere, within the sphere* (inclosed by it), or *outside the sphere* (not inclosed by it).

448. Properties of a Sphere. As in § 134, we have

1. *All radii of the same sphere or of equal spheres are equal.*

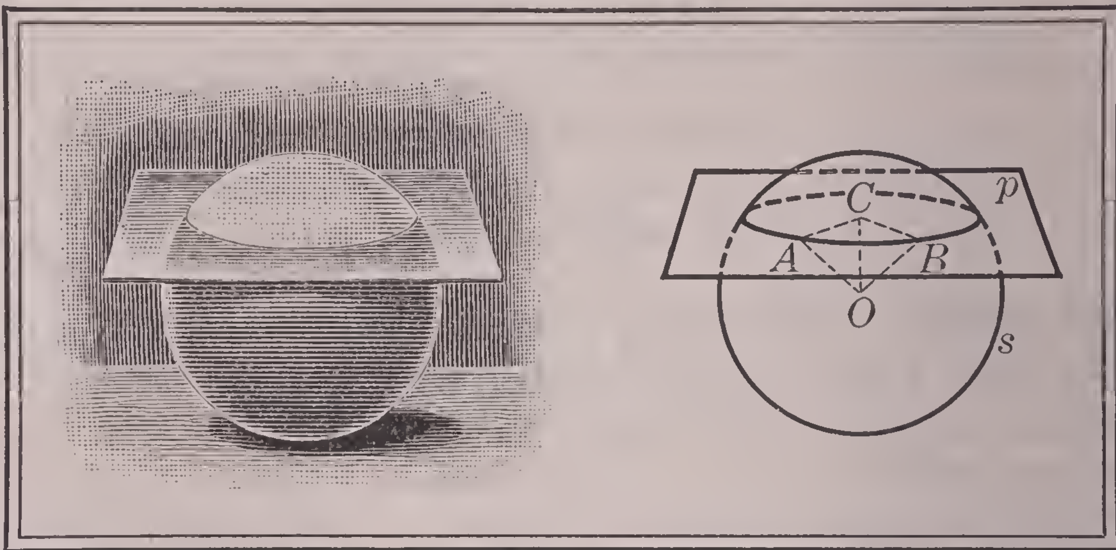
2. *All spheres of equal radii are equal.*

3. *A point is within, on, or outside a sphere according as the distance of the point from the center is less than, equal to, or greater than the radius.*

There are also other properties, such as that a diameter is twice a radius, which are too obvious to require mention.

Proposition 1. Plane Intersecting a Sphere

449. Theorem. *If a plane intersects a sphere, the line of intersection is a circle.*



Given the plane p intersecting the sphere s with center O.

Prove that the line of intersection is a ⊙.

Proof. Let OA, OB be any two radii of s to points on the line of intersection, and let OC be the ⊥ from O to p.

Draw	CA and CB.	Post. 1
Then	$\angle$s ACO and BCO are rt. $\angle$s,	§ 316
	$OC = OC$,	Iden.
and	$OA = OB$.	§ 448, 1
Hence	$\triangle AOC$ is congruent to $\triangle BOC$,	§ 71
and	$CA = CB$.	§ 38

Then any points A and B, and hence all points on the line of intersection, lie on a ⊙. § 134, 6

That is, the line of intersection is a ⊙.

450. Great Circle. The line of intersection of a sphere and a plane passing through the center is called a *great circle* of the sphere.

451. Corollary. *Through any two points on a sphere an arc of a great circle can be drawn.*

For the two points and the center of the sphere determine a plane.

452. Small Circle. The line of intersection of a sphere and a plane which does not pass through the center is called a *small circle* of the sphere.

453. Corollary. *Through any three points on a sphere one and only one circle can be drawn.*

454. Corollary. *A diameter of a sphere perpendicular to the plane of a circle of the sphere passes through the center of the circle.*

455. Poles of a Circle. If a diameter of a sphere is perpendicular to the plane of a circle of the sphere, the ends of the diameter are called the *poles* of the circle.

456. Spherical Distance. The length of the smaller arc of the great circle joining two points on a sphere is called the *spherical distance* between the points, or, where no confusion is likely to arise, simply the *distance.*

457. Polar Distance. The spherical distance from the nearer pole of a circle to any point on the circle is called the *polar distance of the circle.*

The spherical distance of a great circle from either of its poles may be taken as the polar distance of the circle.

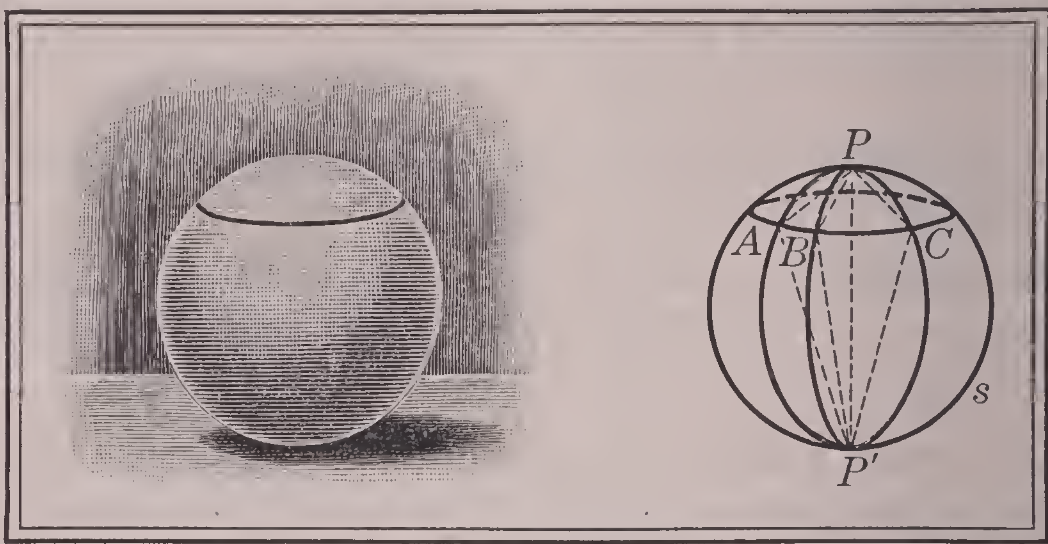
458. Quadrant. One fourth of a great circle is called a *quadrant.*

459. Tangent Lines and Planes. A line or plane that has one and only one point in common with a sphere, however far produced, is said to be *tangent to the sphere*, and the sphere is said to be *tangent to the line* or *plane.*

460. Tangent Spheres. Two spheres which have one and only one point in common are said to be *tangent.*

Proposition 2. Equal Polar Distances

461. Theorem. *All points on a circle of a sphere are equidistant from either pole of the circle.*



Given A, B, C, any points on a ⊙ of the sphere s; P, P', the poles of ⊙ ABC; and PAP', PBP', PCP', arcs of great ⓢ through A, B, C.

Prove that arc PA = arc PB = arc PC,
and that arc $P'A$ = arc $P'B$ = arc $P'C$.

Proof. Draw PP' and the chords PA, PB, PC.

Then PP' is $\perp$ to the plane of ⊙ ABC, § 455
and passes through the center of ⊙ ABC. § 454
Then chord PA = chord PB = chord PC. § 326, 2

Since the planes of great ⓢ pass through the center of s, the ⓢ have equal radii (§ 448, 1), and hence are equal.

$\therefore$ arc PA = arc PB = arc PC. § 140

Similarly, by drawing chords from P' to A, B, C, we have

arc $P'A$ = arc $P'B$ = arc $P'C$.

462. Corollary. *The polar distance of a great circle is a quadrant.*

Exercises. Circles of a Sphere

Draw figures to illustrate each of the following:

1. Great circle.
2. Small circle.
3. Poles of a circle.
4. Spherical distance.
5. Polar distance.
6. Quadrant.

Prove the important properties in Exs. 7–11:

7. *Parallel circles of a sphere have the same poles.*

8. *All great circles of a sphere are equal.*

9. *Every great circle bisects the sphere.*

10. *Two great circles bisect each other.*

11. *If the planes of two great circles are perpendicular to each other, each circle passes through the poles of the other.*

12. A radius of a sphere perpendicular to a chord of the sphere bisects the chord.

13. Equal chords of a sphere are equidistant from the center.

14. Of the chords of a sphere through a point within it, the chords which are perpendicular to the diameter of the sphere through the point are the shortest.

The student has doubtless noticed the analogy between the propositions relating to the circle and those relating to the sphere. If we think of a plane as passing through the center of a sphere and any other point under discusssion, the figure of the corresponding proposition in plane geometry will often appear on this plane.

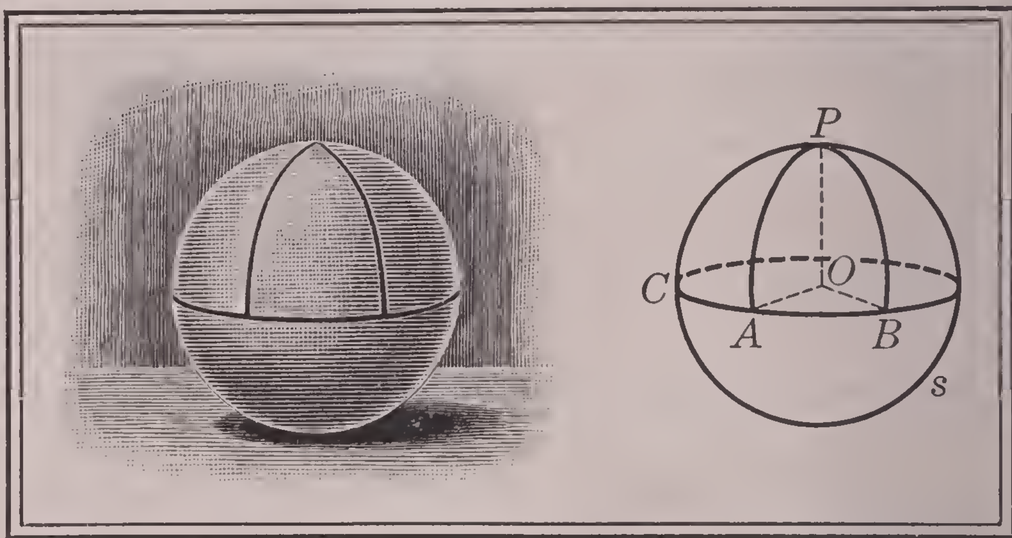
15. An infinite number of spheres can pass through two given points, and their centers lie in a fixed plane.

16. An infinite number of spheres can pass through three given points, and their centers lie on a fixed straight line.

17. The distance between the centers of two spheres of radii r and r' is d. State the condition under which the spheres intersect.

Proposition 3. Pole of a Great Circle

463. Theorem. *A point on a sphere which is at the distance of a quadrant from each of two other points on the sphere, not the ends of a diameter, is a pole of the great circle passing through these points.*



Given the point P on the sphere s; the quadrants PA, PB; and ABC, the great $\odot$ passing through A and B.

Prove that *P is a pole of the $\odot ABC$.*

Proof. From O, the common center of the great $\circledS$ PA, PB, ABC (§ 450), draw OA, OB, OP.

Since arcs PA and PB are fourths of great $\circledS$, § 458

then $\measuredangle$s AOP and BOP are rt. $\measuredangle$s. § 171

Hence OP is $\perp$ to the plane of $\odot ABC$, § 320

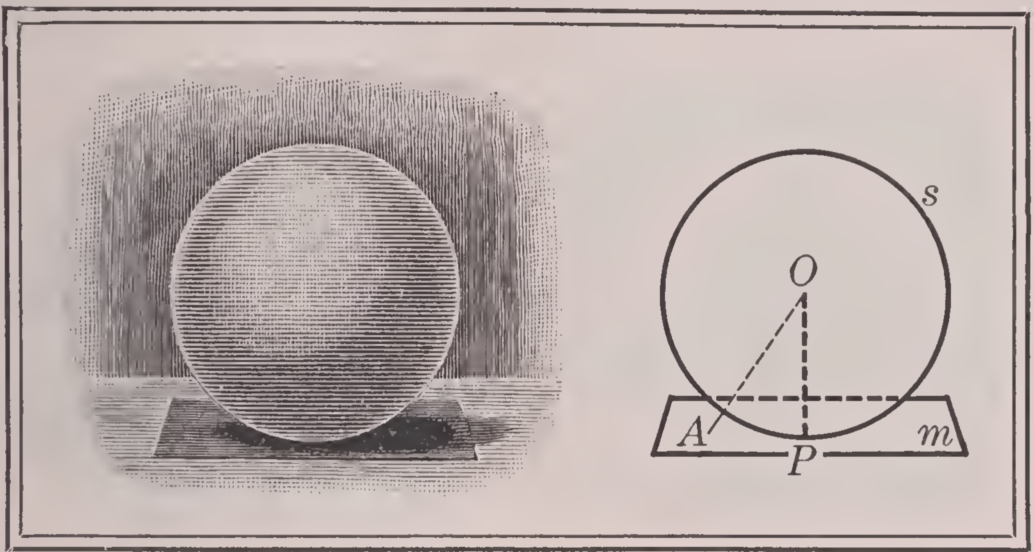
and P is a pole of the $\odot ABC$. § 455

Why may not the two points be the ends of a diameter?

464. Spherical Angle. When two great-circle arcs intersect they are said to form a *spherical angle.* A spherical angle is considered as equal to the plane angle formed by the tangents to the arcs at their point of intersection.

Proposition 4. Tangent Plane

465. Theorem. *If a plane is perpendicular to a radius at its end on the sphere, the plane is tangent to the sphere.*



Given the sphere s with the plane $m \perp$ to the radius OP at P.

Prove that m *is tangent to* s.

Proof. Let A be any point except P in m, and draw OA.

Then, since $OA > OP$ (§ 326, 1), A is outside s. § 448, 3

Then every point in m, except P, is outside s.

Hence m is tangent to s. § 459

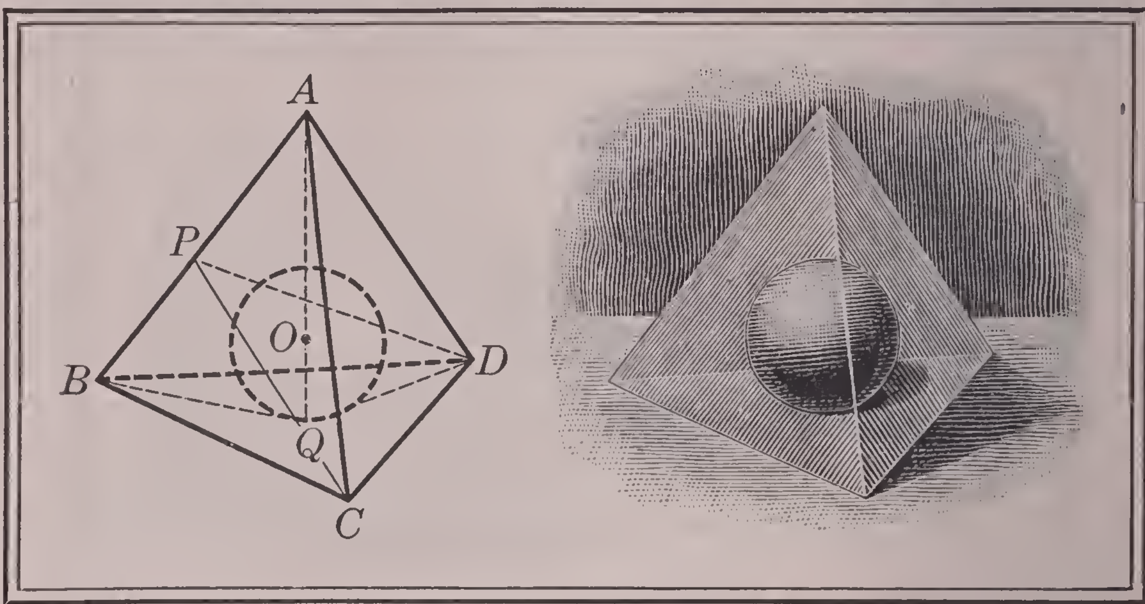
466. Corollary. *If a plane is tangent to a sphere, it is perpendicular to the radius drawn to the point of contact.*

467. Inscribed Sphere. If a sphere is tangent to all the faces of a polyhedron, the sphere is said to be *inscribed in the polyhedron,* and the polyhedron is said to be *circumscribed about the sphere.*

468. Circumscribed Sphere. If all the vertices of a polyhedron lie on a sphere, the sphere is said to be *circumscribed about the polyhedron,* and the polyhedron is said to be *inscribed in the sphere.*

Proposition 5. Tetrahedron and Inscribed Sphere

469. Theorem. *A sphere can be inscribed in any given tetrahedron.*



Given the tetrahedron *ABCD.*

Prove that a sphere can be inscribed in ABCD.

Proof. The face $\angle$s of the dihedral $\angle$s can be bisected. § 103

These bisectors and the corresponding edges of the dihedral $\angle$s determine planes. § 314, 1

These planes bisect the dihedral $\angle$s (§ 349), and are the loci of points equidistant from the faces. § 359

Any two of these bisecting planes, as *ABQ* and *ADQ*, intersect in a st. line *AQ*. § 315

This line *AQ* cuts another bisecting plane, as *CDP*, in some point *O*.

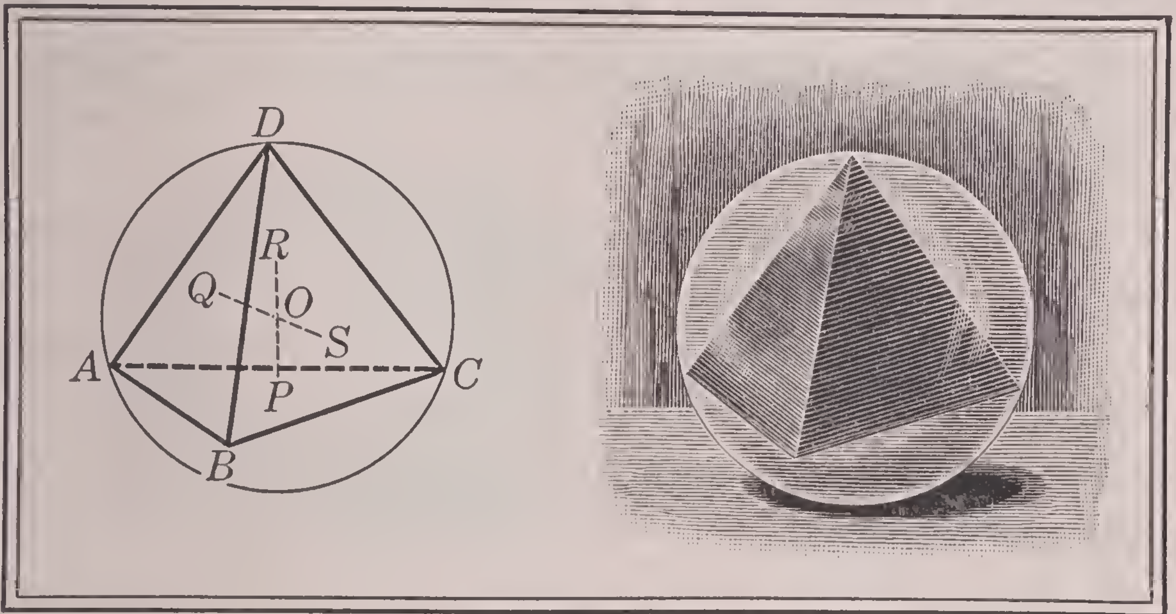
Hence *O*, the common intersection of the three bisecting planes, is equidistant from the four faces of *ABCD*.

Since the $\perp$s from *O* upon the faces are equal, the faces of *ABCD* are tangent to a sphere with center *O*. §§ 446, 465

Hence a sphere can be inscribed in *ABCD*. § 467

Proposition 6. Tetrahedron and Circumscribed Sphere

470. Theorem. *A sphere can be circumscribed about any given tetrahedron.*



Given the tetrahedron ***ABCD***.

Prove that a sphere can be circumscribed about ABCD.

Proof. Let P be the center of the ⊙ circumscribed about face ABC, and Q the center of the ⊙ about face ABD. § 188

Let PR be ⊥ to face ABC, and QS to face ABD. § 324

Then PR is the locus of points equidistant from A, B, C; and QS, of points equidistant from A, B, D. § 328

Now PR and QS both lie in the plane ⊥ to AB at its mid-point (§ 329). If QS is ∥ to PR, it is ⊥ to face ABC (§ 331). But this is impossible, since QS is ⊥ to face ABD which intersects face ABC. Hence PR and QS intersect, as at O.

Since O is equidistant from A, B, C, D,

then A, B, C, D lie on a sphere. § 446

Hence a sphere can be circumscribed about $ABCD$. § 468

471. Corollary. *Through four points not in the same plane one and only one sphere can pass.*

Exercises. Review

1. *The intersection of two spheres is a circle whose plane is perpendicular to the line which joins the centers of the spheres and whose center is on that line.*

2. The four perpendiculars erected at the centers of the circles circumscribed about the faces of a tetrahedron intersect in the same point.

3. The six planes perpendicular to the edges of a tetrahedron at their midpoints intersect in the same point.

4. The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.

5. Circles on the same sphere which have equal polar distances are equal.

6. Equal circles on the same sphere have equal polar distances.

7. Find the locus of points in a plane at a given distance from a given point. Also find the locus of such points in a three-dimensional space.

8. A line tangent to a great circle of a sphere lies in the plane tangent to the sphere at the point of contact.

9. Any line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.

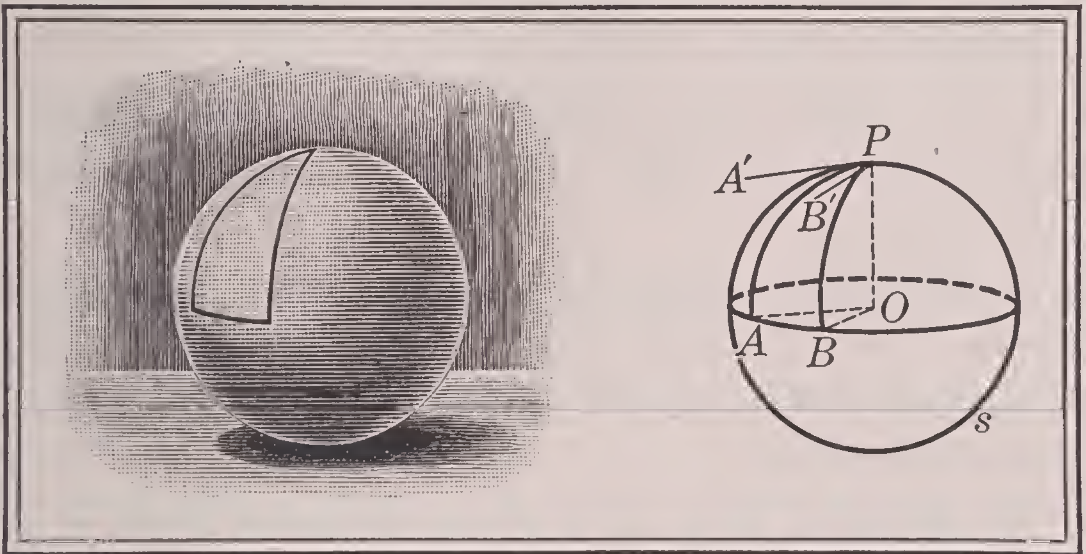
10. Through a given point on a given sphere one and only one plane can be passed tangent to the sphere.

11. Find a point in a plane equidistant from two intersecting lines in the plane, and at a given distance from a given point not in the plane. Discuss the solution for all types of cases.

12. How many points determine a straight line? a circle? a sphere? Prove that two spheres coincide if they have this last number of points in common.

Proposition 7. Measure of a Spherical Angle

472. Theorem. *A spherical angle is measured by the arc of the great circle which has the vertex of the angle as pole and is included between the arms of the angle.*



Given PA, PB, arcs of great Ⓢ of the sphere s intersecting at P; PA', PB', the tangents to these arcs at P; and the arc AB, included between PA and PB, of the great ⊙ with P as pole.

Prove that the spherical $\angle APB$ *is measured by arc* AB.

Proof. Let the planes of the great Ⓢ PA, PB, AB intersect in PO, AO, BO respectively.

Then PB' is $\perp$ to PO, § 147

and OB is $\perp$ to PO. § 171

Hence PB' is $\parallel$ to OB (§ 57), and similarly PA' is $\parallel$ to OA.

$\therefore \angle A'PB' = \angle AOB$. § 343

Since $\angle AOB$ is measured by arc AB, § 171

then $\angle A'PB'$ is measured by arc AB. Ax. 5

$\therefore \angle APB$ is measured by arc AB. § 464

473. Corollary. *A spherical angle has the same measure as the dihedral angle formed by the planes of the two circles.*

Exercises. Review

1. If an arc of a great circle passes through the pole of another great circle, the two circles are perpendicular to each other.

2. If two great circles are perpendicular to each other, each passes through the poles of the other.

3. If the first of two great circles passes through the poles of the second, then the second passes through the poles of the first.

4. If three lines form a triangle, it is possible (§§ 193, 194) to construct four circles tangent to all three lines. Consider the analogous case for spheres and for the four planes which form a tetrahedron.

5. All straight lines at a given distance from a given point are tangent to a certain sphere.

6. If two planes which intersect in the line AB touch a sphere at the points C and D respectively, the line CD is $\perp$ to AB in the sense mentioned in § 335; that is, a plane can be passed through $CD \perp$ to AB.

7. If two unequal spheres intersect, are the tangents to the larger sphere from any point in the plane of their common circle shorter than the tangents from that point to the smaller sphere? State the proposition in correct form and then prove it.

8. Given that the two points P, P' are 16 cm. apart, find the locus of points that are 10 cm. from P and 12 cm. from P'.

9. If the edges of a tetrahedron are tangent to a sphere, the sum of any pair of opposite edges is equal to the sum of any other pair.

Compare this with Ex. 6, page 119.

II. Spherical Polygons

474. Spherical Polygon. A portion of a sphere bounded by three or more arcs of great circles is called a *spherical polygon.*

The difference between the general nature of a spherical polygon and that of a plane polygon is that the former lies on a spherical surface and has arcs of great circles as its sides, while the latter lies on a plane surface and has segments of straight lines as its sides.

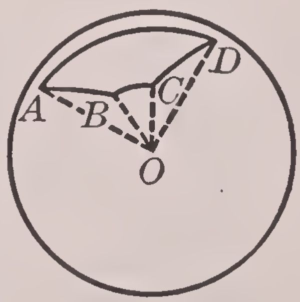
The terms *sides, angles, vertices, diagonal, convex, concave,* and *congruent* are used as with plane polygons.

475. Spherical Triangle. A spherical polygon of three sides is called a *spherical triangle.*

A spherical triangle may be *right, obtuse,* or *acute,* and may also be *equilateral, isosceles,* or *scalene.*

The terms *spheric polygon* and *spheric triangle* are also used.

476. Relation of Polygons to Polyhedral Angles. The planes of the sides of a spherical polygon form a polyhedral angle whose vertex is the center of the sphere, whose face angles are measured by the sides of the polygon, and whose dihedral angles have the same numerical measure as the angles of the polygon.



Thus, the planes of the sides of the polygon $ABCD$ here shown form the polyhedral $\angle\, O\text{-}ABCD$. The face $\measuredangle\, BOA$, COB, $\cdots$ are measured by the sides BA, CB, $\cdots$ of the polygon. The dihedral angle whose edge is OA has the same measure as the spherical $\angle\, BAD$, and so on.

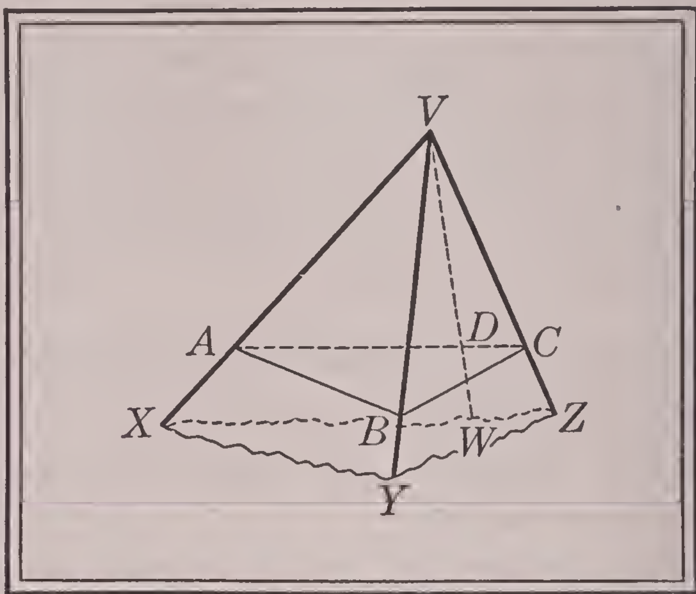
Hence *from any property of polyhedral angles, we may infer an analogous property of spherical polygons, and conversely.*

Since we have considered only convex polyhedral angles in the preceding work, we shall consider only convex spherical polygons.

Because of the relation between polyhedral angles and spherical polygons, we shall first consider the former.

Proposition 8. Two Face Angles

477. Theorem. *The sum of any two face angles of a trihedral angle is greater than the third face angle.*



Given the trihedral $\angle V\text{-}XYZ$ with face $\angle XVZ >$ face $\angle XVY$ or face $\angle YVZ$.

Prove that $\angle XVY + \angle YVZ > \angle XVZ.$

Proof. In the plane XVZ let $\angle XVW = \angle XVY$, and through any point D of VW draw a line cutting VX in A and VZ in C. On VY take $VB = VD$.

Then A, B, C determine a plane. § 314, 1

Since $AV = AV$, $VB = VD$, and $\angle AVB = \angle AVD$, then $\triangle AVB$ is congruent to $\triangle AVD$ (§ 40), and $AB = AD$ (§ 38).

Now $AB + BC > AD + DC.$ Post. 3

$\therefore BC > DC.$ Ax. 7

Then since $VB = VD$, and $VC = VC$,

$\angle BVC > \angle DVC.$ § 127

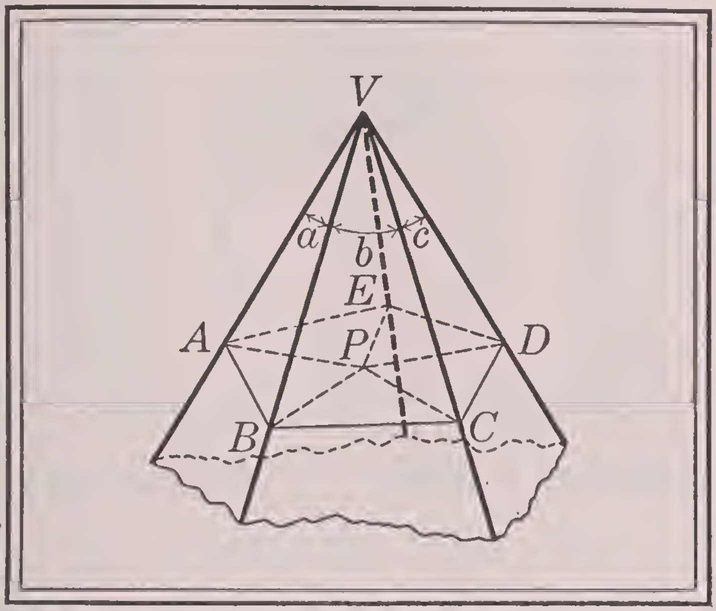
Then $\angle AVB + \angle BVC > \angle AVD + \angle DVC.$ Ax. 7

Hence $\angle AVB + \angle BVC > \angle AVC;$ Ax. 5

that is, $\angle XVY + \angle YVZ > \angle XVZ.$

Proposition 9. Sum of Face Angles

478. Theorem. *The sum of the face angles of a polyhedral angle is less than four right angles.*



Given the polyhedral $\angle V$ with the face $\angle$s $a, b, c, \cdots$.

Prove that $\quad a+b+c+\cdots<4$ *rt.* $\angle$s.

Proof. Let a plane cut all the edges of $\angle V$, thus forming the polygon $ABC\cdots$, and let P be any point within $ABC\cdots$.

Drawing $PA, PB, PC, \cdots$, there are as many $\triangle$s$(PAB, PBC, \cdots)$ as there are faces $(VAB, VBC, \cdots)$.

Hence the sum of the $\angle$s of all the $\triangle$s with vertex V is equal to the sum of the $\angle$s with vertex P. § 65, Ax. 1

Now $\quad \angle EAV+\angle BAV>\angle BAE,$

$\angle VBA+\angle CBV>\angle CBA, \cdots.$ § 477

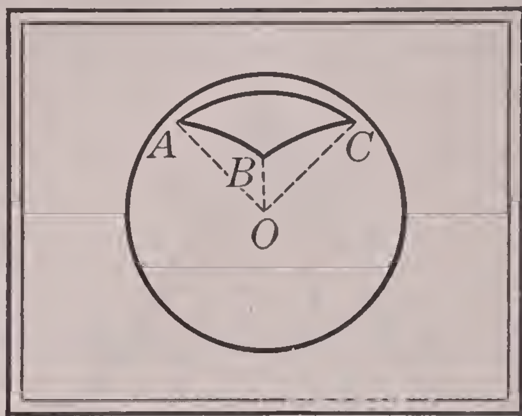
Hence the sum of the $\angle$s at the bases of the $\triangle$s with vertex V is greater than the sum of the $\angle$s at the bases of the $\triangle$s with vertex P. Ax. 8

$\therefore a+b+c+\cdots<\angle APB+\angle BPC+\angle CPD+\cdots,$ Ax. 8

or $\quad a+b+c+\cdots<4$ rt. $\angle$s. § 13

Proposition 10. Side of a Triangle

479. Theorem. *Any side of a spherical triangle is less than the sum of the other two sides.*



Given the spherical $\triangle ABC$.

Prove that $CA < AB + BC$.

Proof. Let O-ABC be the corresponding trihedral $\angle$.

Then $\angle COA < \angle BOA + \angle COB$. § 477

$\therefore CA < AB + BC$. § 476

Exercises. Spherical Triangles

1. Explain how you could proceed to bisect a given great-circle arc.

2. Explain how you could determine the arc that bisects a given spherical angle.

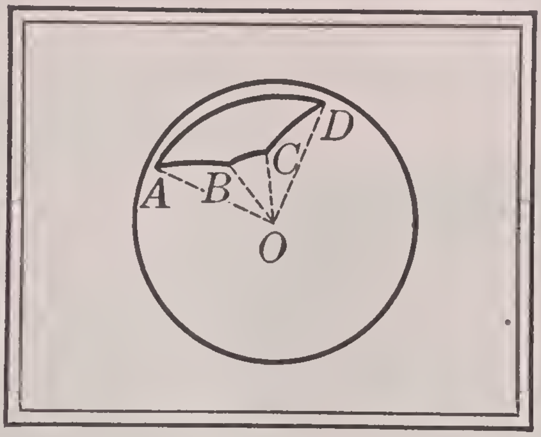
3. Draw a sphere and upon it draw freehand a spherical $\triangle ABC$. With A, B, C as poles draw freehand three great circles and show that these circles divide the sphere into eight triangles.

Assume that the center, diameter, and radius are given.

4. Make a drawing of a sphere and on the sphere show an equilateral spherical triangle, each side of which is 90°. Then draw a triangle with the three vertices as poles.

Proposition 11. Sum of Sides

480. Theorem. *The sum of the sides of a spherical polygon is less than 360°.*



Given the spherical polygon ***ABCD.***

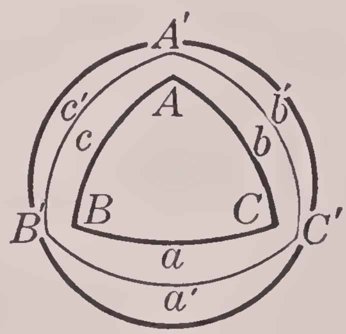
Prove that $AB+BC+CD+DA<360°$.

Proof. Let $O\text{-}ABCD$ be the corresponding polyhedral $\angle$.

Then $\angle BOA+\angle COB+\angle DOC+\angle DOA<360°$. § 478

$\therefore AB+BC+CD+DA<360°$. § 476

481. Polar Triangle. The triangle formed by the arcs of great circles of which the vertices of a given triangle are poles is called the *polar triangle* of the given triangle.



