include infinite serieses, provided they are converging ones; and it may even be extended to any algebraic expressions that can be expanded into such serieses. This class of spheroids comprehends the sphere, the ellipsoid, both sorts of elliptical spheroids of revolution, and an infinite number of other figures, as well such as can be described by the revolving of curves about their axes, as others which cannot be so generated.

In the second chapter of the third book of the *Mécanique Céleste*, LAPLACE has treated of the attractions of spheroids of every kind; and in particular he has given a very ingenious method for computing the attractive forces of that class which in their figures approach nearly to spheres. In studying that work, I discovered that the learned author had fallen into an error in the proof of his fundamental theorem; in consequence of which he has represented his method as applicable to all spheroids whatever, provided they do not differ much from spheres; whereas in truth, when the error of calculation is corrected, and the demonstration made rigorous, his analysis is confined exclusively to that particular kind, described above, which it is proposed to make the subject of this discourse. I have already treated of this matter in a separate paper, in which I have pointed out the source of LAPLACE's mistake, and likewise have strictly demonstrated his method for the instances that properly fall within its scope. In farther considering the same subject, it occurred to me that the investigation in the second chapter of the third book of the *Mécanique Céleste*, however skilfully and ingeniously conceived, is nevertheless indirect, and is besides liable to another objection of still greater weight; it does not exhibit the several terms of the series for the attractive force in separate and independent expressions: it only points out in what manner they may be derived successively, one after another; in so much that the terms of the series near the beginning cannot be found without previously computing all the rest. This remark gave occasion to the following paper, in which it is my design to give a solution of the problem which is not chargeable with the imperfections just mentioned: the analysis is direct, and every term of the series for the attractive force is deduced immediately from the radius of the spheroid. As the ellipsoid, which comprehends both sorts of elliptical spheroids of revolution, falls within the class of figures here treated of, I have derived, as a corollary from my investigation, the formulas for the attractions of that figure which are required in the theory of the earth: this paper therefore will contain all that is useful on the subject of the attractions of spheroids, as far as our knowledge at present extends, deduced by one uniform mode of analysis.

Having mentioned the principal object of this discourse, I must likewise notice a subordinate purpose I have in view; it is to put in a clear light the real grounds of Laplace's method, and of the equivalent method delivered in the following pages; to the accomplishment of which nothing is likely to contribute so much, as a direct and rigorous analysis perspicuously conducted. To promote the same end still farther, by preserving greater order and perspicuity in treating a subject in its own nature very complicated, this paper will be divided into two principal sections: in the first section it is proposed to lay down the analytical propositions on which the investigation is founded: the second section will contain the solution of the problem under consideration.

One more preliminary observation it is proper to add. The problem of attractions contains two cases; when the density of the attracting body is uniform throughout; when it varies according to any given law: it is in the first of these two cases that the chief difficulties occur; and as I have nothing new to add on the second case, I shall here confine my attention to homogeneous spheroids, unit being supposed to denote the density.

I.

Preliminary Investigations.

1. Let μ denote the cosine of an angle, and let

$$f = \left\{ r^2 - 2ra.\mu + a^2 \right\}^{\frac{1}{2}};$$

then the truth of the following equation in partial fluxions will be proved merely by performing the operations indicated, viz.

$$(1 - \mu^2)^n . \left(\frac{d . \frac{a - r\mu}{f^{2n+3}}}{dr} \right) + \left(\frac{d . \frac{(1-\mu^2)^{n+1}}{f^{2n+3}}}{d\mu} \right) = 0.$$

Now put $S = \frac{1}{f^{2n+1}}$; then

$$\left. \begin{aligned} \left(\frac{dS}{da}\right) &= -(2n+1) . \frac{a - r\mu}{f^{2n+3}} \\ \frac{1}{ra} . \left(\frac{dS}{d\mu}\right) &= 2n+1 . \frac{1}{f^{2n+3}} \end{aligned} \right\}$$

therefore, on account of the first equation, we shall obtain by substitution,

$$-ra\,.\,(1-\mu^2)^n\,.\left(\frac{ddS}{dadr}\right)+\left\{\frac{d\,.\left\{(1-\mu^2)^{n+1}\,.\left(\frac{dS}{d\mu}\right)\right\}}{d\mu}\right\}=0.$$

2. Let $\frac{1}{f}$ be reduced into a series of the descending powers of r; then

$$\frac{1}{f}=C^{(0)}\,.\,\frac{1}{r}+C^{(1)}\,.\,\frac{a}{r^2}+C^{(2)}\,.\,\frac{a^2}{r^3}\,\ldots\,+C^{(i)}\,.\,\frac{a^i}{r^{i+1}},\text{ \&c.}$$

and $C^{(i)}$ will be a rational and integral function of μ of i dimensions: substitute this series for S in the equation last found (n being $=0$), and we shall obtain

$$i\,(i+1)\,C^{(i)}+\frac{d\,.\left\{(1-\mu^2)\,.\,\frac{dC^{(i)}}{d\mu}\right\}}{d\mu}=0\,\ldots\ldots\,(1).$$

Again, take the fluxions n times successively in $\frac{1}{f}$ and likewise in the series equivalent to it, making μ the only variable; and we shall get

$$\frac{1}{f^{2n+1}}=\frac{1}{1.3.5\ldots 2n-1}\,.\times\left\{\frac{1}{r^{2n+1}}\,.\,\frac{d^nC^{(n)}}{d\mu}+\frac{a}{r^{2n+2}}\,.\,\frac{d^nC^{(n+1)}}{d\mu^n}\,\ldots\ldots\ldots\right.$$

$$\left.+\frac{a^{i-n}}{r^{i+n+1}}\,.\,\frac{d^nC^{(i)}}{d\mu^n},\text{ \&c.}\right\}:$$

substitute this series for S in the equation of No. 1, and we shall get

$$(i-n)\,(i+n+1)\,.\,(1-\mu^2)^n\,.\,\frac{d^nC^{(i)}}{d\mu^n}+\frac{d\,.\left\{(1-\mu^2)^{n+1}\,.\,\frac{d^{n+1}C^{(i)}}{d\mu^{n+1}}\right\}}{d\mu}$$

$$=0\,\ldots\ldots\ldots\,(2).$$

From this last equation it follows that

$$\int(1-\mu^2)^n\,.\,\frac{d^nC^{(i)}}{d\mu^n}\,.\,d\mu=0$$

when the fluent is taken between the limits $\mu=-1$ and

$\mu = 1$: for the fluent in question is equal to $-\frac{1}{i-n \,.\, i+n+1} \,.\, (1-\mu^2)^{n+1} \,.\, \frac{d^{n+1}C^{(i)}}{d\mu^{n+1}}$, a quantity which is evanescent at both the limits.

