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Uncertainty and complementarity: two key notions in Paul Busch's studies at the heart of quantum mechanics

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Uncertainty and complementarity: two key notions in Paul Busch's studies at the heart of quantum mechanics

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Abstract. The problem of approximate joint measurement of complementary observables, like position and momentum, and the relevance of the uncertainty relations to that question were at the heart of the investigations of Paul Busch into the foundations of quantum mechanics. A good third of his published work dealt with this and closely related questions. This paper is an attempt to survey some of the steps taken in that research starting with Paul's first papers on the subject matter in the early 1980's and reaching its height in recent years.

1. Introduction

The problem of approximate joint measurement of complementary observables and the relevance of the uncertainty relations to that question were at the heart of Paul Busch's investigations into the foundations of quantum mechanics.

The intuitive background of these investigations was defined in the early writings of Niels Bohr, Wolfgang Pauli, and Werner Heisenberg and may be summarised as follows: on one hand, according to Bohr [1, 2] and Pauli [3], position and momentum are complementary observables in the sense that all the experimental arrangements allowing their unambiguous operational definitions and measurements are mutually exclusive. Since mutually exclusive arrangements cannot be applied jointly, such observables cannot be operationally defined and measured together. On the other hand, according to Heisenberg [4], complementary observables, like position and momentum, can be defined and measured jointly if sufficient ambiguities are allowed in their definitions. For the necessary defining ambiguities or measurement inaccuracies $\delta q, \delta p$ for position and momentum Heisenberg gave his famous relation $\delta q \cdot \delta p \sim h$.

The problem arising from these heuristics read and still reads: how to express and possibly confirm or reject these intuitive ideas in (the Hilbert space formulation of) quantum mechanics?

With this contribution I try to survey some steps taken in that research starting with Paul's first papers on the subject matter and reaching its height in recent years. I have been fortunate to follow and participate in much of that work since 1981. A lot of our common work, with many collaborators, is summarised in the monographs *The Quantum Theory of Measurement* [5], *Operational Quantum Physics* [6], and *Quantum Measurement* [7]. I will use freely the standard notions of quantum mechanics, with its minimal interpretation, as developed, for instance, in our last book [7]. Part of the present contribution is based on a paper [8] appearing in the special issue of Foundations of Physics, entitled *Paul Busch: At the Heart of Quantum Mechanics.*

2. First steps

Though important conceptual and mathematical clarifications and extensions of Heisenberg's groundbreaking paper [4] of 1927 appeared almost immediately in the subsequent writings of Kennard [9], Weyl [10], and Robertson [11], the interpretation and meaning of Heisenberg's uncertainty relations, notably their relevance to the joint measurability of the involved observables, remained controversial over several decades to come. A clear modern account of this development is given in [12].

A reason for that controversy was, at least partly, in the adopted mathematical theory of quantum mechanics which contained the result, due to John von Neumann [13], according to which observables, as self-adjoint operators, have a joint observable if and only if they commute, and for commuting observables there are no (nontrivial) uncertainty relations. Another reason, already clearly pointed out by Kennard [9] and further emphasized, for instance, by Popper [14], is that quantum probabilities, measurement outcome distributions, do not reflect as such any measurement errors or inaccuracies.

The possibility to resolve this controversy was slowly emerging in the late 1960s with the works of Ludwig [15], Davies and Lewis [16], and others, where it was recognised that the probabilistic or statistical structure of quantum mechanics is compatible with the assumption that observables are given not only as spectral measures, associated with self-adjoint operators, but, more generally, with semispectral measures, normalised positive operator measures, an operator representation of an observable being just an idealized special case. This extension was soon utilized to describe fuzzy observables and approximate measurements, to mention only the papers of Ali, Emch, and Prugovečki [17, 18] and the monograph of Davies [19]. It is in this context that Paul defended his doctoral thesis "Unbestimmtheitsrelation und simultane Messungen in der Quantentheorie" in Cologne 1982 [20].

This thesis can be seen as a first systematic attempt to distinguish between preparation (statistical) and measurement (individualistic) uncertainty relations, the latter being discussed in terms of fuzzy position and fuzzy momentum observables and exemplified through an elaboration and extension of the Arthurs-Kelly model [21] for an approximate joint measurement of position and momentum. Since the emphasis of the thesis is on an individualistic/realist interpretation of quantum mechanics, it contains also a detailed study of necessary ideality conditions of the involved measurement processes needed in such an interpretation. Leaving aside the question of a realist interpretation, the thesis argues, even only within a model, that the two aspects of the uncertainty relations are closely related with each other, preparation uncertainty relations appearing as a part of the measurement uncertainty relations, reflecting the intuitive idea that the possibilities for measurements cannot overtake the possibilities for preparations. A generic notion of a measurement error adequate for quantum mechanics still waited to be found. Perhaps, a first clear formulation of this task is given in the 1983 monograph of Ludwig [22, pp 197-8].