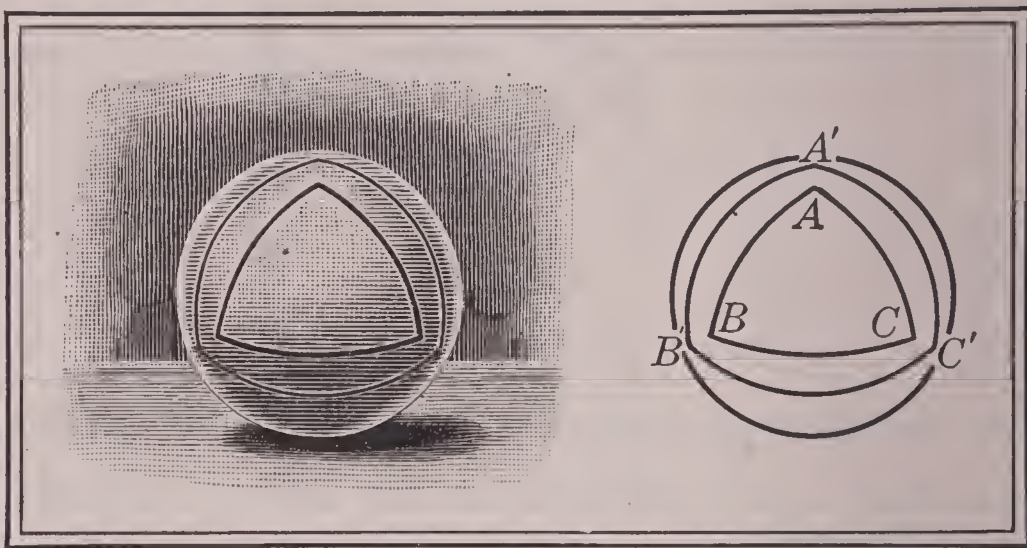
Thus, if A is the pole of the great circle of which a' is an arc, B is the pole of the great circle of which b' is an arc, and C is the pole of the great circle of which c' is an arc, then $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$.

If, with A, B, C as poles, entire great circles are drawn, these circles divide the sphere into eight spherical triangles. Of these eight triangles, that one is the polar of $\triangle ABC$ whose vertex A', corresponding to A, lies on the same side of BC as the vertex A; and similarly for the other vertices.

While it is desirable to have a spherical blackboard on which the student can draw figures, any small ball will serve the purpose.

Proposition 12. Reciprocal Polar Triangles

482. Theorem. *If one spherical triangle is the polar triangle of another, then the second is the polar triangle of the first.*



Given ABC, **a spherical** $\triangle$, **and** $A'B'C'$, **its polar** $\triangle$.

Prove that ABC *is the polar* $\triangle$ *of* $A'B'C'$.

Proof. Since A is the pole of arc $B'C'$,

and C is the pole of arc $A'B'$, § 481

then B' is at the distance of a quadrant from A and C. § 462

$\therefore B'$ is the pole of arc AC. § 463

Similarly, A' is the pole of arc BC,

and C' is the pole of arc AB.

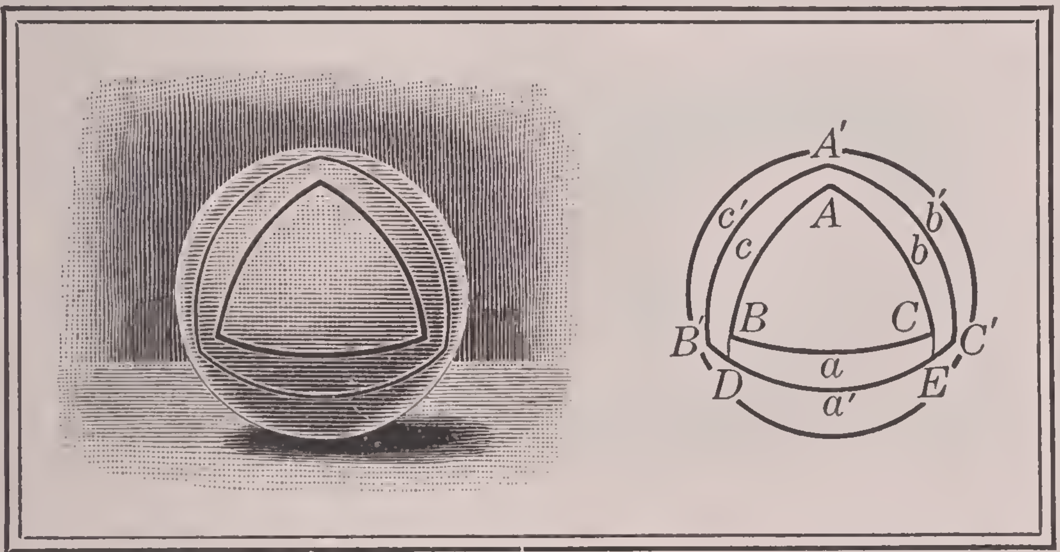
$\therefore ABC$ is the polar $\triangle$ of $A'B'C'$. § 481

The student should notice that we may just as well start with ABC as the polar triangle of $A'B'C'$, and then prove that $A'B'C'$ is the polar triangle of ABC.

It should also be noticed that it is not necessary that either of the triangles should be wholly within the other. For example, if we draw the figures freehand, taking AB as about 100°, AC as about 100°, and BC as about 30°, one triangle will overlap the other.

Proposition 13. Angle and Side Supplementary

483. Theorem. *Any angle of either of two polar triangles is the supplement of the opposite side of the other.*



Given the polar $\triangle$ ABC and $A'B'C'$.

Prove that $\angle A$ *and* a'; $\angle B$ *and* b'; $\angle C$ *and* c'; $\angle A'$ *and* a; $\angle B'$ *and* b; $\angle C'$ *and* c *are respectively supplementary.*

Proof. Let arcs AB and AC produced meet arc $B'C'$ at D and E respectively.

Since B' is the pole of arc AE,

$$\text{arc } B'E = 90°. \qquad \S\ 462$$

Similarly, $\text{arc } DC' = 90°.$

Hence $\text{arc } B'D + \text{arc } DE + \text{arc } DC' = 180°,$ Ax. 1

or $\text{arc } DE + a' = 180°.$ Ax. 5

But $\text{arc } DE$ is the measure of $\angle A$. § 472

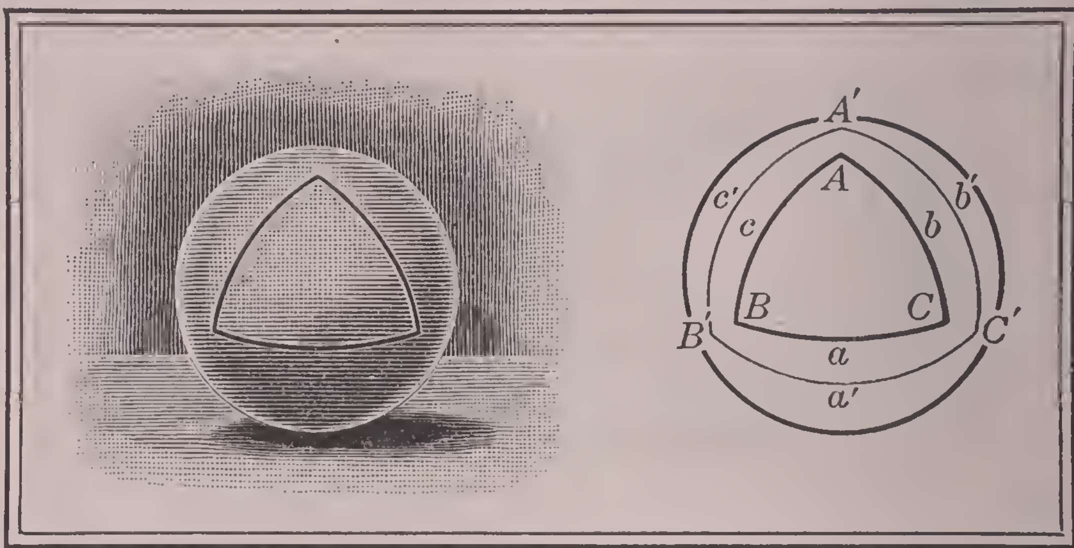
$$\therefore \angle A + a' = 180°. \qquad \text{Ax. } 5$$

Similarly, $\angle B + b' = 180°$, and $\angle C + c' = 180°$; hence $\angle A$ and a'; $\angle B$ and b'; $\angle C$ and c' are respectively supplementary.

In a similar way, by considering the angles of $\triangle A'B'C'$ and producing the sides of $\triangle ABC$, the other relations can be proved.

Proposition 14. Sum of the Angles of a Triangle

484. Theorem. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*



Given the spherical $\triangle ABC$.

Prove that $180° < \angle A + \angle B + \angle C < 540°$.

Proof. Let $A'B'C'$ be the polar $\triangle$ of ABC, with the sides of both $\triangle$ lettered as usual.

Then $\angle A + a' = 180°$, $\angle B + b' = 180°$, $\angle C + c' = 180°$. § 483

$\therefore \angle A + \angle B + \angle C + a' + b' + c' = 540°$, Ax. 1

or $\angle A + \angle B + \angle C = 540° - (a' + b' + c')$. Ax. 2

Now $a' + b' + c' < 360°$. § 480

$\therefore \angle A + \angle B + \angle C > 180°$.

Also, $a' + b' + c' > 0°$.

$\therefore \angle A + \angle B + \angle C < 540°$.

485. Triangles classified as to Right Angles. Since (§ 484) a spherical triangle may have two or even three right angles, and two or even three obtuse angles, we find it convenient to speak of a spherical triangle which has two right angles as *birectangular*, and one which has three right angles as *trirectangular*.

Exercises. Spherical Polygons

1. Two sides of a spherical triangle are respectively $83° 48'$ and $64° 59'$. What is known concerning the number of degrees in the third side?

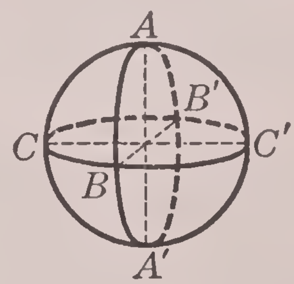
2. Three sides of a spherical quadrilateral are respectively $87° 39'$, $74° 48'$, and $68° 56'$. What is known (§ 480) concerning the number of degrees in the fourth side?

3. If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.

4. In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.

Since the angles are right angles, what two planes are perpendicular to a third plane? What two arcs must therefore pass through the pole of a third arc? Then what two arcs are quadrants? How is the third angle measured?

5. Each side of a trirectangular spherical triangle is a quadrant.



6. Three planes passed through the center of a sphere, each perpendicular to the other two, divide the spherical surface into eight congruent trirectangular triangles.

Find the number of degrees in the sides of a spherical triangle, given the angles of its polar triangle as follows:

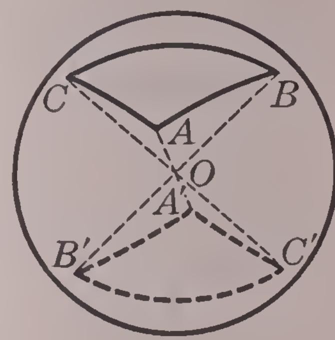
7. $83°$; $78°$; $64°$. **8.** $84° 50'$; $49° 38'$; $104° 40'$.

Find the number of degrees in the angles of a spherical triangle, given the sides of its polar triangle as follows:

9. $69° 42' 38''$; $93° 48' 8''$; $38° 36' 15''$.

10. $72° 48' 26''$; $104° 38' 43''$; $90°$.

486. Symmetric Spherical Triangles. If through the center O of a sphere the diameters AA', BB', CC' are drawn, and the points A, B, C and also the points A', B', C' are joined by arcs of great circles, the spherical $\triangle ABC$ and $A'B'C'$ are called *symmetric spherical triangles.*

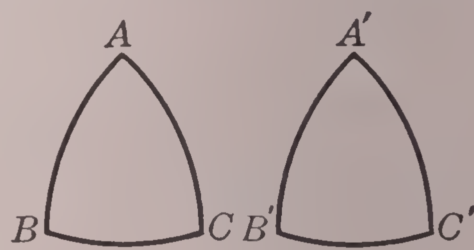


In the same way we may form two symmetric polygons of any number of sides. We may then place the symmetric polygons thus formed in any position we choose upon the sphere.

487. Relation of Symmetric Triangles. Two symmetric triangles are mutually equilateral and mutually equiangular; but in general they are not congruent, since they cannot be made to coincide by superposition. Thus, in the above figure, if the $\triangle ABC$ is made to slide on the sphere until the vertex A falls on A', it is evident that the two triangles cannot be made to coincide since, looked at from the point O, the corresponding parts of the triangles occur in reverse order.

The relation of two symmetric spherical triangles, which is similar to that of a pair of gloves, may be illustrated by cutting the triangles out of the peel of an orange.

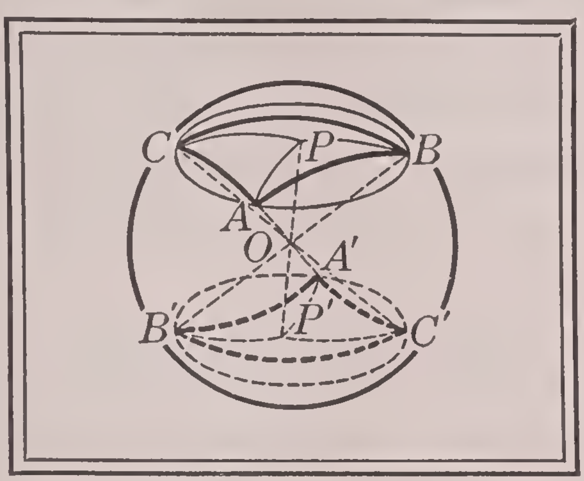
488. Symmetric Isosceles Triangles. Consider, however, the case of the symmetric $\triangle ABC$ and $A'B'C'$ in which $AB = AC$, and $A'B' = A'C'$; that is, the two symmetric triangles are isosceles. Then because AB, AC, $A'B'$, and $A'C'$ are all equal, and the $\angle A$ and A' are equal since they were originally formed by vertical dihedral angles (§ 486), the two triangles can be made to coincide.



If two symmetric spherical triangles are isosceles, they are superposable and hence are congruent.

Proposition 15. Symmetric Triangles

489. Theorem. *Two symmetric spherical triangles are equivalent.*



Given the symmetric spherical ⚠ ABC and $A'B'C'$.

Prove that $\triangle ABC$ *is equivalent to* $\triangle A'B'C'$.

Proof. Let the ⚠ ABC and $A'B'C'$ be placed with their corresponding vertices opposite each to each with respect to the center of the sphere. Post. 5

Let P be the pole of the small ⊙ through A, B, C; P', the pole of the small ⊙ through A', B', C'; and PA, PB, PC, $P'A'$, $P'B'$, $P'C'$, the arcs of great ⓢ.

Now ⚠ PCA and $P'C'A'$ are symmetric. § 486

Also, $\text{arc } PA = \text{arc } PB = \text{arc } PC$,

and $\text{arc } P'A' = \text{arc } P'B' = \text{arc } P'C'$. § 461

Then $\triangle PCA$ is congruent to $\triangle P'C'A'$ (§ 488), and, similarly, ⚠ PAB, $P'A'B'$ and ⚠ PBC, $P'B'C'$ are respectively congruent.

Now $\triangle ABC = \triangle PCA + \triangle PAB + \triangle PBC$,

and $\triangle A'B'C' = \triangle P'C'A' + \triangle P'A'B' + \triangle P'B'C'$. Ax. 10

$\therefore \triangle ABC$ is equivalent to $\triangle A'B'C'$. Ax. 5

If the pole P lies outside $\triangle ABC$, then P' lies outside $\triangle A'B'C'$, and each triangle is equivalent to the sum of two symmetric isosceles triangles diminished by a third. Hence the result is the same as before.

490. Congruent and Symmetric Triangles. In Book I we studied the congruence of triangles because upon these relations the whole structure of plane geometry is built. There are corresponding propositions relating to spherical triangles and to trihedral angles, but we do not need them for the work in the measurement of the sphere. They will therefore be omitted at this point, but will be considered in §§ 535–538, where they may be studied if the opportunity permits.

Exercises. Review

1. If two great-circle arcs intersect, the vertical angles are equal.

2. Every point lying on a great circle which bisects a given arc of another great circle at right angles is equidistant (§ 456) from the ends of the given arc.

3. From the center of a sphere three radii, each perpendicular to the other two, are drawn. Find the number of degrees in the sides and angles of the spherical triangle determined by the ends of the radii.

4. Is it possible to have a spherical triangle with angles of 75°, 80°, and 120°? with angles of 82°, 96°, and 2°? with angles of 110°, 80°, and 5°? with angles of 200°, 150°, and 190°? with angles of 188°, 206°, and 250°? State the reason in each case.

5. The face angles of a polyhedral angle are 90°, 90°, and 90°. State all that you can with respect to the sides and angles of the corresponding spherical triangle.

6. Of what kind of spherical triangle can it be said that the triangle is its own polar triangle?

7. Draw freehand a spherical triangle with angles of 200°, 90°, and 90°.

III. Mensuration

491. Important Measurements. In measuring small tracts of land it is customary to consider the earth as flat, since the results are sufficiently close for practical purposes. When, however, we have to measure large areas, like states or countries, it is necessary to make allowance for the fact that the earth is a sphere.

The most important measurements that we need to consider for such purposes are the lengths of lines on a sphere and the areas of spherical polygons. We also need to know that a great-circle arc is the shortest line on a sphere from one point to another, and to know how to find areas.

For the study of mechanical and astronomical problems we need to be able to make a few other measurements, including that of the volume of a sphere.

492. Area of a Sphere. Two formulas relating to the sphere are often learned in arithmetic or in algebra. One is the formula for the area; namely,

$$A = 4\pi r^2.$$

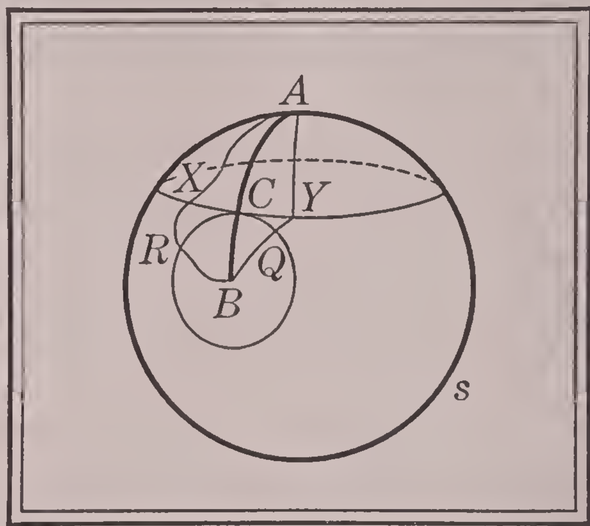
Since πr^2 is the area of a great circle on a sphere of radius r, this formula states the remarkable fact that the area of the sphere is four times the area of a great circle; that is, that the area of a certain curve surface is exactly that of a certain plane surface. We shall later prove that this formula is correct.

493. Volume of a Sphere. The second formula that may already be familiar to the student relates to the volume of a sphere; namely, $V = \frac{4}{3}\pi r^3.$

It is impossible to give a satisfactory explanation of this formula in arithmetic or in algebra, but we shall now be able to give one that is fairly so.

Proposition 16. Shortest Path between Points

494. Theorem. *The shortest path on a sphere between two points is the arc, less than a semicircle, of the great circle joining the two points.*



Given AB, **the arc, less than a semicircle, of the great** ⊙ **joining the points** A **and** B **on the sphere** s.

Prove that AB is the shortest path on the sphere between A and B.

Proof. Let C be any point on arc AB, and let XCY and RCQ be arcs of the small ⓢ which have A and B respectively as poles.

Then if Y is any point except C on arc XCY, and if AY and BY are arcs of great ⓢ, we have

$$AY = AC. \qquad \S\ 461$$

Now $$AY + BY > AC + BC. \qquad \S\ 479$$

Taking away AY from the left member of the inequality and AC, its equal, from the right member, we have

$$BY > BC. \qquad \text{Ax. 7}$$

Now $$BC = BQ. \qquad \S\ 461$$

$$\therefore BY > BQ. \qquad \text{Ax. 5}$$

Hence Y lies outside the ⊙ whose pole is B. § 134, 6

Then, since Y is any point except C on arc XCY, the arcs XCY and RCQ have only the point C in common.

Now let $AXRB$ be any line, which does not pass through C, from A to B on the sphere.

This line cuts the arcs XCY and RCQ in the separate points X and R; and if we revolve the line AX about A as a fixed point until X coincides with C, we shall have a line from A to C equal to the line AX.

In like manner, we can have a line from B to C equal to the line BR.

Hence we can have a line from A to B through C that is equal to the sum of the lines AX and BR.

But such a line is less than the line $AXRB$ by the line XR.

Hence no line which does not pass through C can be the shortest line from A to B; that is, the shortest path from A to B is through C.

But C is any point on the arc AB.

Hence the shortest line from A to B passes through every point of the arc AB, and consequently coincides with the arc AB.

That is, the shortest path from A to B is the arc AB of the great ⊙ through A and B.

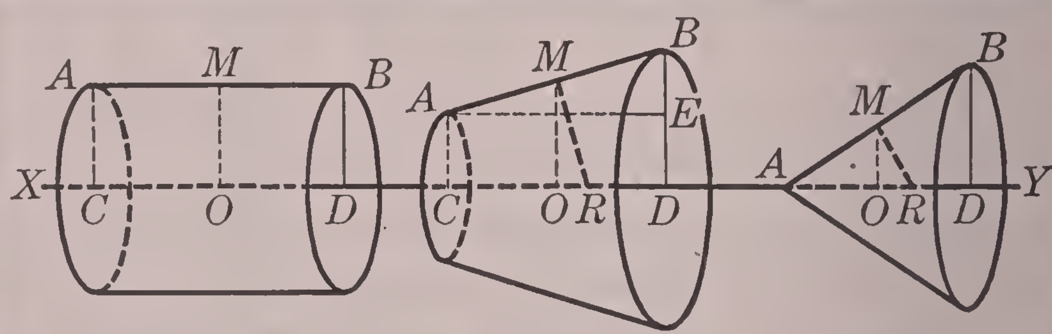
This fact is of great importance in navigation.

495. Geodetic Line. Just as we use straight lines in measuring distances on a plane, we use great-circle arcs in measuring distances on a sphere. Since these arcs are used in geodesy, the science of measuring the earth's surface, they are called *geodetic lines*.

If we examine the map on a globe, we see that the geodetic line from New York to Plymouth, England, goes much farther north than we should think if we looked only at a flat map.

Proposition 17. Surface Generated by a Line

496. Theorem. *If a straight-line segment revolves about an axis in its plane, the area of the surface generated is the product of the projection of the line segment on the axis and the circumference of the circle whose radius is the perpendicular to the line segment at its midpoint included between this point and the axis.*



Case 1 Case 2 Case 3

Given AB, a segment revolving about an axis XY in the same plane; CD, the projection of AB on XY; M, the midpoint of AB; MR, the $\perp$ to AB at M; and S, the area generated by AB.

Prove that $S = CD \cdot 2\pi MR.$

Proof. Let MO be $\perp$ to XY.

1. If AB is $\parallel$ to XY, $CD = AB$, MR coincides with MO, and a rt. cylinder is generated. Hence $S = AB \cdot 2\pi MO = CD \cdot 2\pi MR$ (§ 424).

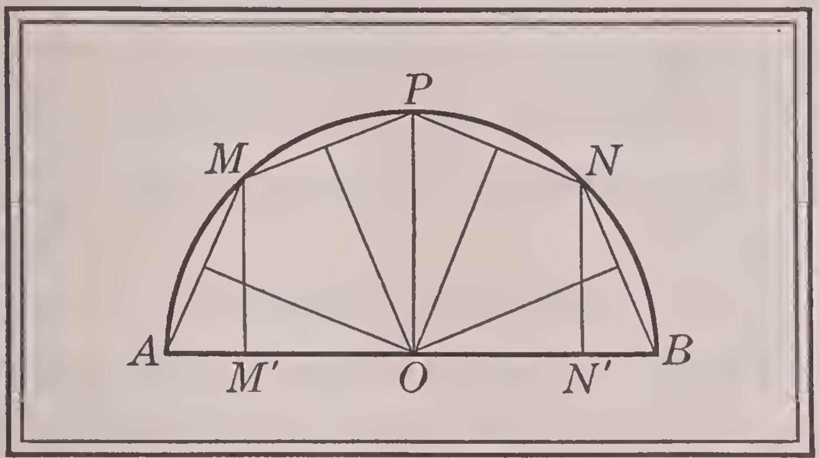
2. If AB is not $\parallel$ to XY and does not cut XY, let AE be $\parallel$ to XY. In the similar $\triangle$ MOR, AEB (§ 210), $MO : AE = MR : AB$ (§ 205); whence $AB \cdot MO = AE \cdot MR = CD \cdot MR$ (§ 198, 1). Then, since AB generates a frustum of a cone of revolution, $S = AB \cdot 2\pi MO = CD \cdot 2\pi MR$ (§ 442).

3. If A lies on XY, $CD = AD$. In the similar $\triangle$ ADB, MOR (§ 210), $AB : MR = AD : MO$ (§ 205); whence $AB \cdot MO = AD \cdot MR = CD \cdot MR$ (§ 198, 1). Now $MO = \frac{1}{2} BD$ (§ 87). Then, since AB generates a cone of revolution, $S = \frac{1}{2} AB \cdot 2\pi BD = AB \cdot 2\pi MO = CD \cdot 2\pi MR$ (§ 438).

The student should give the proof of each case in complete form.

Proposition 18. Area of a Sphere

497. Theorem. *The area of a sphere of radius r is $4\pi r^2$.*



Given a sphere of radius r and area S.

Prove that $S = 4\pi r^2$.

Proof. Let the sphere be generated by the revolution of the semicircle APB about AB, let $AMPNB$ be a regular inscribed semipolygon of $2n$ sides, and let $\perp$s from the center O be constructed to the equal chords AM, MP, $\cdots$.

Then these $\perp$s bisect the chords (§ 141) and are equal (§ 150).

Let r' be the length of each of these equal $\perp$s, and let MM' and NN' be $\perp$ to AB.

Since arc $AP =$ arc BP (§ 140), PO also is $\perp$ to AB.

Then the area generated by AM is $AM' \cdot 2\pi r'$,

the area generated by MP is $M'O \cdot 2\pi r'$, $\cdots$. § 496

Letting S' represent the area of the surface generated by $AMPNB$, we have $S' = AB \cdot 2\pi r'$. Ax. 1

Now if the number of sides is indefinitely increased,

$$r' \to r,$$ § 303, 3

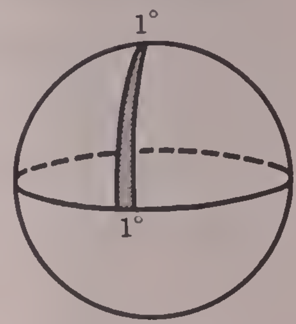
and hence $$AB \cdot 2\pi r' \to AB \cdot 2\pi r.$$ § 301, 1

But, always, $$S' = AB \cdot 2\pi r' = 4\pi r r',$$ § 496

and obviously $$S' \to S.$$

$$\therefore S = 4\pi r^2.$$ § 301, 2

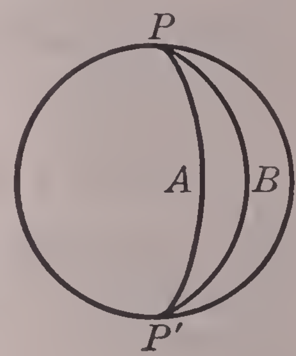
498. Spherical Degree. Just as we take a unit of length, commonly 1°, in measuring an arc, so we take a unit of area in measuring a spherical figure. This unit, called a *spherical degree,* is a spherical triangle of which two sides are quadrants and the third side is an arc of 1°. Since the triangle is evidently $\frac{1}{360}$ of a hemisphere, we see that



A spherical degree is $\frac{1}{720}$ *of a sphere.*

The angles of the unit triangle are evidently angles of 90°, 90°, and 1°.

499. Lune. A portion of a sphere bounded by the halves of two great circles is called a *lune.*



A lune is therefore part of a spherical surface. It may, by extending the definition of polygon, be considered as a polygon of two sides.

Since the two angles formed by the sides of a lune are equal, either angle may be taken as *the angle* of the lune. In this figure, $\angle APB$ may be taken as the angle of the lune $PAP'B$.

500. Area of a Lune. Since the area of a lune whose angle is 1° is twice a spherical degree, and the area of a lune increases at the same rate as the angle, we see that

The number of spherical degrees in the area of a lune is twice the number of degrees in the angle of the lune.

501. Spherical Excess. The number of degrees by which the sum of the angles of a spherical polygon of n sides exceeds $(n-2)180$ is the *spherical excess* of the polygon.

In a spherical triangle, we have $n-2=3-2=1$. Hence the spherical excess of a triangle is the number of degrees by which the sum of its angles exceeds 180.

For example, if the angles of a spherical triangle are 80°, 90°, and 100°, the spherical excess of the triangle is 90. If the angles of a spherical polygon are 150°, 155°, 90°, 55°, 100°, the spherical excess is $150+155+90+55+100-(5-2)180$, or $550-540$, or 10.

Exercises. Areas

1. *The area of a sphere is the product of the diameter and the circumference of a great circle.*

2. *The areas of two spheres are to each other as the squares of their radii, or as the squares of their diameters.*

3. The area of a sphere is 72 sq. in. Find the area of a lune whose angle is 10°.

It is evidently $\frac{20}{720}$ of the area of the sphere.

4. The area of a sphere is 1440 sq. in. Find the number of square inches in a spherical degree.

5. If the number of square inches in a spherical degree is 15, what is the area of the sphere?

6. The area of a lune is 10 sq. cm. and the angle of the lune is 36°. Find the area of the sphere.

7. The area of a lune is two spherical degrees. Find the angle of the lune.

8. The angle of a lune is 15° 30′. Find the area of the lune in spherical degrees.

9. A lune with an angle of 20° is on a sphere which has a radius of 7 in. Find the number of square inches in the area of the lune.

In this and the following exercises, take $\pi = 3\frac{1}{7}$.

Find the areas of spheres with radii as follows:

10. 4.9 in. **11.** 3.5 in. **12.** 2 ft. 4 in. **13.** 6 ft. 5 in.

Find the areas of spheres with diameters as follows:

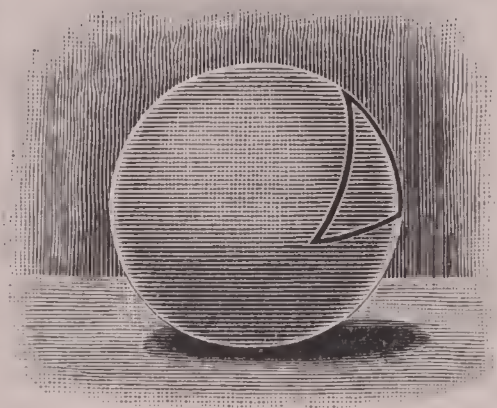
14. 42 in. **15.** 5.6 in. **16.** 6.3 in. **17.** 4 ft. 8 in.

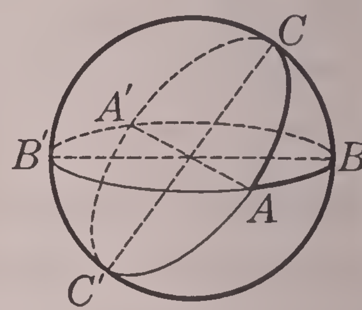
Find the radii of spheres with areas as follows:

18. 616 sq. in. **19.** 2464 sq. in. **20.** 15,400 sq. mm.

Proposition 19. Area of a Triangle

502. Theorem. *The area of a triangle of spherical excess E on a sphere of radius r is* $\frac{1}{180}E\pi r^2$.





Given ABC, a $\triangle$ on a sphere of radius r; E, the spherical excess of the $\triangle$; and S, the area of the $\triangle$.

Prove that $S = \frac{1}{180}E\pi r^2$.

Proof. Let the sides of the $\triangle ABC$ be produced to form great ⓢ, and let AA', BB', CC' be diameters.

Then $\triangle\!\!\!s$ $AB'C'$ and $A'BC$ are symmetric. § 486

$\therefore \triangle AB'C'$ is equivalent to $\triangle A'BC$. § 489

Hence we have

$S + \triangle AB'C' =$ lune of $\angle A = 2A$ spherical degrees,

$S + \triangle AB'C =$ lune of $\angle B = 2B$ spherical degrees,

$S + \triangle AC'B =$ lune of $\angle C = 2C$ spherical degrees. § 500

$\therefore 2S + \frac{1}{2}$ sphere $= 2(A+B+C)$ spherical degrees, Ax. 1

or $2S + 360$ spherical degrees $= 2(A+B+C)$; Ax. 5

whence $S = A+B+C-180$. Ax. 4

Hence the area of $\triangle ABC$ in spherical degrees is E. § 501

But each spherical degree is $\frac{1}{720}$ of $4\pi r^2$. § 498

Hence $S = \frac{1}{720}E \cdot 4\pi r^2 = \frac{1}{180}E\pi r^2$.

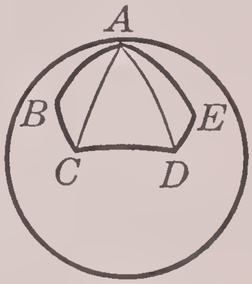
503. Corollary. *A spherical triangle is equivalent to a lune whose angle is half the spherical excess of the triangle.*

In the proof of § 502 we showed that $S = A + B + C - 180$, and the number of spherical degrees in the area of a lune is twice the number of degrees in the angle of the lune (§ 500).

504. Corollary. *In a polygon of area S and spherical excess E on a sphere of radius r,*

$$S = \tfrac{1}{180} E\pi r^2.$$

In the spherical polygon here shown, by drawing all the diagonals from any vertex, we have a △ with each of the sides as base except the two which meet at the vertex; that is, there are $(n-2)$ ▵.



Since (§ 502) the area of each △ is $\frac{1}{180}\pi r^2$ times the excess of the number of degrees in each △ over 180, the sum of the areas of the ▵ is $\frac{1}{180}\pi r^2$ times the excess of the sum of all the ∠s over $(n-2)180$.

But (§ 501) the spherical excess of the polygon is the sum of the ∠s less $(n-2)180$, and hence

$$S = \tfrac{1}{180} E\pi r^2.$$

505. Corollary. *A spherical polygon is equivalent to a lune whose angle is half the spherical excess of the polygon.*

This follows from §§ 503 and 504.

506. Significance of the Formula of § 504. It should be noticed that E in the formula of § 504 depends upon the number of sides in the polygon, since it is always the number of degrees in the sum of the angles minus $(n-2)180$.

In the case of a lune there are two sides, and hence $(n-2)180 = 0$. Thus the spherical excess is simply the sum of the two equal angles, or twice the angle of the lune.

Hence in a lune of $\angle L$, we have $S = \frac{2}{180} L\pi r^2 = \frac{1}{90} L\pi r^2$.

If $L = 360°$, that is, if the lune covers the entire sphere, $S = \frac{360}{90}\pi r^2 = 4\pi r^2$, as found in § 497.

The advantage in using §§ 502 and 504 instead of §§ 503 and 505 is apparent, since in the former we have a single formula for the area without reference to the lune.

Exercises. Areas of Spherical Polygons

1. Find the area of a triangle with angles of 110°, 100°, and 95° on a sphere with a radius of 6 in.

We have $E = 110 + 100 + 95 - 180 = 125,$

and $S = \frac{1}{180} E\pi r^2 = \frac{125}{180} \cdot \frac{22}{7} \cdot 6 \cdot 6 = 78\frac{4}{7}.$

Hence, the area of the triangle is 78.57 sq. in.

In all cases take $3\frac{1}{7}$, or $\frac{22}{7}$, for π except when otherwise directed.

2. Find the area of a polygon with angles of 100°, 110°, 120°, and 170°, on a sphere with a radius of 63 in.

In this case $E = 100 + 110 + 120 + 170 - 2 \times 180 = 140,$

and $S = \frac{1}{180} E\pi r^2 = \frac{140}{180} \cdot \frac{22}{7} \cdot 63 \cdot 63 = 9702.$

Hence the area of the polygon is 9702 sq. in.

3. Taking the radius of the earth as 4000 mi., find the area of the earth.

4. A triangle on the earth's surface has one vertex at the north pole and the others on the equator, one at 30° W. and the other at 20° E. Considering the earth as a sphere with a radius of 4000 mi., find the area of the triangle.