If we consider $\frac{d^o C^{(i)}}{d\mu^o}$ as a symbolical representation of $C^{(i)}$, the equation (1) will be included in the equation (2); whence it is easy to infer that whatever is proved of $\frac{d^n C^{(i)}}{d\mu^n}$ by the help of the equation (2) may be transferred to $C^{(i)}$ by putting $n = o$; a remark that will enable us to consult brevity, and of which we shall freely avail ourselves.

3. It is now proposed to find the value of $C^{(i)}$ in a series of the powers of μ.* The equation (2), by expanding its last term, will become

$$i\,(i+1)\,C^{(i)} - 2\mu \,.\, \frac{dC^{(i)}}{d\mu} + (1-\mu^2) \,.\, \frac{ddC^{(i)}}{d\mu^2} = o:$$

let the series

$$A^{(o)}\,\mu^i + A^{(1)}\,\mu^{i-2} + A^{(2)}\,\mu^{i-4} \ldots\ldots + A^{(s)} \,.\, \mu^{i-2s} \ldots + \&c.$$

be assumed as equivalent to $C^{(i)}$; then by substituting and equating the coefficient of μ^{i-2s} to o, we shall get

$$A^{(s)} = -\frac{(i-2s+2)\,(i-2s+1)}{2s\,(2i-2s+1)} \,.\, A^{(s-1)};$$

and, by putting $s = 1$, $s = 2$, &c. successively, we shall hence be able to determine the proportions of all the coefficients to the first one $A^{(o)}$, which must be investigated from other considerations. Now $C^{(i)}$ is the coefficient of $\frac{a^i}{r^{i+1}}$ in the ex-

* Méc. Céleste Liv. 3e, No. 15.

pansion of $\frac{1}{f} = \frac{1}{(r^2+a^2)^{\frac{1}{2}}} \cdot \left(1 - \frac{2ra \cdot \mu}{r^2+a^2}\right)^{-\frac{1}{2}}$; and, by the binomial theorem, the term containing μ^i will be $= \frac{1.3.5 \ldots\ldots 2i-1}{2.4.6 \ldots\ldots 2i} \times \frac{2^i r^i a^i \mu^i}{(r^2+a^2)^{i+\frac{1}{2}}} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i}, \frac{a^i \cdot \mu^i}{r^{i+1}} \cdot \left(1 + \frac{a^2}{r^2}\right)^{-(i+\frac{1}{2})}$; whence it is plain that $A^{(o)} = \frac{1.3.5 \ldots\ldots 2i-1}{1.2.3 \ldots\ldots i}$: consequently,

$$C^{(i)} = \frac{1.3.5 \ldots 2i-1}{1.2.3 \ldots i} \times \left\{\mu^i - \frac{i(i-1)}{2(2i-1)} \cdot \mu^{i-2} + \frac{i(i-1)(i-2)(i-3)}{2.4.(2i-1)(2i-3)} \cdot \mu^{i-4} - \&c.\right\}$$

If we take the fluxions n times successively in the last formula, we shall obtain

$$\frac{d^n C^{(i)}}{d\mu^n} = \frac{1.3.5 \ldots\ldots 2i-1}{1.2.3 \ldots\ldots i-n} \cdot \left\{\mu^{i-n} - \frac{i-n \cdot i-n-1}{2.2i-1} \cdot \mu^{i-n-2} + \frac{i-n \cdot i-n-1 \cdot i-n-2 \cdot i-n-3}{2.4 \cdot 2i-1 \cdot 2i-3} \cdot \mu^{i-n-4} - \&c.\right\}.$$

When $i-n$ is an even number, $\frac{d^n C^{(i)}}{d^n}$ will contain a part, equal to

$$\pm \frac{1.3.5 \ldots i+n+1}{2.4.6 \ldots i-n},$$

independent of μ; and when $i-n$ is an odd number, the same quantity will contain a part, equal to

$$\pm \frac{1.3.5 \ldots\ldots i+n}{2.4.6 \ldots\ldots i-n-1} \cdot \mu,$$

multiplied by μ only: these two parts of the value of $\frac{d^n C^{(i)}}{d\mu^n}$ we shall afterwards have occasion to refer to.

4. It is proposed to investigate the fluent of

$$(1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu,$$

between the limits $\mu = -1$ and $\mu = 1$; supposing P to be a rational and integral function of μ.

On account of the equation (2), we get

$$\int (1-\mu^2) \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = -\frac{1}{i-n \cdot i+n+1} \cdot \int P \cdot d \cdot \left\{ (1-\mu^2)^{n+1} \cdot \frac{d^{n+1} C^{(i)}}{d\mu^{n+1}} \right\};$$

and, by integrating by parts,

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = -\frac{1}{i-n \cdot i+n+1} \cdot (1-\mu^2)^{n+1} \cdot \frac{d^{n+1} C^{(i)}}{d\mu^{n+1}} \cdot P + \frac{1}{i-n \cdot i+n+1} \cdot \int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1} C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu:$$

and, by rejecting that part of the fluent which is evanescent at both the limits, we have

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = \frac{1}{i-n \cdot i+n+1} \cdot \int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1} C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu.$$