3. Complementarity as a lack of joint tests

In spite of the numerous essays Bohr wrote on the topic, he never gave an explicit definition of the notion of complementarity, nor wrote an extensive treatise on the subject matter, a fact which may explain the abundance of the secondary literature on this theme. Leaving that aside, let us note that complementarity can be, and has been expressed in quantum mechanics (as well as in more general probabilistic theories) in several alternative ways in terms of measurement outcome probabilities, observables, instruments, or measurement schemes, see, e.g. [6, 7, 22]. In [23, 24] a formulation of the *complementarity* of observables as a *lack of joint tests* was advanced. Here it will be expressed directly on the level of effects constituting the observables, as further developed more recently in [8].

For any two effects $E, F \in \mathcal{E}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \,|\, 0 \leq A \leq I\}$, the yes-outcome of a yes-no measurement of a (nontrivial) dichotomic observable $\{0, A, I-A, I\}$, with $A \leq E, A \leq F$, gives probabilistic information on both E and F. Such a measurement is a *joint test* of E and F. The bigger A the better joint test though, unless E or F is a projection, the set of their common lower bounds

$$\text{l.b.}\{E, F\} = \{A \in \mathcal{E}(\mathcal{H}) \mid A \leq E, A \leq F\}$$

need not have a greatest element. No such tests exist exactly when E and F are disjoint:

$$\text{l.b.}\{E, F\} = \{0\}, \text{ that is, } E \wedge F = 0.$$

For any two observables $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and $\mathsf{F} : \mathcal{B} \to \mathcal{L}(\mathcal{H})$ their lack of joint tests may vary between the two extremes,

$$\begin{aligned} &\text{l.b.}\{\mathsf{E}(X), \mathsf{F}(Y)\} = \{0\} \text{ for all } X, Y, \ \mathsf{E}(X) \neq I \neq \mathsf{F}(Y), \\ &\text{l.b.}\{\mathsf{E}(X), \mathsf{F}(Y)\} \neq \{0\} \text{ for all } X, Y, \ \mathsf{E}(X) \neq 0 \neq \mathsf{F}(Y), \end{aligned}$$

giving rise to the question of where to put the definition of complementarity? To decide on that, recall that for an operational definition of an observable $\mathsf{E} : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ it suffices to know the effects $\mathsf{E}(X)$ for a semiring $\mathcal{R}$ which *generates* the σ-algebra $\mathcal{A} \subset 2^\Omega$ and *covers* the value set Ω in the sense of countable (disjoint) union. Hence, we define:

> Observables E and F are *complementary* if $\mathsf{E}(X) \wedge \mathsf{F}(Y) = 0$ for all $X \in \mathcal{R}, Y \in \mathcal{S}$ (such that $\mathsf{E}(X) \neq I \neq \mathsf{F}(Y)$) for some generating and covering semirings $\mathcal{R} \subset \mathcal{A}$ and $\mathcal{S} \subset \mathcal{B}$.

As an immediate observation we note that complementary observables have no joint measurements. In fact, if M is a joint measurement of E and F, that is, $\mathsf{E}(X) = \mathsf{M}(X, \Xi), \mathsf{F}(Y) = \mathsf{M}(\Omega, Y)$ for all $X \in \mathcal{A}, Y \in \mathcal{B}$, then $\mathsf{M}(X, Y)$ is a common lower bound of $\mathsf{E}(X)$ and $\mathsf{F}(Y)$ for all X, Y; thus, if E and F are complementary, then $I = \mathsf{M}(\Omega, \Xi) = \mathsf{M}(\cup X_i, \cup Y_j) = \sum \mathsf{M}(X_i, Y_j) = 0$, for some disjoint covers $(X_i) \subset \mathcal{R}$ and $(Y_j) \subset \mathcal{S}$ of the value spaces Ω and Ξ, respectively. This is a contradiction.

The study of the complementarity of observables reduces to the study of the set of the lower bounds of their effects. Let us recall that if one of the effects E and F is a projection, then the set $\text{l.b.}\{E, F\}$ contains the greatest elements $E \wedge F$ (in the order of $\mathcal{E}(\mathcal{H})$ [25]) and if both are projections then $E \wedge F$ is equal to their greatest lower bound in the projection lattice, the projection onto the closed subspace $E(\mathcal{H}) \cap F(\mathcal{H})$, the intersection of the ranges of E and F. In general, the range $E(\mathcal{H}) = \text{ran}(E)$ of an effect E need not be closed. If P_E denotes the support projection of E, that is, the projection onto $\mathcal{H}_E = (\ker E)^\perp = \overline{\text{ran}(E)}$, we have $E \leq P_E$, so that, in general, the set $\text{l.b.}\{E, F\}$ is bounded from above by the projection $P_E \wedge P_F$.