5. In making a survey of part of a continent a triangle was laid out with angles of 60°, 100°, and 20° 6′. Find the area of the triangle to the nearest 1000 sq. mi.

Find the areas of triangles with angles as follows on spheres of the given radii:

6. 130°, 100°, 95°; $r = 7$ in. 7. 110°, 100°, 40°; $r = 35$ in.

Taking $\pi = 3.1416$, find the areas of spherical polygons with angles as follows on spheres of the given radii:

8. 136°, 154°, 70°, 90°; $r = 16$ in.

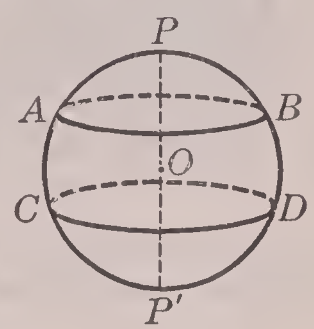
9. 145°, 150°, 90°, 100°, 130°; $r = 30$ in.

10. 175°, 168°, 88°, 142°, 100°, 90°; $r = 40$ in.

507. Zone. A portion of a sphere included between two parallel planes is called a *zone.*

If a great circle revolves about its diameter as an axis, any arc of the circle generates a zone.

It must be remembered that *sphere* means the same as spherical surface, so that a zone, like a lune, is a surface. Thus on the earth we have the torrid zone, which is that part of the earth's surface included between the planes of the tropics of Cancer and Capricorn.



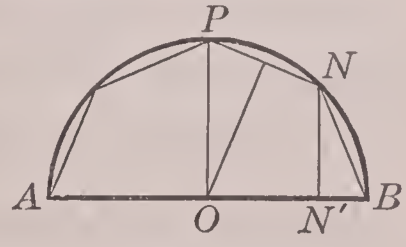
The circles made by the planes are called the *bases* of the zone, and the distance between the planes is called the *altitude* of the zone.

If one of the planes is tangent to the sphere and the other plane cuts the sphere, the zone is called a *zone of one base.* If both planes are tangent to the sphere, the zone is a complete sphere.

508. Corollary. *In a zone of area S and altitude h on a sphere of radius r,*

$$S = 2\pi rh.$$

If we apply the reasoning of § 497 to the zone generated by the revolution of arc PN, we have



$$S = ON' \times 2\pi r.$$

But ON' is the altitude h.

Hence $S = 2\pi rh.$

For example, if the radius is 14 in. and the altitude of the zone is 5 in.,

$$S = 2\pi rh = 2 \cdot \tfrac{22}{7} \cdot 14 \cdot 5 \text{ sq. in.} = 440 \text{ sq. in.}$$

509. Areas on a Sphere. The most important formulas for the areas on a sphere may be summarized as follows:

Sphere,	$S = 4\pi r^2.$
Triangle,	$S = \frac{1}{180} E\pi r^2.$
Polygon,	$S = \frac{1}{180} E\pi r^2.$
Lune of $\angle L$,	$S = \frac{1}{90} L\pi r^2.$
Zone,	$S = 2\pi rh.$

Exercises. Areas

1. The area of a zone of one base is equivalent to the area of the circle whose radius is the chord of the generating arc.

2. Two zones on the same sphere are to each other as their altitudes.

3. If the earth's radius is 4000 mi., and the altitude of the torrid zone is 3200 mi., what is the area of this zone?

On a sphere whose radius is 14 in., find the areas of lunes whose angles are as follows:

4. $60°$. 5. $90°$. 6. $42° 30'$. 7. $32° 20'$.

8. Find the area of a lune whose angle is $40°$ on a sphere with a radius of 28 in.

9. Find the area of a lune whose angle is $85°$ on a sphere with a diameter of 21 in.

Taking $\pi = \frac{22}{7}$, find the areas of spherical polygons with angles as follows on spheres of the given diameters:

10. $140°, 90°, 60°, 120°$; $d = 20$ in.

11. $170°, 160°, 95°, 30°, 100°$; $d = 32$ in.

Taking $\pi = 3.1416$, find the areas of spherical polygons with angles as follows on spheres of the given circumferences:

12. $140°, 120°, 100°, 130°, 100°$; $C = 6.2832$ in.

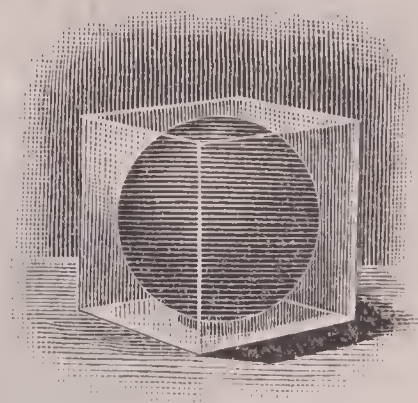
13. $120°, 140°, 130°, 80°, 160°, 135°$; $C = 18.8496$ in.

Taking $\pi = 3.14$, find the areas of spherical polygons with angles as follows on spheres of the given areas:

14. $70°, 160°, 90°, 120°$; $S = 600$ sq. in.

15. $65° 30', 140° 50', 95° 34', 138° 50'$; $S = 600$ sq. in.

510. Sphere Inscribed in a Cube. It is readily seen that the lines joining the centers of the opposite faces of a cube are each equal to an edge of the cube, that they all intersect in one point, and that this point is equidistant from all six faces. It therefore seems obvious that a sphere can be inscribed in a cube.



The formal proof of this fact can easily be given, but the student will probably see that its truth is so evident as to render a proof unnecessary.

Similarly, we speak of a sphere as inscribed in a cylinder, but no formal definition is necessary.

511. Volume of a Sphere. The volume inclosed by a sphere is called the *volume of the sphere.*

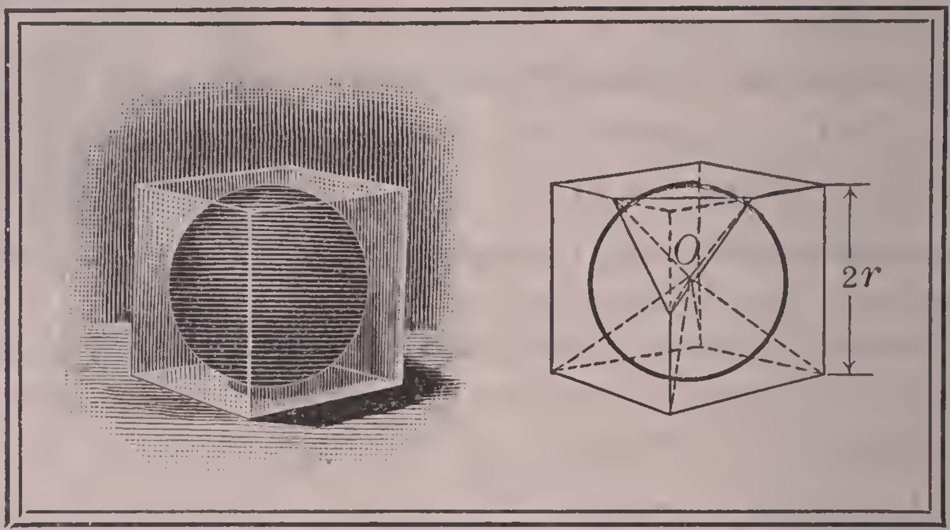
512. Sphere as a Limit. Suppose that a plane is tangent to a sphere at the point on the sphere determined by the line joining any vertex of the circumscribed cube to the center of the sphere. It is then apparent that since some of the cube has been cut off, the volume of the circumscribed polyhedron thus formed is nearer than that of the cube to the volume of the sphere.

This process may be continued for all the vertices of the cube, repeated for all the vertices of the circumscribed polyhedron thus formed, again repeated, and so on indefinitely. The volume of the circumscribed polyhedron thus approaches the volume of the sphere. That is, if V is the volume of the sphere, and V' is the volume of the circumscribed polyhedron of n faces, as described above, then $V' \rightarrow V$ as $n \rightarrow \infty$.

Similarly, if A is the area of the polyhedron of n faces, as described above, and S is the area of the sphere, then $A \rightarrow S$ as $n \rightarrow \infty$.

Proposition 20. Volume of a Sphere

513. Theorem. *The volume of a sphere of radius r is* $\frac{4}{3}\pi r^3$.



Given a sphere of radius r **and volume** V.

Prove that $V = \frac{4}{3}\pi r^3$.

Proof. Let the sphere be inscribed in a cube (§ 510) whose edge is $2r$. Then the lines joining the center to the vertices of the cube are the edges of six pyramids of altitude r whose bases are the faces of the cube.

One such pyramid is shown in the figure.

The volume of each pyramid is a face of the cube multiplied by $\frac{1}{3}r$ (§ 407), and the volume of the six pyramids, or of the whole cube, is the area of the surface of the cube multiplied by $\frac{1}{3}r$ (Ax. 1).

Now let planes be tangent to the sphere at the points where the edges of the pyramids cut the sphere.

One such plane is shown in the figure.

We then have a circumscribed solid whose volume V', although greater than the volume V of the sphere, is nearer V than is the volume of the circumscribed cube.

Also the area A of the circumscribed solid is nearer the area S of the sphere than is the area of the cube.

Proceeding as before, let the center of the sphere be connected with the vertices of the new polyhedron. These connecting lines are the edges of pyramids of altitude r whose bases are the faces of the polyhedron.

Then the sum of the volumes of these pyramids is the area of the circumscribed polyhedron multiplied by $\frac{1}{3}r$; that is, as before,

$$V' = A \cdot \tfrac{1}{3}r.$$

If we continue to increase indefinitely the number of faces of the circumscribed polyhedron, we see that

$$V' \to V \text{ and } A \to S. \qquad \S\ 512$$

$$\therefore V = S \cdot \tfrac{1}{3}r. \qquad \S\ 301, 2$$

But $$S = 4\pi r^2. \qquad \S\ 497$$

Hence $$V = 4\pi r^2 \cdot \tfrac{1}{3}r, \qquad \text{Ax. } 5$$

or $$V = \tfrac{4}{3}\pi r^3.$$

Exercises. Volume of a Sphere

1. *The volume of a sphere is the product of the area of its surface and one third of its radius.*

2. *The volume of a sphere of diameter d is $\frac{1}{6}\pi d^3$.*

3. *The volumes of two spheres are to each other as the cubes of their radii or as the cubes of their diameters.*

4. If the radius of the earth is 4000 mi., and if the atmosphere extends 50 mi. above the surface of the earth, what is the volume of the atmosphere?

5. If a solid iron ball 4 in. in diameter weighs 9 lb., what is the weight of a spherical iron shell which is 2 in. thick and has an external diameter of 20 in.?

Exercises. Area and Volume of a Sphere

1. How many square feet of lead are needed to cover a hemispherical dome which is 66 ft. in circumference?

The student should always use $\frac{22}{7}$ as the value of π, unless otherwise directed.

2. A hollow ball 8 ft. in diameter surmounts the dome of a church. Making no allowance for the support, how much will it cost to gild the surface of the ball at 10¢ per square inch?

3. Taking the circumference of the earth as 25,000 mi., find the area of the surface to the nearest million square miles.

Find the volumes of spheres whose radii are:

4. 4.2 in.	6. 7 in.	8. $3\frac{1}{2}$ in.	10. 5.6 ft.
5. 6.3 in.	7. 14 in.	9. $10\frac{1}{2}$ in.	11. 4 ft. 1 in.

12. The diameter of a spherical basket ball is 10 in. Allowing 50 sq. in. for waste, how many square inches of leather are needed to cover it?

13. The distance across the top of a bowl in the shape of a portion of a sphere is 14 in. and the greatest depth is 7 in. Allowing $7\frac{1}{2}$ gal. to the cubic foot, how many pints of water does the bowl hold?

14. If the numbers expressing the area and the volume of a sphere are the same and the units of measure are the square inch and the cubic inch respectively, what is the diameter of the sphere?

15. The weights of two spheres are in the ratio 2 : 5 and the weights of 1 cu. in. of each of the substances of which they are composed are in the ratio 7 : 2. Find the ratio of the diameters.

IV. General Review

Exercises. Polyhedrons

1. The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face meet in a point P which divides each line so that the ratio of the shorter segment to the whole line is $1:4$.

The point P is called the *center of gravity* of the tetrahedron.

2. In Ex. 1, the lines which join the midpoints of the opposite edges of the tetrahedron are each bisected by the center of gravity.

3. The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the areas of the faces including the dihedral angle.

4. Show how to cut a cube by a plane so that the section shall be a regular hexagon.

5. If the face angles at the vertex of a triangular pyramid are all right angles, if the areas of the lateral faces are A, B, and C respectively, and if the area of the base is D, then $A^2 + B^2 + C^2 = D^2$.

6. Show how to cut a tetrahedron by a plane so that the section shall be a parallelogram.

7. The altitude of a regular tetrahedron is equal to the sum of the four perpendiculars drawn from any point within the tetrahedron to the four faces.

8. Draw figures to show how to cut a cube so as to have a section of three sides; of four sides; of five sides; of as many more sides as possible.

9. The section of a regular octahedron made by a plane parallel to and midway between any pair of opposite faces is a regular hexagon.

Exercises. Formulas Relating to the Sphere

Deduce formulas for the following:

1. The area S of a zone of height $\frac{1}{2}r$ on a sphere of radius r.

In the exercises upon this page, and in similar cases, the unit of area is the square of the unit of length, and the unit of volume is the cube of the unit of length. That is, if we think of r as feet, then S will be square feet and V will be cubic feet.

2. The volume V of a sphere in terms of C, the circumference; in terms of S, the area of the sphere.

3. The radius r of a sphere in terms of V, the volume.

4. The area S of the zone, on a sphere of radius r, which is illuminated by an electric light at the height h above the surface.

5. The diameter d of a sphere in terms of S, the area of the sphere.

6. The altitude h of a zone of area S on a sphere of volume V.

7. The volume of the metal in a spherical iron shell of which the internal radius is r, and the thickness of the metal is t.

8. The weight of a spherical metal shell in which the inside radius is r, the thickness of the metal is t, and the weight of a cubic unit of the metal is w.

9. The diameter of the sphere upon which a zone of area S has an altitude h.

10. The area S' of a zone of altitude h upon a sphere of area S.

11. If R and r are the radii of the spheres circumscribed about and inscribed in a regular tetrahedron of edge $2a$, then $R = 3r = \frac{1}{2}a\sqrt{6}$.

Exercises. Cylinders

1. The volume of a right circular cylinder is the product of the lateral area and half the radius.

2. The volume of a right circular cylinder is the product of the area of the rectangle, which generates the cylinder by revolving about one of its sides, multiplied by the circumference of the circle generated by the point of intersection of the diagonals of the rectangle.

3. If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is the product of the total area and one third the radius.

4. By what number must the dimensions of a cylinder of revolution be multiplied to obtain a similar cylinder of revolution (Ex. 9, page 358) whose entire surface area is twice the first? n times the first?

5. What is the multiplier in Ex. 4 if the volume of the second cylinder is to be twice that of the first? is to be n times that of the first?

6. Compare the volumes of the solids generated by the successive revolution of a rectangle of base b and altitude h about two adjacent sides.

7. Find the radius of a right circular cylinder in which the number of cubic units of volume is equal to the number of square units of the area of the entire surface.

8. Find to the nearest square centimeter the area of the total surface of a cylinder whose altitude is 7.6 cm. and the diameter of whose base is 4.2 cm.

9. Find to the nearest cubic centimeter the volume of a cylinder whose altitude is 6 cm. and which fits exactly into a right prism whose base is a square that is 1.8 cm. on a side.

Exercises. Cones and Pyramids

1. The altitude of a cone of revolution is 24 in. and the radius of the base is 10 in. Find the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone. Find also the number of degrees in the angle of this sector.

The second result may be expressed either in degrees with a decimal fraction, or in degrees, minutes, and seconds.

2. The volume of a regular pyramid is the product of one third its lateral area and the perpendicular distance from the center of the base to any lateral face.

3. Find the volume of a pyramid whose base is 30 sq. in. and one of whose lateral edges, which makes an angle of 45° with the plane of the base, is 5 in. long.

4. A pyramid is cut by a plane parallel to the base and bisecting the altitude. What is the ratio of the volume of the pyramid cut off to that of the entire pyramid?

5. Consider Ex. 4 for the case of a cone.

6. The height of a regular hexagonal pyramid is 6 in. and one edge of the base is 1 in. Find the volume and also find the volume of the pyramid cut off by a plane 4 in. from the base and parallel to it.

7. One of the lateral edges of a regular hexagonal pyramid is 6 in., and the radius of the circle circumscribed about the base is 1 in. Find the altitude, the volume, the lateral area, and the area of the total surface.

8. If a right triangle of hypotenuse h and sides a and b revolves about h as an axis, what is the volume of the solid thus generated?

9. If the radius of the base of a right circular cone is r and the angle at the vertex is 120°, what is the volume?

Exercises. Spheres, Cylinders, and Cones

1. *The area of a sphere is two thirds the area of the total surface of the circumscribed cylinder.*

2. *The volume of a sphere is two thirds the volume of the circumscribed cylinder.*

Exs. 1 and 2 were discovered by Archimedes, one of the greatest mathematicians of Greece, about 250 B.C.

3. A sphere of radius 6 in. and a right circular cone of the same radius stand on a plane. If the height of the cone is equal to the diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections.

4. In a cylindric jar 8 in. in diameter, water is standing to a depth of 6 in. If an iron ball 4 in. in diameter is dropped into the jar, what is then the depth of the water?

5. On the base of a right circular cone a hemisphere is constructed outside the cone. Given that the area of the hemisphere is equal to that of the cone, and that the radius is r, find the slant height of the cone, the inclination of the slant height to the base, and the volume of the entire solid.

6. Find the area of a sphere inscribed in a cylinder of volume $\frac{1}{4}\pi d^3$, where d is the diameter of the sphere.

7. Find the volume of a sphere inscribed in a cylinder of area $\pi d(2 + d)$.

8. A sphere of radius r is inscribed in a cylinder. Find the volume of the cylinder not occupied by the sphere.

9. A cylinder is circumscribed about a hemisphere, and a cone is inscribed in the cylinder so as to have its vertex on the upper base and to have its base in common with the lower base of the cylinder. Prove that the volumes of the cone, the hemisphere, and the cylinder are proportional to 1, 2, 3.

Exercises. Portions of the Surface of a Sphere

1. If the altitude of the north temperate zone is 1800 mi., what is the area of the zone?

In Exs. 1–7 take 4000 mi. as the radius of the earth.

2. How far in one direction can a man see from an ocean steamer if his eye is 50 ft. above the water?

3. How many square miles of the earth's surface can be seen from an airplane at an elevation of 10,000 ft.?

4. At what height above the earth must a man be in order to see one eighth of the surface?

5. What fractional part of the earth's surface could be seen if an observer were at the height of the earth's radius above the sea?

6. If the ocean area is three fourths of the earth's surface and the average depth of the water is 2 mi., what is the volume of water in the oceans?

7. In a lighthouse on an isolated rock the light is placed 168 ft. above the surface of the sea and can be seen from any point within a circle reaching to the horizon. Find the number of square miles of the earth's surface inclosed by this circle.

Find the areas of triangles with angles as follows on spheres of the given radii:

8. $120°, 110°, 90°$; $r = 7$ in.

9. $105°, 105°, 80°$; $r = 7$ in.

10. $120°, 95°, 90°$; $r = 14$ in.

11. $90°, 90°, 90°$; $r = 91$ in.

Find the areas of polygons with angles as follows on spheres of the given radii:

12. $140°, 150°, 80°, 80°$; $r = 14$ in.

13. $120°, 100°, 185°, 80°, 100°$; $r = 21$ in.

Exercises. Spherical Polygons and Polyhedral Angles

1. The planes which are perpendicular to the three faces of a trihedral angle and bisect the face angles meet in a straight line. After proving the proposition state the corresponding one relating to a spherical triangle.

It is unnecessary to prove the latter, since it follows by § 476.

2. The planes passing through the edges of a trihedral angle and perpendicular to the opposite faces meet in a straight line. Consider also, as in Ex. 1, the corresponding proposition relating to a spherical triangle.

3. Find the area of a spherical triangle, given that the perimeter of its polar triangle is 297° and that the radius of the sphere is 10 in.

4. Find the spherical excess of a spherical triangle whose angles are 87°, 92°, and 106°; of a spherical quadrilateral whose angles are 145°, 92°, 75°, and 125°.

5. If the two polygons of Ex. 4 are both on a sphere of radius 10 in., what is the area of each?

On a sphere of radius 7 in., find the areas of spherical triangles with angles as follows:

6. 92°, 93°, 94°.	**8.** 100°, 100°, 100°.	**10.** 32°, 48°, 130°.
7. 98°, 102°, 116°.	**9.** 98°, 102°, 120°.	**11.** 68°, 37°, 140°.

On a sphere of radius 14 in., find the areas of spherical polygons with angles as follows:

12. 80°, 90°, 100°, 110°.	**14.** 96°, 72°, 116°, 130°.
13. 72°, 88°, 110°, 120°.	**15.** 100°, 100°, 100°, 100°.

16. Discuss the case of the area of a spherical triangle whose angles are 200°, 280°, and 60° on a sphere whose radius is 10 in.

Exercises. Miscellaneous Exercises

1. If a cube and a sphere have equal volumes, what is the ratio of the radius of the sphere to the edge of the cube?

2. Given that the diagonal of a cube is $4\sqrt{3}$ in., find the radius of the sphere whose area is equal to that of the cube.

3. The radius of the base of a right circular cylinder is r and the altitude of the cylinder is h. Find the radius and the volume of a sphere whose area is equivalent to the lateral surface of the cylinder.

4. If the area of a zone of one base is n times the area of the circle which forms this base, the altitude of the zone is equal to the diameter of the sphere multiplied by $(n-1)/n$. Discuss the special case in which $n=1$.

5. Find to the nearest 0.1 in. how far from the center of a sphere of radius 8 in. a plane should be passed so as to cut from the sphere a circle 154 sq. in. in area.

6. Find the ratio of the volume of a sphere to the volume of the inscribed cube.

7. Consider Ex. 6 for the circumscribed cube.

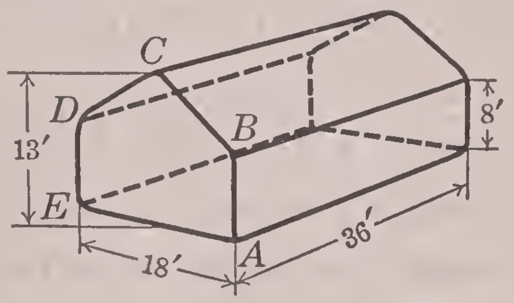
8. Find the ratio of the volume of the cube inscribed in a sphere to that of the cube circumscribed about the sphere.

9. Find the difference between the volumes of two cubes, one inscribed in a sphere of radius 1 in. and the other circumscribed about it.

10. Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 20 ft. high, given that the bases of the frustum are squares 20 ft. and 12 ft. respectively on a side, and the base of the prism is the section of the frustum parallel to the bases and midway between them.

Omit Ex. 10 if § 408 was not taken.

11. In certain parts of the United States, stacks of hay are shaped roughly like a barn, as shown in this figure. The farmers use this rule for finding the approximate number of tons: Take the "overthrow" (the length $ABCDE$); subtract the width; divide by 2, and call the result the height. Multiply this height by the product of the length and width, and call the result the cubic contents. Divide this by 412 to find the number of tons of wild hay; by 450 for timothy; and by 512 for alfalfa or clover. Consider the accuracy of the rule for finding the volume in the case here shown.



12. Using the rule and the dimensions of the figure in Ex. 11, find the approximate number of tons in a stack of wild hay; of timothy; of clover.

13. The square of the diagonal of a cube is how many times the square of an edge?

14. Find the ratio of the sum of the squares of all the edges of a cube to the sum of the squares of all the diagonals of the faces; of all the diagonals of the cube itself.

15. Find the length of the perpendicular from a vertex of a regular octahedron of edge e to the plane determined by the four adjacent vertices.

16. The shortest distance between two opposite edges of a regular tetrahedron is equal to half the diagonal of the square constructed on an edge.

17. The sum of the squares of the edges of any tetrahedron is four times the sum of the squares of all the lines joining the midpoints of opposite sides.

18. Find the volume of the regular tetrahedron of which the sum of the areas of the faces is $4f$.

19. The six planes that pass through the six edges of a tetrahedron and bisect the respective opposite edges meet in a point.

20. A cubic foot of copper is drawn into a wire 2000 ft. long. Find the diameter of the wire.

21. Find the volume of a pyramid with equal lateral edges e and a base which is an equilateral triangle of side s.

22. Consider Ex. 21 for the case in which the base is a regular octagon of side s.

23. The base of a regular pyramid of volume V and height h is a square. Find the length of a lateral edge.

24. In a cube a plane passes through the midpoints of three edges that meet in the same vertex. Given that the volume of the cube is e^3, find the volume of the tetrahedron thus cut off.

25. An iron casting is in the form of a right circular cone upon the base of which is a hemisphere of the same radius outside the cone. If the casting, which is 7 in. in diameter and $9\frac{1}{2}$ in. long, is placed in a cylindric can $8\frac{3}{4}$ in. in diameter and 10 in. high, filled with water, how much water remains in the can?

26. Water is flowing into a tank through a pipe 2.1 in. in diameter at the rate of 3 ft. (linear) per second. Allowing 231 cu. in. to a gallon, how much water will flow into the tank in 1 hr.?

27. If the centers of two intersecting spheres are 5 in. apart, and the radii of the spheres are 3 in. and 4 in. respectively, what is the area of the circle formed by their intersection?

28. A sphere 1 ft. in diameter is cut from a cube of lead 1 ft. on an edge. If the pieces cut off are melted and cast into another sphere, what is the diameter of this sphere?

SUPPLEMENT

I. Incommensurable Cases

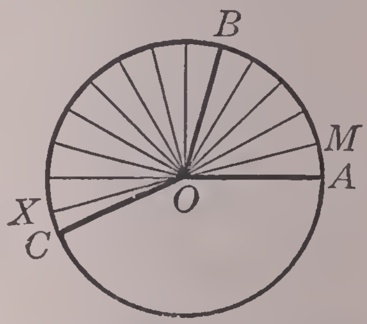
514. Subjects Treated. In the study of geometry there are many topics that might be taken in addition to those found in any textbook. The theorems and problems already given in this text are standard propositions which are looked upon as fundamental, and are usually required as preliminary to more advanced work. These propositions and a reasonable number of originals selected from the exercises, will be all that most classes have time to consider. It occasionally happens, however, that a class is able to do more than this, and then more exercises may be selected from the large number supplied, and a few additional topics may be studied. For this latter purpose the supplementary work is added, but its study should not be undertaken at the expense of the fundamental propositions and exercises. The work on practical mensuration (§§ 543–550), however, may be taken with advantage in place of some of the less important propositions in the text.

The subjects treated in the following pages include the incommensurable cases of certain propositions, additional propositions in the mensuration of solids, a few general theorems relating to similar polyhedrons, and some work on congruent spherical triangles and on practical mensuration. There are also added a few of those recreations of geometry that add a peculiar interest to the subject, and a brief sketch of the history of geometry, which all students are advised to read as a matter of general information.

515. Central Angles. In § 170 it was proved for the commensurable case that central angles have the same ratio as their arcs. We shall now prove the theorem for the incommensurable case.

That is, in the figure here shown in which the $\angle$s AOB and BOC and their arcs AB and BC are incommensurable, we have to prove that

$$\frac{\angle BOC}{\angle AOB}=\frac{\text{arc } BC}{\text{arc } AB}.$$



Divide $\angle AOB$ into any number of equal parts and apply one of these parts, as $\angle AOM$, to $\angle BOC$ as many times as possible. Since the angles are incommensurable, there is a remainder, $\angle XOC$, less than one of the parts.

If we increase the number of parts into which $\angle AOB$ is divided, the size of a part can be decreased indefinitely.

That is, $\quad \angle AOM \rightarrow 0,$

and hence $\quad \angle XOC \rightarrow 0.$

Then $\quad \angle BOX \rightarrow \angle BOC$

and $\quad \text{arc } BX \rightarrow \text{arc } BC.$

Since $\quad \angle AOB$ and arc AB are constants,

then $$\frac{\angle BOX}{\angle AOB} \rightarrow \frac{\angle BOC}{\angle AOB},$$

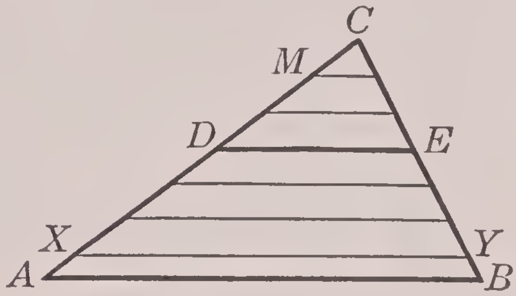
and $$\frac{\text{arc } BX}{\text{arc } AB} \rightarrow \frac{\text{arc } BC}{\text{arc } AB}. \qquad \text{§ 301, 1}$$

But $$\frac{\angle BOX}{\angle AOB}=\frac{\text{arc } BX}{\text{arc } AB}. \qquad \text{§ 170}$$

$$\therefore \frac{\angle BOC}{\angle AOB}=\frac{\text{arc } BC}{\text{arc } AB}. \qquad \text{§ 301, 2}$$

That is, the central angles have the same ratio as their arcs, even though the angles are incommensurable.

516. Sides of a Triangle. In § 201 it was proved for the commensurable case that a line parallel to one side of a triangle divides the other sides proportionally. We shall now prove the theorem for the incommensurable case.



Divide CD into any number of equal parts and apply one of these parts, as CM, to DA as many times as possible. Since CD and DA are incommensurable, there is a remainder, such as XA, which is less than one of the parts.

Construct $XY \parallel$ to AB. § 107

Then $$\frac{DX}{CD} = \frac{EY}{CE}.$$ § 201

If we increase the number of parts into which CD is divided, the length of a part can be decreased indefinitely.

That is, $CM \rightarrow 0$,

and hence $XA \rightarrow 0$,

and $YB \rightarrow 0$.

Then $DX \rightarrow DA$,

and $EY \rightarrow EB$.

Now $$\frac{DX}{CD} \rightarrow \frac{DA}{CD},$$

and $$\frac{EY}{CE} \rightarrow \frac{EB}{CE},$$ § 301, 1

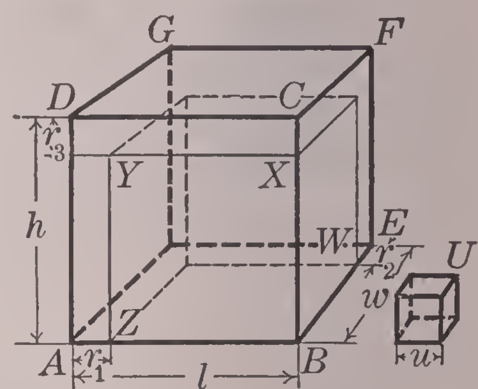
But $$\frac{DX}{CD} = \frac{EY}{CE}.$$ Proved

$$\therefore \frac{DA}{CD} = \frac{EB}{CE}.$$ § 301, 2

That is, the sides are divided proportionally, even though their segments are incommensurable.

517. Volume of a Rectangular Parallelepiped. Referring to the case discussed in § 391, let us suppose that the edges of the rectangular parallelepiped are all incommensurable.

Let U, in the figure below, be the unit of volume and let u be the unit of length. Then if u is applied to AB as many times as possible, there is a remainder r_1 less than u. Similarly, there is a remainder r_2 on the width BE, and a remainder r_3 on the height AD.



Now if we let the unit U decrease indefinitely, r_1, r_2, and r_3 also decrease indefinitely; that is, as $U \rightarrow 0$, then $u \rightarrow 0$, $r_1 \rightarrow 0$, $r_2 \rightarrow 0$, and $r_3 \rightarrow 0$.

Since r_1, r_2, and r_3 all approach zero as a limit, we see that $BZ \rightarrow l$, $BW \rightarrow w$, and $BX \rightarrow h$, as U is taken continually smaller and smaller.

Let P be the volume of the rectangular parallelepiped with the dimensions l, w, h, and let P' be the volume of the one with the dimensions BZ, BW, BX.

Now, as $\quad U \rightarrow 0,$

$$P' \rightarrow P.$$

From previous discussions of limits we shall assume, as seems evidently to be the case, that

$$BZ \cdot BW \cdot BX \rightarrow lwh.$$

But $\quad P' = BZ \cdot BW \cdot BX. \qquad$ § 391

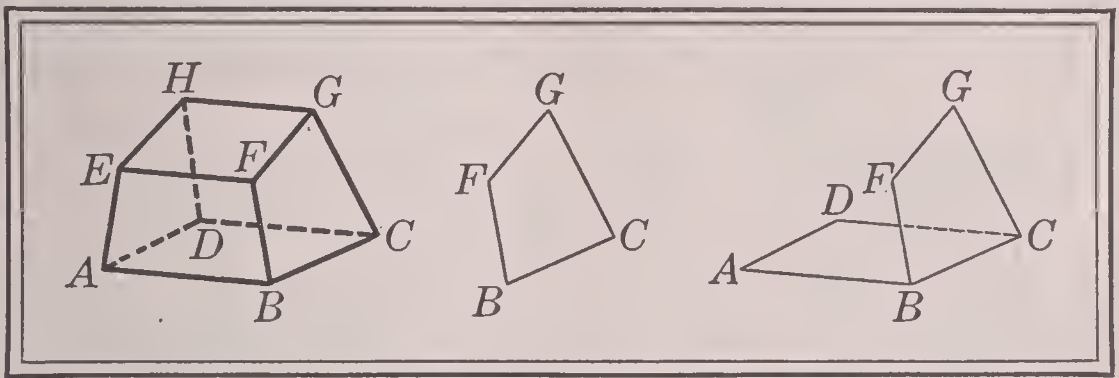
$$\therefore P = lwh. \qquad \text{§ 301, 2}$$

No proof of this case is satisfactory for a textbook of this type. If rigorous, the proof is too difficult for an elementary class; if simple, it lacks scientific accuracy. The fact that the elementary proofs often given are open to serious scientific criticism has led most careful writers to outline merely the general nature of the proof as has been done above. Teachers are advised to require only that the above discussion be read understandingly.

II. Polyhedrons

Proposition 1. The Polyhedron Theorem

518. Theorem. *In a polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.*



Given AG, a polyhedron; e, the number of edges; v, the number of vertices; and f, the number of faces.

Prove that $e + 2 = v + f.$

Proof. For one face, as $BCGF$, $e = v$.

Adding a second face, as $ABCD$, there is formed a surface of two faces which has one edge (BC), and two vertices (B and C), common to the two faces.

Hence for two faces, $e = v + 1.$

Adding a third face $ABFE$, adjoining each of the first two, this face will have two edges (AB, BF) and three vertices (A, B, F) in common with the surface of two faces.

Hence for three faces, $e = v + 2.$

Similarly, for four faces, $e = v + 3$, and so on.