In this last equation the expressions on both sides are entirely similar; and therefore by a repetition of the same operations we shall obtain

$$\int (1-\mu^2)^{n+1} \cdot \frac{d^{n+1} C^{(i)}}{d\mu^{n+1}} \cdot \frac{dP}{d\mu} \cdot d\mu = \frac{1}{i-n-1 \cdot i+n+2} \cdot \int (1-\mu^2)^{n+2} \cdot \frac{d^{n+2} C^{(i)}}{d\mu^{n+2}} \cdot \frac{d^2 P}{d\mu^2} \cdot d\mu:$$

and exterminating the integral common to both these equations, we shall get

$$\int (1-\mu^2)^n \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot P \cdot d\mu = \frac{1}{i-n \cdot i-n-1} \times \frac{1}{i+n+1 \cdot i+n+2} \times \int (1-\mu^2)^{n+2} \cdot \frac{d^{n+2} C i}{d\mu^{n+2}} \cdot \frac{d^2 P}{d\mu^2} \cdot d\mu.$$

It is evident we may continue the like operations as far as we please: for abridging expressions let

$\sigma = i - n \,.\, i - n - 1 \,.\, i - n - 2 \ldots\ldots i - n - m + 1;$

$\tau = i + n + 1 \,.\, i + n + 2 \,.\, i + n + 3 \ldots\ldots i + n + m;$

then after m successive operations we shall get,

$$\int (1 - \mu^2)^n \,.\, \frac{d^n C^i}{d\mu^n} \,.\, P \,.\, d\mu = \frac{1}{\sigma . \tau} \times \int (1 - \mu^2)^{n+m} \,.\, \frac{d^{n+m} C^{(i)}}{d\mu^{n+m}} \,.\, \frac{d^m P}{d\mu^m} \,.\, d\mu.$$

If m, less then $i - n$, denote the dimensions of P, then $\frac{d^m P}{d\mu^m}$ will be a constant quantity, and the fluent on the right-hand side will be $= o$ (No. 2): hence this theorem, viz.

"If P be a rational and integral function of μ, and of less "dimensions than $i - n$, then

$$\int (1 - \mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, P \,.\, d\mu = o$$

"when the whole fluent is taken between the limits $\mu = -1$ "and $\mu = 1$."

If the dimensions of P be not less than $i - n$, put $m = i - n$, and for $\frac{d^i C^{(i)}}{d\mu^i}$ write its value, $1.3.5 \ldots . 2i - 1$ (3); and the preceding formula will become

$$\int (1 - \mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, P \,.\, d\mu = \frac{i - n + 1 \,.\, i - n + 2 \ldots\ldots i + n}{2.4.6 \ldots\ldots\ldots\ldots 2i} \times \int (1 - \mu^2)^i \,.\, \frac{d^{i-n} P}{d\mu^{i-n}} \,.\, d\mu:$$

and hence, $\beta^{(n)} = \frac{1}{i-n+1 \,.\, i-n+2 \ldots\ldots i+n}$, we have

$$\beta^{(n)} \,.\int (1 - \mu^2)^n \,.\, \frac{d^n C^{(i)}}{d\mu^n} \,.\, P \,.\, d\mu = \frac{\int (1 - \mu^2)^i \,.\, \frac{d^{i-n} P}{d\mu^{i-n}} \,.\, d\mu}{2.4.6 \ldots\ldots\ldots\ldots 2i}.$$

By means of the last formula the fluent in question will be reduced to the integration of expressions of this kind, viz. $\mu^{s-1} . (1-\mu^2)^i . d\mu$; a research with which mathematicians are familiar. In the first place when s is even; then, considering the definite fluent between the limits $\mu = -1$ and $\mu = 1$; we have

$$\int \mu^{s-1} . (1-\mu^2)^i . d\mu = o:$$

and indeed, supposing P to be any odd function of μ, we have more generally $\int P . d\mu = o$, between the same limits. In the second place when s is odd; then, taking the definite fluent as before,

$$\int \mu^{s-1} . (1-\mu^2)^i . d\mu = \frac{2}{s} . \frac{2i}{2i+s} . \frac{2(i-1)}{2i+s-2} . \frac{2(i-3)}{2i+s-4} \ldots\ldots \frac{2}{2+s}.$$

The observations that have already been made are sufficient to point out in what manner the expressions of the fluents under consideration may be formed with great practical commodiousness.

5. Let μ, μ', γ denote the cosines of the three sides of a spherical triangle; and let φ be the angle opposite to the side whose cosine is γ: then, according to what is taught in spherical trigonometry,

$$\gamma = \mu\mu'^2 + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} . \cos. \varphi:$$

suppose farther that $f = \left\{r^2 - 2ra . \gamma + a^2\right\}^{\frac{1}{2}}$,
and let

$$\frac{1}{f} = Q^{(0)} . \frac{1}{r} + Q^{(1)} . \frac{a}{r^2} + Q^{(2)} . \frac{a^2}{r^3} \ldots\ldots + Q^{(i)} . \frac{a^i}{r^{i+1}} \ \&c.$$

it is required to expand $Q^{(i)}$, which is the same function of γ that $C^{(i)}$ is of μ, into a series of the cosines of φ and its multiples.*

* Méc. Cél. Liv. 3e, No. 15.