4. The case of Q and P

The canonical position-momentum pair (Q, P), as studied in the thesis [20], is the prototype example of complementary observables, their complementarity can be derived from their Fourier equivalence and is manifested in the relation:

$$\mathsf{Q}(X) \wedge \mathsf{P}(Y) = 0, \tag{4.1}$$

for all bounded $X, Y \in \mathcal{B}(\mathbb{R})$, independently of the Lebesgue measures $\ell(X), \ell(Y)$ of the sets. Their Fourier equivalence gives equally well $\mathsf{Q}(X) \wedge \mathsf{P}(Y)^\perp = \mathsf{Q}(X)^\perp \wedge \mathsf{P}(Y) = 0$ for those sets, underlying together with (4.1) the non-Boolean structure of the projection lattice $\mathcal{P}(\mathcal{H})$. For the complements of such sets one instead has

$$\mathsf{Q}(\mathbb{R} \setminus X) \wedge \mathsf{P}(\mathbb{R} \setminus Y) \neq 0, \tag{4.2}$$

a result going back to [26, 27]. In addition, Q and P have both totally noncommutative spectral projections (associated with half-lines) as well as mutually commutative spectral projections (associated with appropriate periodic sets), cases of interest also to Busch *et al.* [28]. In view of the importance of these results in the quantum probability calculus as developed in [29], they were collectively referred to as the Jauch theorem in [30].

Many more examples of complementary observables are known and extensively discussed in [8], to mention number N and canonical phase Φ, any two (rotated) quadratures $\mathsf{Q}_\alpha, \mathsf{Q}_\beta$, as well as the pairs $(\mathsf{Q}_\alpha, \mathsf{H})$, where H is an energy observable, with the operator $H = \frac{1}{2m}P^2 + V(Q)$, where V is any function such that H has a purely discrete spectrum.

The equal importance of the complementary observables for the full description of (the state of) the system was underlined in some of Bohr's writings. Hence, it is worth recalling that, due to the informational incompleteness of the pair (Q, P), as shown by Bargmann and reported in [31, p 92], some complementary information is hereby lacking for a full state description. Any triple $(\mathsf{Q}, \mathsf{P}, \mathsf{H})$, with a discrete energy H, is pairwise complementary but none is known to be informationally complete. However, any set of rotated quadratures $\{\mathsf{Q}_\theta \,|\, \theta \in J \subset [0, \pi)\}$ is pairwise complementary and informationally complete if J is a dense subset of $[0, \pi)$ [32].

5. Breaking complementarity of Q and P: generalized Jauch theorem

The fuzzy position and momentum observables studied in [20] were of the convolution form $\mu * \mathsf{Q} = \mathsf{Q}_\mu, \nu * \mathsf{P} = \mathsf{P}_\nu$, with the effects

$$\mathsf{Q}_\mu(X) = \int (\mu * \chi_X)(q)\, d\mathsf{Q}(q), \tag{5.1}$$

$$\mathsf{P}_\nu(Y) = \int (\nu * \chi_Y)(p)\, d\mathsf{P}(p), \tag{5.2}$$

and with the probability measures μ and ν having the densities $f = |\varphi|^2$ and $g = |\psi|^2$ (defined by unit vectors φ, ψ in $L^2(\mathbb{R})$). It was further assumed that the standard deviations of μ and ν, $\Delta(\mu), \Delta(\nu)$, describe the measurement inaccuracies. From [18, 19] it was known that if ψ is the Fourier transform of φ, that is, $\psi = \hat{\varphi}$, then the fuzzy observables Q_μ and P_ν can be obtained as the marginal observables of a (covariant) phase space observable G^T, $T = |\varphi\rangle\langle\varphi|$, with the effects

$$\mathsf{G}^T(X \times Y) = \frac{1}{2\pi} \int_{X \times Y} W_{q,p} T W_{q,p}^*\, dq dp, \tag{5.3}$$

where $W_{q,p}$ are the Weyl operators. In this case

$$0 \neq \mathsf{G}^T(X \times Y) \in \text{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\},$$

for all X, Y, meaning that the thus introduced fuzziness has broken the complementarity of Q and P. Moreover, now $\Delta(\mu)\Delta(\nu) = \Delta(\mathsf{Q}, \varphi)\Delta(\mathsf{P}, \varphi) \geq \frac{1}{2}\hbar$, indicating that the assumed measurement inaccuracies are related to the preparation of the joint measurement modelled by the phase space observable G^T.

This model led to the study of the question of how much inaccuracy in the form of convolutions is to be introduced in position and momentum to break their complementarity and possibly allow their joint measurements [24, 30].