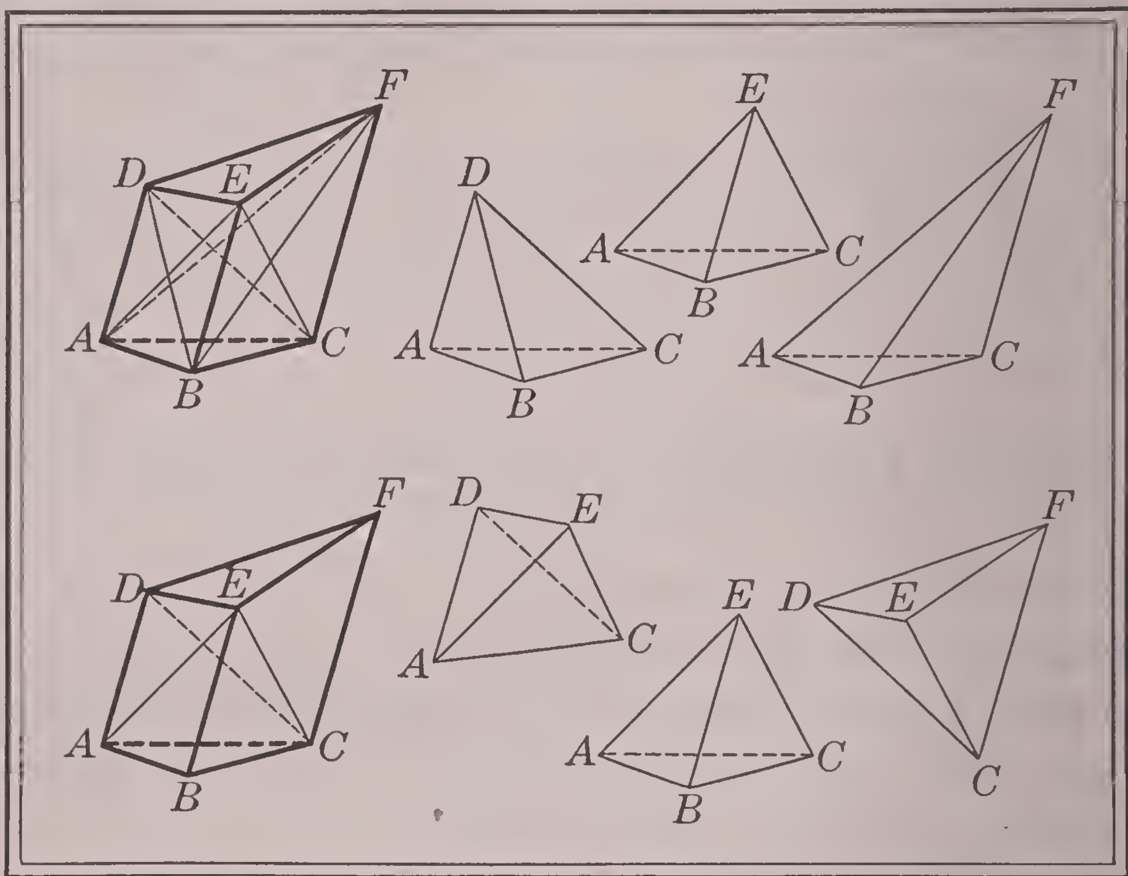
Hence for $(f-1)$ faces, $e = v + (f - 1) - 1.$

Now the addition of the next face, which is the last one, will not increase the number of edges or vertices.

Hence for f faces, $e = v + f - 2$, or $e + 2 = v + f.$

Proposition 2. Truncated Triangular Prism

519. Theorem. *A truncated triangular prism is equivalent to the sum of three pyramids whose common base is the base of the prism and whose vertices are the three vertices of the inclined section.*



Given the truncated triangular prism with base ***ABC*** and inclined section ***DEF***, and divided into the three pyramids ***D-ABC***, ***E-ABC***, and ***F-ABC***.

Prove that ABC-DEF is equivalent to the sum of the three pyramids D-ABC, E-ABC, and F-ABC.

Proof. Dividing the truncated prism *ABC-DEF* into the pyramids *E-ACD*, *E-ABC*, *E-CFD*, as shown in the second group of figures, we shall now show that these pyramids are equivalent to those in the first group.

Now pyramid $E\text{-}ACD =$ pyramid $B\text{-}ACD$, § 407

because they have the common base ACD and equal altitudes, since the vertices E and B lie on EB which is ∥ to the plane ACD.

But the pyramid $B\text{-}ACD$ may be regarded as having the base ABC and the vertex D; that is, as pyramid $D\text{-}ABC$.

$\therefore$ pyramid $E\text{-}ACD =$ pyramid $D\text{-}ABC$.

The pyramids $E\text{-}ABC$ are the same in each division of the prism; that is, they have the base ABC and the vertex E.

Now $\triangle CFD = \triangle CFA$, § 245

because they have the common base CF and equal altitudes, since their vertices lie on AD which is ∥ to CF.

Then pyramid $E\text{-}CFD =$ pyramid $B\text{-}CFA$, § 407

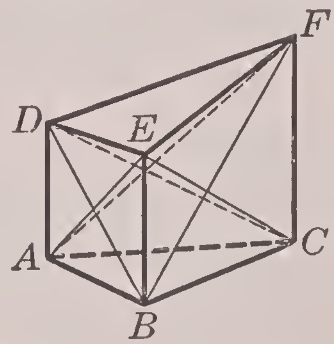
because they have equivalent bases (the △ CFD and CFA) and equal altitudes, since EB is ∥ to the plane ACFD.

But the pyramid $B\text{-}CFA$ may be regarded as having the base ABC and the vertex F; that is, as pyramid $F\text{-}ABC$.

$\therefore$ pyramid $E\text{-}CFD =$ pyramid $F\text{-}ABC$.

Hence the truncated triangular prism $ABC\text{-}DEF$ is equivalent to the sum of the three pyramids whose common base is ABC and whose vertices are D, E, and F.

520. Corollary. *The volume of a truncated right triangular prism is one third the product of the base and the sum of the lateral edges.*



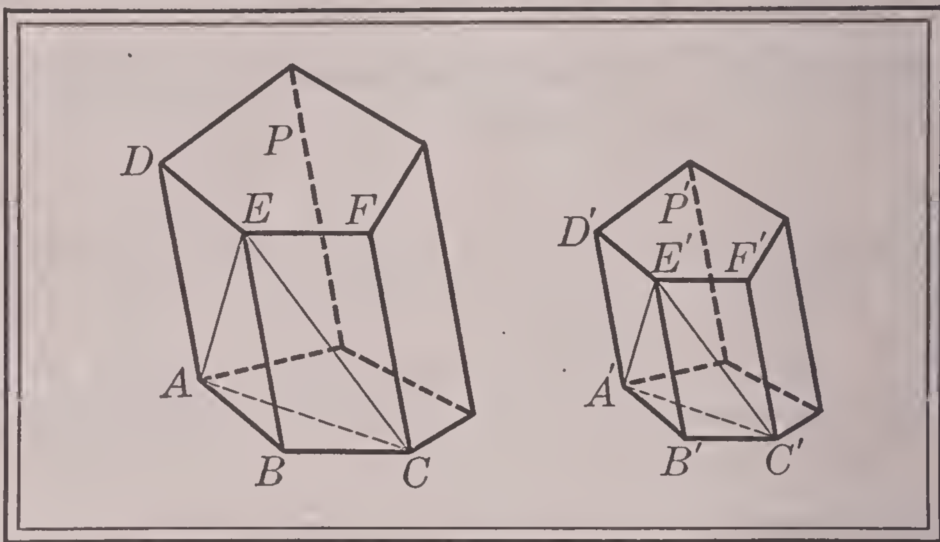
Since the lateral edges DA, EB, FC are $\perp$ to the base ABC, they are the altitudes of the three pyramids whose sum is equivalent to $ABC\text{-}DEF$.

It is interesting to consider the special case in which $\triangle DEF$ is ∥ to $\triangle ABC$.

521. Similar Polyhedrons. Polyhedrons which have the same number of faces, respectively similar and similarly placed, and which have their corresponding polyhedral angles equal, are called *similar polyhedrons.*

Proposition 3. Similar Polyhedrons

522. Theorem. *Two similar polyhedrons can be separated into the same number of tetrahedrons, similar each to each and similarly placed.*



Given the similar polyhedrons P and P'.

Prove that P and P' can be separated into the same number of tetrahedrons, similar each to each and similarly placed.

Proof. Let E and E' be corresponding vertices.

By drawing corresponding diagonals, as AC, $A'C'$, let all the faces of P and P', except those which include the $\angle$s E and E', be divided into corresponding $\triangle$s.

Also, let a plane, as EAC, pass through E and each diagonal of the faces of P, and a plane, as $E'A'C'$, through E' and each corresponding diagonal of P'.

Any two corresponding tetrahedrons $E\text{-}ABC$ and $E'\text{-}A'B'C'$ have the faces ABC, EAB, EBC similar respectively to the faces $A'B'C'$, $E'A'B'$, $E'B'C'$. § 225

Since $$\frac{AE}{A'E'} = \frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{EC}{E'C'},$$ § 205

then face EAC is similar to face $E'A'C'$. § 214

Then the corresponding trihedral ∡s of the tetrahedrons are equal. § 367

Hence $E\text{-}ABC$ is similar to $E'\text{-}A'B'C'$. § 521

If $E\text{-}ABC$ and $E'\text{-}A'B'C'$ are removed, the polyhedrons which are left remain similar; for the new faces EAC and $E'A'C'$ have just been proved similar, the modified faces AED and $A'E'D'$, ECF and $E'C'F'$, are similar (§ 225), and the modified polyhedral ∡s E and E', A and A', C and C' remain equal each to each, since the corresponding parts taken from these ∡s are equal. This process of removing similar tetrahedrons can be continued as necessary.

Hence P and P' can be separated into the same number of tetrahedrons, similar each to each and similarly placed.

523. Corollary. *The corresponding edges of similar polyhedrons are proportional.*

This follows from the definitions of §§ 521 and 205.

524. Corollary. *Any two corresponding lines in two similar polyhedrons have the same ratio as any two corresponding edges.*

For these lines may be shown to be sides of similar polygons.

525. Corollary. *Two corresponding faces of similar polyhedrons are proportional to the squares of any two corresponding edges.*

This follows from §§ 521 and 251.

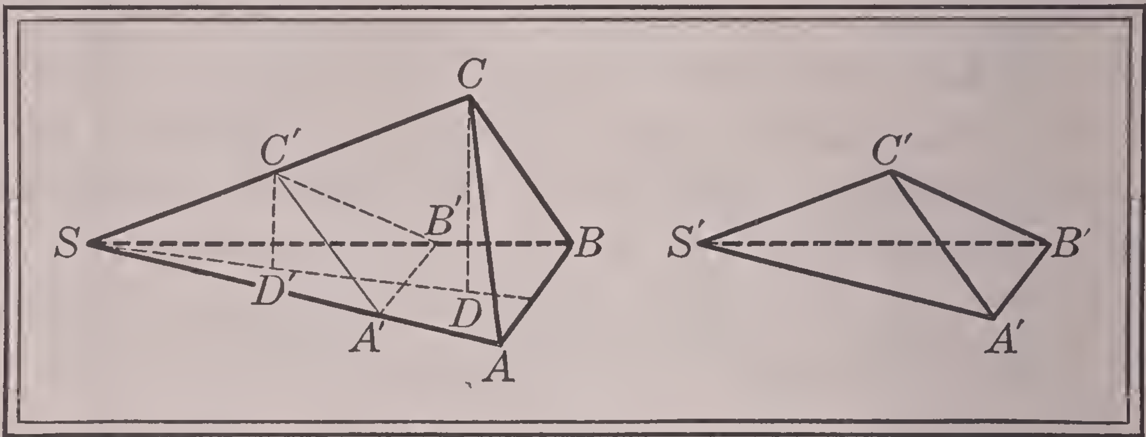
526. Corollary. *The areas of the entire surfaces of two similar polyhedrons are proportional to the squares of any two corresponding edges.*

527. Corollary. *The areas of two similar cylinders, or of two similar cones, are proportional to the squares of any two corresponding lines.*

Consider limits, and apply § 526.

Proposition 4. Ratio of Tetrahedrons

528. Theorem. *The volumes of two tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.*



Given S-ABC and S'-$A'B'C'$, two tetrahedrons with trihedral $\angle S =$ trihedral $\angle S'$; and V and V', the volumes.

Prove that $$\frac{V}{V'} = \frac{SA \cdot SB \cdot SC}{S'A' \cdot S'B' \cdot S'C'}.$$

Proof. Place tetrahedron S'-$A'B'C'$ upon S-ABC so that trihedral $\angle S$ shall coincide with its equal, $\angle S'$.

Let CD and $C'D'$ be $\perp$s to the plane SAB, and let the plane of CD and $C'D'$ intersect SAB in $SD'D$.

The faces SAB and $SA'B'$ may be taken as the bases and CD and $C'D'$ as the altitudes of the triangular pyramids C-SAB and C'-$SA'B'$ respectively.

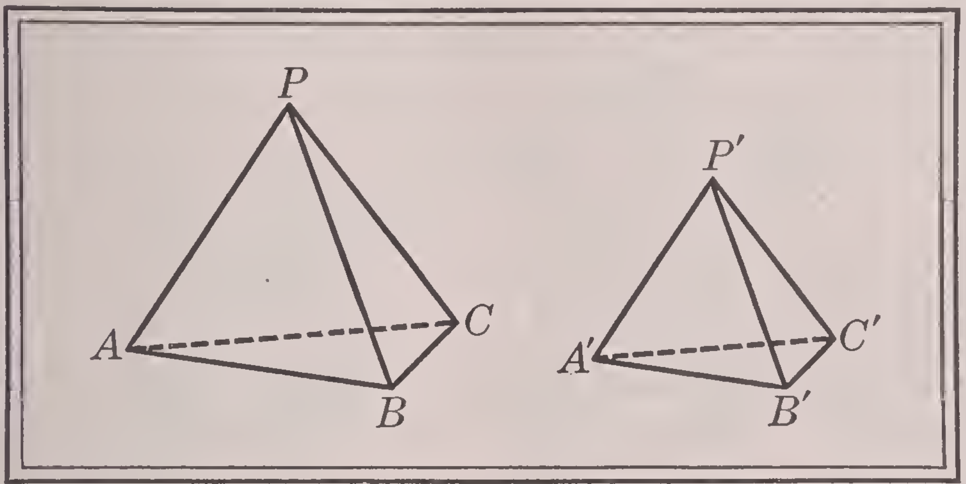
Then $$\frac{V}{V'} = \frac{\frac{1}{3} SAB \cdot CD}{\frac{1}{3} SA'B' \cdot C'D'} = \frac{SAB}{SA'B'} \cdot \frac{CD}{C'D'}. \qquad \S\ 406$$

But $\dfrac{SAB}{SA'B'} = \dfrac{SA \cdot SB}{SA' \cdot SB'}$ (§ 249), and $\dfrac{CD}{C'D'} = \dfrac{SC}{SC'}$ (§ 205).

$$\therefore \frac{V}{V'} = \frac{SA \cdot SB \cdot SC}{SA' \cdot SB' \cdot SC'} = \frac{SA \cdot SB \cdot SC}{S'A' \cdot S'B' \cdot S'C'}. \qquad \text{Ax. 5}$$

Proposition 5. Ratio of Tetrahedrons

529. Theorem. *The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.*



Given $P\text{-}ABC$ and $P'\text{-}A'B'C'$, two similar tetrahedrons; V and V', the volumes; and PB and $P'B'$, two corresponding edges.

Prove that $$\frac{V}{V'} = \frac{\overline{PB}^3}{\overline{P'B'}^3}.$$

Proof. Since $P\text{-}ABC$ is similar to $P'\text{-}A'B'C'$, Given

the corresponding polyhedral $\measuredangle$s are equal. § 521

Hence $$\frac{V}{V'} = \frac{PB \cdot PC \cdot PA}{P'B' \cdot P'C' \cdot P'A'},$$ § 528

or $$\frac{V}{V'} = \frac{PB}{P'B'} \cdot \frac{PC}{P'C'} \cdot \frac{PA}{P'A'}.$$

But $$\frac{PB}{P'B'} = \frac{PC}{P'C'} = \frac{PA}{P'A'}.$$ § 523

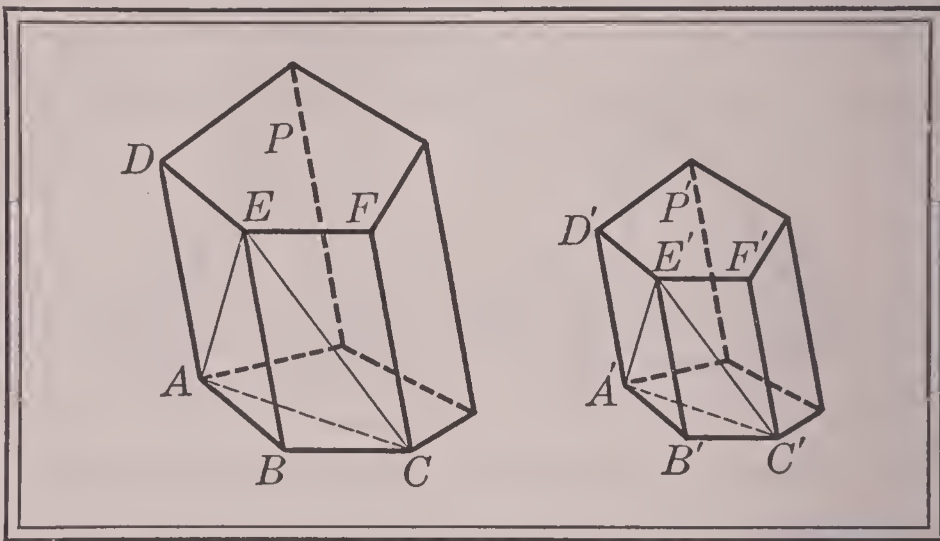
Substituting $\frac{PB}{P'B'}$ for its equals, we have

$$\frac{V}{V'} = \frac{PB}{P'B'} \cdot \frac{PB}{P'B'} \cdot \frac{PB}{P'B'},$$ Ax. 5

or $$\frac{V}{V'} = \frac{\overline{PB}^3}{\overline{P'B'}^3}.$$

Proposition 6. Ratio of Polyhedrons

530. Theorem. *The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.*



Given P and P', two similar polyhedrons; V and V', the volumes; and EB and $E'B'$, any two corresponding edges.

Prove that $V:V' = \overline{EB}^3 : \overline{E'B'}^3$.

Proof. Let P and P' be separated into tetrahedrons similar each to each and similarly placed (§ 522), and let their respective volumes be $V_1, V_2, V_3, \cdots, V_1', V_2', V_3', \cdots$.

Then $V_1 : V_1' = \overline{EB}^3 : \overline{E'B'}^3$,

$V_2 : V_2' = \overline{EB}^3 : \overline{E'B'}^3$, and so on. § 529

$\therefore V_1 + V_2 + V_3 + \cdots : V_1' + V_2' + V_3' + \cdots = \overline{EB}^3 : \overline{E'B'}^3$. § 198, 8

But $V_1 + V_2 + V_3 + \cdots = V$, and $V_1' + V_2' + V_3' + \cdots = V'$.

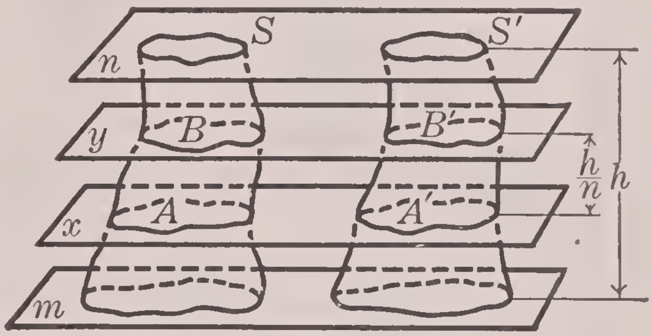
$\therefore V : V' = \overline{EB}^3 : \overline{E'B'}^3$. Ax. 5

531. Corollary. *The volumes of two similar cylinders, or of two similar cones, are proportional to the cubes of any two corresponding lines.*

Consider limits, and apply § 530.

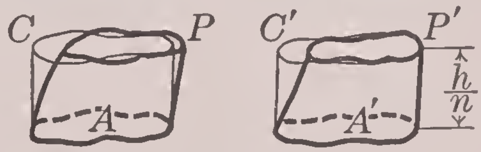
532. Cavalieri's Theorem. In connection with the work in mensuration it is desirable to call attention to a theorem which was set forth by an Italian mathematician, Bonaventura Cavalieri (1598–1647), nearly three centuries ago. Since a complete proof requires some knowledge of the calculus, the theorem is here treated informally.

Theorem. *If two solids lie between parallel planes, and if sections made by any plane parallel to the given planes are equivalent, the solids are equivalent.*



That is, if the two solids S and S', lie between the parallel planes m and n, and if the planes x, y, $\cdots$ cut the solids S and S' so that $A = A'$, $B = B'$, $\cdots$, the solids S and S' are equivalent.

Let P and P' be the portions lying between x and y, $\cdots$, and let the altitude of P and P' be one nth of the altitude h of S and S'. On the bases A, A'; B, B'; $\cdots$ suppose right cylinders or prisms C and C' to stand, each with the altitude h/n. Then $C = C'$, for any value of n.



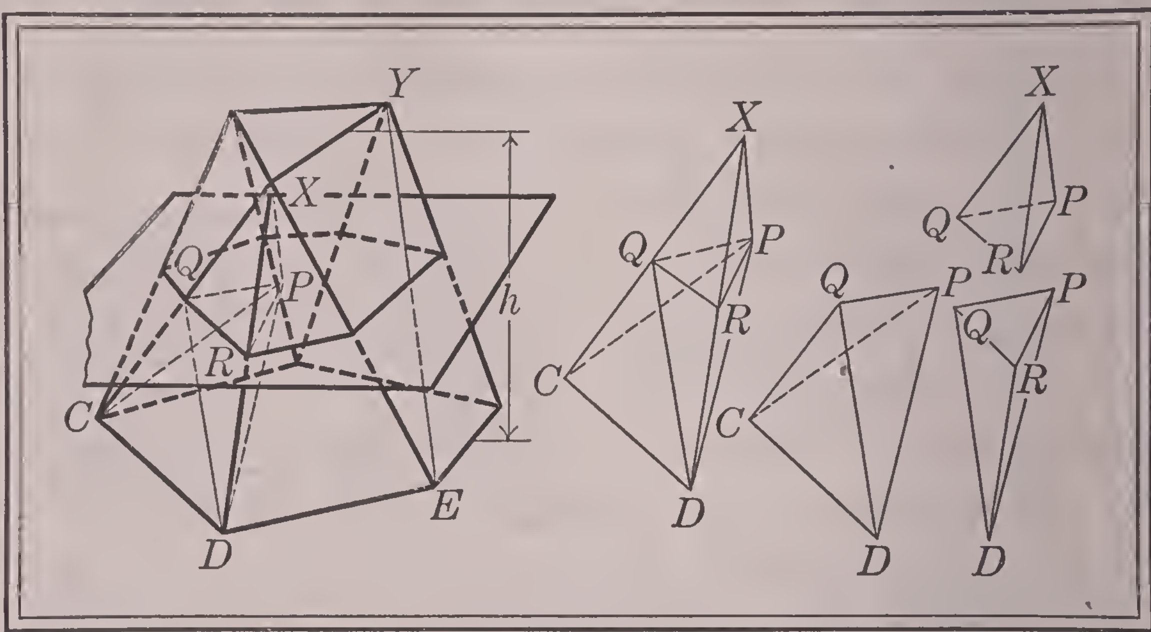
As $n \to \infty$ we see that the sum of the C's approaches S and the sum of the C''s approaches S'. Since each C is always equivalent to each C', we may assume that $S = S'$.

533. Prismoid. A polyhedron which has for its bases two polygons in parallel planes, and for its lateral faces triangles or trapezoids with one side common to one base and the opposite vertex or side common to the other base, is called a *prismoid*.

The *midsection* of such a polyhedron is the section which is parallel to the base and bisects the altitude.

Proposition 7. Prismoid Formula

534. Theorem. *The volume of a prismoid is the product of one sixth the altitude by the sum of the bases and four times the midsection.*



Given *CDE* $\cdots$-*XY* $\cdots$, a prismoid; *V*, the volume; *B*, *B'*, and *M*, the areas of the bases and midsection; and *h*, the altitude.

Prove that $V = \frac{1}{6} h(B + B' + 4M)$.

Proof. If any lateral face is a trapezoid, let it be divided into two $\triangle$ by a diagonal, as *EY*.

Let any point *P* in the midsection be joined to the vertices of the polyhedron and of the midsection, thus separating the prismoid into pyramids which have their vertices at *P*, and which have as their respective bases the lower base, the upper base, and the lateral faces of the prismoid.

The pyramid *P-XCD*, which we may call a lateral pyramid of volume V_p, is composed of the pyramids *P-XQR*, *P-QDR*, *P-QCD* of volumes V_1, V_2, V_3 respectively.

Now *P-XQR* may be regarded as having the vertex *X* and base *PQR*, and *P-QDR* the vertex *D* and base *PQR*.

Then in $P\text{-}XQR$ $\qquad V_1 = \frac{1}{6}h \cdot PQR,$

and in $P\text{-}QDR$ $\qquad V_2 = \frac{1}{6}h \cdot PQR.$ § 406

The pyramids $P\text{-}QCD$ and $P\text{-}QDR$ have the same vertex P, but, since the base CD of $\triangle QCD$ is twice the base QR of $\triangle QDR$ (§ 87), and these $\triangle$s have the same altitude (§ 533), the base QCD is twice the base QDR (§ 246).

Hence the pyramid $P\text{-}QCD$ is equivalent to twice the pyramid $P\text{-}QDR$; or, in pyramid $P\text{-}QCD$,

$$V_3 = \tfrac{2}{6}h \cdot PQR. \qquad \text{Ax. 3}$$

Hence in pyramid $P\text{-}XCD$, which is composed of $P\text{-}XQR$, $P\text{-}QDR$, and $P\text{-}QCD$,

$$V_p = \tfrac{4}{6}h \cdot PQR. \qquad \text{Ax. 1}$$

Similarly, the volume of each lateral pyramid is $\frac{4}{6}h$ times the area of that part of the midsection included within it; and hence for the sum of all the lateral pyramids,

$$V_p = \tfrac{4}{6}h \cdot M. \qquad \text{Ax. 1}$$

The volume of the pyramid with base $CDE \cdots$ is $\frac{1}{6}hB$, and that of the pyramid with base $XY \cdots$ is $\frac{1}{6}hB'$. § 406

$$\therefore V = \tfrac{1}{6}h(B + B' + 4M). \qquad \text{Ax. 1}$$

Exercises. The Prismoid

Deduce the following formulas as cases of a prismoid:

1. Cube, $V = e^3$.
2. Prism, $V = Bh$.
3. Pyramid, $V = \frac{1}{3}Bh$.
4. Parallelepiped, $V = Bh$.
5. Frustum of a pyramid, $V = \frac{1}{3}h\left(B + B' + \sqrt{BB'}\right)$.
6. The area of the upper base of a prismoid is 5 sq. in.; of the lower base, 9 sq. in.; of the midsection, 7 sq. in.; and the altitude is 4 in. Find the volume.
7. Consider Ex. 6 when each measurement is doubled.

Exercises. Review

1. Given that the base of a regular pyramid is an equilateral triangle of side s and that the slant height is l, find the altitude and the volume of the pyramid.

2. Consider Ex. 1 when the base is a square of side s.

3. Given that the base of a regular pyramid is a square of side s and that the area of each lateral face is A, find the volume of the pyramid.

4. Find the volume of a truncated right triangular prism whose base has an area of 7 sq. in., and whose lateral edges are $1\frac{1}{2}$ in., $1\frac{7}{8}$ in., and $2\frac{5}{8}$ in. respectively.

5. In two tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other, the edges of this trihedral angle in the first are 3 in., 4 in., and 5 in. respectively, and those of the corresponding angle of the other are 5 in., 6 in., and 7 in. respectively. Find the ratio of the volumes of the tetrahedrons.

6. How many faces are there in a crystal which has five vertices and nine edges?

7. What part of a cube is cut off by a plane passing through the vertex B' in the upper base and the diagonal AC in the lower base?

8. Two similar polyhedrons have the edges e_1 and e_2 of the first corresponding to e_1' and e_2' of the second. If $e_1 = 4$ in., $e_2 = 7$ in., and $e_1' = 5.6$ in., how long is e_2'?

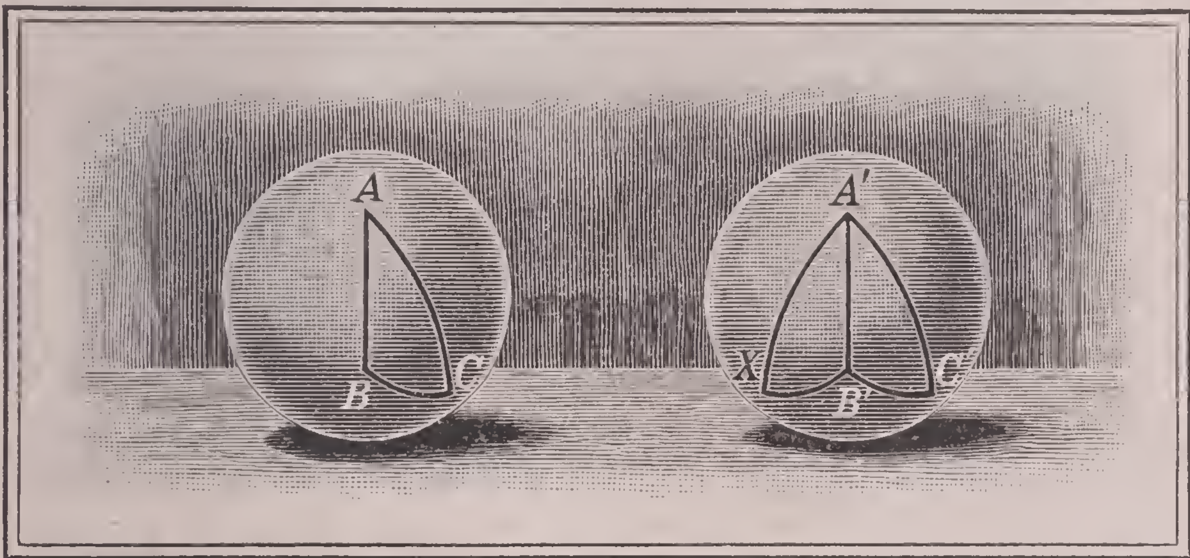
9. By the aid of Cavalieri's Theorem, prove § 405.

10. A wedge has for its base a rectangle l inches long and w inches wide. The cutting edge is e inches long, and is parallel to the base. The distance from e to the base is d inches. Write a formula for the volume of the wedge. Apply this formula to the case of $l = 5$, $w = 2$, $e = 3$, $d = 4$.

III. Spherical Triangles

Proposition 8. Two Sides and Included Angle

535. Theorem. *If two triangles on the same sphere or on equal spheres have two sides and the included angle of one equal respectively to the corresponding parts of the other, the triangles are either congruent or symmetric.*



Given ABC and $A'B'C'$, two $\triangle$ on the same sphere or on equal spheres, with $AB = A'B'$, $AC = A'C'$, $\angle A = \angle A'$, and similarly arranged; and the $\triangle ABC$ and $A'B'X$ with $AB = A'B'$, $AC = A'X$, $\angle A = \angle A'$, and arranged in reverse order.

Prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$,
and that $\triangle ABC$ *is symmetric with respect to* $\triangle A'B'X$.

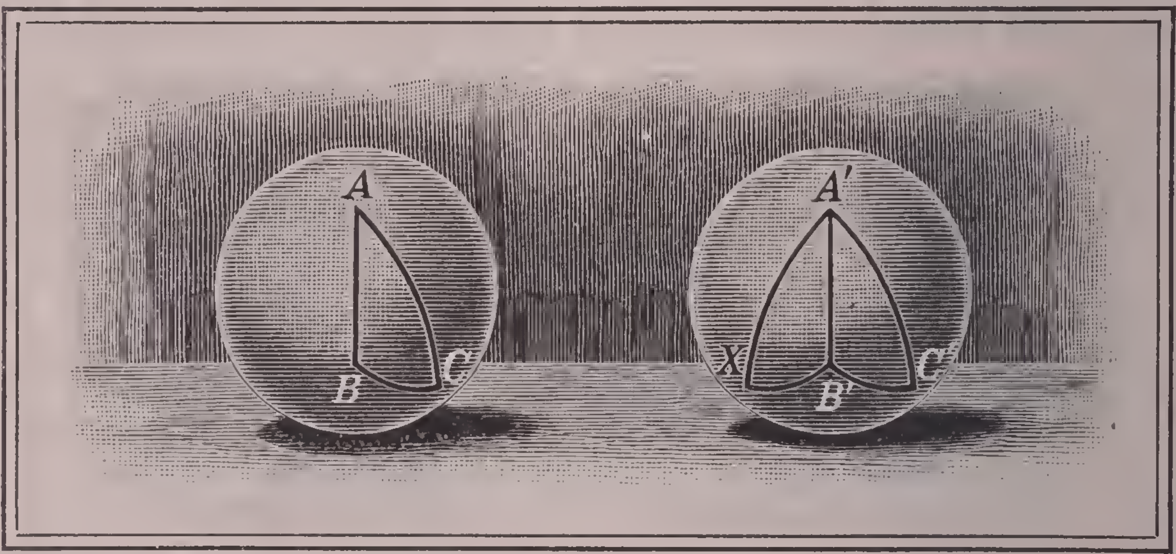
Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$. Post. 5

By a proof similar to that of the corresponding case in plane geometry (§ 40), show that $\triangle ABC$ is congruent to $\triangle A'B'C'$.

Let $\triangle A'B'C'$ be symmetric with respect to $\triangle A'B'X$. Now show that $\triangle A'B'X$ and $A'B'C'$ have $A'B' = A'B'$, $A'X = A'C'$, $\angle XA'B' = \angle C'A'B'$, and are arranged in reverse order (§ 487), and hence that $\triangle ABC$ and $A'B'C'$ have $AB = A'B'$, $AC = A'C'$, $\angle A = \angle C'A'B'$ (Ax. 5), and are similarly arranged. Then show that $\triangle ABC$ and $A'B'C'$ are congruent, as above, and hence that $\triangle ABC$ and $A'B'X$ are symmetric (Ax. 5).

Proposition 9. Two Angles and Included Side

536. Theorem. *If two triangles on the same sphere or on equal spheres have two angles and the included side of one equal respectively to the corresponding parts of the other, the triangles are either congruent or symmetric.*



Given ABC and $A'B'C'$, two $\triangle$ on the same sphere or on equal spheres, with $\angle A = \angle A'$, $\angle C = \angle C'$, $AC = A'C'$, and similarly arranged; and the $\triangle ABC$ and $A'B'X$ with $\angle A = \angle A'$, $\angle C = \angle X$, $AC = A'X$, and arranged in reverse order.

Prove that $\triangle ABC$ *is congruent to* $\triangle A'B'C'$,

and that $\triangle ABC$ *is symmetric with respect to* $\triangle A'B'X$.

Proof. Place $\triangle ABC$ upon $\triangle A'B'C'$. Post. 5

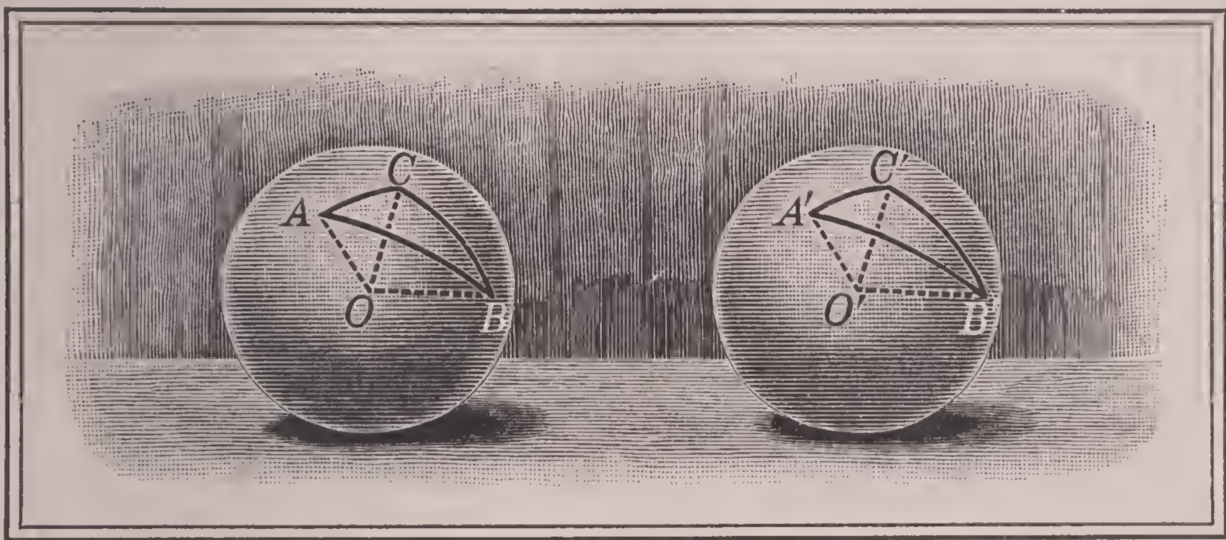
By a proof similar to the corresponding case in plane geometry (§ 44), show that $\triangle ABC$ is congruent to $\triangle A'B'C'$.

Let $\triangle A'B'C'$ be symmetric with respect to $\triangle A'B'X$.

Now show that $\triangle A'B'X$ and $A'B'C'$ have $\angle XA'B' = \angle C'A'B'$, $\angle X = \angle C'$, $A'X = A'C'$, and are arranged in reverse order (§ 487), and hence that $\triangle ABC$ and $A'B'C'$ have $\angle A = \angle C'A'B'$, $\angle C = \angle C'$, and $AC = A'C'$ (Ax. 5), and are similarly arranged. Then show that $\triangle ABC$ and $A'B'C'$ are congruent, as proved above, and hence that $\triangle ABC$ is symmetric with respect to $\triangle A'B'X$ (Ax. 5).