LAPLACE has proved that every one of the coefficients in the series for $\frac{1}{f}$ will satisfy an equation in partial fluxions which is thus generally expressed for $Q^{(i)}$, viz.

$$i\,(i+1)\,.\,Q^{(i)} + \left\{\frac{d\,.\left\{(1-\mu^2)\,.\left(\frac{dQ^{(i)}}{d\mu}\right)\right\}}{d\mu}\right\} + \frac{\left(\frac{ddQ^{(i)}}{d\varphi^2}\right)}{1-\mu^2} = 0.$$

This is a fundamental equation in his investigation, and it is necessary for effecting the expansion here proposed: but we shall refer to LAPLACE's work for the demonstration of it.*

It is plain that $Q^{(i)}$, when it is considered as a function of μ and the cosines of φ and its multiples, may be thus represented, viz.

$$Q^{(i)} = H^{(0)} + (1-\mu^2)^{\frac{1}{2}}\,.\,H^{(1)}\,.\,\cos.\,\varphi + (1-\mu^2)^{\frac{2}{2}}\,.\,H^{(2)}\,.\,\cos.\,2\varphi + \&c.$$

the general term of the series being $(1-\mu^2)^{\frac{n}{2}}\,.\,H^{(n)}\,.\,\cos.\,n\varphi$, which ought to satisfy LAPLACE's equation in partial fluxions: now, having actually substituted that quantity in the equation mentioned, and having divided all the terms by $\cos.\,n\varphi$, I have found,

$$(i-n)\,(i+n+1)\,.\,(1-\mu^2)^{\frac{n}{2}}\,.\,H^{(n)} - 2\,(n+1)\,\mu\,(1-\mu^2)^{\frac{n}{2}}\,.\,\frac{dH^{(n)}}{d\mu} + (1-\mu^2)^{\frac{n}{2}+1}\,.\,\frac{ddH^{(n)}}{d\mu^2} = 0:$$

and, after having multiplied all the terms by $(1-\mu^2)^{\frac{n}{2}}$, the result will be equivalent to this equation, viz.

$$(i-n)\,(i+n+1)\,(1-\mu^2)^n\,.\,H^{(n)} + \frac{d.\left\{(1-\mu^2)^{n+1}\,.\,\frac{dH^{(n)}}{d\mu}\right\}}{d\mu} = 0,$$

whence it follows (equat. 2.) that $H^{(n)} = B^{(n)}\,.\,\frac{d^nC^{(i)}}{d\mu^n}$, where

* Méc. Cél. No. 9, Liv. 3e, and No. 11, Liv. 2d.

$B^{(n)}$ denotes a quantity that does not contain μ; therefore the general term of the series for $Q^{(i)}$ is $B^{(n)} . (1-\mu^2)^{\frac{n}{2}} . \frac{d^n C^{(i)}}{d\mu^n}$. cos. $n\phi$: but as μ and μ' enter alike into the expression of $Q^{(i)}$, it is clear that they will be both equally concerned in every term of its expansion: therefore the general term of the series will be,

$$\beta^{(n)} . (1-\mu^2)^{\frac{n}{2}} . (1-\mu'^2)^{\frac{n}{2}} . \frac{d^n C^{(i)}}{d\mu^n} . \frac{d^n C'^{(i)}}{d\mu'^n} . \cos. n\phi,$$

where $C'^{(i)}$ is put to denote the same function of μ' that $C^{(i)}$ does of μ; and $\beta^{(n)}$ is a quantity that contains neither μ nor μ', and which can only be a numeral coefficient, and is all that now remains unknown.

In order to determine $\beta^{(n)}$, we must follow the process of LAPLACE.* It is to be observed that $Q^{(i)}$ is the coefficient of $\frac{a^i}{r^{i+1}}$ in the expansion of the radical $\left\{r^2 - 2ra . \gamma + a^2\right\}^{-\frac{1}{2}} =$

$$\left\{r^2 - 2ra . (\mu\mu' + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} . \cos. \phi) + a^2\right\}^{-\frac{1}{2}};$$

which, when the squares and other higher powers are neglected, will be equal to

$$\left\{r^2 - 2ra . \cos. \phi + a^2\right\}^{-\frac{1}{2}} + ra . \mu\mu' . \left\{r^2 - 2ra . \cos. \phi + a^2\right\}^{-\frac{3}{2}};$$

from the first term of this expression are derived all the parts of the expansion of the radical $\left\{r^2 - 2ra . \gamma + a^2\right\}^{-\frac{1}{2}}$ which are independent on μ and μ'; and from the second term of it are derived all those parts which contain only $\mu\mu'$, without the squares and higher powers: now if we determine the parts

* Méc. Cél. Liv. 3e, No. 15.

mentioned by the actual expansion of the two radicals, and likewise determine the corresponding parts of $Q^{(i)}$ by means of the formulas in No. 3; the comparison of the equivalent expressions will determine the values of the coefficients required.

To execute the operations alluded to, let c denote the number whose hyperbolic logarithm is unit; then

$$\left\{r^2 - 2ra \,.\, \cos.\,\varphi + a^2\right\}^{-s} = \left(r - a \,.\, c^{\varphi\sqrt{-1}}\right)^{-s} . \left(r - a \,.\, c^{-\varphi\sqrt{-1}}\right)^{-s};$$

and if we represent the expansions of the two binomials by the serieses

$$\frac{1}{r^s} + A^{(1)} . \frac{a \,.\, c^{\varphi\sqrt{-1}}}{r^{1+s}} + A^{(2)} . \frac{a^2 \,.\, c^{2\varphi\sqrt{-1}}}{r^{2+s}} + \&c.$$

$$\frac{1}{r^s} + A^{(1)} . \frac{a \,.\, c^{-\varphi\sqrt{-1}}}{r^{1+s}} + A^{(2)} . \frac{a^2 \,.\, c^{-2\varphi\sqrt{-1}}}{r^{2+s}} + \&c.$$

we shall obtain the expansion of the radical by multiplying the two serieses: let p and q denote the ranks of any two terms in both serieses, then the part of the expansion derived from the multiplication of the aforesaid parts, will be

$$2A^{(p)} . A^{(q)} . \frac{a^{p+q}}{r^{p+q+2s}} . \left\{\frac{c^{(p-q)\varphi\sqrt{-1}} + c^{-(p-q)\varphi\sqrt{-1}}}{2}\right\};$$

$$\text{or, } 2A^{(p)} . A^{(q)} . \frac{a^{p+q}}{r^{p+q+2s}} . \cos.\,(p-q) . \varphi.$$