The first question is answered in what Paul called generalised Jauch theorem, [30, Theorem $(\widetilde{J})$], a result generalising the equivalence of the statements $\text{l.b.}\{\mathsf{Q}(X), \mathsf{P}(Y)\} \neq \{0\}$ and $\text{ran}(\mathsf{Q}(X)) \cap \text{ran}(\mathsf{P}(Y)) \neq \{0\}$, to the effects $\mathsf{Q}_\mu(X)$ and $\mathsf{P}_\nu(Y)$:

$$\text{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\} \iff \text{ran}(\sqrt{\mathsf{Q}_\mu(X)}) \cap \text{ran}(\sqrt{\mathsf{P}_\nu(Y)}) \neq \{0\}. \tag{5.4}$$

A partial answer to the second question came from the observation that

$$\mathsf{Q}_\mu(X) \leq \mathsf{Q}(\mathrm{supp}(\mu * \chi_X)),\ \ \mathsf{P}_\nu(Y) \leq \mathsf{P}(\mathrm{supp}(\nu * \chi_Y)), \tag{5.5}$$

showing that if the spectral projections $\mathsf{Q}(\mathrm{supp}(\mu * \chi_X))$ and $\mathsf{P}(\mathrm{supp}(\nu * \chi_Y))$ are disjoint then also the effects $\mathsf{Q}_\mu(X)$ and $\mathsf{P}_\nu(Y)$ are disjoint, independently of the standard deviations of μ and ν. The converse question whether $\mathsf{Q}_\mu(X) \wedge \mathsf{P}_\nu(Y) = 0$ also implies $\mathsf{Q}(\mathrm{supp}(\mu * \chi_X)) \wedge \mathsf{P}(\mathrm{supp}(\nu * \chi_Y)) = 0$ was left open in [30] (but was to be answered in the negative, see below).

Result (5.4) extends to arbitrary effects. Indeed, [33, Theorem 3] characterises the range of the square root of an effect in terms of the rank-1 operators contained in it: for any effect $E \in \mathcal{E}(\mathcal{H})$ and for any rank-1 operator $|\varphi\rangle\langle\varphi|$, $\|\varphi\| \leq 1$, one has

$$\varphi \in \mathrm{ran}\sqrt{E} \iff \exists \lambda > 0 \text{ s.t. } \lambda|\varphi\rangle\langle\varphi| \leq E. \tag{5.6}$$

This then shows that for any $E, F \in \mathcal{E}(\mathcal{H})$,

$$\mathrm{l.b.}\{E, F\} \neq \{0\} \iff \mathrm{ran}\sqrt{E} \cap \mathrm{ran}\sqrt{F} \neq \{0\}. \tag{5.7}$$

A direct study of either of these equivalent conditions may be difficult in concrete cases, as already exemplified by the study of the various cases of $\mathsf{Q}(X)$ and $\mathsf{P}(Y)$, and a further complication may arise since the range (of the square root) of an effect need not be closed. To study complementarity, it is thus useful to have some upper bound estimates for (5.7). From $\mathrm{ran}\sqrt{E} \subset \overline{\mathrm{ran}\sqrt{E}} = \mathcal{H}_E$, one immediately gets one: if $\mathcal{H}_E \cap \mathcal{H}_F = \{0\}$, that is, $P_E \wedge P_F = 0$, then also $E \wedge F = 0$. However, the converse implication does not hold: there are effects E, F which are disjoint though $P_E \wedge P_F \neq 0$ [8, Proposition 7].

The fuzzy position and momentum effects are of a special form of being (nontrivial) functions of a spectral measure. In general, if $E = \int h \, d\mathsf{A}$ for some spectral measure $\mathsf{A} : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ and a Borel measurable function $h : \mathbb{R} \to [0, 1]$, then a direct calculation shows that $P_E \leq \mathsf{A}(\mathrm{supp}(h))$, extending the observation (5.5). Again, there are (even nontrivial) examples showing that this inequality may be strict [8, Lemma 4].

For any two effects $E, F \in \mathcal{E}(\mathcal{H})$ of the above form $E = \int h d\mathsf{A}$, $F = \int k d\mathsf{B}$, one thus has the equivalent conditions:

$$\begin{aligned}
\mathrm{l.b.}\{E, F\} \ &\subseteq \ \{D \in \mathcal{E}(\mathcal{H}) \,|\, D \leq P_E \wedge P_F\} \\
&\subseteq \ \{D \in \mathcal{E}(\mathcal{H}) \,|\, D \leq \mathsf{A}(\mathrm{supp}(h)) \wedge \mathsf{B}(\mathrm{supp}(k))\}, \\
\mathrm{ran}\sqrt{E} \cap \mathrm{ran}\sqrt{F} \ &\subseteq \ \mathcal{H}_E \cap \mathcal{H}_F \subseteq \mathsf{A}(\mathrm{supp}(h))(\mathcal{H}) \cap \mathsf{B}(\mathrm{supp}(k))(\mathcal{H}),
\end{aligned}$$

where all the inclusions may be proper.