Proposition 10. Mutually Equilateral Triangles

537. Theorem. *If two triangles on the same sphere or on equal spheres are mutually equilateral, they are mutually equiangular and are either congruent or symmetric.*



Given ABC, $A'B'C'$, two $\triangle$ on the same sphere or on equal spheres, with $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$.

Prove that $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, *and that* $\triangle ABC$ *and* $A'B'C'$ *are either congruent or symmetric.*

Proof. Let O and O' be the centers of the spheres, and let a plane pass through each pair of vertices of each $\triangle$ and the center of its sphere.

Then in the trihedral $\angle$ at O and O' the corresponding face $\angle$ are respectively equal. § 137

Then the trihedral $\angle O$ and O' are equal (§ 367), and hence the corresponding dihedral $\angle$ are respectively equal (§ 364).

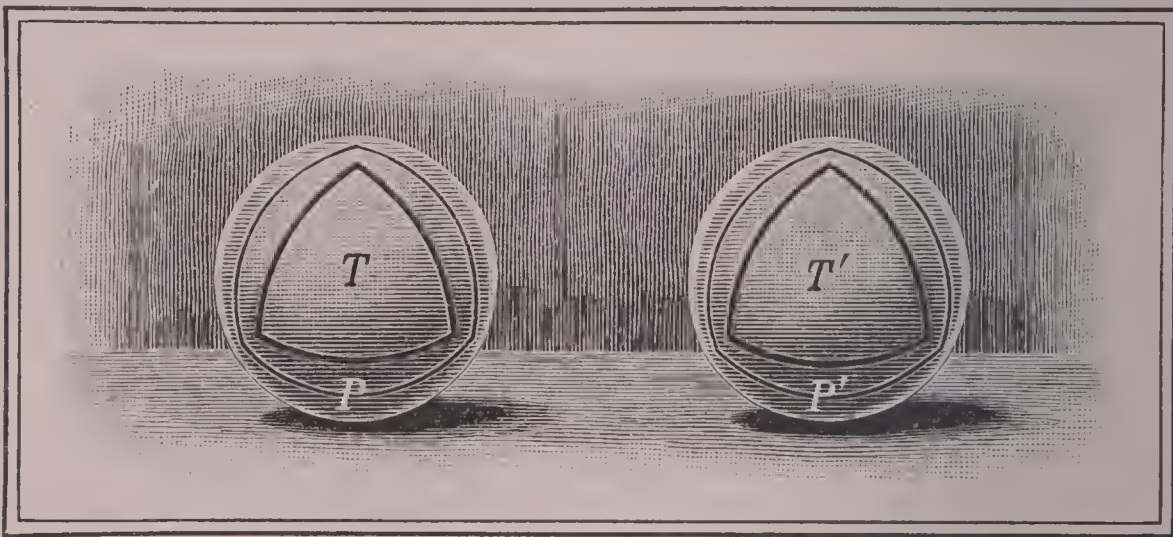
$$\therefore \angle A = \angle A', \angle B = \angle B', \angle C = \angle C'. \qquad \text{§ 473}$$

Hence the $\triangle$ are either congruent or symmetric. § 536

In the above figure the parts are arranged in the same order, so that the triangles are congruent. The parts might be arranged in reverse order, as in the $\triangle ABC$ and $A'B'X$ in the figure of § 536, in which case the $\triangle ABC$ and $A'B'C'$ would be symmetric.

Proposition 11. Mutually Equiangular Triangles

538. Theorem. *If two triangles on the same sphere or on equal spheres are mutually equiangular, they are mutually equilateral, and are either congruent or symmetric.*



Given T and T', two mutually equiangular ⚠ on the same sphere or on equal spheres.

Prove that T and T′ are mutually equilateral, and that they are either congruent or symmetric.

Proof. Let $\triangle P$ be the polar $\triangle$ of $\triangle T$, and let $\triangle P'$ be the polar $\triangle$ of $\triangle T'$.

Since	⚠ T and T' are mutually equiangular,	Given
	the polar ⚠ P and P' are mutually equilateral.	§ 483
Hence	the polar ⚠ P and P' are mutually equiangular.	§ 537
Now	⚠ T and T' are the polar ⚠ of ⚠ P and P'.	§ 482
Hence	⚠ T and T' are mutually equilateral.	§ 483
$\therefore$	⚠ T and T' are either congruent or symmetric.	§ 537

The statement that mutually equiangular spherical triangles are mutually equilateral, and are either congruent or symmetric, is true only when they are on the same sphere or on equal spheres. When the spheres are unequal, the spherical triangles are unequal.

Proposition 12. Isosceles Triangle

539. Theorem. *In an isosceles spherical triangle the angles opposite the equal sides are equal; and conversely, if two angles of a spherical triangle are equal, the sides opposite these angles are equal.*

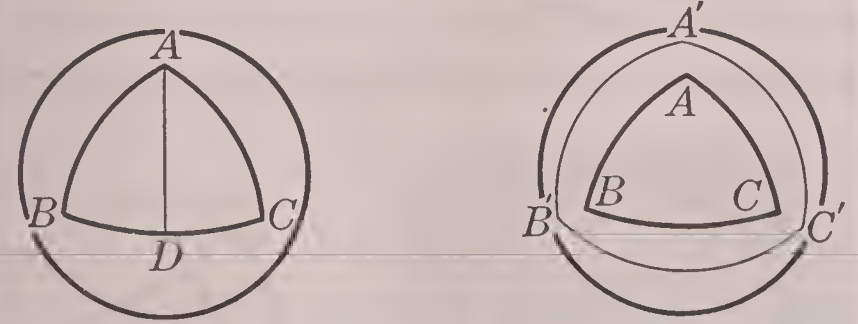


Fig. 1 Fig. 2

1. Given the isosceles spherical $\triangle ABC$ with $AC = AB$.

Prove that $\angle B = \angle C$.

Proof. Let AD be the arc of a great ⊙ which bisects $\angle A$.

Then $AB = AC$, Given

$AD = AD$, Iden.

and $\angle BAD = \angle CAD$. Const.

Hence ⊿s BDA and CDA are symmetric. § 535

$\therefore \angle B = \angle C$. § 487

2. Given the spherical $\triangle ABC$ with $\angle B = \angle C$.

Prove that $AC = AB$.

Proof. Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$.

Then $A'C' + \angle B = 180°$ and $A'B' + \angle C = 180°$. § 483

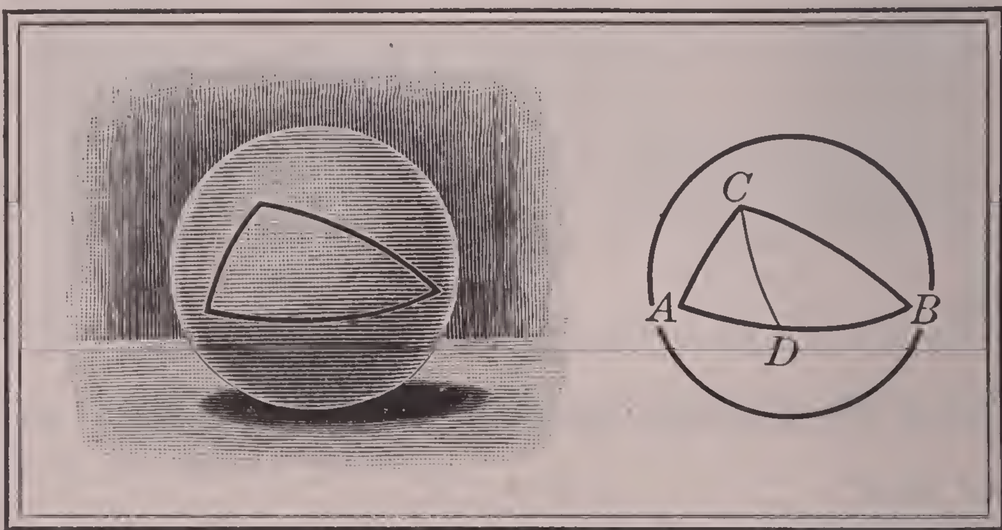
$\therefore A'C' = A'B'$. Axs. 5, 2

Then, by 1, $\angle B' = \angle C'$.

$\therefore AC = AB$. § 483, Ax. 5

Proposition 13. Unequal Parts

540. Theorem. *If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the side opposite the greater angle is the greater; and conversely, if two sides are unequal, the angles opposite these sides are unequal, and the angle opposite the greater side is the greater.*



1. Given the spherical $\triangle ABC$ with $\angle C > \angle B$.

Prove that $AB > AC$.

Proof. Let CD, the arc of a great ⊙, make $\angle DCB = \angle B$.

Then $DB = DC$. § 539, 2

Now $AD + DC > AC$. § 479

$\therefore AD + DB > AC$, or $AB > AC$. Ax. 5

2. Given the spherical $\triangle ABC$ with $AB > AC$.

Prove that $\angle C > \angle B$.

Proof. Using the indirect method, if $\angle C = \angle B$, then $AB = AC$ (§ 539, 2), which is impossible, since $AB > AC$; and if $\angle C < \angle B$, then, by 1, $AB < AC$, which is also impossible.

$\therefore \angle C > \angle B$.

Exercises. Review

1. Find the volume of a truncated right triangular prism whose lateral edges are 1 in., $1\frac{3}{4}$ in., and $2\frac{1}{8}$ in. respectively, and whose base is an equilateral triangle 3 sq. in. in area.

2. The volume of any truncated triangular prism is one third the product of a right section and the sum of the lateral edges.

3. In two tetrahedrons which have a trihedral angle of one equal to a trihedral angle of the other, the edges of these angles are 2 in., $2\frac{1}{2}$ in., 3 in. in the first tetrahedron and 3 in., $3\frac{1}{2}$ in., $4\frac{1}{4}$ in. in the second. Find the ratio of the volumes of the tetrahedrons.

4. A polyhedron of eight faces and six vertices has how many edges?

5. Consider the possibility of a crystal with four faces and two edges; with six faces and four edges.

6. Consider the possibility of a four-edged polyhedron.

7. The volume of the first of two similar tetrahedrons is 32 cu. in., and to an edge 2 in. long in the first there corresponds an edge $2\frac{1}{2}$ in. long in the second. Find the volume of the second tetrahedron.

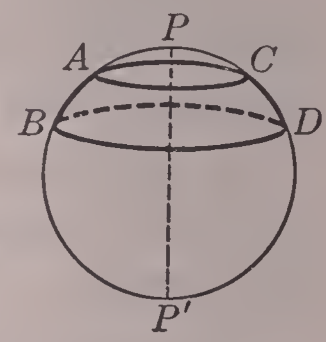
8. Given that the volumes of two similar polyhedrons are in the ratio 1 : 8, find the length of the edge of the first that corresponds to one 3 in. long in the second.

9. Find the volume of a prismoid in which the areas of the bases are 7 sq. in. and 4 sq. in. respectively, the area of the midsection is 5 sq. in., and the height is 8 in.

10. Show that the formula for the volume of a prismoid also applies to the cylinder and the cone.

11. Sketch the figure and state the proposition relating to trihedral angles that follows from each of §§ 535–539.

541. Spherical Segment. If a sphere is cut by two parallel planes, the solid thus formed between the planes is called a *spherical segment.*



The parallel sections are called the *bases.*

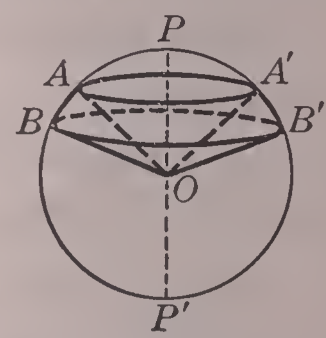
The formula for the volume V of a segment is

$$V = \frac{1}{6}\pi h(3r^2 + 3r'^2 + h^2),$$

where r and r' are the radii of the bases and h is the altitude of the segment. This formula is given here only for reference, as its proof is too difficult.

If one of the parallel planes is tangent to the sphere, the segment is called a *spherical segment of one base.*

542. Spherical Sector. The solid generated by the revolution of a sector of a circle about a diameter of the circle as an axis is called a *spherical sector.*



In this figure the solid is generated by the revolution of the sector AOB about the diameter PP'. The zone generated by the arc AB is called the *base* of the spherical sector.

The formula for the volume V of such a solid is

$$V = \frac{1}{3}rS,$$

where S is the area of the base, and r is the radius of the sphere.

Exercises. Spherical Segments and Sectors

1. Find the volume of a spherical segment whose bases are 6 in. and 8 in. in diameter, and whose altitude is 2 in.

2. If the diameter of a sphere is 14 in. and the altitude of the zone forming the base of a spherical sector is 3 in., what is the volume of the sector?

3. By regarding a spherical segment of one base as the difference between a spherical sector (whose base is a zone of one base) and a cone, show that the formula for the volume is $V = \frac{1}{3}\pi h^2(3r - h)$, where r is the radius of the sphere and h is the altitude of the segment.

IV. Practical Mensuration

543. Nature of the Work. In this country the demand for supplementary exercises in practical mensuration is increasing. The work which is required is based not merely upon the demonstrative geometry of the plane and of the simpler solids as set forth in this text, but also upon the actual measurements of lines and upon the trigonometry of the right triangle as it is now taught in connection with algebra in many schools. Material of this kind may safely replace certain of the propositions of plane geometry, such as those which involve inequalities, certain theorems in relation to the circle, and certain parts of Book IV, and it may also take the place of various propositions in Book VI. It is here offered as optional work for the use of those teachers who wish to modify the standard course in demonstrative geometry, as given in this text, by the introduction of a moderate amount of work in advanced mensuration.

No attempt has been made to include any exercises on the mensuration of the conic sections, which is more advantageously treated in the calculus or in connection with the propositions of analytic geometry. Furthermore, such work is not so practical for the general student as that which relates to the more common plane and solid figures.

As giving proper training in space perception, without involving the logic of demonstration, it is believed that teachers will find this material of great value. It is generally conceded that plane geometry furnishes a sufficient amount of training in deductive reasoning for an initial course, and that the value of solid geometry lies chiefly in its presentation of spatial relations. Such a presentation is made more vital by work of the nature and extent set forth in the following pages.

544. Symbols and Formulas of Plane Geometry. The following symbols and formulas are needed in the exercises:

SYMBOLS

a = apothem	A = area	r = radius
b = base	C = circumference	s = side
d = diameter	$\angle A$ = angle of polygon	$\pi = \frac{22}{7}$, 3.1416
h = altitude	$\angle O$ = central angle	$1/\pi = 0.3183$

Primes indicate that there are two parts of the same name, such as the bases b and b' of a trapezoid.

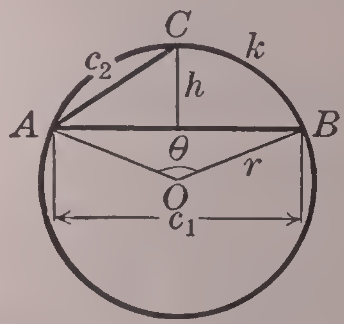
FORMULAS TO BE MEMORIZED

Parallelogram, $A = bh$ Trapezoid, $A = \frac{1}{2}h(b + b')$

Triangle, $A = \frac{1}{2}bh$ Circumference, $C = 2\pi r = \pi d$

Area of a circle, $A = \pi r^2 = \frac{1}{4}\pi d^2$

FORMULAS FOR REFERENCE



Chord AB, $c_1 = 2r\sin\frac{1}{2}\theta$

Chord AC, $c_2 = 2r\sin\frac{1}{4}\theta$

Arc ACB, $k = \frac{1}{3}(8c_2 - c_1)$

The result by this formula is approximate.

Sector $OACB$, $A = \frac{1}{360}\pi r^2\theta$

Equilateral triangle, $A = 0.4330\,s^2$

$\angle A = 60°$ $\angle O = 120°$ $r = 0.5774\,s$ $a = 0.2887\,s$

Regular pentagon, $A = 1.7205\,s^2$

$\angle A = 108°$ $\angle O = 72°$ $r = 0.8506\,s$ $a = 0.6882\,s$

Regular hexagon, $A = 2.5980\,s^2$

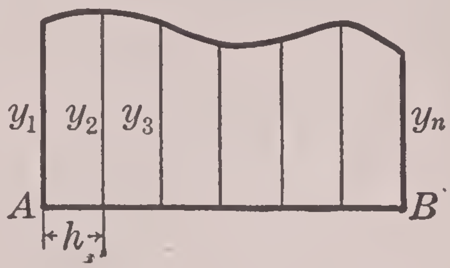
$\angle A = 120°$ $\angle O = 60°$ $r = s$ $a = 0.8660\,s$

Regular octagon, $A = 4.8284\,s^2$

$\angle A = 135°$ $\angle O = 45°$ $r = 1.3065\,s$ $a = 1.2071\,s$

SIMPSON'S RULE FOR AREAS

To find the area between the x axis and a continuous curve, *Simpson's Rule,* which is named for its inventor, is often used. In the figure here shown the base line AB (x axis) is divided into an even number of equal parts so that there is an odd number of ordinates $y_1, y_2, y_3, \cdots, y_n$. The approximate area is then found by the following formula:



$$A = \tfrac{1}{3}h[(y_1+y_n)+2(y_3+y_5+y_7+\cdots)+4(y_2+y_4+y_6+\cdots)]$$

This formula may be expressed in words as follows:

To find the approximate area between a continuous curve and the x axis, add the extreme ordinates, twice the sum of the other odd ordinates, and four times the sum of the even ordinates, and then multiply the result by one third the common distance between the successive ordinates.

The approximation is closer if the curve is undulating (wave-like) as in the figure shown above. The formula and rule need not be memorized.

In the work in practical mensuration the symbols (′) and (″) are used for feet and inches respectively.

The application of the rule may be illustrated by the case of a curve which has the ordinates 16.5′, 21′, 24′, 25′, 29.5′, 33′, 30′, 28.5′, 28.5′, 29′, 30′, 25.5′, 21.5′, 20′, 20.5′, and a common distance (h) between the ordinates of 7.68′.

We then have $y_1 + y_n = 16.5' + 20.5' = 37'.$

$$2(y_3+y_5+y_7+\cdots) = 2(24'+29.5'+30'+28.5'+30'+21.5') = 327'.$$

$$4(y_2+y_4+y_6+\cdots) = 4(21'+25'+33'+28.5'+29'+25.5'+20') = 728'.$$

Hence $A = \frac{1}{3}\times 7.68(37+327+728)$ sq. ft. $= 2795.52$ sq. ft.

545. Symbols and Formulas of Solid Geometry. In addition to the symbols and formulas given in § 544, the following are needed in the exercises:

Symbols

V = volume	S = area of curve surface
B = area of base	l = slant height

E = spherical excess of a polygon

In the case of cylinders and cones, only cylinders and cones of revolution are considered.

Formulas to be Memorized

Cone,	$S = \pi r l$	$V = \frac{1}{3}\pi r^2 h$
Cylinder,	$S = 2\pi r h$	$V = \pi r^2 h$
Sphere,	$S = 4\pi r^2$	$V = \frac{4}{3}\pi r^3$
Zone,	$S = 2\pi r h$	
Spherical polygon,	$S = \frac{1}{180} E\pi r^2$	
Prism or cylinder,		$V = Bh$
Pyramid or cone,		$V = \frac{1}{3} Bh$

It is unnecessary to give formulas for the lateral area of a prism and for the area of the total surface of a cone or a cylinder, as these areas can be found by taking the sum of other known areas.

Formulas for Reference

Frustum of a pyramid, $V = \frac{1}{3} h(B + B' + \sqrt{BB'})$

Frustum of a cone, $S = \pi l(r + r')$

$$V = \frac{1}{3} h(B + B' + \sqrt{BB'}) = \frac{1}{3}\pi h(r^2 + r'^2 + rr')$$

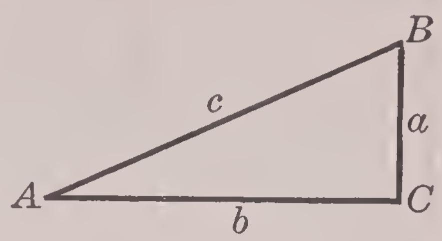
Spherical sector, $V = \frac{1}{3} rS$

Here S is the area of the zone forming the base of the sector.

Spherical segment of one base, $V = \frac{1}{3}\pi h^2(3r - h)$

Instead of using this formula the student may consider the spherical segment as the difference between a spherical sector and a cone.

546. Trigonometry Presupposed. The exercises assume a knowledge of four functions of an angle and the ability to use these functions in solving the right triangle.



The four functions which are needed are as follows:

$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}; \text{ whence } a = c \sin A$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}; \text{ whence } b = c \cos A$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b}; \text{ whence } a = b \tan A$$

$$\cot A = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}; \text{ whence } b = a \cot A$$

Tables of these functions are given on pages 454–461.

547. Trigonometric Formulas. The following formulas are inserted for reference:

$$\sin^2 A + \cos^2 A = 1 \qquad \sin A = \sqrt{1 - \cos^2 A}$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\cot A} \qquad \cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A}$$

548. Use of the Tables. In the tables on pages 454–461 the functions are given for every 0.1°, or for every 6′. In the columns of differences the difference for every 1′ is given. For example, to find sin 55° 20′, find sin 55° 18′ on page 455, and to it add 3 (for 0.0003) found under 2′; that is,

$$\sin 55° 20' = 0.8221 + 0.0003 = 0.8224$$

In finding cos 35° 45′, for example, since the cosine decreases as the angle increases, we subtract the difference (indicated in the table as a minus difference). Hence

$$\cos 35° 45' = 0.8121 - 0.0005 = 0.8116$$

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	.0349	.0366	.0384	.0401	.0419	.0436	.0454	.0471	.0488	.0506	3	6	9	12	15
3	.0523	.0541	.0558	.0576	.0593	.0610	.0628	.0645	.0663	.0680	3	6	9	12	15
4	.0698	.0715	.0732	.0750	.0767	.0785	.0802	.0819	.0837	.0854	3	6	9	12	14
5	.0872	.0889	.0906	.0924	.0941	.0958	.0976	.0993	.1011	.1028	3	6	9	12	14
6	.1045	.1063	.1080	.1097	.1115	.1132	.1149	.1167	.1184	.1201	3	6	9	12	14
7	.1219	.1236	.1253	.1271	.1288	.1305	.1323	.1340	.1357	.1374	3	6	9	12	14
8	.1392	.1409	.1426	.1444	.1461	.1478	.1495	.1513	.1530	.1547	3	6	9	12	14
9	.1564	.1582	.1599	.1616	.1633	.1650	.1668	.1685	.1702	.1719	3	6	9	12	14
10	.1736	.1754	.1771	.1788	.1805	.1822	.1840	.1857	.1874	.1891	3	6	9	11	14
11	.1908	.1925	.1942	.1959	.1977	.1994	.2011	.2028	.2045	.2062	3	6	9	11	14
12	.2079	.2096	.2113	.2130	.2147	.2164	.2181	.2198	.2215	.2233	3	6	9	11	14
13	.2250	.2267	.2284	.2300	.2317	.2334	.2351	.2368	.2385	.2402	3	6	8	11	14
14	.2419	.2436	.2453	.2470	.2487	.2504	.2521	.2538	.2554	.2571	3	6	8	11	14
15	.2588	.2605	.2622	.2639	.2656	.2672	.2689	.2706	.2723	.2740	3	6	8	11	14
16	.2756	.2773	.2790	.2807	.2823	.2840	.2857	.2874	.2890	.2907	3	6	8	11	14
17	.2924	.2940	.2957	.2974	.2990	.3007	.3024	.3040	.3057	.3074	3	6	8	11	14
18	.3090	.3107	.3123	.3140	.3156	.3173	.3190	.3206	.3223	.3239	3	6	8	11	14
19	.3256	.3272	.3289	.3305	.3322	.3338	.3355	.3371	.3387	.3404	3	5	8	11	14
20	.3420	.3437	.3453	.3469	.3486	.3502	.3518	.3535	.3551	.3567	3	5	8	11	14
21	.3584	.3600	.3616	.3633	.3649	.3665	.3681	.3697	.3714	.3730	3	5	8	11	14
22	.3746	.3762	.3778	.3795	.3811	.3827	.3843	.3859	.3875	.3891	3	5	8	11	14
23	.3907	.3923	.3939	.3955	.3971	.3987	.4003	.4019	.4035	.4051	3	5	8	11	14
24	.4067	.4083	.4099	.4115	.4131	.4147	.4163	.4179	.4195	.4210	3	5	8	11	13
25	.4226	.4242	.4258	.4274	.4289	.4305	.4321	.4337	.4352	.4368	3	5	8	11	13
26	.4384	.4399	.4415	.4431	.4446	.4462	.4478	.4493	.4509	.4524	3	5	8	10	13
27	.4540	.4555	.4571	.4586	.4602	.4617	.4633	.4648	.4664	.4679	3	5	8	10	13
28	.4695	.4710	.4726	.4741	.4756	.4772	.4787	.4802	.4818	.4833	3	5	8	10	13
29	.4848	.4863	.4879	.4894	.4909	.4924	.4939	.4955	.4970	.4985	3	5	8	10	13
30	.5000	.5015	.5030	.5045	.5060	.5075	.5090	.5105	.5120	.5135	3	5	8	10	13
31	.5150	.5165	.5180	.5195	.5210	.5225	.5240	.5255	.5270	.5284	2	5	7	10	12
32	.5299	.5314	.5329	.5344	.5358	.5373	.5388	.5402	.5417	.5432	2	5	7	10	12
33	.5446	.5461	.5476	.5490	.5505	.5519	.5534	.5548	.5563	.5577	2	5	7	10	12
34	.5592	.5606	.5621	.5635	.5650	.5664	.5678	.5693	.5707	.5721	2	5	7	10	12
35	.5736	.5750	.5764	.5779	.5793	.5807	.5821	.5835	.5850	.5864	2	5	7	9	12
36	.5878	.5892	.5906	.5920	.5934	.5948	.5962	.5976	.5990	.6004	2	5	7	9	12
37	.6018	.6032	.6046	.6060	.6074	.6088	.6101	.6115	.6129	.6143	2	5	7	9	12
38	.6157	.6170	.6184	.6198	.6211	.6225	.6239	.6252	.6266	.6280	2	5	7	9	11
39	.6293	.6307	.6320	.6334	.6347	.6361	.6374	.6388	.6401	.6414	2	4	7	9	11
40	.6428	.6441	.6455	.6468	.6481	.6494	.6508	.6521	.6534	.6547	2	4	7	9	11
41	.6561	.6574	.6587	.6600	.6613	.6626	.6639	.6652	.6665	.6678	2	4	7	9	11
42	.6691	.6704	.6717	.6730	.6743	.6756	.6769	.6782	.6794	.6807	2	4	6	9	11
43	.6820	.6833	.6845	.6858	.6871	.6884	.6896	.6909	.6921	.6934	2	4	6	8	11
44	.6947	.6959	.6972	.6984	.6997	.7009	.7022	.7034	.7046	.7059	2	4	6	8	10

All the above sines are less than 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7083	.7096	.7108	.7120	.7133	.7145	.7157	.7169	.7181	2	4	6	8	10
46	.7193	.7206	.7218	.7230	.7242	.7254	.7266	.7278	.7290	.7302	2	4	6	8	10
47	.7314	.7325	.7337	.7349	.7361	.7373	.7385	.7396	.7408	.7420	2	4	6	8	10
48	.7431	.7443	.7455	.7466	.7478	.7490	.7501	.7513	.7524	.7536	2	4	6	8	10
49	.7547	.7559	.7570	.7581	.7593	.7604	.7615	.7627	.7638	.7649	2	4	6	8	9
50	.7660	.7672	.7683	.7694	.7705	.7716	.7727	.7738	.7749	.7760	2	4	6	7	9
51	.7771	.7782	.7793	.7804	.7815	.7826	.7837	.7848	.7859	.7869	2	4	5	7	9
52	.7880	.7891	.7902	.7912	.7923	.7934	.7944	.7955	.7965	.7976	2	4	5	7	9
53	.7986	.7997	.8007	.8018	.8028	.8039	.8049	.8059	.8070	.8080	2	3	5	7	9
54	.8090	.8100	.8111	.8121	.8131	.8141	.8151	.8161	.8171	.8181	2	3	5	7	8
55	.8192	.8202	.8211	.8221	.8231	.8241	.8251	.8261	.8271	.8281	2	3	5	7	8
56	.8290	.8300	.8310	.8320	.8329	.8339	.8348	.8358	.8368	.8377	2	3	5	6	8
57	.8387	.8396	.8406	.8415	.8425	.8434	.8443	.8453	.8462	.8471	2	3	5	6	8
58	.8480	.8490	.8499	.8508	.8517	.8526	.8536	.8545	.8554	.8563	2	3	5	6	8
59	.8572	.8581	.8590	.8599	.8607	.8616	.8625	.8634	.8643	.8652	1	3	4	6	7
60	.8660	.8669	.8678	.8686	.8695	.8704	.8712	.8721	.8729	.8738	1	3	4	6	7
61	.8746	.8755	.8763	.8771	.8780	.8788	.8796	.8805	.8813	.8821	1	3	4	6	7
62	.8829	.8838	.8846	.8854	.8862	.8870	.8878	.8886	.8894	.8902	1	3	4	5	7
63	.8910	.8918	.8926	.8934	.8942	.8949	.8957	.8965	.8973	.8980	1	3	4	5	6
64	.8988	.8996	.9003	.9011	.9018	.9026	.9033	.9041	.9048	.9056	1	3	4	5	6
65	.9063	.9070	.9078	.9085	.9092	.9100	.9107	.9114	.9121	.9128	1	2	4	5	6
66	.9135	.9143	.9150	.9157	.9164	.9171	.9178	.9184	.9191	.9198	1	2	3	5	6
67	.9205	.9212	.9219	.9225	.9232	.9239	.9245	.9252	.9259	.9265	1	2	3	4	6
68	.9272	.9278	.9285	.9291	.9298	.9304	.9311	.9317	.9323	.9330	1	2	3	4	5
69	.9336	.9342	.9348	.9354	.9361	.9367	.9373	.9379	.9385	.9391	1	2	3	4	5
70	.9397	.9403	.9409	.9415	.9421	.9426	.9432	.9438	.9444	.9449	1	2	3	4	5
71	.9455	.9461	.9466	.9472	.9478	.9483	.9489	.9494	.9500	.9505	1	2	3	4	5
72	.9511	.9516	.9521	.9527	.9532	.9537	.9542	.9548	.9553	.9558	1	2	3	4	4
73	.9563	.9568	.9573	.9578	.9583	.9588	.9593	.9598	.9603	.9608	1	2	2	3	4
74	.9613	.9617	.9622	.9627	.9632	.9636	.9641	.9646	.9650	.9655	1	2	2	3	4
75	.9659	.9664	.9668	.9673	.9677	.9681	.9686	.9690	.9694	.9699	1	1	2	3	4
76	.9703	.9707	.9711	.9715	.9720	.9724	.9728	.9732	.9736	.9740	1	1	2	3	3
77	.9744	.9748	.9751	.9755	.9759	.9763	.9767	.9770	.9774	.9778	1	1	2	3	3
78	.9781	.9785	.9789	.9792	.9796	.9799	.9803	.9806	.9810	.9813	1	1	2	2	3
79	.9816	.9820	.9823	.9826	.9829	.9833	.9836	.9839	.9842	.9845	1	1	2	2	3
80	.9848	.9851	.9854	.9857	.9860	.9863	.9866	.9869	.9871	.9874	0	1	1	2	2
81	.9877	.9880	.9882	.9885	.9888	.9890	.9893	.9895	.9898	.9900	0	1	1	2	2
82	.9903	.9905	.9907	.9910	.9912	.9914	.9917	.9919	.9921	.9923	0	1	1	2	2
83	.9925	.9928	.9930	.9932	.9934	.9936	.9938	.9940	.9942	.9943	0	1	1	1	2
84	.9945	.9947	.9949	.9951	.9952	.9954	.9956	.9957	.9959	.9960	0	1	1	1	1
85	.9962	.9963	.9965	.9966	.9968	.9969	.9971	.9972	.9973	.9974	0	0	1	1	1
86	.9976	.9977	.9978	.9979	.9980	.9981	.9982	.9983	.9984	.9985	0	0	1	1	1
87	.9986	.9987	.9988	.9989	.9990	.9990	.9991	.9992	.9993	.9993	0	0	0	1	1
88	.9994	.9995	.9995	.9996	.9996	.9997	.9997	.9997	.9998	.9998	0	0	0	0	0
89	.9998	.9999	.9999	.9999	.9999	1.000	1.000	1.000	1.000	1.000	0	0	0	0	0

The precise value of all sines except sin 90° is less than 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	− Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	1.000	1.000	1.000	1.000	1.000	1.000	.9999	.9999	.9999	.9999	0	0	0	0	0
1	.9998	.9998	.9998	.9997	.9997	.9997	.9996	.9996	.9995	.9995	0	0	0	0	0
2	.9994	.9993	.9993	.9992	.9991	.9990	.9990	.9989	.9988	.9987	0	0	0	0	0
3	.9986	.9985	.9984	.9983	.9982	.9981	.9980	.9979	.9978	.9977	0	0	1	1	1
4	.9976	.9974	.9973	.9972	.9971	.9969	.9968	.9966	.9965	.9963	0	0	1	1	1
5	.9962	.9960	.9959	.9957	.9956	.9954	.9952	.9951	.9949	.9947	0	1	1	1	1
6	.9945	.9943	.9942	.9940	.9938	.9936	.9934	.9932	.9930	.9928	0	1	1	1	2
7	.9925	.9923	.9921	.9919	.9917	.9914	.9912	.9910	.9907	.9905	0	1	1	2	2
8	.9903	.9900	.9898	.9895	.9893	.9890	.9888	.9885	.9882	.9880	0	1	1	2	2
9	.9877	.9874	.9871	.9869	.9866	.9863	.9860	.9857	.9854	.9851	0	1	1	2	2
10	.9848	.9845	.9842	.9839	.9836	.9833	.9829	.9826	.9823	.9820	1	1	2	2	3
11	.9816	.9813	.9810	.9806	.9803	.9799	.9796	.9792	.9789	.9785	1	1	2	2	3
12	.9781	.9778	.9774	.9770	.9767	.9763	.9759	.9755	.9751	.9748	1	1	2	3	3
13	.9744	.9740	.9736	.9732	.9728	.9724	.9720	.9715	.9711	.9707	1	1	2	3	3
14	.9703	.9699	.9694	.9690	.9686	.9681	.9677	.9673	.9668	.9664	1	1	2	3	4
15	.9659	.9655	.9650	.9646	.9641	.9636	.9632	.9627	.9622	.9617	1	2	2	3	4
16	.9613	.9608	.9603	.9598	.9593	.9588	.9583	.9578	.9573	.9568	1	2	2	3	4
17	.9563	.9558	.9553	.9548	.9542	.9537	.9532	.9527	.9521	.9516	1	2	3	4	4
18	.9511	.9505	.9500	.9494	.9489	.9483	.9478	.9472	.9466	.9461	1	2	3	4	5
19	.9455	.9449	.9444	.9438	.9432	.9426	.9421	.9415	.9409	.9403	1	2	3	4	5
20	.9397	.9391	.9385	.9379	.9373	.9367	.9361	.9354	.9348	.9342	1	2	3	4	5
21	.9336	.9330	.9323	.9317	.9311	.9304	.9298	.9291	.9285	.9278	1	2	3	4	5
22	.9272	.9265	.9259	.9252	.9245	.9239	.9232	.9225	.9219	.9212	1	2	3	4	6
23	.9205	.9198	.9191	.9184	.9178	.9171	.9164	.9157	.9150	.9143	1	2	3	5	6
24	.9135	.9128	.9121	.9114	.9107	.9100	.9092	.9085	.9078	.9070	1	2	4	5	6
25	.9063	.9056	.9048	.9041	.9033	.9026	.9018	.9011	.9003	.8996	1	3	4	5	6
26	.8988	.8980	.8973	.8965	.8957	.8949	.8942	.8934	.8926	.8918	1	3	4	5	6
27	.8910	.8902	.8894	.8886	.8878	.8870	.8862	.8854	.8846	.8838	1	3	4	5	7
28	.8829	.8821	.8813	.8805	.8796	.8788	.8780	.8771	.8763	.8755	1	3	4	6	7
29	.8746	.8738	.8729	.8721	.8712	.8704	.8695	.8686	.8678	.8669	1	3	4	6	7
30	.8660	.8652	.8643	.8634	.8625	.8616	.8607	.8599	.8590	.8581	1	3	4	6	7
31	.8572	.8563	.8554	.8545	.8536	.8526	.8517	.8508	.8499	.8490	2	3	5	6	8
32	.8480	.8471	.8462	.8453	.8443	.8434	.8425	.8415	.8406	.8396	2	3	5	6	8
33	.8387	.8377	.8368	.8358	.8348	.8339	.8329	.8320	.8310	.8300	2	3	5	6	8
34	.8290	.8281	.8271	.8261	.8251	.8241	.8231	.8221	.8211	.8202	2	3	5	7	8
35	.8192	.8181	.8171	.8161	.8151	.8141	.8131	.8121	.8111	.8100	2	3	5	7	8
36	.8090	.8080	.8070	.8059	.8049	.8039	.8028	.8018	.8007	.7997	2	3	5	7	9
37	.7986	.7976	.7965	.7955	.7944	.7934	.7923	.7912	.7902	.7891	2	4	5	7	9
38	.7880	.7869	.7859	.7848	.7837	.7826	.7815	.7804	.7793	.7782	2	4	5	7	9
39	.7771	.7760	.7749	.7738	.7727	.7716	.7705	.7694	.7683	.7672	2	4	6	7	9
40	.7660	.7649	.7638	.7627	.7615	.7604	.7593	.7581	.7570	.7559	2	4	6	8	9
41	.7547	.7536	.7524	.7513	.7501	.7490	.7478	.7466	.7455	.7443	2	4	6	8	10
42	.7431	.7420	.7408	.7396	.7385	.7373	.7361	.7349	.7337	.7325	2	4	6	8	10
43	.7314	.7302	.7290	.7278	.7266	.7254	.7242	.7230	.7218	.7206	2	4	6	8	10
44	.7193	.7181	.7169	.7157	.7145	.7133	.7120	.7108	.7096	.7083	2	4	6	8	10