When $i - n$ is an even number, we have only to make $p + q = i$, and $p - q = n$, and $s = \frac{1}{2}$; and we shall get

$$2 \times \frac{1.3.5 \ldots i-n-1}{2.4.6 \ldots i-n} \times \frac{1.3.5 \ldots i+n-1}{2.4.6 \ldots i+n} . \cos.\,n\varphi,$$

for the part of the coefficient of $\frac{a^i}{r^{i+1}}$, or of $Q^{(i)}$, which is multiplied by cos. $n\varphi$ and clear of μ and μ'; but the like part of $\beta^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot (1-\mu'^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \frac{d^n C'^{(i)}}{d\mu^n}$. cos. $n\varphi$ (which is the whole expression of the part of $Q^{(i)}$ multiplied by cos. $n\varphi$) obtained by the help of the formulas in No. 3, is

$$\beta^{(n)} \cdot \left(\frac{1.3.5....i+n-1}{2.4.6....\ i-n}\right)^2 \cdot \cos.\ n\varphi:$$

therefore by equating the equivalent expressions, we get

$$\beta^{(n)} = \frac{2}{i-n+1\ .\ i-n+2.......i+n}.$$

When $i-n$ is odd, make $p+q=i-1$; $p-q=n$; $s=\frac{3}{2}$: then we will obtain

$$2 \times \frac{1.3.5....i+n}{2.4.6....i+n-1} \cdot \frac{1.3.5....i-n}{2.4.6....i-n-1} \cdot \mu\mu' \cdot \cos.\ n\varphi$$

for the part of $Q^{(i)}$, or of the coefficient of $\frac{a^i}{r^{i+1}}$, which is multiplied by $\mu\mu'$. cos, $n\varphi$: but the like part of $\beta^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot (1-\mu'^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \frac{d^n C'^{(i)}}{d\mu^n}$. cos. $n\varphi$, obtained by the formulas in No. 3, is

$$\beta^{(n)} \cdot \left(\frac{1.3.5....i+n}{2.4.6....i-n-1}\right)^2 \cdot \mu\mu' \cdot \cos.\ n\varphi:$$

whence we get, in this case also,

$$\beta^{(n)} = \frac{2}{i-n+1\ .\ i-n+2..........i+n}.$$

Now if we write $2\beta^{(n)}$ in the place of $\beta^{(n)}$; that is, if we henceforth put (as in No. 4)

$$\beta^{(n)} = \frac{1}{i-n+1\ .\ i-n+2\ .\ i-n+3.......i+n};$$

then all the terms of the expansion we are seeking, will be found by making $n=1$, $n=2$, $n=3$, &c. successively, and it will be thus expressed, viz.

$$Q^{(i)} = C^{(i)} . C'^{(i)} + 2\beta^{(1)} . (1-\mu^2)^{\frac{1}{2}} . (1-\mu'^2)^{\frac{1}{2}} . \frac{dC^{(i)}}{d\mu} . \frac{dC'^{(i)}}{d\mu'} . \cos . \phi$$
$$+ 2\beta^{(2)} . (1-\mu^2)^{\frac{2}{2}} . (1-\mu'^2)^{\frac{2}{2}} . \frac{d^2C^{(i)}}{d\mu^2} . \frac{d^2C'^{(i)}}{d\mu'^2} . \cos . 2\phi$$
$$\vdots$$
$$+ 2\beta^{(n)} . (1-\mu^2)^{\frac{n}{2}} . (1-\mu'^2)^{\frac{n}{2}} . \frac{d^nC^{(i)}}{d\mu^n} . \frac{d^nC'^{(i)}}{d\mu'^n} . \cos . n\phi$$
&c.

II.

Investigation of the Attractions of Spheroids of a particular Kind.

6. Instead of seeking immediately the attraction of a spheroid in any proposed direction, it will be more advantageous to investigate (as Laplace has done) the value of the expression (to be henceforth denoted by V) which is the sum of the quotients produced by dividing all the molecules of the mass of the spheroid by their respective distances from the attracted point. For such is the nature of the analytical expression now mentioned, that if it be first transformed into a function of three rectangular co-ordinates one of which is parallel to a line given by position, and the fluxion with regard to this co-ordinate be taken; the coefficient of the partial fluxion after its sign is changed, will denote the attractive force which acts parallel to the given line. In order to demonstrate this property of the function V, we shall suppose that x, y, z denote the co-ordinates of the molecule dM, and a, b, c, the co-ordinates of the attracted point: then

$$V = \int \frac{dM}{\left\{(a-x)^2 + (b-y)^2 + (c-z)^2\right\}^{\frac{1}{2}}};$$

the fluent being understood to be extended to all the molecules of the mass of the spheroid: now if the fluxion of this expression be taken, making a the only variable, we shall have

$$-\left(\frac{dV}{da}\right)=\int\frac{(a-x)\,.\,dM}{\left\{(a-x)^2+(b-y)^2+(c-z)^2\right\}^{\frac{3}{2}}}:$$

where the expression on the right hand side is the attractive force parallel to a, as will readily appear by decomposing the direct attractions of all the molecules into the partial attractions parallel to the co-ordinates. But, besides enabling us to find the attractive force in any proposed direction, the function V has another advantage; for it is this function, and not the expressions of the attractive forces, which enters into the equation of the surface of a body, wholly or partly fluid, in a state of equilibrium.*

The expression for V, exhibited above is not of a commodious form, and on this account it becomes necessary to transform it. Let $x = R' \cos.\theta'$; $y = R' \sin.\theta' \cos.\varpi'$; and $z = R' \sin.\theta' \sin.\varpi'$; then will R′ be the line drawn from the molecule dM to the origin of the co-ordinates; θ' will be the angle which R′ makes with the axis of x; and ϖ' the angle which the projection of R′ upon the plane to which x is perpendicular, makes with a line given by position in the same plane: from the assumed values of x, y, z, it is easy to derive these new values, viz.

$$x = R' \cos.\theta' = \sqrt{R'^2 - y^2 - z^2}$$
$$y = R' \sin.\theta' \cos.\varpi' = \sqrt{R'^2 \sin.^2\theta' - z^2}$$
$$z = R' \sin.\theta' \sin.\varpi':$$

and, by taking the fluxions so as to make x vary with R′, y with θ', and z with ϖ'; which will leave dx, dy, dz, as well as dR', $d\theta'$, $d\varpi'$, unrelated and independent on one another as the