To close this section let us return briefly to the fuzzy position and fuzzy momentum studied in [20]. By now it is known that any observable which shares the symmetry properties of position (translation covariance and boost invariance) is of the form $\mu * \mathsf{Q}$ for some probability measure μ. Similarly, any observable which shares the symmetry properties of momentum (boost covariance, translation invariance) is of the form $\nu * \mathsf{P}$ for some ν [34]. Moreover, any such pair $(\mu * \mathsf{Q}, \nu * \mathsf{P})$ is jointly measurable exactly when they can be obtained as the margins of some covariant phase space observable G^T as defined in (5.3) [35]. In addition, any observable M which is covariant under the phase space translations (translations and boosts) is of that form for some $T \geq 0, \mathrm{tr}\, T = 1$, a result going back to Holevo [36] and Werner [37] (for alternative revised proofs, see [38,39]). Further, any such an observable can be realised by a modified Arthurs-Kelly model (eight-port homodyne detection scheme) of an approximate joint measurement of position and momentum [40].

For any unsharp position $\mu * \mathsf{Q}$ and momentum $\nu * \mathsf{P}$ we now have

$$\text{l.b.}\{\mathsf{Q}_\mu(X), \mathsf{P}_\nu(Y)\} \neq \{0\} \;\Leftrightarrow \text{ran}\left(\sqrt{\mathsf{Q}_\mu(X)}\right) \cap \text{ran}\left(\sqrt{\mathsf{P}\nu(Y)}\right) \neq \{0\}.$$

Since, for instance, $\text{supp}(\chi_X * \mu) \subset \overline{\overline{X} + \text{supp}(\mu)}$, one observes that $\mu * \mathsf{Q}$ and $\nu * \mathsf{P}$ remain complementary if the unsharpness measures μ and ν have bounded supports, independently of the sizes of $\Delta(\mu)$ and $\Delta(\nu)$. On the other hand, the pair $(\mu * \mathsf{Q}, \nu * \mathsf{P})$ is jointly measurable if and only if they can be obtained as the margins of a phase space observable G^T, in which case $\Delta(\mu)\Delta(\nu) = \Delta(\mathsf{Q}, T)\Delta(\mathsf{P}, T) \geq \frac{\hbar}{2}$. Still, the question of the possible relation of μ, or $\Delta(\mu)$, to the actual measurement accuracy is not explicitly addressed. Finally, the question left open in [30] is answered in the following [8, Proposition 13]:

> For any bounded intervals $X, Y \subset \mathbb{R}$ with lengths d_X, d_Y satisfying $d_X d_Y \leq \pi/2$, there exist probability measures μ, ν with finite variance, such that the effects $\mathsf{Q}_\mu(X)$ and $\mathsf{P}_\nu(Y)$ are complementary, but $\mathsf{Q}(\text{supp}(\mu * \chi_X)) \wedge \mathsf{P}(\text{supp}(\nu * \chi_Y)) \neq 0$.

6. Approximate measurements and measurement uncertainty

The notion of a fuzzy or approximate observable, like $\mu * \mathsf{Q}$, has been used in the above discussed investigations to model an approximate measurement of the given observable, like Q, with the idea that the probability measure μ accounts for the approximation. This is well justified since the convolving measure simply adds state independent noise in the measurement outcome statistics, like $(\mu * \mathsf{Q})_\rho = \mu * \mathsf{Q}_\rho$. Still, no systematic theory of an approximate measurement of an observable is given here.

With the revised operational tools of quantum measurement theory, the study of the measurement uncertainty relations, including noise and disturbance, got a new boost in the turn of the millennium. Part of this new wave of interest was triggered by the work of Ozawa [41, 42] on the error-disturbance relations. The proposed notions of Ozawa were independently criticised in the papers of Busch *et al.* [43] and Werner [44].

Common to [43, 44] was a search for operationally meaningful measures of measurement error, noise, and disturbance, with the idea that such notions should be built on comparing the outcomes and statistics of the target observable, the observable one intends to measure, and the approximating observable, the observable that is actually implemented by the measurement scheme – an idea advanced already by Ludwig [22, pp 197-8] in 1983 but missing in Ozawa's work. Following this idea several measures of error (noise and disturbance) have been proposed and developed, among them are the notions of metric error, calibration error, and error bar width, studied also in many writings of Paul and his collaborators [45–50], and [7, Chpt. 13].

A breakthrough step was taken in the 2004 paper of Werner [44] where a metric error $\Delta_1(\widetilde{\mathsf{E}}, \mathsf{E})$ in measuring the approximator $\widetilde{\mathsf{E}}$ in place of the target E was defined as the worst-case (state independent) limit of the Monge distance d_1 between the distributions $\widetilde{\mathsf{E}}_\rho$ and E_ρ:

$$\Delta_1(\widetilde{\mathsf{E}}, \mathsf{E}) = \sup_\rho d_1(\widetilde{\mathsf{E}}_\rho, \mathsf{E}_\rho).$$

In subsequent work [47, 49, 50] this was extended to use the Wasserstein distance of order α, $1 \leq \alpha < \infty$, that is, replacing d_1 with

$$d_\alpha(\widetilde{\mathsf{E}}_\rho, \mathsf{E}_\rho) = \inf_\gamma \left(\int |x-y|^\alpha \, d\gamma(x,y)\right)^{\frac{1}{\alpha}},$$