The precise value of all cosines except cos 0° is less than 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	− Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	.7071	.7059	.7046	.7034	.7022	.7009	.6997	.6984	.6972	.6959	2	4	6	8	10
46	.6947	.6934	.6921	.6909	.6896	.6884	.6871	.6858	.6845	.6833	2	4	6	8	11
47	.6820	.6807	.6794	.6782	.6769	.6756	.6743	.6730	.6717	.6704	2	4	6	9	11
48	.6691	.6678	.6665	.6652	.6639	.6626	.6613	.6600	.6587	.6574	2	4	7	9	11
49	.6561	.6547	.6534	.6521	.6508	.6494	.6481	.6468	.6455	.6441	2	4	7	9	11
50	.6428	.6414	.6401	.6388	.6374	.6361	.6347	.6334	.6320	.6307	2	4	7	9	11
51	.6293	.6280	.6266	.6252	.6239	.6225	.6211	.6198	.6184	.6170	2	5	7	9	11
52	.6157	.6143	.6129	.6115	.6101	.6088	.6074	.6060	.6046	.6032	2	5	7	9	12
53	.6018	.6004	.5990	.5976	.5962	.5948	.5934	.5920	.5906	.5892	2	5	7	9	12
54	.5878	.5864	.5850	.5835	.5821	.5807	.5793	.5779	.5764	.5750	2	5	7	9	12
55	.5736	.5721	.5707	.5693	.5678	.5664	.5650	.5635	.5621	.5606	2	5	7	10	12
56	.5592	.5577	.5563	.5548	.5534	.5519	.5505	.5490	.5476	.5461	2	5	7	10	12
57	.5446	.5432	.5417	.5402	.5388	.5373	.5358	.5344	.5329	.5314	2	5	7	10	12
58	.5299	.5284	.5270	.5255	.5240	.5225	.5210	.5195	.5180	.5165	2	5	7	10	12
59	.5150	.5135	.5120	.5105	.5090	.5075	.5060	.5045	.5030	.5015	3	5	8	10	13
60	.5000	.4985	.4970	.4955	.4939	.4924	.4909	.4894	.4879	.4863	3	5	8	10	13
61	.4848	.4833	.4818	.4802	.4787	.4772	.4756	.4741	.4726	.4710	3	5	8	10	13
62	.4695	.4679	.4664	.4648	.4633	.4617	.4602	.4586	.4571	.4555	3	5	8	10	13
63	.4540	.4524	.4509	.4493	.4478	.4462	.4446	.4431	.4415	.4399	3	5	8	10	13
64	.4384	.4368	.4352	.4337	.4321	.4305	.4289	.4274	.4258	.4242	3	5	8	11	13
65	.4226	.4210	.4195	.4179	.4163	.4147	.4131	.4115	.4099	.4083	3	5	8	11	13
66	.4067	.4051	.4035	.4019	.4003	.3987	.3971	.3955	.3939	.3923	3	5	8	11	13
67	.3907	.3891	.3875	.3859	.3843	.3827	.3811	.3795	.3778	.3762	3	5	8	11	13
68	.3746	.3730	.3714	.3697	.3681	.3665	.3649	.3633	.3616	.3600	3	5	8	11	14
69	.3584	.3567	.3551	.3535	.3518	.3502	.3486	.3469	.3453	.3437	3	5	8	11	14
70	.3420	.3404	.3387	.3371	.3355	.3338	.3322	.3305	.3289	.3272	3	5	8	11	14
71	.3256	.3239	.3223	.3206	.3190	.3173	.3156	.3140	.3123	.3107	3	6	8	11	14
72	.3090	.3074	.3057	.3040	.3024	.3007	.2990	.2974	.2957	.2940	3	6	8	11	14
73	.2924	.2907	.2890	.2874	.2857	.2840	.2823	.2807	.2790	.2773	3	6	8	11	14
74	.2756	.2740	.2723	.2706	.2689	.2672	.2656	.2639	.2622	.2605	3	6	8	11	14
75	.2588	.2571	.2554	.2538	.2521	.2504	.2487	.2470	.2453	.2436	3	6	8	11	14
76	.2419	.2402	.2385	.2368	.2351	.2334	.2317	.2300	.2284	.2267	3	6	8	11	14
77	.2250	.2233	.2215	.2198	.2181	.2164	.2147	.2130	.2113	.2096	3	6	9	11	14
78	.2079	.2062	.2045	.2028	.2011	.1994	.1977	.1959	.1942	.1925	3	6	9	11	14
79	.1908	.1891	.1874	.1857	.1840	.1822	.1805	.1788	.1771	.1754	3	6	9	11	14
80	.1736	.1719	.1702	.1685	.1668	.1650	.1633	.1616	.1599	.1582	3	6	9	11	14
81	.1564	.1547	.1530	.1513	.1495	.1478	.1461	.1444	.1426	.1409	3	6	9	12	14
82	.1392	.1374	.1357	.1340	.1323	.1305	.1288	.1271	.1253	.1236	3	6	9	12	14
83	.1219	.1201	.1184	.1167	.1149	.1132	.1115	.1097	.1080	.1063	3	6	9	12	14
84	.1045	.1028	.1011	.0993	.0976	.0958	.0941	.0924	.0906	.0889	3	6	9	12	14
85	.0872	.0854	.0837	.0819	.0802	.0785	.0767	.0750	.0732	.0715	3	6	9	12	14
86	.0698	.0680	.0663	.0645	.0628	.0610	.0593	.0576	.0558	.0541	3	6	9	12	15
87	.0523	.0506	.0488	.0471	.0454	.0436	.0419	.0401	.0384	.0366	3	6	9	12	15
88	.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89	.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

All the above cosines are less than 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	0.0000	.0017	.0035	.0052	.0070	.0087	.0105	.0122	.0140	.0157	3	6	9	12	15
1	0.0175	.0192	.0209	.0227	.0244	.0262	.0279	.0297	.0314	.0332	3	6	9	12	15
2	0.0349	.0367	.0384	.0402	.0419	.0437	.0454	.0472	.0489	.0507	3	6	9	12	15
3	0.0524	.0542	.0559	.0577	.0594	.0612	.0629	.0647	.0664	.0682	3	6	9	12	15
4	0.0699	.0717	.0734	.0752	.0769	.0787	.0805	.0822	.0840	.0857	3	6	9	12	15
5	0.0875	.0892	.0910	.0928	.0945	.0963	.0981	.0998	.1016	.1033	3	6	9	12	15
6	0.1051	.1069	.1086	.1104	.1122	.1139	.1157	.1175	.1192	.1210	3	6	9	12	15
7	0.1228	.1246	.1263	.1281	.1299	.1317	.1334	.1352	.1370	.1388	3	6	9	12	15
8	0.1405	.1423	.1441	.1459	.1477	.1495	.1512	.1530	.1548	.1566	3	6	9	12	15
9	0.1584	.1602	.1620	.1638	.1655	.1673	.1691	.1709	.1727	.1745	3	6	9	12	15
10	0.1763	.1781	.1799	.1817	.1835	.1853	.1871	.1890	.1908	.1926	3	6	9	12	15
11	0.1944	.1962	.1980	.1998	.2016	.2035	.2053	.2071	.2089	.2107	3	6	9	12	15
12	0.2126	.2144	.2162	.2180	.2199	.2217	.2235	.2254	.2272	.2290	3	6	9	12	15
13	0.2309	.2327	.2345	.2364	.2382	.2401	.2419	.2438	.2456	.2475	3	6	9	12	15
14	0.2493	.2512	.2530	.2549	.2568	.2586	.2605	.2623	.2642	.2661	3	6	9	12	16
15	0.2679	.2698	.2717	.2736	.2754	.2773	.2792	.2811	.2830	.2849	3	6	9	13	16
16	0.2867	.2886	.2905	.2924	.2943	.2962	.2981	.3000	.3019	.3038	3	6	9	13	16
17	0.3057	.3076	.3096	.3115	.3134	.3153	.3172	.3191	.3211	.3230	3	6	10	13	16
18	0.3249	.3269	.3288	.3307	.3327	.3346	.3365	.3385	.3404	.3424	3	6	10	13	16
19	0.3443	.3463	.3482	.3502	.3522	.3541	.3561	.3581	.3600	.3620	3	7	10	13	16
20	0.3640	.3659	.3679	.3699	.3719	.3739	.3759	.3779	.3799	.3819	3	7	10	13	17
21	0.3839	.3859	.3879	.3899	.3919	.3939	.3959	.3979	.4000	.4020	3	7	10	13	17
22	0.4040	.4061	.4081	.4101	.4122	.4142	.4163	.4183	.4204	.4224	3	7	10	14	17
23	0.4245	.4265	.4286	.4307	.4327	.4348	.4369	.4390	.4411	.4431	3	7	10	14	17
24	0.4452	.4473	.4494	.4515	.4536	.4557	.4578	.4599	.4621	.4642	4	7	11	14	18
25	0.4663	.4684	.4706	.4727	.4748	.4770	.4791	.4813	.4834	.4856	4	7	11	14	18
26	0.4877	.4899	.4921	.4942	.4964	.4986	.5008	.5029	.5051	.5073	4	7	11	15	18
27	0.5095	.5117	.5139	.5161	.5184	.5206	.5228	.5250	.5272	.5295	4	7	11	15	18
28	0.5317	.5340	.5362	.5384	.5407	.5430	.5452	.5475	.5498	.5520	4	8	11	15	19
29	0.5543	.5566	.5589	.5612	.5635	.5658	.5681	.5704	.5727	.5750	4	8	12	15	19
30	0.5774	.5797	.5820	.5844	.5867	.5890	.5914	.5938	.5961	.5985	4	8	12	16	20
31	0.6009	.6032	.6056	.6080	.6104	.6128	.6152	.6176	.6200	.6224	4	8	12	16	20
32	0.6249	.6273	.6297	.6322	.6346	.6371	.6395	.6420	.6445	.6469	4	8	12	16	20
33	0.6494	.6519	.6544	.6569	.6594	.6619	.6644	.6669	.6694	.6720	4	8	13	17	21
34	0.6745	.6771	.6796	.6822	.6847	.6873	.6899	.6924	.6950	.6976	4	9	13	17	21
35	0.7002	.7028	.7054	.7080	.7107	.7133	.7159	.7186	.7212	.7239	4	9	13	18	22
36	0.7265	.7292	.7319	.7346	.7373	.7400	.7427	.7454	.7481	.7508	5	9	14	18	23
37	0.7536	.7563	.7590	.7618	.7646	.7673	.7701	.7729	.7757	.7785	5	9	14	18	23
38	0.7813	.7841	.7869	.7898	.7926	.7954	.7983	.8012	.8040	.8069	5	9	14	19	24
39	0.8098	.8127	.8156	.8185	.8214	.8243	.8273	.8302	.8332	.8361	5	10	15	20	24
40	0.8391	.8421	.8451	.8481	.8511	.8541	.8571	.8601	.8632	.8662	5	10	15	20	25
41	0.8693	.8724	.8754	.8785	.8816	.8847	.8878	.8910	.8941	.8972	5	10	16	21	26
42	0.9004	.9036	.9067	.9099	.9131	.9163	.9195	.9228	.9260	.9293	5	11	16	21	27
43	0.9325	.9358	.9391	.9424	.9457	.9490	.9523	.9556	.9590	.9623	6	11	17	22	28
44	0.9657	.9691	.9725	.9759	.9793	.9827	.9861	.9896	.9930	.9965	6	11	17	23	29

All tangents of angles less than 45° are less than 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	+ Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.0000	.0035	.0070	.0105	.0141	.0176	.0212	.0247	.0283	.0319	6	12	18	24	30
46	1.0355	.0392	.0428	.0464	.0501	.0538	.0575	.0612	.0649	.0686	6	12	18	25	31
47	1.0724	.0761	.0799	.0837	.0875	.0913	.0951	.0990	.1028	.1067	6	13	19	25	32
48	1.1106	.1145	.1184	.1224	.1263	.1303	.1343	.1383	.1423	.1463	7	13	20	26	33
49	1.1504	.1544	.1585	.1626	.1667	.1708	.1750	.1792	.1833	.1875	7	14	21	28	34
50	1.1918	.1960	.2002	.2045	.2088	.2131	.2174	.2218	.2261	.2305	7	14	22	29	36
51	1.2349	.2393	.2437	.2482	.2527	.2572	.2617	.2662	.2708	.2753	8	15	23	30	38
52	1.2799	.2846	.2892	.2938	.2985	.3032	.3079	.3127	.3175	.3222	8	16	24	31	39
53	1.3270	.3319	.3367	.3416	.3465	.3514	.3564	.3613	.3663	.3713	8	16	25	33	41
54	1.3764	.3814	.3865	.3916	.3968	.4019	.4071	.4124	.4176	.4229	9	17	26	34	43
55	1.4281	.4335	.4388	.4442	.4496	.4550	.4605	.4659	.4715	.4770	9	18	27	36	45
56	1.4826	.4882	.4938	.4994	.5051	.5108	.5166	.5224	.5282	.5340	10	19	29	38	48
57	1.5399	.5458	.5517	.5577	.5637	.5697	.5757	.5818	.5880	.5941	10	20	30	40	50
58	1.6003	.6066	.6128	.6191	.6255	.6319	.6383	.6447	.6512	.6577	11	21	32	43	53
59	1.6643	.6709	.6775	.6842	.6909	.6977	.7045	.7113	.7182	.7251	11	23	34	45	56
60	1.7321	.7391	.7461	.7532	.7603	.7675	.7747	.7820	.7893	.7966	12	24	36	48	60
61	1.8040	.8115	.8190	.8265	.8341	.8418	.8495	.8572	.8650	.8728	13	26	38	51	64
62	1.8807	.8887	.8967	.9047	.9128	.9210	.9292	.9375	.9458	.9542	14	27	41	55	68
63	1.9626	.9711	.9797	.9883	.9970	**.0057**	.0145	.0233	.0323	.0413	15	29	44	58	73
64	2.0503	.0594	.0686	.0778	.0872	.0965	.1060	.1155	.1251	.1348	16	31	47	63	78
65	2.1445	.1543	.1642	.1742	.1842	.1943	.2045	.2148	.2251	.2355	17	34	51	68	85
66	2.2460	.2566	.2673	.2781	.2889	.2998	.3109	.3220	.3332	.3445	18	37	55	73	92
67	2.3559	.3673	.3789	.3906	.4023	.4142	.4262	.4383	.4504	.4627	20	40	60	79	99
68	2.4751	.4876	.5002	.5129	.5257	.5386	.5517	.5649	.5782	.5916	22	43	65	87	108
69	2.6051	.6187	.6325	.6464	.6605	.6746	.6889	.7034	.7179	.7326	24	47	71	95	119
70	2.7475	.7625	.7776	.7929	.8083	.8239	.8397	.8556	.8716	.8878	26	52	78	104	130
71	2.9042	.9208	.9375	.9544	.9714	.9887	**.0061**	.0237	.0415	.0595	29	58	87	116	144
72	3.0777	.0961	.1146	.1334	.1524	.1716	.1910	.2106	.2305	.2506	32	64	96	129	161
73	3.2709	.2914	.3122	.3332	.3544	.3759	.3977	.4197	.4420	.4646	36	72	108	144	180
74	3,4874	.5105	.5339	.5576	.5816	.6059	.6305	.6554	.6806	.7062	41	81	122	163	204
75	3.7321	.7583	.7848	.8118	.8391	.8667	.8947	.9232	.9520	.9812					
76	4.0108	.0408	.0713	.1022	.1335	.1653	.1976	.2303	.2635	.2972					
77	4.3315	.3662	.4015	.4373	.4737	.5107	.5483	.5864	.6252	.6646					
78	4.7046	.7453	.7867	.8288	.8716	.9152	.9594	**.0045**	.0504	.0970	Use ordinary interpolation.				
79	5.1446	.1929	.2422	.2924	.3435	.3955	.4486	.5026	.5578	.6140					
80	5.6713	.7297	.7894	.8502	.9124	.9758	**.0405**	.1066	.1742	.2432					
81	6.3138	.3859	.4596	.5350	.6122	.6912	.7720	.8548	.9395	**.0264**					
82	7.1154	.2066	.3002	.3962	.4947	.5958	.6996	.8062	.9158	**.0285**					
83	8.1443	.2636	.3863	.5126	.6427	.7769	.9152	**.0579**	.2052	.3572					
84	9.5144	.6768	.8448	**.0187**	.1988	.3854	.5789	.7797	.9882	**.2048**					
85	11.430	11.66	11.91	12.16	12.43	12.71	13.00	13.30	13.62	13.95					
86	14.301	14.67	15.06	15.46	15.89	16.35	16.83	17.34	17.89	18.46					
87	19.081	19.74	20.45	21.20	22.02	22.90	23.86	24.90	26.03	27.27					
88	28.636	30.14	31.82	33.69	35.80	38.19	40.92	44.07	47.74	52.08					
89	57.290	63.66	71.62	81.85	95.49	114.6	143.2	191.0	286.5	573.0					

Heavy-face type indicates that the integral part is to be increased by 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	— Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0	∞	573.0	286.5	191.0	143.2	114.6	95.49	81.85	71.62	63.66	Use ordinary interpolation.				
1	57.290	52.08	47.74	44.07	40.92	38.19	35.80	33.69	31.82	30.14					
2	28.636	27.27	26.03	24.90	23.86	22.90	22.02	21.20	20.45	19.74					
3	19.081	18.46	17.89	17.34	16.83	16.35	15.89	15.46	15.06	14.67					
4	14.301	13.95	13.62	13.30	13.00	12.71	12.43	12.16	11.91	11.66					
5	11.430	.2048	**.9882**	.7797	.5789	.3854	.1988	.0187	**.8448**	.6768					
6	9.5144	.3572	.2052	.0579	**.9152**	.7769	.6427	.5126	.3863	.2636					
7	8.1443	.0285	**.9158**	.8062	.6996	.5958	.4947	.3962	.3002	.2066					
8	7.1154	.0264	**.9395**	.8548	.7720	.6912	.6122	.5350	.4596	.3859					
9	6.3138	.2432	.1742	.1066	.0405	**.9758**	.9124	.8502	.7894	.7297					
10	5.6713	.6140	.5578	.5026	.4486	.3955	.3435	.2924	.2422	.1929					
11	5.1446	.0970	.0504	.0045	**.9594**	.9152	.8716	.8288	.7867	.7453					
12	4.7046	.6646	.6252	.5864	.5483	.5107	.4737	.4373	.4015	.3662					
13	4.3315	.2972	.2635	.2303	.1976	.1653	.1335	.1022	.0713	.0408					
14	4.0108	**.9812**	.9520	.9232	.8947	.8667	.8391	.8118	.7848	.7583					
15	3.7321	.7062	.6806	.6554	.6305	.6059	.5816	.5576	.5339	.5105	41	81	122	163	203
16	3.4874	.4646	.4420	.4197	.3977	.3759	.3544	.3332	.3122	.2914	36	72	108	144	180
17	3.2709	.2506	.2305	.2106	.1910	.1716	.1524	.1334	.1146	.0961	32	64	96	129	161
18	3.0777	.0595	.0415	.0237	.0061	**.9887**	.9714	.9544	.9375	.9208	29	58	87	116	144
19	2.9042	.8878	.8716	.8556	.8397	.8239	.8083	.7929	.7776	.7625	26	52	78	104	130
20	2.7475	.7326	.7179	.7034	.6889	.6746	.6605	.6464	.6325	.6187	24	47	71	95	119
21	2.6051	.5916	.5782	.5649	.5517	.5386	.5257	.5129	.5002	.4876	22	43	65	87	108
22	2.4751	.4627	.4504	.4383	.4262	.4142	.4023	.3906	.3789	.3673	20	40	60	79	99
23	2.3559	.3445	.3332	.3220	.3109	.2998	.2889	.2781	.2673	.2566	18	37	55	73	92
24	2.2460	.2355	.2251	.2148	.2045	.1943	.1842	.1742	.1642	.1543	17	34	51	68	85
25	2.1445	.1348	.1251	.1155	.1060	.0965	.0872	.0778	.0686	.0594	16	31	47	63	78
26	2.0503	.0413	.0323	.0233	.0145	.0057	**.9970**	.9883	.9797	.9711	15	29	44	58	73
27	1.9626	.9542	.9458	.9375	.9292	.9210	.9128	.9047	.8967	.8887	14	27	41	55	68
28	1.8807	.8728	.8650	.8572	.8495	.8418	.8341	.8265	.8190	.8115	13	26	38	51	64
29	1.8040	.7966	.7893	.7820	.7747	.7675	.7603	.7532	.7461	.7391	12	24	36	48	60
30	1.7321	.7251	.7182	.7113	.7045	.6977	.6909	.6842	.6775	.6709	11	23	34	45	56
31	1.6643	.6577	.6512	.6447	.6383	.6319	.6255	.6191	.6128	.6066	11	21	32	43	53
32	1.6003	.5941	.5880	.5818	.5757	.5697	.5637	.5577	.5517	.5458	10	20	30	40	50
33	1.5399	.5340	.5282	.5224	.5166	.5108	.5051	.4994	.4938	.4882	10	19	29	38	48
34	1.4826	.4770	.4715	.4659	.4605	.4550	.4496	.4442	.4388	.4335	9	18	27	36	45
35	1.4281	.4229	.4176	.4124	.4071	.4019	.3968	.3916	.3865	.3814	9	17	26	34	43
36	1.3764	.3713	.3663	.3613	.3564	.3514	.3465	.3416	.3367	.3319	8	16	25	33	41
37	1.3270	.3222	.3175	.3127	.3079	.3032	.2985	.2938	.2892	.2846	8	16	24	31	39
38	1.2799	.2753	.2708	.2662	.2617	.2572	.2527	.2482	.2437	.2393	8	15	23	30	38
39	1.2349	.2305	.2261	.2218	.2174	.2131	.2088	.2045	.2002	.1960	7	14	22	29	36
40	1.1918	.1875	.1833	.1792	.1750	.1708	.1667	.1626	.1585	.1544	7	14	21	28	34
41	1.1504	.1463	.1423	.1383	.1343	.1303	.1263	.1224	.1184	.1145	7	13	20	26	33
42	1.1106	.1067	.1028	.0990	.0951	.0913	.0875	.0837	.0799	.0761	6	13	19	25	32
43	1.0724	.0686	.0649	.0612	.0575	.0538	.0501	.0464	.0428	.0392	6	12	18	25	31
44	1.0355	.0319	.0283	.0247	.0212	.0176	.0141	.0105	.0070	.0035	6	12	18	24	30

Heavy-face type indicates that the integral part is to be decreased by 1.

°	0.0°	0.1°	0.2°	0.3°	0.4°	0.5°	0.6°	0.7°	0.8°	0.9°	– Differences				
	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45	1.0000	**.9965**	.9930	.9896	.9861	.9827	.9793	.9759	.9725	.9691	6	11	17	23	29
46	0.9657	.9623	.9590	.9556	.9523	.9490	.9457	.9424	.9391	.9358	6	11	17	22	28
47	0.9325	.9293	.9260	.9228	.9195	.9163	.9131	.9099	.9067	.9036	5	11	16	21	27
48	0.9004	.8972	.8941	.8910	.8878	.8847	.8816	.8785	.8754	.8724	5	10	16	21	26
49	0.8693	.8662	.8632	.8601	.8571	.8541	.8511	.8481	.8451	.8421	5	10	15	20	25
50	0.8391	.8361	.8332	.8302	.8273	.8243	.8214	.8185	.8156	.8127	5	10	15	20	24
51	0.8098	.8069	.8040	.8012	.7983	.7954	.7926	.7898	.7869	.7841	5	9	14	19	24
52	0.7813	.7785	.7757	.7729	.7701	.7673	.7646	.7618	.7590	.7563	5	9	14	18	23
53	0.7536	.7508	.7481	.7454	.7427	.7400	.7373	.7346	.7319	.7292	5	9	14	18	23
54	0.7265	.7239	.7212	.7186	.7159	.7133	.7107	.7080	.7054	.7028	4	9	13	18	22
55	0.7002	.6976	.6950	.6924	.6899	.6873	.6847	.6822	.6796	.6771	4	9	13	17	21
56	0.6745	.6720	.6694	.6669	.6644	.6619	.6594	.6569	.6544	.6519	4	8	13	17	21
57	0.6494	.6469	.6445	.6420	.6395	.6371	.6346	.6322	.6297	.6273	4	8	12	16	20
58	0.6249	.6224	.6200	.6176	.6152	.6128	.6104	.6080	.6056	.6032	4	8	12	16	20
59	0.6009	.5985	.5961	.5938	.5914	.5890	.5867	.5844	.5820	.5797	4	8	12	16	20
60	0.5774	.5750	.5727	.5704	.5681	.5658	.5635	.5612	.5589	.5566	4	8	12	15	19
61	0.5543	.5520	.5498	.5475	.5452	.5430	.5407	.5384	.5362	.5340	4	8	11	15	19
62	0.5317	.5295	.5272	.5250	.5228	.5206	.5184	.5161	.5139	.5117	4	7	11	15	18
63	0.5095	.5073	.5051	.5029	.5008	.4986	.4964	.4942	.4921	.4899	4	7	11	15	18
64	0.4877	.4856	.4834	.4813	.4791	.4770	.4748	.4727	.4706	.4684	4	7	11	14	18
65	0.4663	.4642	.4621	.4599	.4578	.4557	.4536	.4515	.4494	.4473	4	7	11	14	18
66	0.4452	.4431	.4411	.4390	.4369	.4348	.4327	.4307	.4286	.4265	3	7	10	14	17
67	0.4245	.4224	.4204	.4183	.4163	.4142	.4122	.4101	.4081	.4061	3	7	10	14	17
68	0.4040	.4020	.4000	.3979	.3959	.3939	.3919	.3899	.3879	.3859	3	7	10	13	17
69	0.3839	.3819	.3799	.3779	.3759	.3739	.3719	.3699	.3679	.3659	3	7	10	13	17
70	0.3640	.3620	.3600	.3581	.3561	.3541	.3522	.3502	.3482	.3463	3	7	10	13	16
71	0.3443	.3424	.3404	.3385	.3365	.3346	.3327	.3307	.3288	.3269	3	6	10	13	16
72	0.3249	.3230	.3211	.3191	.3172	.3153	.3134	.3115	.3096	.3076	3	6	10	13	16
73	0.3057	.3038	.3019	.3000	.2981	.2962	.2943	.2924	.2905	.2886	3	6	9	13	16
74	0.2867	.2849	.2830	.2811	.2792	.2773	.2754	.2736	.2717	.2698	3	6	9	13	16
75	0.2679	.2661	.2642	.2623	.2605	.2586	.2568	.2549	.2530	.2512	3	6	9	12	16
76	0.2493	.2475	.2456	.2438	.2419	.2401	.2382	.2364	.2345	.2327	3	6	9	12	15
77	0.2309	.2290	.2272	.2254	.2235	.2217	.2199	.2180	.2162	.2144	3	6	9	12	15
78	0.2126	.2107	.2089	.2071	.2053	.2035	.2016	.1998	.1980	.1962	3	6	9	12	15
79	0.1944	.1926	.1908	.1890	.1871	.1853	.1835	.1817	.1799	.1781	3	6	9	12	15
80	0.1763	.1745	.1727	.1709	.1691	.1673	.1655	.1638	.1620	.1602	3	6	9	12	15
81	0.1584	.1566	.1548	.1530	.1512	.1495	.1477	.1459	.1441	.1423	3	6	9	12	15
82	0.1405	.1388	.1370	.1352	.1334	.1317	.1299	.1281	.1263	.1246	3	6	9	12	15
83	0.1228	.1210	.1192	.1175	.1157	.1139	.1122	.1104	.1086	.1069	3	6	9	12	15
84	0.1051	.1033	.1016	.0998	.0981	.0963	.0945	.0928	.0910	.0892	3	6	9	12	15
85	0.0875	.0857	.0840	.0822	.0805	.0787	.0769	.0752	.0734	.0717	3	6	9	12	15
86	0.0699	.0682	.0664	.0647	.0629	.0612	.0594	.0577	.0559	.0542	3	6	9	12	15
87	0.0524	.0507	.0489	.0472	.0454	.0437	.0419	.0402	.0384	.0367	3	6	9	12	15
88	0.0349	.0332	.0314	.0297	.0279	.0262	.0244	.0227	.0209	.0192	3	6	9	12	15
89	0.0175	.0157	.0140	.0122	.0105	.0087	.0070	.0052	.0035	.0017	3	6	9	12	15

All cotangents of angles greater than 45° are less than 1.

POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2 601	132 651	7.141	3.708
2	4	8	1.414	1.260	52	2 704	140 608	7.211	3.733
3	9	27	1.732	1.442	53	2 809	148 877	7.280	3.756
4	16	64	2.000	1.587	54	2 916	157 464	7.348	3.780
5	25	125	2.236	1.710	55	3 025	166 375	7.416	3.803
6	36	216	2.449	1.817	56	3 136	175 616	7.483	3.826
7	49	343	2.646	1.913	57	3 249	185 193	7.550	3.849
8	64	512	2.828	2.000	58	3 364	195 112	7.616	3.871
9	81	729	3.000	2.080	59	3 481	205 379	7.681	3.893
10	100	1 000	3.162	2.154	60	3 600	216 000	7.746	3.915
11	121	1 331	3.317	2.224	61	3 721	226 981	7.810	3.936
12	144	1 728	3.464	2.289	62	3 844	238 328	7.874	3.958
13	169	2 197	3.606	2.351	63	3 969	250 047	7.937	3.979
14	196	2 744	3.742	2.410	64	4 096	262 144	8.000	4.000
15	225	3 375	3.873	2.466	65	4 225	274 625	8.062	4.021
16	256	4 096	4.000	2.520	66	4 356	287 496	8.124	4.041
17	289	4 913	4.123	2.571	67	4 489	300 763	8.185	4.062
18	324	5 832	4.243	2.621	68	4 624	314 432	8.246	4.082
19	361	6 859	4.359	2.668	69	4 761	328 509	8.307	4.102
20	400	8 000	4.472	2.714	70	4 900	343 000	8.367	4.121
21	441	9 261	4.583	2.759	71	5 041	357 911	8.426	4.141
22	484	10 648	4.690	2.802	72	5 184	373 248	8.485	4.160
23	529	12 167	4.796	2.844	73	5 329	389 017	8.544	4.179
24	576	13 824	4.899	2.884	74	5 476	405 224	8.602	4.198
25	625	15 625	5.000	2.924	75	5 625	421 875	8.660	4.217
26	676	17 576	5.099	2.962	76	5 776	438 976	8.718	4.236
27	729	19 683	5.196	3.000	77	5 929	456 533	8.775	4.254
28	784	21 952	5.292	3.037	78	6 084	474 552	8.832	4.273
29	841	24 389	5.385	3.072	79	6 241	493 039	8.888	4.291
30	900	27 000	5.477	3.107	80	6 400	512 000	8.944	4.309
31	961	29 791	5.568	3.141	81	6 561	531 441	9.000	4.327
32	1 024	32 768	5.657	3.175	82	6 724	551 368	9.055	4.344
33	1 089	35 937	5.745	3.208	83	6 889	571 787	9.110	4.362
34	1 156	39 304	5.831	3.240	84	7 056	592 704	9.165	4.380
35	1 225	42 875	5.916	3.271	85	7 225	614 125	9.220	4.397
36	1 296	46 656	6.000	3.302	86	7 396	636 056	9.274	4.414
37	1 369	50 653	6.083	3.332	87	7 569	658 503	9.327	4.431
38	1 444	54 872	6.164	3.362	88	7 744	681 472	9.381	4.448
39	1 521	59 319	6.245	3.391	89	7 921	704 969	9.434	4.465
40	1 600	64 000	6.325	3.420	90	8 100	729 000	9.487	4.481
41	1 681	68 921	6.403	3.448	91	8 281	753 571	9.539	4.498
42	1 764	74 088	6.481	3.476	92	8 464	778 688	9.592	4.514
43	1 849	79 507	6.557	3.503	93	8 649	804 357	9.644	4.531
44	1 936	85 184	6.633	3.530	94	8 836	830 584	9.695	4.547
45	2 025	91 125	6.708	3.557	95	9 025	857 375	9.747	4.563
46	2 116	97 336	6.782	3.583	96	9 216	884 736	9.798	4.579
47	2 209	103 823	6.856	3.609	97	9 409	912 673	9.849	4.595
48	2 304	110 592	6.928	3.634	98	9 604	941 192	9.899	4.610
49	2 401	117 649	7.000	3.659	99	9 801	970 299	9.950	4.626
50	2 500	125 000	7.071	3.684	100	10 000	1 000 000	10.000	4.642

Exercises. Mensuration of Plane Figures

1. Construct a triangle with sides of 2.3″, 3.2″, and 3.8″ respectively. Measure any altitude and find the area of the triangle. Check the work by using another altitude.

In measuring line segments use a pair of dividers to transfer the lengths to a graduated ruler.

2. Draw a circle with a radius of 4.8″, take a point 8″ from the center, and from this point draw a tangent. Find the length of the tangent by measuring and check it by § 222.

3. Construct an equilateral triangle which shall have an area of 3 sq. in. Check the work by measuring the base and the altitude and finding the area from these results.

4. If a phonograph record 10″ in diameter costs $1.25, and if the price is based upon the total area of the disk, how much should a record 12″ in diameter cost?

5. Find the area of a parallelogram of which two sides are 2″ and 3″ respectively, and the included angle is 52° 30′. Check the result by measuring an altitude.

6. Two sides of a parallelogram are 10′ and 12′ respectively and the included angle is 45°. Find the area and the length of each diagonal.

7. If the diagonal of the $\square ABCD$ is 3.2″ and $\angle BAC = 32°$, what is the area of the rectangle? Check the result by drawing the figure and measuring the sides.

8. The diagonals of a rhombus are 50″ and 34″ respectively. Find the length of each side, the size of each angle, and the area.

In finding the size of one angle, we have $\tan x = \frac{25}{17} = 1.4706$. Looking for 1.4706 in the table of tangents, we find on page 459 that 1.4706 lies between 1.4659 and 1.4715, the tangents of 55.7° and 55.8° respectively. Since accuracy to the nearest 0.1° is sufficient and 1.4706 is nearer 1.4715 than it is 1.4659, we see that $x = 55.8°$.

9. Within a square which is 2″ on a side inscribe the largest possible equilateral triangle and find its area.

The vertices of the triangle must lie on the sides of the square.