* Méc. Cél. Liv. 3, No. 4.

case requires, we shall have

$$dx = \frac{R'dR'}{\sqrt{R'^2 - y^2 - z^2}} = \frac{dR'}{\cos.\theta'}$$

$$dy = \frac{R'^2 \sin.\theta' \cos.\theta' . d\theta'}{\sqrt{R'^2 \sin.^2\theta' - z^2}} = \frac{R' \cos.\theta' . d\theta'}{\cos.\varpi'}$$

$$dz = R' \sin.\theta' \cos.\varpi' . d\varpi':$$

consequently (the density being denoted by unit) $dM = dx . dy . dz = R'^2 . dR' . d\theta' \sin.\theta' . d\varpi$. Farther, let $a = r \cos.\theta$, $b = r \sin.\theta \cos.\varpi$, $c = r \sin.\theta \sin.\varpi$; then, by substitution,

$$* V = \iiint \frac{R'^2 . dR' . d\theta' \sin.\theta' . d\varpi'}{\sqrt{r^2 - 2rR' . (\cos.\theta \cos.\theta' + \sin.\theta \sin.\theta' \cos.(\varpi' - \varpi)) + R'^2}}$$

and if we put $\cos.\theta = \mu$, $\cos.\theta' = \mu'$; then

$$\left.\begin{array}{l} V = \iiint \frac{R'^2 . dR' . d\mu' . d\varpi'}{\sqrt{r^2 - 2rR' . \gamma + R'^2}} \\ \gamma = \mu\mu' + \sqrt{1-\mu^2} . \sqrt{1-\mu'^2} . \cos.(\varpi' - \varpi) \end{array}\right\} (4).$$

7. When the attracted point is without the surface, the expression for V, in order to embrace the whole mass of the spheroid, must be integrated from $R' = o$, to $R' = R$, R denoting what R' becomes at the surface; from $\mu' = -1$ to $\mu' = 1$; and from $\varpi' = o$ to $\varpi' = 2\pi$, 2π being the circumference when the radius is unit. In this case V must be reduced into a series containing the descending powers of r, which we may thus represent, viz.

$$V = \frac{B^{(o)}}{r} + \frac{B^{(1)}}{r^2} + \frac{B^{(2)}}{r^3} \ldots\ldots + \frac{B^{(i)}}{r^{i+1}} \ldots\ldots \&c.$$

and if we expand the radical in the last expression of V into a similar series, and use $Q^{(i)}$ to denote the same thing as formerly in No. 4, we shall get, by equating the corresponding terms,

* Méc. Cél. Liv. 3e, No. 8.

$$B^{(i)} = \iiint R'^{i+2} \, . \, dR' \, . \, d\mu' \, . \, d\varpi' \, . \, Q^{(i)}.^*$$

In this expansion $B^{(0)}$, in every case is equal to the mass of the spheroid: and with regard to the second term, LAPLACE has remarked that it may be made to disappear by fixing the origin of R', which is an arbitrary point, in the centre of gravity of the spheroid. To prove this, we have

$$B^{(1)} = \iiint R'^3 \, . \, dR' \, . \, d\mu' \, . \, d\varpi' \, . \, Q^{(1)}:$$

but $dM = R'^2 . dR' . d\mu' . d\varpi'$; and $R' . Q^{(1)} = R' . \gamma = \mu \times R'\mu' + \sqrt{1-\mu^2} . \cos. \varpi \times R' . \sqrt{1-\mu'^2} . \cos. \varpi' + \sqrt{1-\mu^2} . \sin. \varpi \times R' . \sqrt{1-\mu'^2} . \sin. \varpi' = \mu \times x + \sqrt{1-\mu^2} . \cos. \varpi \times y + \sqrt{1-\mu^2} . \sin. \varpi \times z$; where x, y, z denote as before the coordinates of the molecule dM: therefore, by substitution,

$$B^{(1)} = \mu \times \int x . dM + \sqrt{1-\mu^2} . \cos. \varpi \times \int y . dM + \sqrt{1-\mu^2} . \sin. \varpi \times \int z . dM:$$

now, if all the planes to which x, y, z, are perpendicular pass through the centre of gravity; then, by the nature of that point, $\int x . dM = 0$; $\int y . dM = 0$; $\int z . dM = 0$: therefore $B^{(1)} = 0.$†

In the expression of $B^{(i)}$ none of the integrations can be executed in a general manner, excepting that relative to dR': let R denote what R' becomes at the surface of the spheroid; then

$$B^{(i)} = \frac{1}{i+3} . \iint R^{i+3} . d\mu' . d\varpi' . Q^{(i)}.$$

8. When the attracted point is within the spheroid, the value of V will be represented by a series of the ascending powers of r: let

* Méc. Cél. Liv. 3e, No. 9. † Ibid. Liv. 3e, No. 12.

$$V = b^{(0)} + b^{(1)} . r + b^{(2)} . r^2 + b^{(3)} . r^3 + \&c.;$$

then by expanding the radical in the formula (4) into a series of a similar form, and equating the corresponding terms, we shall get

$$b^{(i)} = \iiint \frac{dR' . d\mu' . d\varpi' . Q^{(i)}}{R^{i-1}} .*$$

In this value of $b^{(i)}$, the integration with regard to dR' cannot be executed from $R' = o$, as in the former case; because this expansion of V necessarily supposes that the attracted point is included within all the attracting matter: let R be what R′ becomes at the surface of the spheroid, which is the outer surface bounding the attracting matter, and let ρ be the radius of the inner surface; then, with respect to the matter between the two surfaces, and for a point within them both, we shall have

$$b^{(i)} = \frac{1}{i-2} . \iint \left\{ \frac{1}{\rho^{i-2}} - \frac{1}{R^{i-2}} \right\} . d\mu' . d\varpi' . Q^{(i)} . \quad (6).$$

In the case of $i = 2$, the expression of the coefficient takes a particular form: for

$$b^{(2)} = \iiint \frac{dR' . d\mu' . d\varpi' . Q^{(2)}}{R'};$$

and, by integrating,

$$b^{(2)} = \iint \left\{ \log. R' - \log. \rho \right\} . d\mu' . d\varpi' . Q^{(2)}.$$

Let us now seek an expression of the force with which the whole spheroid attracts a point within the surface. For this purpose we shall suppose ρ to denote the radius of a sphere which completely envelops the spheroid: and we shall determine; first, the value of V, relatively to the matter between the spheroid and the sphere; secondly, its value, relatively to

* Méc. Cél. Liv. 3e, No. 13.

the whole sphere: then the difference of these values will be the quantity proposed to be investigated.