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where the infimum (actually minimum) is taken over all the couplings γ of the (real) probability measures $\widetilde{\mathsf{E}}_\rho$ and E_ρ. If, for instance, E is a spectral measure and $\widetilde{\mathsf{E}} = \mu * \mathsf{E}$, then

$$\Delta_\alpha(\widetilde{\mathsf{E}}, \mathsf{E}) = d_\alpha(\mu, \delta_0) = (\int |x|^\alpha \, d\mu(x))^{\frac{1}{\alpha}},$$

where δ_0 is the point measure at 0. For $\alpha = 2$, this gives $\Delta_2(\widetilde{\mathsf{E}}, \mathsf{E}) \geq \Delta(\mu)$, with equality whenever $\int x \, d\mu(x) = 0$.

The paper [44] also developed an averaging method to construct, under appropriate conditions, a covariant observable from a given observable. For approximate joint measurements of position and momentum this all led to the following fundamental result [47, 49], see also [7, Chpt. 15], a result which can be seen as a completion of the study of the problem faced in [20], now in the frame of the minimal interpretation of quantum mechanics.

Theorem 1. Let M be any phase space observable (covariant or not), with the marginal observables $\mathsf{M}_1, \mathsf{M}_2$, and $1 \leq \alpha, \beta < \infty$. If the measurement errors $\Delta_\alpha(\mathsf{M}_1, \mathsf{Q})$ and $\Delta_\beta(\mathsf{M}_2, \mathsf{P})$ both are finite then there is a covariant phase space observable G^T such that $\Delta_\alpha(\mathsf{G}^T_1, \mathsf{Q}) \leq \Delta_\alpha(\mathsf{M}_1, \mathsf{Q})$ and $\Delta_\beta(\mathsf{G}^T_2, \mathsf{P}) \leq \Delta_\beta(\mathsf{M}_2, \mathsf{P})$, and thus

$$\begin{aligned} \Delta_\alpha(\mathsf{M}_1, \mathsf{Q})\Delta_\beta(\mathsf{M}_2, \mathsf{P}) &\geq \Delta_\alpha(\mathsf{G}^T_1, \mathsf{Q})\Delta_\beta(\mathsf{G}^T_2, \mathsf{P}) \\ &= d_\alpha(\mathsf{Q}_{\Pi T\Pi, \delta_0}) d_\beta(\mathsf{P}_{\Pi T\Pi, \delta_0}) \geq c_{\alpha\beta}\hbar, \end{aligned} \tag{6.1}$$

where Π is the parity operator, and the constant $c_{\alpha\beta}$ is connected to the ground state energy $g_{\alpha\beta}$ of the Hamiltonian $H_{\alpha\beta}$, the closure of the essentially selfadjoint operator $|Q|^\alpha + |P|^\beta$, by the equation

$$c_{\alpha\beta} = \alpha^{\frac{1}{\beta}} \beta^{\frac{1}{\alpha}} (\frac{g_{\alpha\beta}}{\alpha + \beta})^{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

The latter inequality in (6.1) is an equality exactly when $\Pi T\Pi$ arise from the ground state of $H_{\alpha\beta}$ by phase space translation and dilation.

For $\alpha = \beta = 2$, $H_{\alpha\beta}$ is twice the harmonic oscillator Hamiltonian with the ground state energy $g_{22} = 1$, giving rise to the familiar lower bound $\hbar/2$. In the case of finite measurement errors, the errors are bounded from below by the preparation uncertainties characterizing the measurement scheme implementing the approximate joint measurement G^T; for $\alpha = \beta = 2$, one has $\Delta_2(\mathsf{M}_1, \mathsf{Q}) \geq \Delta(\mathsf{Q}_{\Pi T\Pi})$ and $\Delta_2(\mathsf{M}_2, \mathsf{P}) \geq \Delta(\mathsf{P}_{\Pi T\Pi})$.

There is now an increasing flow of papers analysing in one or another form something like a measurement uncertainty region for two (or more) incompatible observables E_i, with the value spaces $(\Omega_i, \mathcal{A}_i)$,

$$\mathsf{MU}(\Omega_1, \Omega_2) = \{(\Delta_1(\mathsf{M}_1, \mathsf{E}_1), \Delta_2(\mathsf{M}_2, \mathsf{E}_2)) \mid \mathsf{M} : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})\}, \tag{6.2}$$

and with various measures Δ_1, Δ_2 of uncertainty. During the past five years also Paul was highly active in this research, as illustrated, e.g., by the papers [51, 52].