10. Express the results of § 234 and § 235 trigonometrically without using b'.

By the aid of Simpson's Rule find the area between the curve and the x axis in Exs. 11 and 12:

11. Ordinates: 14′, 16′, 17′, 15′, 13′, 12′, 14′; common distance between the successive ordinates, 4′.

12. Ordinates: 0″, 1.3″, 2.4″, 3.5″, 4.6″, 5.7″, 6.8″, 5.07″, 4.06″, 3.05″, 2.04″; common distance between the successive ordinates, 2.4″.

13. Show that the distance m in miles to the horizon from a point f feet above the surface of the sea is given approximately by the formula $m = \frac{1}{2}\sqrt{6f}$.

Use § 253 and take the radius of the earth as 4000 mi.

14. If the top of a ship's mast is 60′ above the sea, how far must the ship sail before it disappears below the horizon?

15. A sign painter who is laying out a clock face wishes to show the time as 8 18. Draw a circle with a radius of 2″ and show accurately the position of the hands.

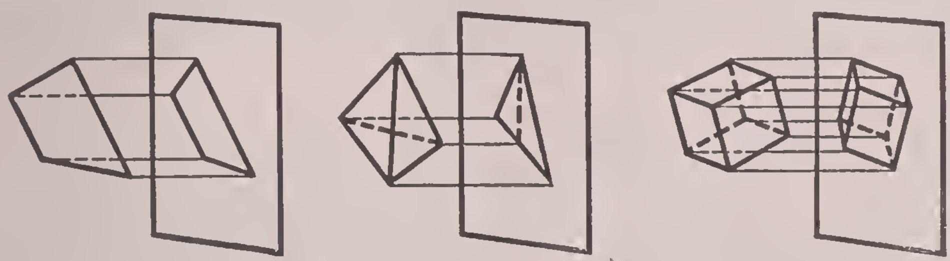
16. A hexagonal nut $ABCDEF$ is $\frac{5}{8}''$ on a side. Find the "distance across the flats" (that is, from AB to DE), and the "distance across the corners" (that is, AD).

The quoted terms are used by machinists.

17. The approximate area A of a segment of a circle which is cut off by a chord c and which has a height h is given by the formula $A = \frac{2}{3}hc + h^3/2c$. What is the approximate cross-section area of water flowing through a horizontal pipe, when $c = 26''$ and $h = 12.5''$?

549. Drawing of Solid Figures. The student is expected to draw the necessary figures with the help of ruler, compasses, and protractor. The measurement of line segments is allowed in all cases.

Solids are represented as if projected upon a plane. We may conveniently imagine the projections as shadows cast by wire models of the solids, as in the following figures:

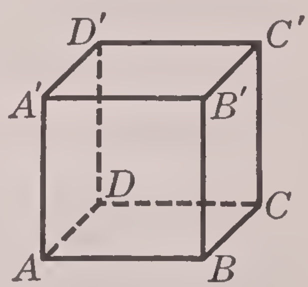


We may think of the shadow as cast by the sun's rays, which for practical purposes are considered parallel. If the rays are perpendicular to the plane of projection, the projection is known as *orthogonal projection*, the word "orthogonal" meaning right-angled.

The first figure above shows the orthogonal projection of a quadrilateral, the second that of a tetrahedron, and the third that of a cube.

550. Oblique Projection. If the rays which cast the shadows are oblique to the plane of projection, the figure formed is said to be in *oblique projection.*

A convenient method of representing figures in oblique projection is illustrated by the cube here shown. In this case the rear and front faces ($ABB'A'$ and $DCC'D'$) are squares, the lines DA, CB, $C'B'$, $D'A'$ are parallel, $\angle D'C'B' = 45°$, and $C'B'$ is half of $C'D'$. All angles in such a projection are therefore 45°, 90°, or 135°.

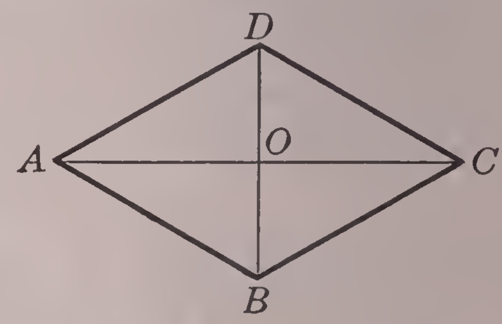


This type of oblique projection, sometimes called *cabinet projection,* will be used in the exercises which follow. The student is expected to be able to reproduce the figures and to draw others of the same general nature.

It is sometimes convenient to take $\angle DCB$ as 30° or as 60°, and to take $B'C'$ as three fourths of $C'D'$, but this is not often necessary.

Exercises. Mensuration of Solids

1. This figure shows the orthogonal projection upon a parallel plane of a rhombus forming the base of a given right parallelepiped. It therefore shows the base in correct proportions. Measure $\angle BAD$ with the protractor and measure the side AB. From these measurements compute the length of each diagonal, and check the results by actual measurement.

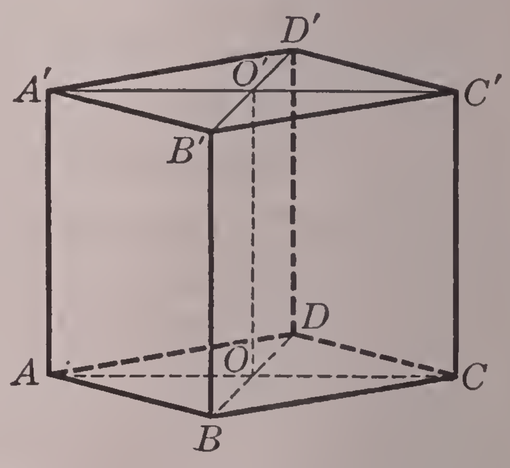


If $\angle BAD = 60°$ and $AB = 0.75''$, we have $\angle BAC = \angle CAD = 30°$ and $BD = AB = 0.75''$. Then, since $\angle AOB = 90°$, we have $AO = AB \cos 30° = 0.75'' \times 0.8660$. Hence $AC = 1.5 \times 0.8660'' = 1.299''$, or $1.3''$.

2. In the figure of Ex. 1 if $AB = 14.5''$ and $\angle BAD = 56°$, what are the lengths of AC and BD?

The student should reproduce the figure of Ex. 1 for these proportions. In most exercises it will be sufficient to compute dimensions to the nearest $0.1''$ and angles to the nearest $0.1°$.

3. This figure shows the right parallelepiped of Ex. 1 projected as described in § 550. If the lateral edge AA' is $17.8''$ and the base has the dimensions given in Ex. 2, what is the area of the total surface of the parallelepiped?

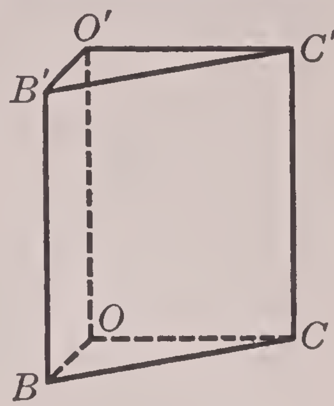


The student should draw the figure for these dimensions, and similarly in each of the exercises where new dimensions are given.

4. In Ex. 3 compute the length of the longest diagonal of the parallelepiped; of the shortest diagonal.

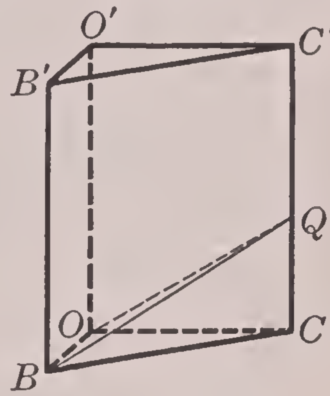
5. In Ex. 3 compute the volume of the parallelepiped.

6. This figure shows one of the triangular prisms formed by passing planes through the diagonals of the figure of Ex. 3. Using the measurements given in Exs. 2 and 3, compute the area of the total surface of the prism.



By referring to the figure of Ex. 1 it is seen that $\angle BOC = 90°$. In this and all similar exercises, any results obtained in a previous exercise may be used in the solution.

7. This figure shows the triangular prism of Ex. 6 cut by a plane through BO and a point Q on CC'. Given that $\angle COQ = 30°$, compute the area of $\triangle OCQ$.



Since, as in all figures in these exercises, the projection used is that of § 550, the student will find that the $\triangle OCQ$ appears in correct proportions. Now, using the length of OC found in Ex. 2, we have $CQ = OC \tan COQ$.

8. In Ex. 7 find the areas of $\triangle\triangle BQO$ and BCQ, and the area of the total surface of the tetrahedron $Q\text{-}BCO$.

It will be helpful to draw $\triangle\triangle BQO$ and BCQ in correct proportions. It is not necessary to draw $\triangle BCO$ since this triangle appears in correct proportions in the figure drawn for Ex. 2.

9. In Ex. 7 find the area of the total surface of the truncated prism $B'C'O'\text{-}BQO$.

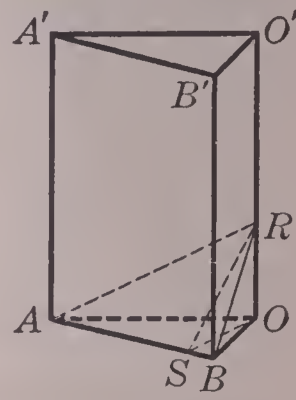
10. Show that the area of $\triangle BCO$ in Ex. 8 is equal to the area of $\triangle BQO$ multiplied by $\cos COQ$.

First show that $OC = OQ \cos COQ$, and then use this value in finding the area of $\triangle BCO$.

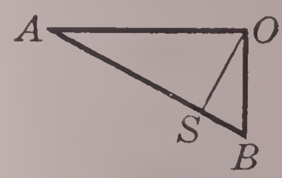
11. In the figure of Ex. 7 given that $OC = 12.4''$ and $CQ = 10.4''$, find the angle which the plane BQO makes with the base.

Since OB is $\perp$ to OC, $\angle COQ$ is the plane angle of $\angle C\text{-}BO\text{-}Q$.

12. This figure shows the prism A'-ABO formed by passing planes through the diagonals of the figure of Ex. 3. In this prism a plane is passed through AB and a point R on OO'. Also, OS is $\perp$ to AB so that $\angle OSR$ is the plane angle of the dihedral $\angle O$-BA-R. Given that the prism has the same dimensions as in Exs. 2 and 3 (that is, that $AB = 14.5''$, $\angle BAO = 28°$, and $AA' = 17.8''$), and that $\angle OSR = 45°$, find the lengths of OS and OR.



Draw the $\triangle ABO$ in correct proportions and construct $OS \perp$ to AB, as shown in this figure. Then, using the length of OA found in Ex. 2, we have $OS = OA \sin BAO$. Hence the length of OS can be found. Then, since $\angle OSR = 45°$, we have $OR = OS$.



13. In Ex. 12 find the areas of $\triangle AOR$, ABR, and BOR.

Draw the $\triangle ABR$ and BOR in correct proportions.

14. In Ex. 12 find $\angle OBR$ and the length of BR; find $\angle OAR$ and the length of AR.

15. In Ex. 13 show that the area of $\triangle ABO$ is equal to the area of $\triangle ABR$ multiplied by $\cos OSR$.

16. In Ex. 12 find the volume of the tetrahedron R-ABO and the volume of the truncated prism $A'B'O'$-ABR.

Ex. 5 may be used in finding the volume of the prism A'-ABO.

17. In Ex. 12, suppose that $\angle OSR = 30°$, and find the lengths of OS and OR.

In this case OS and OR are not equal.

18. Consider Ex. 13 for the case of Ex. 17.

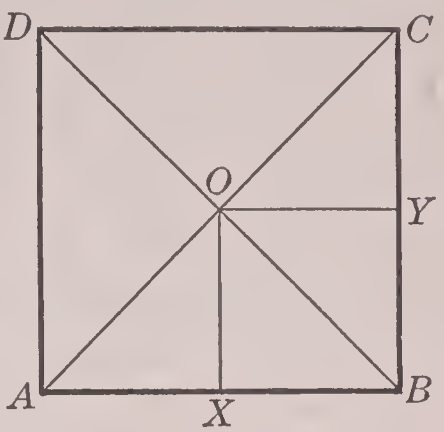
19. Consider Ex. 14 for the case of Ex. 17.

20. Consider Ex. 12, letting $\angle BAO = \frac{1}{2}\angle OSR = 15°$.

21. Consider Ex. 13 for the case of Ex. 20.

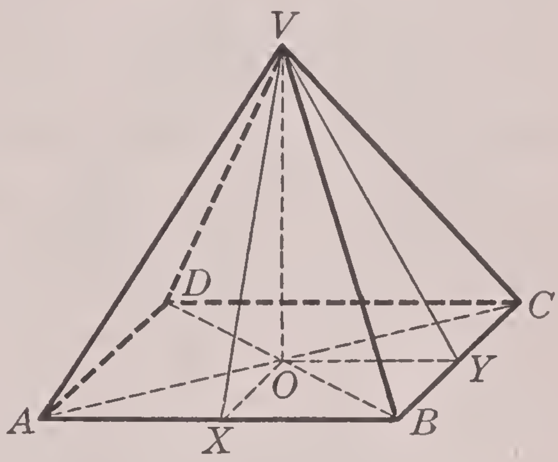
22. Consider Ex. 14 for the case of Ex. 20.

23. This figure shows the orthogonal projection of a regular pyramid with a square base upon a plane parallel to the base. Given that $AB = 24''$, compute the lengths of the diagonals AC and BD, and the lengths of the $\perp$s OX and OY upon AB and BC respectively.



The student should notice that in this projection the vertex V of the pyramid coincides with O, the center of the base.

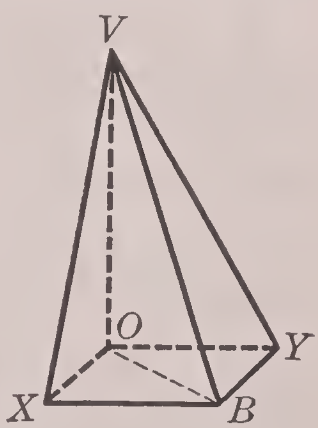
24. This figure shows the regular pyramid of Ex. 23 projected as described in § 550. Given that the altitude of the pyramid is $20.8''$, compute the volume.



25. In Ex. 24 compute the length of a lateral edge BV of the pyramid, and the $\angle VBO$ which this edge makes with the base.

Draw in correct proportions the triangle made by passing a plane through VO and the vertex B.

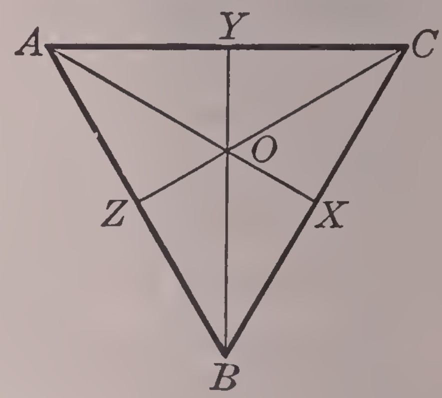
26. This figure shows the pyramid cut from the figure of Ex. 24 by passing planes through VO and OX, and through VO and OY. Compute the length of the lateral edge VY and the $\angle VYO$ which VY makes with the base.



27. In Ex. 24 compute the area of the total surface of the original pyramid, and show that the area of the base is equal to the lateral area multiplied by cos VYO.

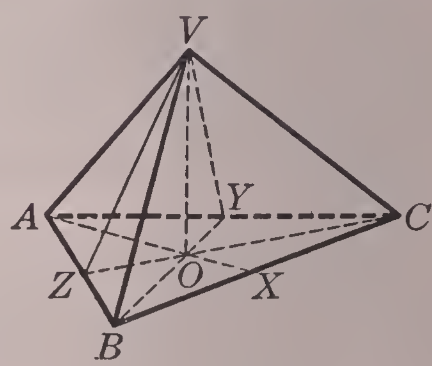
The edge VY in Ex. 26 is the slant height of the original pyramid.

28. This figure shows the base of a regular pyramid projected orthogonally upon a parallel plane. In the figure, AX, BY, and CZ are the altitudes from the vertices A, B, and C respectively. If $AB = 10''$, what is the area of the base of the pyramid?



The special formula of § 544 for the area of an equilateral triangle should be used.

29. This figure shows the pyramid of Ex. 28 projected as described in § 550. If the lateral edges make an angle of 45° with the base (that is, $\angle OBV = 45°$), what is the altitude of the pyramid?

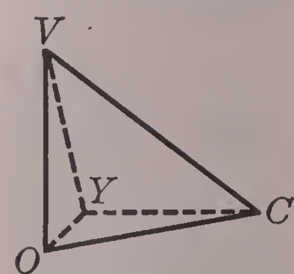


Draw the $\triangle BOV$ in correct proportions. Remember that the medians of a triangle meet in a point two thirds of the distance from the vertex to the opposite side.

30. In Ex. 29 compute the volume of the pyramid.

31. In Ex. 29 compute the slant height VZ and the angle made by the lateral faces with the base.

32. This figure shows the pyramid formed by passing planes through VO and OC and VO and OY in the figure of Ex. 29. Compute the area of the total surface.

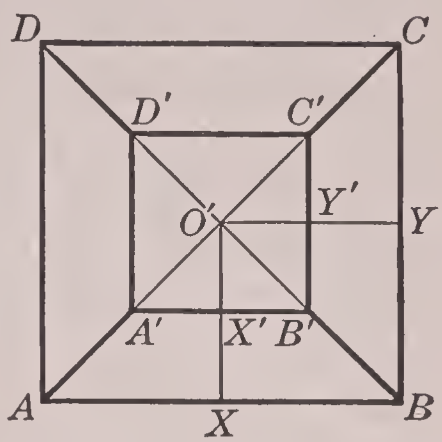


33. In Ex. 29 how far from the vertex V will a plane parallel to the base cut off a pyramid of half the volume?

Since the volumes of two similar solids are proportional to the cubes of any two corresponding lines, what relation exists between the altitude of the small pyramid and that of the original pyramid?

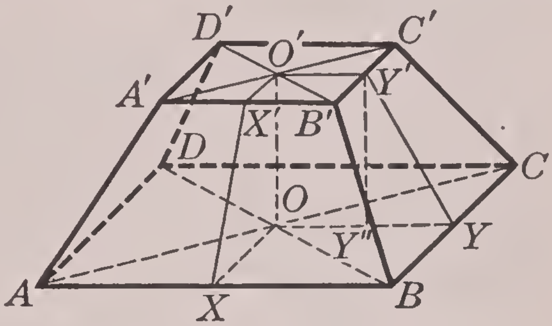
34. If the pyramid in Ex. 29 weighs 20 oz., how far from the vertex V should a plane parallel to the base be passed so as to cut off a small pyramid which shall weigh 5 oz.?

35. This figure shows the orthogonal projection of a frustum of a regular pyramid with a square base upon a plane parallel to the bases. In the figure, $O'X'$ and $O'Y'$ are $\perp$ to $A'B'$ and $B'C'$ respectively. Given that $AB = 14''$ and $A'B' = 7''$, find the areas of the upper and lower bases of the frustum.



36. This figure shows the frustum of Ex. 35 projected as described in § 550. Given that the lateral faces make an angle of 60° with the base, compute the altitude OO' and the slant height YY'.

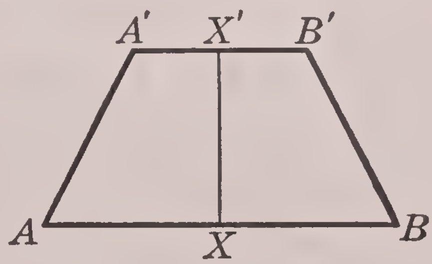
In the figure the section $OYY'O'$ appears in correct proportions.



37. In Ex. 36 compute the volume of the frustum by the formula given in § 545.

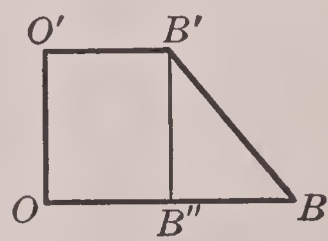
38. In Ex. 36 find the area of the total surface of the frustum.

Draw the face $ABB'A'$ in correct proportions as here shown.



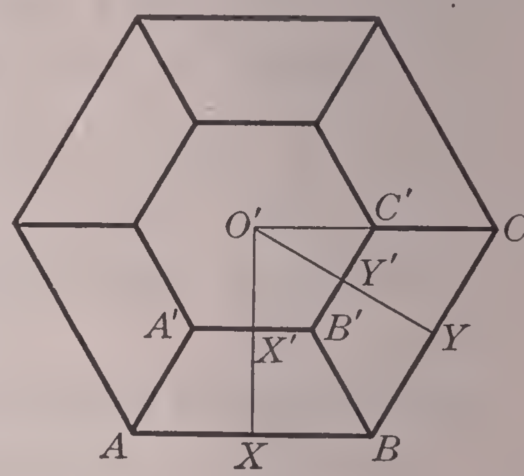
39. In Ex. 36 compute the length of BB' and the angles which the lateral edges make with the lower base and with the upper base.

Draw the section $OBB'O'$ in correct proportions as here shown.



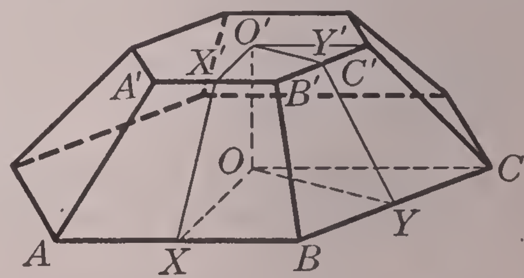
40. Draw the figure and compute the area of the total surface of the solid formed by passing planes through OO' and OX and through OO' and OY in the figure of Ex. 36.

41. This figure shows the orthogonal projection of a frustum of a regular hexagonal pyramid upon a plane parallel to the bases. Given that $O'X'$ is $\perp$ to $A'B'$, $AB = 20''$, and $A'B' = 10''$, find the areas of the upper and lower bases.



The student should use the special formula of § 544 for the area of a regular hexagon. After finding the area of the upper base, that of the lower base can be found by an easy multiplication.

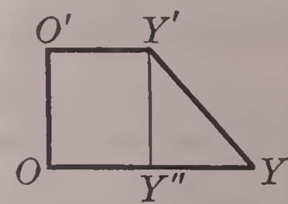
42. This figure shows the frustum of Ex. 41 projected as described in § 550. If the lateral edges make an angle of 45° with the lower base, what is the altitude of the frustum?



43. In Ex. 42 compute the volume of the frustum.

The student should use the formula in § 545 and the results of Ex. 41.

44. In Ex. 42 compute the slant height of the frustum and the angles which the lateral faces make with the upper and lower bases.



Draw the section $OYY'O'$ in correct proportions as here shown.

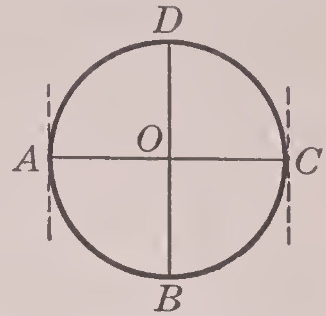
45. In Ex. 42 find the number of degrees in each angle of a lateral face.

Draw the face $ABB'A'$ in correct proportions.

46. In Ex. 42 compute the area of the total surface of the frustum.

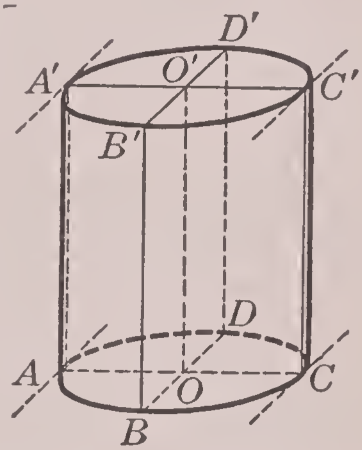
47. Draw the figure and compute the area of the total surface of the solid formed by passing planes through OO' and OX and OO' and OC in the figure of Ex. 42.

48. This figure shows the base of a cylinder of revolution projected orthogonally upon a parallel plane. Given that the diameter of the cylinder is 2.8″, compute the circumference and the area of the base.



The tangent lines are helpful in drawing the projection of the base, as explained in Ex. 49.

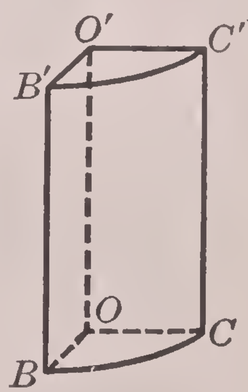
49. This figure shows the cylinder of Ex. 48 projected as described in § 544. If the altitude of the cylinder is 3.5″, what is the volume?



It should be noticed that the projections of the upper and lower bases are ellipses. In drawing the figure for Ex. 49, the student should draw these ellipses freehand.

50. In Ex. 49 compute the area of the curve surface; of the total surface.

51. This figure shows one fourth of the cylinder of Ex. 49, formed by passing planes through OO' and OB and OO' and OC. Compute the area of the total surface of this part of the cylinder.

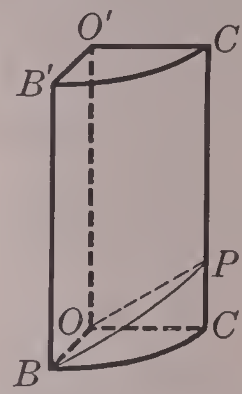


The difficulties of measuring the complete cylinder in projection can usually be avoided by taking one fourth of the cylinder as here shown.

52. *The locus of points at a distance r from a given straight line is the curve surface of a cylinder of revolution which has the radius r and the given line as axis.*

53. *The locus of points whose distances from a given plane and a given line perpendicular to that plane are in a fixed ratio is the curve surface of a cone of revolution which has the given line for its axis and the point of intersection of the line and the plane for its vertex.*

54. This figure shows the section of the figure of Ex. 51 made by a plane passed through OB at an angle of 30° with the base of the cylinder and cutting the element CC' at P. Find the area of the section OBP.

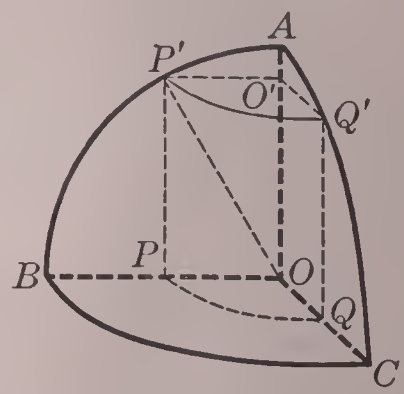


Since OB is $\perp$ to the section $OCC'O'$, the dihedral angle between the plane OBP and the base is measured by $\angle COP$. In Ex. 9 we proved that, in the figure of Ex. 7, $\triangle BCO = \triangle BQO \cdot \cos COQ$, and hence we may legitimately assume that a similar relation holds for this figure. Then since $\angle COP$ and the area of BCO are known, the area of OBP can be found.

55. In Ex. 54 compute the area of the curve surface BCP and of the entire surface of the solid $POBC$.

By analogy to the diagonals of a parallelogram, BP bisects the cylindric surface which has the radius OC and the height CP.

56. This figure shows the intersection of a fourth of a cylindric surface and an eighth of a sphere whose center O is on the axis of the cylindric surface. If the radius of the cylindric surface is half the radius of the sphere, the height of the cylindric surface is what part of the radius of the sphere?

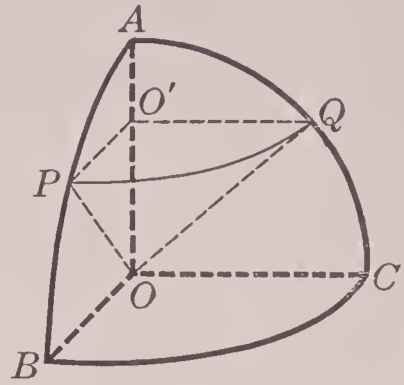


Since $OP = \frac{1}{2} OP'$, what is the size of $\angle P'OP$?

57. Assuming the figure of Ex. 56 completed to show the entire hemisphere and the cylindric surface, find the ratio of the area of the cylindric surface to the area of the hemisphere.

58. In the figure of Ex. 57 given that the radius of the sphere is 4.2″, find the area of the zone whose upper base is the intersection of the cylindric surface and the hemisphere.

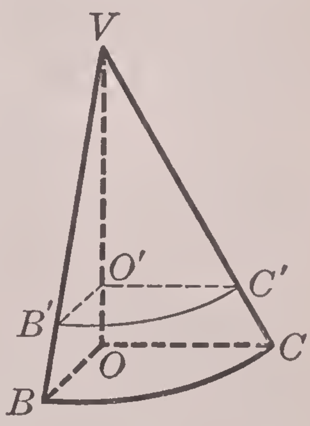
59. This figure shows the intersection of a fourth of a cone of revolution whose vertex is at O with an eighth of a sphere of center O. Given that the radius of the sphere is 6.3″ and that $\angle QOO' = 50°$, find the radius of the circular section made by the conic surface on the sphere.



60. Assuming the figure of Ex. 59 completed to show the entire hemisphere and the cone, find the ratio of the volume of the cone to that of the hemisphere.

61. In Ex. 60 find the volume of the spherical segment which lies above the intersection of the conic surface and the hemisphere.

62. The figure below shows one fourth of a cone of revolution. Given that the radius of the base is 3.5″ and that $\angle OVC = 30°$, compute the altitude of the cone and the angle which an element makes with the base.



63. In Ex. 62 compute the volume of the entire cone and the area of the curve surface.

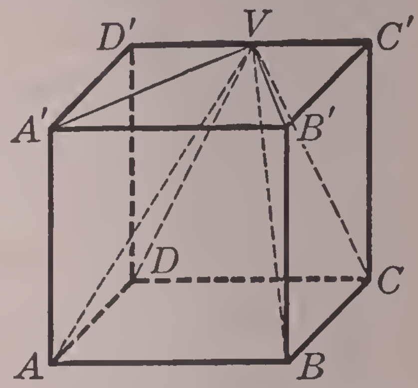
64. In Ex. 62 how far from the vertex of the cone must the plane $B'C'O'$ be passed in order to bisect the volume of the entire cone?

65. In Ex. 64 compute the volume of the frustum by the formula in § 545 and check by taking half the first result of Ex. 63.

66. Consider Ex. 64, supposing that the plane $B'C'O'$ is to cut off a frustum equivalent to one third the cone.

67. In Ex. 66 compute the volume of the frustum and check as in Ex. 65.

68. This figure shows a cube 18″ on an edge projected as described in § 550. The midpoint V of $C'D'$ is the vertex of an inscribed pyramid. Compute the volume and the lateral edges of this pyramid.



Draw the orthogonal projection of the face $A'B'C'D'$ as in Ex. 23, and then draw the $\triangle A'VA$ in correct proportions.

69. In Ex. 68 draw each face of the pyramid in correct proportions, and compute the altitude of each one.

70. In Ex. 68 compute the area of the total surface of the pyramid.

71. In Ex. 68 compute the size of each angle of $\triangle VAD$; of $\triangle VDC$; of $\triangle VAB$.

72. In Ex. 68 suppose that a plane bisecting the cube is passed parallel to the base $ABCD$, thus cutting off a frustum of the pyramid. Draw the figure and compute the volume and the area of the total surface of the frustum.

73. If another pyramid with the base $BCC'B$ and the vertex A is inscribed in the cube of Ex. 68, how does its volume compare with that of the pyramid $V\text{–}ABCD$?

In each case, state the locus of points in space which satisfy the following conditions, and draw the figure:

74. Equidistant from two intersecting planes, and at a distance r from a fixed point O.

75. At a distance d from a line through two fixed points A and O, and at a distance r greater than d from O.

76. Equidistant from two planes which intersect in the line AB and at a distance r from AB.

77. At a distance D from a given plane m and at a distance r from a line PQ which is perpendicular to m.

78. A regular pyramid with a square base $8''$ on a side has an altitude of $10''$. Compute the volume of the solid and the area of the total surface.

79. Consider Ex. 78 for a regular pentagonal pyramid with the same side and altitude.

80. Consider Ex. 78 for a regular hexagonal pyramid with the same side and altitude.

81. Consider Ex. 78 for a regular octagonal pyramid with the same side and altitude.

82. The lower base of a frustum of a regular pyramid is a square $6''$ on a side, the altitude of the frustum is $8''$, and the side of the upper base is half that of the lower base. Compute the volume of the frustum.

83. In Ex. 82 compute the area of the total surface.

84. Consider Exs. 82 and 83 for a frustum of a regular pentagonal pyramid with the same dimensions.

85. Consider Exs. 82 and 83 for a frustum of a regular hexagonal pyramid with the same dimensions.

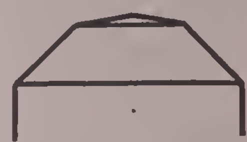
86. Consider Exs. 82 and 83 for a frustum of a regular octagonal pyramid with the same dimensions.

87. A water tank open at the top is to be built of sheet iron in the form of a frustum of a cone. The diameter of the lower base is to be $14'$, that of the upper base $10\frac{1}{2}'$, and the tank is to contain 1500 cu. ft. If an additional 15% of the surface area is allowed for waste and overlapping, how many square feet of sheet iron are required to build the tank?

88. A steel container is in the form of a cylinder with a hemispherical top. If the inside length of the cylinder is $4'$ and the diameter is $1'6''$, what is the volume?

89. In Ex. 88 find the area of the entire inside surface.

90. A cylindric silo 30′ in diameter is covered by the roof shown in this figure. The lower part of the roof is in the form of a frustum of a cone whose upper base has a diameter of 15′ and whose sloping sides make an angle of 45° with the lower base. The upper part is a conic surface which makes an angle of 10° with the upper base of the frustum. Making an allowance of 20% for waste, compute the number of square feet of material required to cover this roof.



91. Find the diameter of a solid sphere formed by melting and recasting the metal in two solid spheres of lead which are 2″ and 3″ respectively in diameter.

92. If a spherical drop of water is vaporized into a spray of 1000 equal spheres, but the total volume of water is unchanged, by what per cent is the total surface increased?

93. A cross section of a hollow cast-iron pillar 10′ long is a triangle whose sides are 6″, 8″, and 10″, and the thickness of the metal is 1″. If cast iron weighs 450 lb. per cubic foot, how much does the pillar weigh?

94. A cubic foot of brass is drawn out into a wire 0.1″ in diameter. Find the length of the wire.

95. During a rainfall of 0.5″, to what depth will a circular well 5′ 6″ in diameter be filled by the drainage from a rectangular asphalted court which is 30′ 6″ × 26′ 5″?

96. If a plane cuts the axis of a right circular cylinder which is 14″ in diameter at an angle of 45°, what is the area of the section thus formed?

If A is the area, then $A \cos 45°$ is equal to what area (Ex. 54)?

97. A hole 1″ square is cut horizontally and symmetrically through a vertical wooden cylinder which is $\sqrt{2}$ inches in diameter. Find the approximate amount of wood removed in cutting the hole.

98. The radii of the bases of a frustum of a right circular cone are 11″ and 12.5″ respectively, and the height of the frustum is 5″. In the original cone, find the area of the curve surface, the area of the total surface, the altitude, and the volume.

99. The area of the total surface of a polyhedron weighing 64 lb. is 340 sq. in. Find the surface of a similar polyhedron made of the same material and weighing 1000 lb.

100. The bottom and top diameters of a tub are to be 24″ and 30″ respectively. What depth should be allowed so that the tub shall hold just 30 gal. (1 gal. = 231 cu. in.)?

101. A pump has a cylindric barrel 4″ in diameter, and the volume of water in 1′ of the length is pumped at each stroke. Find the number of strokes necessary to fill a tub in the form of a frustum of a cone 3′ in diameter at the bottom, 4′ in diameter at the top, and 1′6″ high.

102. It is desired to double the capacity of a cylindric boiler, but to keep the diameter and height in the same ratio. By what per cent will the total area be increased?

103. If the area of the surface of a spherical balloon is doubled, by what per cent is the circumference increased? the diameter? the volume? the radius?

104. Find the least amount of wood which it is necessary to waste in cutting a cube out of a wooden sphere 4″ in diameter.

105. Find the volume of the largest sphere that can be cut from a cone of revolution 14″ high and 12″ in diameter.

106. The specifications for a brass sphere 1″ in diameter require that the alloy contain one part of zinc to two parts of copper by volume. Given that 1 cu. ft. of copper weighs 550 lb. and that 1 cu. ft. of zinc weighs 428 lb., find the weight of the sphere.