With regard to the first value of V, it is to be observed that R is here the radius of the inner, and ρ that of the outer surface; therefore (6),

$$b^{(i)} = \frac{1}{i-2} \cdot \iint \left\{ \frac{1}{R^{i-2}} - \frac{1}{\rho^{i-2}} \right\} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}.$$

But I say that $\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = o$, when the fluent is extended between the proper limits: for μ, μ', and γ are the cosines of the three sides of a spherical triangle, and $\varpi' - \varpi$ is the angle of the same triangle opposite to the side whose cosine is γ; and if we put ψ to denote the angle opposite to the side whose cosine is μ'; then since the fluxion of the spherical surface may be either $d\mu' \cdot d\varpi'$ or $d\gamma \cdot d\psi$; therefore, when the fluents are extended to the whole surface of the sphere, we shall have

$$\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = \int Q^{(i)} \cdot d\gamma \cdot d\psi = 2\pi \cdot \int Q^{(i)} \cdot d\gamma:$$

but $\int Q^{(i)} \cdot d\gamma$, between the limits $\gamma = -1$ and $\gamma = 1$, is $= o$ (No. 2): therefore $\int Q^{(i)} \cdot d\mu' \cdot d\varpi' = o$.

Consequently the preceding expression of $b^{(i)}$ will become simply

$$b^{(i)} = \frac{1}{i-2} \cdot \iint \frac{Q^{(i)} \cdot d\mu \cdot d\varpi'}{R^{i-2}}:$$

and the value of V, relative to the shell of matter between the spheroid and sphere will be expressed by this series, viz.

$$\begin{aligned} V = \tfrac{1}{2} \iint (\rho^2 - R^2) \cdot d\mu' \cdot d\varpi' - r \cdot \iint R \cdot d\mu' \cdot d\varpi' \cdot Q^{(1)} \\ - r^2 \cdot \iint \log. R \cdot d\mu' \cdot d\varpi' \cdot Q^{(2)} \\ + r^3 \cdot \iint \frac{Q^{(3)} \cdot d\mu' \cdot d\varpi'}{R^2} \\ + \&c. \end{aligned}$$

As to the value of V for the whole sphere, it is composed of two parts: one relative to the matter within the attracted point, which is a sphere whose radius is the distance of that point from the centre; and the other, relative to the remaining matter of the sphere: the value of the first part is $= \frac{4\pi}{3} . r^2$; the value of the second part is $= \frac{1}{2} \int\int (\rho^2 - r^2) \times d\mu' . d\varpi'$: therefore the whole value of V is $= \frac{4\pi}{3} . r^2 + \frac{1}{2} \int\int (\rho^2 - r^2) . d\mu' . d\varpi'$.

By taking the difference of these two values, we get

$$\begin{aligned} V = -\frac{2\pi}{3} . r^2 + \frac{1}{2} . \int\int R^2 . d\mu' . d\varpi' . Q^{(0)} \\ + r . \int\int R . d\mu' . d\varpi' . Q^{(1)} \\ + r^2 . \int\int \log . R . d\mu' . d\varpi' . Q^{(2)} \\ - \frac{r^3}{1} . \int\int \frac{Q^{(3)} . d\mu' . d\varpi'}{R} \\ - \frac{r^4}{2} . \int\int \frac{Q^{(4)} . d\mu' . d\varpi'}{R^2} \\ \&c. \end{aligned}$$

this is the value of V when the attracted point is within the spheroid; and the terms in it that are unknown depend only on the radius of the surface, as in the case when the attracted point is without the surface.

9. We now proceed to the application of the formulas that have been investigated. And in the first place we shall consider a spheroid differing little from a sphere: in which case $R = a . (1 + \alpha . y')$, α denoting a coefficient so small that its square and other higher powers may be neglected; and y' a rational and integral function of μ', $\sqrt{1 - \mu'^2} . \cos. \varpi'$ and

$\sqrt{1-\mu'^2}$. sin. ϖ'. It is to be understood that $a \,.\, (1 + \alpha \,.\, y)$ denotes that radius of the spheroid which, produced if necessary, passes through the attracted point; and y is what y' becomes when $\mu' = \mu$ and $\varpi' = \varpi$.

Supposing the attracted point to be without the surface, we have No. 7,

$$\left. \begin{array}{l} V = \frac{B^{(0)}}{r} + \frac{B^{(1)}}{r^2} + \frac{B^{(2)}}{r^3} \ldots\ldots + \frac{B^{(i)}}{r^{i+1}} \ldots \&c. \\ \qquad\qquad B^{(i)} = \frac{1}{i+3} \,.\, \iint R^{i+3} \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}. \end{array} \right\}$$

and by substituting $a \,.\, (1 + \alpha \,.\, y')$ for R and retaining only quantities of the first order with regard to α, we shall get,

$$B^{(i)} = \frac{a^{i+3}}{i+3} \,.\, \iint Q^{(i)} \,.\, d\mu' \,.\, d\varpi' + \alpha \,.\, a^{i+3} \,.\, \iint y' \,.\, d\mu' \,.\, d\varpi' \, Q^{(i)} :$$

but, as has already been proved (No. 8), $\iint Q^{(i)} \,.\, d\mu' \,.\, d\varpi' = 0$: therefore

$$B^{(i)} = \alpha \,.\, a^{i+3} \,.\, \iint y' \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)} :$$

thus the value of $B^{(i)}$ depends upon the integral $\iint y' \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(i)}$, which may be found by means of the analytical formulas in the first part of this discourse, as we now proceed to show.