7. Fragility of complementarity

For any two effects E, F the set of their common lower bounds is characterized by the intersection of the ranges of their square roots. This set can easily be modified with a small perturbation of one of the effects. Indeed, we have [8, Proposition 11]:

For any effect E and for any $\lambda, p \in (0, 1)$ define $E_{\lambda,p} = \lambda E + (1-\lambda)pI$. Then $\operatorname{ran}\sqrt{E_{\lambda,p}} = \mathcal{H}$.

This shows that the complementarity of any two observables $\mathsf{E}_1, \mathsf{E}_2$ is immediately broken by adding trivial noise in one of the observables, say, $\mathsf{E}_1 \to \widetilde{\mathsf{E}}_1 = \lambda\mathsf{E}_1 + (1-\lambda)\mathsf{T}_1$, with the trivial observable $X \mapsto \mathsf{T}_1(X) = \mu_1(X)I$ defined by the probability measure μ_1.

In fact, any two observables E_1 and E_2 can even be made compatible by mixing them with trivial noise,

$$\begin{aligned}\widetilde{\mathsf{E}}_1 &= \lambda\mathsf{E}_1 + (1-\lambda)\mathsf{T}_1, \\ \widetilde{\mathsf{E}}_2 &= \gamma\mathsf{E}_2 + (1-\gamma)\mathsf{T}_2,\end{aligned}$$

and choosing the weights $0 < \lambda, \gamma < 1$ appropriately, for instance, $\gamma = 1-\lambda$. See, for instance, [53, 54].

In studying position-momentum uncertainty, this way of breaking complementarity and making them compatible is, however, not very useful since the errors grow without bound. This can indirectly be concluded from [44, 49] together with [54] but can also be seen directly [55], as reported below.

Lemma 1. Let $\widetilde{\mathsf{Q}} = \lambda\mathsf{Q} + (1-\lambda)\mathsf{T}$ be an approximate position obtained by mixing Q with a trivial observable T with a weight $0 < \lambda < 1$. Then $\Delta_\alpha(\widetilde{\mathsf{Q}}, \mathsf{Q}) = \infty$ for all $1 \le \alpha < \infty$.

Proof. Let μ be the probability measure defining T. We use the Kantorovich duality theorem [56, Theorem 5.10] (in the form [49, Lemma 5]) to compute the distance $\Delta_\alpha(\widetilde{\mathsf{Q}}, \mathsf{Q})^\alpha$. To that end, fix any pair of positive continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ with compact supports satisfying the condition

$$f(y) - g(x) \le |x-y|^\alpha \text{ for all } x, y. \tag{7.1}$$

Assume, in addition, that $f(0) > 0$ (if necessary, replace both functions with the translates $f(y+y_0)$ and $g(x+y_0)$ where $f(y_0) > 0$). Fix a compact set K containing the supports of f and g. Then, for any $r > 0$, also the scaled functions $f_r(y) = rf(r^{-1/\alpha}y)$ and $g_r(x) = rg(r^{-1/\alpha}x)$ satisfy (7.1) and their supports are contained in the set $r^{1/\alpha}K$ extending along r. For each $r > 0$ choose a unit vector $\phi_r \in L^2(\mathbb{R})$ such that its support is outside the set $r^{1/\alpha}K$. Thus, $\langle \phi_r | \mathsf{Q}[f_r]\phi_r \rangle = \int f_r \, d\mathsf{Q}_{\phi_r,\phi_r} = 0$ and $\left\langle \phi_r | \widetilde{\mathsf{Q}}[f_r]\phi_r \right\rangle = (1-\lambda)\mu[f_r]$ for each r. Thus for each $r > 0$

$$\Delta_\alpha(\widetilde{\mathsf{Q}}, \mathsf{Q})^\alpha \ge \left\langle \phi_r | (\widetilde{\mathsf{Q}}[f_r] - \mathsf{Q}[g_r])\phi_r \right\rangle = (1-\lambda)\mu[f_r] = r(1-\lambda)\int f(r^{-1/\alpha}y)d\mu(y).$$

Since the continuous bounded function $y \mapsto f(r^{-1/\alpha}y)$ converges pointwise to the constant function $f(0)$, the dominated convergence theorem implies that $\lim_{r\to\infty} \int f(r^{-1/\alpha}y)d\mu(y) = \int f(0)d\mu(y) = f(0) > 0$, showing that the right hand side of the above inequality approaches infinity along with $r \to \infty$. The Kantorovich duality theorem thus gives $\Delta_\alpha(\widetilde{\mathsf{Q}}, \mathsf{Q}) = \infty$. □

This result shows that though the trivial perturbation may be arbitrarily small on the level of effects, it is not at all small in the sense of the Wasserstein distance of observables.