107. The wooden part of a top consists of a conic frustum with a hemispherical end. The greatest and least diameters of the frustum are 3.5″ and 0.5″, and the slant height is 3.5″. Find the total volume.

108. If an iron sphere 4″ in diameter is placed in a conic vessel which is full of water and whose altitude and diameter are each 5″, how much water will run over?

109. Find the volume of the largest sphere that can be cut from a metal cone whose base has a diameter of 7″ and whose slant height is 7″.

110. If a circular hole 1″ in diameter is bored through a sphere 2″ in diameter and the axis of the hole passes through the center of the sphere, what is the volume of the part of the sphere that is left?

111. A cylinder which is 24″ in diameter and 16″ in height is inscribed in a sphere. Find the area and the volume of the sphere.

112. Prove that the volume of a regular octahedron is $0.47s^3$, approximately, and that the area of the total surface is $3.46s^2$, approximately, where s is an edge of the solid. From these results find the area and the volume of a regular octahedron 3″ on an edge.

113. A stone post is to be surmounted by a sphere 14″ in diameter, and in order to give the sphere a base upon which to stand a segment of one base 2″ high is cut from the sphere. Find the volume of the segment of the sphere that is placed on the post.

114. If the exposed surface of the segment of the sphere in Ex. 113 is polished, how many square inches are polished?

115. Find the volume of a spherical sector of a sphere of radius r, given that the area of the zone which forms the base of the spherical sector is $4\pi r^2$. Draw the figure.

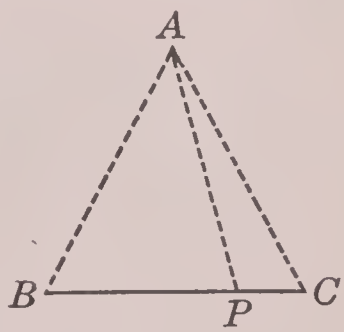
V. Recreations

551. Fallacies. Below are given a few curious problems and interesting fallacies, generally based upon incorrect constructions or statements, which should be undertaken, if time allows, simply as recreations.

1. Any point on a line bisects it.

In the figure below let BC be any line and P any point on it.

Construct an isosceles $\triangle ABC$ upon BC as base, and draw AP.



Since $\angle B = \angle C$,

$AB = AC$,

and $AP = AP$,

then $\triangle ABP$ is congruent to $\triangle ACP$.

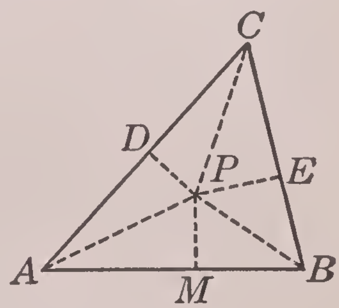
Hence $BP = PC$,

or any point on a line bisects it.

2. Every triangle is isosceles.

Let ABC be any $\triangle$ in which AC is not equal to BC.

Bisect $\angle C$ and construct the $\perp$ bisector of AB, letting it meet the bisector of $\angle C$ at P. They must meet, for if they were $\parallel$, the bisector of $\angle C$ would be $\perp$ to AB and hence would bisect it, thus coinciding with the $\perp$ bisector MP. This would be possible only if $AC = BC$, which is contrary to what is assumed above.



Draw $PD \perp$ to AC and $PE \perp$ to BC.

Then, since CP bisects $\angle C$, we have $PD = PE$; and since MP is the $\perp$ bisector of AB, then $AP = BP$.

Then $\triangle APD$ is congruent to $\triangle BPE$, and hence $AD = BE$.

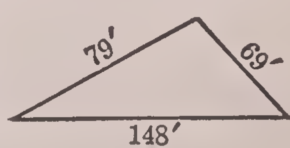
Similarly, $\triangle PDC$ is congruent to $\triangle PEC$, and hence $DC = EC$.

Adding, $AD + DC = BE + EC$, or $AC = BC$.

Hence every $\triangle$ is isosceles.

3. Find the area of this triangle to the nearest 0.1 sq. ft.

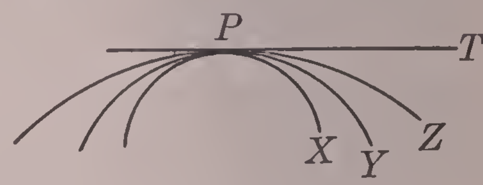
You may use the formula in Ex. 1, page 194, even though you have not proved it. If you prefer, draw the figure to scale, measure the altitude, and then apply § 244.



4. If $A > B$ and $B > C$, then it follows that $A = B = C$.

In this figure the arcs of the ⓢ are all tangent to PT at P. Then, if $A = \angle XPT$, $B = \angle YPT$, and $C = \angle ZPT$,

$$A > B > C.$$

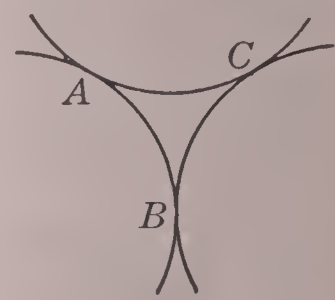


Now the $\angle$ between two ⓢ is defined as the $\angle$ between their tangents at a common point (see page 261).

But the $\angle$ between the tangents at P of any two of these ⓢ is 0, and hence

$$A = B = C.$$

5. Construct a triangle such that the sum of the interior angles is less than 180°.

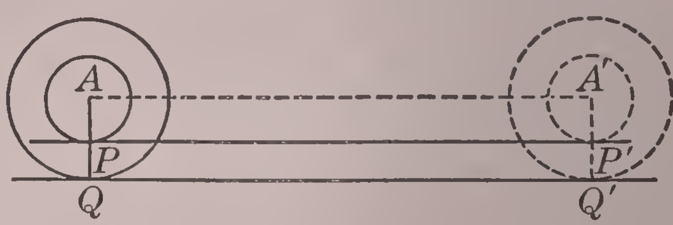


The three ⓢ of which the arcs are here shown are tangent at A, B, C.

Then, as in Ex. 4, the $\angle$ between the tangents at a common point of any two ⓢ is 0.

Hence the sum of the ∡ of the $\triangle$ formed by the tangents at A, B, C is 0.

6. All circles, however large, have equal circumferences.



Let two ⓢ of unequal radii AP and AQ be fastened together, and let them roll along from A to A'.

Then P reaches P' when Q reaches Q'.

Since the ⓢ have rolled along equal distances, their circumferences must be equal.

7. Two coins A and B of the same size are placed upon a table so that A is tangent to B. If B is kept fixed and A is rolled around B, always remaining tangent to B, how many revolutions does A make in rolling once around B?

Play fairly; give your answer and reason for it before experimenting.

8. A man who had a window 2 ft. wide and 4 ft. high wished to double its area. He did so, and still the window was only 2 ft. wide and 4 ft. high. How was this possible?

9. The sum of the parallel sides of a trapezoid is zero.

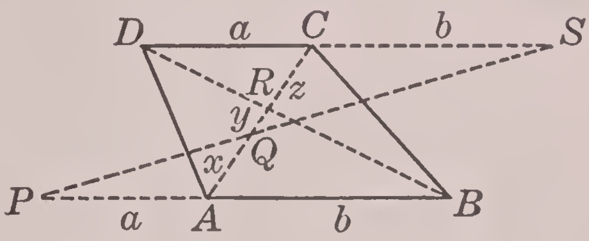
In the figure below let $ABCD$ be the trapezoid with bases AB (or b) and CD (or a).

Now let DC be produced to S and BA to P so that $CS = b$ and $AP = a$.

Then $\triangle PAQ$ is similar to $\triangle SCQ$,

and $\triangle CDR$ is similar to $\triangle ABR$.

Hence $$\frac{a}{b} = \frac{x}{y+z} = \frac{z}{x+y}.$$



Then $$\frac{x}{z} = \frac{y+z}{x+y};$$

whence $$\frac{x-z}{z-x} = \frac{z}{x+y}.$$

But $$\frac{x-z}{z-x} = -1.$$

Then, by substitution, $$\frac{a}{b} = -1,$$

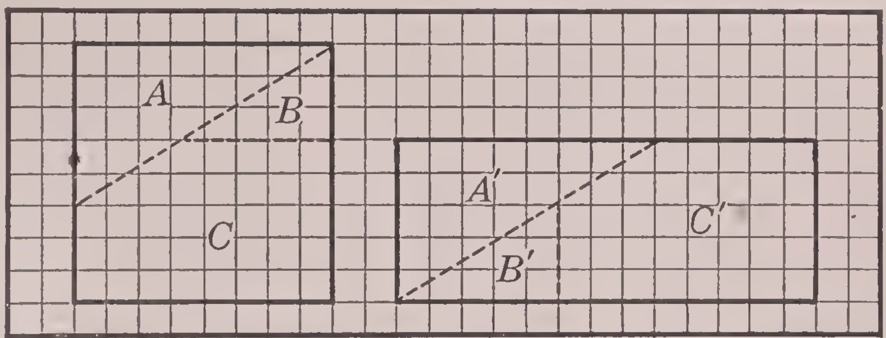
or, multiplying by b, $$a = -b;$$

whence we have $$a + b = 0.$$

10. Any number, however large, is equal to zero.

On a piece of squared paper mark out a square which shall be 8 by 8, and then draw lines dividing it into three parts A, B, C, as shown.

Then mark out a ▭ which shall be 5 by 13, and divide it into three parts such that $A' = A$, $B' = B$, and $C' = C$, as shown.



The number of small squares in the large square is 8×8, or 64, and the number of small squares in the ▭ is 5×13, or 65.

Hence $$65 = 64,$$

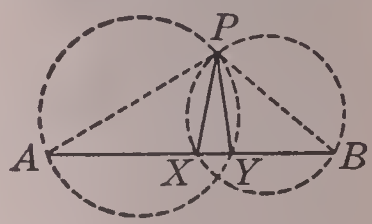
or $$1 = 0.$$

Multiplying these equals by any number, say 25, we have

$$25 = 0,$$

and hence any number is equal to zero.

11. From any point outside a line two perpendiculars can be constructed to the line.



Let AB be any line and P any point not on AB, and draw PA, PB.

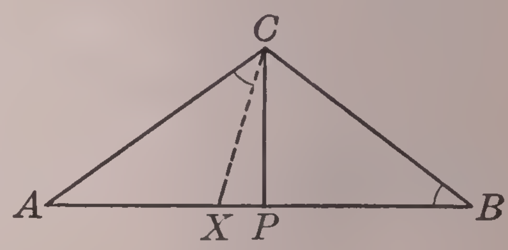
With PA and PB as diameters construct ⓢ intersecting AB at Y and X respectively, and draw PX, PY.

Then $\angle PXA = 90°$, § 173

and $\angle BYP = 90°$. § 173

Hence both PX and PY are $\perp$ to AB.

12. The whole of a line is equal to one of its parts.



In this $\triangle$, CP is $\perp$ to AB, and CX is drawn so as to make $\angle ACX = \angle B$.

Then $\triangle AXC$ is similar to $\triangle ACB$.

Hence these ⟁ are proportional to the squares of corresponding sides, and, since they have equal altitudes, to their bases also.

Then $$\frac{\triangle ACB}{\triangle AXC} = \frac{\overline{BC}^2}{\overline{CX}^2} = \frac{AB}{AX},$$ §§ 250, 246

and hence $$\frac{\overline{BC}^2}{AB} = \frac{\overline{CX}^2}{AX}.$$ § 198, 4

Then $$\frac{\overline{AC}^2 + \overline{AB}^2 - 2AB \cdot AP}{AB} = \frac{\overline{AC}^2 + \overline{AX}^2 - 2AX \cdot AP}{AX},$$ § 234

or $$\frac{\overline{AC}^2}{AB} + AB - 2AP = \frac{\overline{AC}^2}{AX} + AX - 2AP;$$

whence $$\frac{\overline{AC}^2}{AB} - AX = \frac{\overline{AC}^2}{AX} - AB,$$

or $$\frac{\overline{AC}^2 - AB \cdot AX}{AB} = \frac{\overline{AC}^2 - AB \cdot AX}{AX}.$$

Hence $AB = AX$, § 198, 2

or the whole of a line is equal to one of its parts.

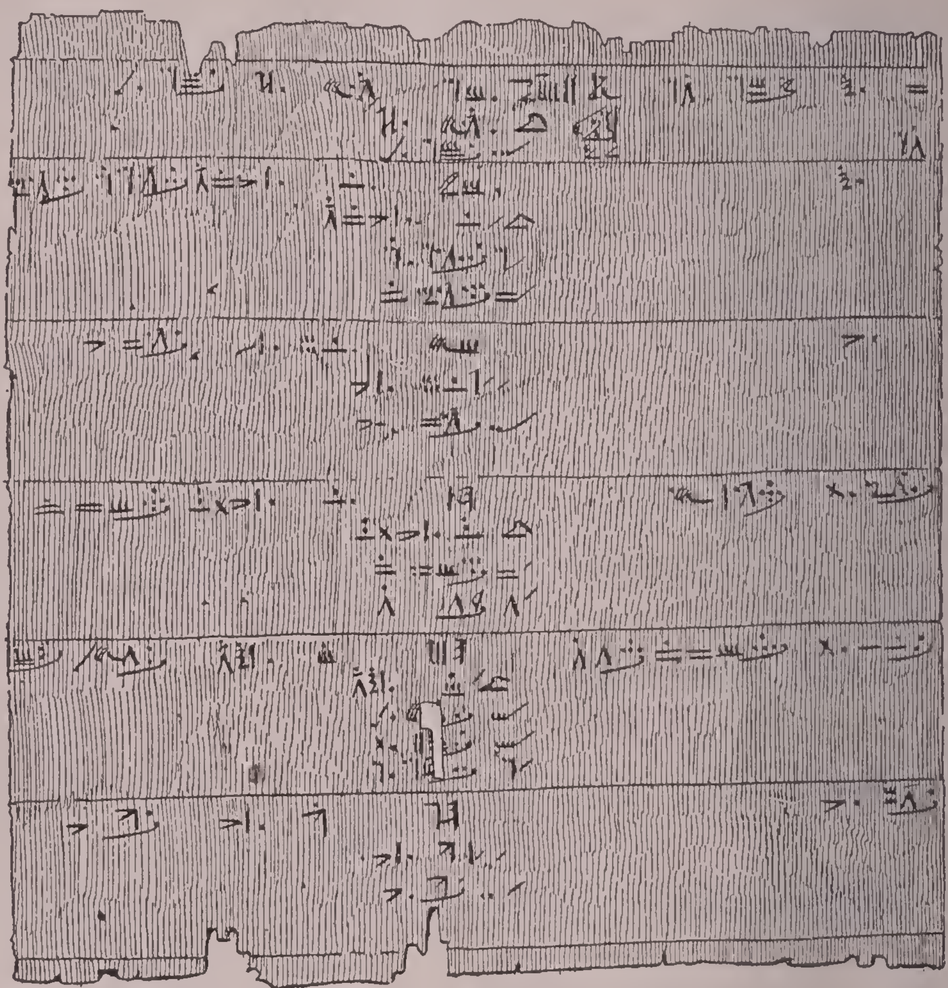
13. Show how to arrange six matches so that each match shall touch four others.

VI. History of Geometry

552. Ancient Geometry. The geometry of very ancient peoples was largely the mensuration of simple areas and volumes such as is taught to children in elementary arithmetic today. They learned how to find the area of a rectangle, and in the oldest mathematical records that we have there is some discussion of triangles and of the volumes of solids.

Our earliest documents relating to geometry have come to us from Babylon and Egypt. Those from Babylon were written, about 2000 B.C., on small clay tablets (some of them about the size of the hand) which were afterwards baked in the sun. They show that the Babylonians of that period knew something of land measures and perhaps had advanced far enough to compute the area of a trapezoid. For the mensuration of the circle they later used, as did the early Hebrews, the value $\pi = 3$.

The first definite knowledge that we have of Egyptian mathematics comes to us from two manuscripts copied on papyrus, a kind of paper used in the countries about the Mediterranean in early times. One of these manuscripts was made by one Aah-mesu (the Moon-born), commonly called Ahmes, who flourished probably about 1550 B.C. The original from which he copied, written about 2000 B.C., has been lost, but the papyrus of Ahmes, written over three thousand years ago, is still preserved and is now in the British Museum. In this manuscript, which is devoted chiefly to fractions and to a crude algebra, is found some work on mensuration. While there is some doubt as to the translation of some of the statements, apparently the curious rules given include the ones that the area of an isosceles triangle is half the product of the base and one of the equal sides, and that the area of a trapezoid with bases b, b' and nonparallel sides each equal to a is $\frac{1}{2}a(b+b')$. One noteworthy advance appears, however, where Ahmes gives a rule for finding the area of a circle, substantially as follows: Multiply the square on the radius by $(\frac{16}{9})^2$.



Part of the Ahmes Papyrus

The oldest extensive book on mathematics in the world, a papyrus roll written by Ahmes about 1550 B.C.

This is equivalent to taking for π the value 3.1605, and is the earliest known case of so close an approximation.

The second ancient Egyptian manuscript, which may have antedated slightly the work of Ahmes, is now in Russia. It is on mensuration and apparently contains one interesting case of the mensuration of a solid.

553. Early Greek Geometry. From Egypt, and possibly from Babylon, geometry passed to the shores of Asia Minor and Greece. The scientific study of the subject begins with Thales, one of the Seven Wise Men of the early Greek civilization. Born at Miletus about 624 B.C., he died there about 548 B.C. He founded at Miletus a school of mathematics and philosophy, known as the Ionic School. How elementary the knowledge of geometry was at that time may be understood from the fact that tradition attributes to Thales only about four propositions.

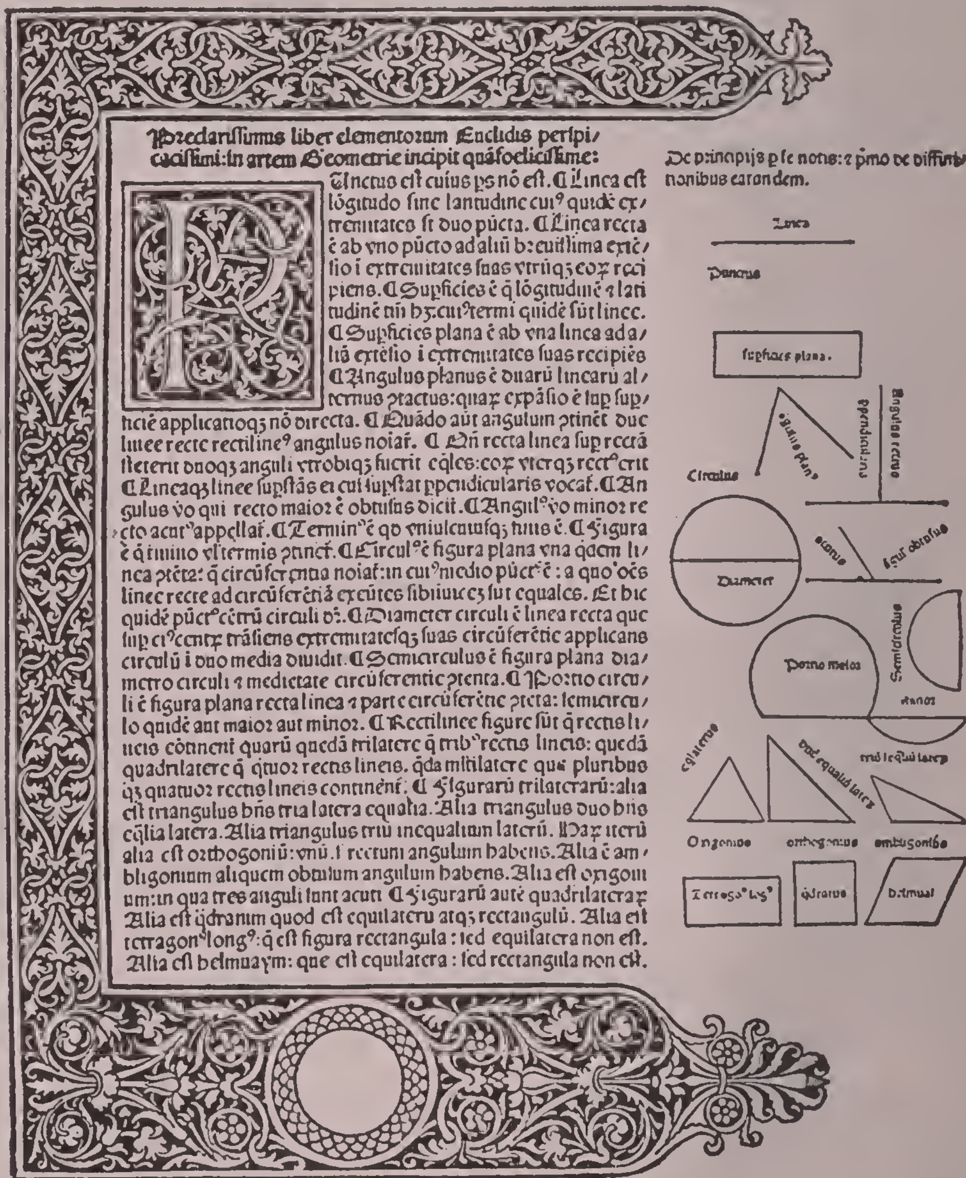
Pythagoras

A coin of Samos, one of the oldest known portrait medals of a mathematician

The greatest pupil of Thales, and one of the most remarkable men of antiquity, was Pythagoras, born probably on the island of Samos, just off the coast of Asia Minor, about the year 580 B.C. Pythagoras set forth as a young man to travel. He went to Miletus and studied under Thales, probably spent several years in Egypt, and very likely went to Babylon. He then founded a school at Crotona, in Italy. He is said to have been the first to demonstrate the proposition in geometry that the square on the hypotenuse of a right triangle is equivalent to the sum of the squares on the other two sides.

554. Euclid. The first great textbook on geometry, and the most famous one that has ever appeared, was written by Euclid, who taught mathematics in the great university at Alexandria, Egypt, about 300 B.C. Alexandria, named in honor of Alexander the Great, was then practically a Greek city, as it was ruled by the Greeks.

Euclid's work is known as the *Elements*, and, in common with all ancient works, the leading divisions were called "books," as is seen in the Bible and in the works of such Latin writers as Cæsar and Vergil. This is why we speak today of the various books of geometry. In this work Euclid placed all the leading propositions of plane geometry that were then known, and arranged them



First Page of Euclid's Elements

From the first printed edition, Venice, 1482

in a logical order. Most geometries of any importance since his time have been based upon this great work of Euclid, and improvements in the sequence, symbols, and wording have been made as occasion demanded.

555. Geometry in the East. The East did little for geometry, although contributing considerably to algebra. The first great Hindu writer was Aryabhatta, who was born in 476 A.D. He gave the very close approximation for π which we express in modern notation as 3.1416. The Arabs, about the time of the Arabian Nights tales (800 A.D.), did much for mathematics by translating the Greek authors into their own language and by bringing learning from India. Indeed, it is to the Arab mathematicians of the ninth and tenth centuries that modern Europe owes its first knowledge of the *Elements* of Euclid. The Arabs, however, contributed nothing of importance to geometry.

556. Geometry in Europe. In the twelfth century Euclid was translated from the Arabic into Latin, since Greek manuscripts were not then at hand, or were neglected because of ignorance of the language. The leading translators were Adelard of Bath (1120), an English monk who had learned Arabic in Spain or in Egypt; Gherardo of Cremona, an Italian monk of the twelfth century; and Johannes Campanus (about 1250), chaplain to Pope Urban IV.

In the Middle Ages in Europe nothing worthy of note was added to the geometry of the Greeks. The first Latin edition of Euclid's *Elements* was printed in 1482, and the first English edition in 1570.

557. Important Propositions. A few facts concerning some of the important propositions will be found of interest.

The theorem which asserts that the base angles of an isosceles triangle are equal is said to have been first proved by Thales, about 575 B.C. This theorem represented the usual limit of instruction in geometry in the Middle Ages, and probably on this account was called the *pons asinorum* (the bridge of fools); that is, it formed a kind of bridge across which fools could not pass. Roger Bacon, about 1250, called it the *fuga miserorum* (the flight of the miserable ones) because they fled at the sight of it.

The second of the congruence theorems is also attributed to Thales, who is said to have used it in measuring the distance from the shore to a ship.

The proposition which relates to the sum of the angles of a triangle is referred to by one of the later Greek writers in these words: "The ancients investigated the theorem of the two right angles in each individual species of triangle, — first in the equilateral, again in the isosceles, and afterwards in the scalene triangle." It is interesting to see that we do not have to take this long method of proving this simple proposition today. It is said that one of the earlier writers, Eudemus, who lived about 335 B.C., attributed the theorem to the Pythagoreans.

Perhaps the earliest records of the Pythagorean Theorem are found in Egyptian and Chinese works which are of uncertain dates, but were apparently written before 1000 B.C., or long before Pythagoras lived. In the Chinese work the statement reads: "Square the first side and the second side and add them together; then the square root is the hypotenuse." The theorem, however, was not proved in either of these works.

558. The Three Famous Problems. The Greeks very early found three problems which they could not solve. The first was that of trisecting any given angle,—the trisection problem; the second was that of constructing a square equivalent to a given circle,—the quadrature problem; and the third was that of constructing a cube that should have a volume twice that of a given cube,—the duplication problem. All three are easily solved if we allow other instruments than the ruler and compasses, but they cannot be solved by the use of these two instruments alone.

VII. Suggestions to Teachers

559. Difficulties of the Student. Among the difficulties and failures which are encountered by the student, the following demand special attention:

1. *Failure to comprehend the purpose of geometry.* At the beginning of Book I a special effort should be made to have the student appreciate the pleasure of "standing upon the vantage ground of truth" and the meaning of a real, deductive, scientific proof of a statement. A reasonable number of references to geometric forms found in the schoolroom, the use of such genuine applications of geometric forms as are within the ability of the student to comprehend, and the transfer of the method of geometric reasoning to simple problems of life will be found helpful. On the other hand, much time can be wasted by dwelling upon forms which, while interesting pictorially, have no significant relation to demonstrative geometry and are of no particular value as constructions.

2. *Failure to comprehend the technical language.* Students are often discouraged because they do not clearly see the meaning of such terms as median, isosceles, hexagon, and rhombus. This difficulty is easily removed, when it is met, by substituting for the term itself a statement of its meaning. In general, the teacher should use as simple terms as possible, particularly in the first part of the work. On this account it is better to use a familiar term such as "corresponding" instead of "homologous," to speak of "what is given" rather than of the "hypothesis," and to speak of "what is to be proved" rather than of the "conclusion," especially as this last term is applied to a statement at the beginning rather than at the end of a proof. Simplicity of language and of symbols is a great asset.

3. *The idea that geometry is to be memorized.* This difficulty can be best overcome by paying particular attention to the first few propositions. The teacher should develop these propositions carefully by questioning the class before the theorems are given, and thus lead the students to feel that they are discovering the proofs for themselves. The students will then come to prefer independent work and will thereafter read the proof in the text as a model of style rather than as a necessary aid. A second valuable aid is the introduction of a number of simple exercises with each of the first few propositions. Several of these may be given as sight work, with rapid demonstration by the student at the blackboard. In this way it will be found unnecessary to resort to such a doubtful device as that of changing the letters or figures from those which have been carefully worked out for the text.

4. *Failure to follow a proof given at the blackboard.* A prominent cause for this failure is the habit which students often form of reading lines and angles by their letters without pointing to the figures at the same time. No one can follow with ease a demonstration filled with expressions like "$\angle AOB = \angle PRX$." It is much better to say "angle m" and point to it, and it is still better to point to a line segment and say "this line segment" instead of using letters to represent it. Letters are more helpful in written work than in oral explanation. Teachers who recognize this fact will not be disturbed by such convenient lettering as ABC and $A'B'C'$.

It is also helpful to a class if the student who is demonstrating a proposition begins his proof by saying, "The general plan of the proof is $\cdots$" and states the plan. For example, he may say, "The general plan of this proof is to show that this triangle is congruent to that one." This method is beneficial not only to the class but to the one who is giving the proof. The statement of the plan of the proof has been given in the fundamental propositions in Books I and II, and after this the student is supposed to state the plan in each case for himself.

5. *Failure to state with precision what is given and what is to be proved.* Time devoted to this difficulty is well spent if it leads the student to acquire the habit of precise statement of these parts of a proof in the first exercises which he meets. A considerable part of the student's difficulty lies in the failure to acquire this habit, and it must be acquired early if at all.

6. *Failure to draw the figure when the statement is read.* It is always a great aid to draw the figure with reasonable neatness, and in as general a form as the circumstances require, while the theorem, problem, corollary, or exercise is being read. Such a habit tends to make the statement clear at once and to emphasize precisely what is given and precisely what is required.

560. Methods of Attack. A great deal has been written upon the methods of attacking an exercise or a proposition. For a beginner, however, it is desirable to keep prominently in mind only two methods:

1. *Analysis.* The student should early acquire the habit of saying, "I can prove this if I can prove that; I can prove that if I can prove this third fact," and so on until he reaches some statement which he can prove. He should then reverse his reasoning and give the proof step by step in proper geometric form.

2. *Indirect Method.* In case the analysis does not lead to the proof the student should say, "Suppose that the fact I am to prove is not true, what follows?" thus taking the indirect method described in § 56.

561. Great Basal Propositions. The teacher will find it helpful to call attention from time to time, and especially at the close of each book, to the great basal propositions of geometry as emphasized in this text. Geometry is peculiar among all branches of mathematics, and indeed among all the sciences, in its dependence upon a strong chain of truths and upon the deductive reasoning which results therefrom. The great basal propositions, the links in this chain, should therefore be thoroughly mastered.

562. Language and Symbolism. Teachers are advised not only to use as simple language as possible, as already mentioned, but to avoid new and unusual terms, especially those which do not have general international sanction. In the same way it is desirable to avoid local or personal prejudices in favor of symbols which are not generally recognized. Such symbols are easily created, and they have some advantage as pieces of shorthand; but it should be remembered that students are being prepared to read general mathematical works, and that for this purpose they can best be helped by using only recognized language and symbols. Such symbolism as *s. a. s.* for "two sides and the included angle" soon runs into the ridiculous, since such forms are neither necessary nor generally helpful.

It is well to remember, too, that there is considerable advantage in lettering and in reading a figure counterclockwise, particularly when a reflex angle is met; but this is not an arbitrary rule to be followed in all cases. If two figures are symmetrically arranged like the triangles in § 47, it is much clearer to read the corresponding letters in the same order, as ABC and $A'B'C'$, even though in one case they are read clockwise. In other words, any such arbitrary rule should be broken without hesitation if there is a decided gain in so doing.

In the matter of symbolism the teacher will find it much better to use $\angle A$, or simply a, instead of $\angle 2$. The use of the numeral is unnecessary, and there is always some confusion in seeming to give a numerical value to an angle which probably has an entirely different value from the one stated.

In general, the teacher will find it helpful to letter figures systematically, as has been done in the text. For example, on account of the ease with which corresponding parts may be recognized, it is more helpful to letter two congruent triangles as ABC and $A'B'C'$ than as DPX and LSV, particularly as it is neither necessary nor desirable to use the letters in giving oral proofs.

563. Discussions. One of the most valuable features in the solution of a problem is the discussion of special cases. This is, in general, left for the teacher to initiate. Nearly every problem has some special case of interest which often leads to the discovery that the solution is impossible under certain circumstances. No text can reasonably be expected to discuss all these special cases, but the class should be encouraged to discover the most interesting ones.

In particular, it is highly desirable to generalize each figure under discussion by studying the various shapes assumed when some point or some line of the figure is moved about in a plane.

564. Rôle of Postulates. The teacher will recognize that it is possible to reduce the number of postulates in any school geometry. This, however, is not desirable. The question to be answered in this connection is, What is best for the student at his stage of mental development? In general, within reason, a statement that seems obvious to a student may safely, at first reading, be taken as a postulate, but it should be proved when the feeling of the need for demonstration arises.

565. Solid Geometry. The introduction to solid geometry should be made slowly. Since the student has been accustomed to seeing only plane figures, the drawing of a solid figure in the flat is confusing at first. The best way for the teacher to anticipate this difficulty is to have a few pieces of cardboard, a few knitting needles filed to sharp points, a pine board about a foot square, and some small corks. The cardboard can be used to illustrate planes, and can be arranged to show parallel planes, perpendicular planes, or planes intersecting obliquely. The knitting needles may be stuck into the board to illustrate lines perpendicular or oblique to a plane. If two or more lines are to meet in a point, the needles may be held together by sticking them into one of the small corks. The figures given in the text can also be illustrated in this manner. Such homely apparatus, which costs almost nothing and is put together in class, seems much more real and is usually more satisfactory than the models which are sold by dealers.

To have a model for each proposition, or even to have a photograph or a stereoscopic picture, is, however, a poor educational policy. The pupil must learn very early to visualize a solid from the flat figure in outline, just as a builder or a mechanic learns to read his working drawings. The drawing of the different projections of a solid as required in connection with the work on practical mensuration (§§ 543–550) is particularly helpful in this respect.

The logical processes used in solid geometry do not differ essentially from those used in plane geometry, and for this reason the treatment of the subject is less extended than that found in Books I–V. For beginners it is the custom to increase somewhat the number of assumptions and to reduce the number of propositions, as compared with the work in plane geometry.

VIII. Important Formulas

566. Notation. The following notation is used in the formulas of plane geometry:

a = apothem	h = height, altitude
A = area	p = perimeter
a, b, c = sides of $\triangle ABC$	r = radius
b, b' = bases	s = semiperimeter of $\triangle ABC$; that is, $s = \frac{1}{2}(a+b+c)$
C = circumference	
d = diameter	$\pi = 3\frac{1}{7}, \frac{22}{7}$, 3.14, or 3.1416.

567. Formulas for Lines. The following are important:

Right triangle,	$a^2 + b^2 = c^2$	§§ 218, 252
Circle,	$C = 2\pi r$	
	$C = \pi d$	§ 307
Radius of circle,	$r = \dfrac{C}{2\pi}$	
Equilateral triangle,	$h = \frac{1}{2} b\sqrt{3}$	
Side of square,	$b = \sqrt{A}$	

568. Areas of Plane Figures. The following are important:

Rectangle,	$A = bh$	§ 241
Parallelogram,	$A = bh$	§ 243
Triangle,	$A = \frac{1}{2} bh$	§ 244
	$A = \sqrt{s(s-a)(s-b)(s-c)}$	p. 194
Equilateral triangle,	$A = \frac{1}{4} b^2\sqrt{3}$	
Trapezoid,	$A = \frac{1}{2} h(b + b')$	§ 247
Regular polygon,	$A = \frac{1}{2} ap$	§ 282
Circle,	$A = \frac{1}{2} rC$	§ 308
	$A = \pi r^2$	§ 309

569. Notation. In addition to that of § 566, the following notation is used in the formulas of solid geometry:

B = area of base	L = lateral area
e = element, lateral edge	M = area of midsection
E = spherical excess	S = area of curve surface
l = slant height	V = volume

570. Areas of Solid Figures. The following are important:

Prism,	$L = ep$	§ 376
Regular pyramid,	$L = \frac{1}{2} lp$	§ 401
Frustum of regular pyramid,	$L = \frac{1}{2} l(p + p')$	§ 402
Cylinder of revolution,	$S = 2 \pi rh$	§ 424
Cone of revolution,	$S = \frac{1}{2} lC$	§ 438
	$S = \pi rl$	§ 439
Frustum of cone of revolution,	$S = \frac{1}{2} l(C + C')$	§ 441
Sphere,	$S = 4 \pi r^2$	§ 497
Spherical polygon,	$S = \frac{1}{180} E \pi r^2$	§ 504
Zone,	$S = 2 \pi rh$	§ 508

571. Volumes. The following are important:

Parallelepiped,	$V = Bh$	§ 393
Prism or cylinder,	$V = Bh$	§§ 395, 425
Cylinder of revolution,	$V = \pi r^2 h$	§ 426
Pyramid or cone,	$V = \frac{1}{3} Bh$	§§ 407, 443
Cone of revolution,	$V = \frac{1}{3} \pi r^2 h$	§ 444
Frustum of pyramid or cone,	$V = \frac{1}{3} h(B + B' + \sqrt{BB'})$	§ 408
Frustum of cone of revolution,	$V = \frac{1}{3} \pi h(r^2 + r'^2 + rr')$	§ 445
Prismoid,	$V = \frac{1}{6} h(B + B' + 4M)$	§ 534
Sphere,	$V = \frac{4}{3} \pi r^3$	§ 513
	$V = \frac{1}{6} \pi d^3$	

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