8. Where do we stand?

The problem of approximate joint measurability of complementary observables and the relevance of the uncertainty relations to that question was a major subject in Paul's investigations into the foundations of quantum mechanics. Perhaps, the first poetic formulation of the problem is due to Wolfgang Pauli in a letter to Werner Heisenberg, dated 19 October 1926 [57]:

> *Man kann die Welt mit dem p-Auge und man kann sie mit dem q-Auge ansehen, aber wenn man beide Augen zugleich aufmachen will*[,] *dann wird man irre.*

Paul's equally poetic answer to Pauli is given in the Epilogue of [6]:

> *We hope to have demonstrated that one can safely open a pair of complementary 'eyes' simultaneously. He who does so may even 'see more' than with one eye only. The means of observation being part of the physical world, Nature Herself protects him from seeing too much and at the same time protects Herself from being questioned too closely: quantum reality, as it emerges under physical observation, is intrinsically unsharp. It can be forced to assume sharp contours – real properties – by performing repeatable measurements. But sometimes unsharp measurements will be both, less invasive and more informative.*

The notion of joint test of a pair of effects, which is behind our formulation of the notion of complementarity of observables is a weaker notion than the notion of their joint measurability (coexistence). Even the pairwise joint measurability of all the effects constituting the observables does not guarantee their joint measurability (see, e.g. [58]). Hence, it is natural that one may break the complementarity of observables without allowing their approximate joint measurement.

The complementarity of position and momentum is manifested in the relation $\mathsf{Q}(X)\wedge\mathsf{P}(Y) = 0$, equivalently, $\mathrm{ran}(\mathsf{Q}(X)) \cap \mathrm{ran}(\mathsf{P}(Y)) = \{0\}$, for all bounded sets X, Y. To break this relation one needs to introduce inaccuracies (unsharpness, fuzziness) in the measurements of the observables to get jointly measurable or at least jointly testable approximators $\widetilde{\mathsf{Q}}$ and $\widetilde{\mathsf{P}}$ with $\mathrm{ran}\sqrt{\widetilde{\mathsf{Q}}(X)} \cap \mathrm{ran}\sqrt{\widetilde{\mathsf{P}}(Y)} \neq \{0\}$ (possibly) for all X, Y. This is the content of Paul's generalized Jauch theorem.

There are natural measurement schemes which lead to jointly testable, though possibly incompatible approximators $\widetilde{\mathsf{Q}} = \mathsf{Q}_\mu$ and $\widetilde{\mathsf{P}} = \mathsf{P}_\nu$. The quality of these approximations can be quantified using, for instance, the Wasserstein 2-distances $\Delta_2(\mathsf{Q}_\mu, \mathsf{Q}) \geq \Delta(\mu)$ and $\Delta_2(\mathsf{P}_\nu, \mathsf{P}) \geq \Delta(\nu)$, but there need be no correlation between the errors $\Delta(\mu)$ and $\Delta(\nu)$. Such approximators are jointly measurable exactly when the error measures μ and ν are Fourier related, in which case $\Delta(\mu)\Delta(\nu) \geq \frac{\hbar}{2}$. Moreover, in this case the errors can be traced back to the preparation uncertainties of the position and momentum of the probe (measuring apparatus). This is essentially the result which led Paul to formulate his answer to Pauli.

A simple way to break the complementarity of any two observables is mixing trivial noise in the statistics of one of them. But, for instance, in the case of position or momentum such an approximation is always extremely bad. Indeed, for instance, $\Delta_2(\mathsf{Q}_p, \mathsf{Q}) = \infty$ for any $\mathsf{Q}_p = p\mathsf{Q} + (1-p)\mathsf{T}_1$, $p < 1$.

As concerns position and momentum, the problem formulated in the opening sentence of this paper is now solved by the following result. Consider any biobservable B, that is, an observable with two independent outcomes, as an approximate joint observable of Q and P. If the degrees of approximations $\Delta_2(\mathsf{B}_1, \mathsf{Q})$ and $\Delta_2(\mathsf{B}_2, \mathsf{P})$ are finite then there is also a covariant phase space observable G^T which serves no worse joint approximator, with

$$\Delta_2(\mathsf{B}_1, \mathsf{Q})\Delta_2(\mathsf{B}_2, \mathsf{P}) \geq \Delta_2(\mathsf{G}^T_1, \mathsf{Q})\Delta_2(\mathsf{G}^T_2, \mathsf{P}) \geq \Delta(\mathsf{Q}_{\Pi T\Pi})\Delta(\mathsf{P}_{\Pi T\Pi}) \geq \frac{\hbar}{2}.$$

This is just a reformulation of Theorem 1.

The notion of joint measurability of two (or more) observables is well understood and it has various equivalent characterisations, see, for instance, [7, Theorem 11.1]. However, as noted above, there are obvious weakenings of this notion based on the joint measurability and joint testability of pairs of the effects constituting the observables. A further, perhaps extreme weakening of this concept is the joint measurability of two (or more) observables understood as any measurement the statistics of which allows one to reconstruct the statistics of the observables in question. Then any two observables can be measured together using tomographic methods (state reconstruction). In the case of Q and P there are even non-tomographic methods (applying informationally incomplete measurements) to measure Q and P together in the sense of "statistical postprocessing" [59].

There is no end in sight in this story.

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