In the first place, when y' is a rational and integral function of μ' only without ϖ', which will be the case in spheroids of revolution: substitute for $Q^{(i)}$ its developement in No. 5, writing $\varpi' - \varpi$ for ϕ; integrate from $\varpi' = 0$ to $\varpi' = 2\pi$, observing that the fluents of all the terms which contain the cosines of $\varpi' - \varpi$ are of the same magnitude at both the limits, and therefore they will add nothing to the value of the integral taken between these limits: then we shall have simply

$$\int\int y' . d\mu' . d\varpi' . Q^{(i)} = 2\pi \ C^{(i)} . \int y' . d\mu' . C'^{(i)}:$$

to execute the remaining integration we have only to apply the method of No. 4: let the integral $\int\int y' . d\mu' . d\varpi' . Q^{(i)}$ be denoted by $2\pi \times U^{(i)}$; then by the method alluded to,

$$U^{(i)} = C^{(i)} \times \frac{\int (1-\mu'^2)^i . \frac{d^i y'}{d\mu'^i} . d\mu'}{2.4.6 \ldots\ldots\ldots 2i}.$$

If y' be a rational and integral function of μ', $\sqrt{1-\mu'^2}$. cos. ϖ' and $\sqrt{1-\mu'^2}$. sin. ϖ'; it must be transformed into a series of the sines and cosines of ϖ' and its multiples; then

$$y' = M^{(0)} + (1-\mu'^2)^{\frac{1}{2}} . M^{(1)} . \cos. \varpi' + (1-\mu'^2)^{\frac{2}{2}} . M^{(2)} . \cos. 2\varpi' \ \&c.$$
$$+ (1-\mu'^2)^{\frac{1}{2}} . N^{(1)} . \sin. \varpi' + (1-\mu'^2)^{2} . N^{(2)} . \sin. 2\varpi' \ \&c.$$

the general term of the series being $(1-\mu'^2)^{\frac{n}{2}} . M^{(n)} . \cos. n\varpi' + (1-\mu'^2)^{\frac{n}{2}} . N^{(n)} . \sin. n\varpi'$, where $M^{(n)}$ and $N^{(n)}$ denote rational and integral functions of μ'; and here the integral in question will consist of as many parts as there are independent functions contained in y'. In order to find the part of the integral resulting from the general term, we must multiply that term into the expansion of $Q^{(i)}$ investigated in No. 5; and in combining these two expressions we may omit all the terms which, after multiplication, would contain the sines and cosines of the multiples of ϖ'; because these, when they are integrated with regard to $d\varpi'$, will be of the same value at both the limits, on which account they will produce nothing in the value of the integral: this being observed, the only term of $Q^{(i)}$ which it is necessary to retain is that one containing cos. $n\,(\varpi'-\varpi)$, which may be thus written,

$$2\beta^{(n)} \cdot (1-\mu^2)^{\frac{n}{2}} \cdot (1-\mu'^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \frac{d^n C'^{(i)}}{d\mu'^n} \cdot \left\{ \cos.\, n\varpi \,.\, \cos.\, n\varpi' + \sin.\, n\varpi \,.\, \sin.\, n\varpi' \right\};$$

and by combining this with the general term of y', there will result the following expression which is clear of the sines and cosines of variable angles, viz.

$$(1-\mu^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \left\{ \cos.\, n\varpi \times \iint \beta^{(n)} \cdot (1-\mu'^2)^n \cdot \frac{d^n C'^{(i)}}{d\mu'^n} \cdot M^{(n)} \cdot d\mu' \cdot d\varpi' + \sin.\, n\varpi \times \iint \beta^{(n)} \cdot (1-\mu'^2)^n \cdot \frac{d^n C^{(i)}}{d\mu'^n} \cdot N^{(n)} \cdot d\mu' \cdot d\varpi' \right\}:$$

this expression again comes under the method of No. 4; let the integral $\iint y' \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}$ be denoted, as before, by $2\pi \cdot U^{(i)}$; then the part of $U^{(i)}$ derived from the general term of y', will, by the method alluded to, be thus expressed,

$$(1-\mu^2)^{\frac{n}{2}} \cdot \frac{d^n C^{(i)}}{d\mu^n} \cdot \left\{ \cos.\, n\varpi \times \frac{\int (1-\mu'^2)^i \cdot \frac{d^{i-n} M^{(n)}}{d\mu'^{i-n}} \cdot d\mu'}{2.4.6 \ldots\ldots\ldots\ldots 2i} + \sin.\, n\varpi \times \frac{\int (1-\mu'^2)^i \cdot \frac{d^{i-n} N^{(n)}}{d\mu'^{i-n}} \cdot d\mu'}{2.4.6 \ldots\ldots\ldots\ldots 2i} \right\}:$$

and if all the parts of $U^{(i)}$ be computed successively by means of this formula, the complete value of that quantity will be found by collecting them all into one sum.

Having thus determined the value of the integral $\iint y' \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)}$, denoted by $2\pi \cdot U^{(i)}$, we have

$$B^{(i)} = \alpha \cdot 2\pi \cdot a^{i+3} \cdot U^{(i)}:$$

but it is to be observed, with regard to the case of $i = o$, that

$B^{(0)} = \frac{1}{3} \iint R^3 . d\mu' . d\varpi' = \frac{a^3}{3} . \iint d\mu' . d\varpi' + \alpha . a^3 . \iint y' . d\mu' . d\varpi' . Q^{(0)} = \frac{4\pi}{3} . a^3 + \alpha . 2\pi . a^3 . U^{(0)}$. Therefore the value of V for a point without the surface of the spheroid, will be found by this series, viz.

$$V = \frac{4\pi . a^3}{3r} + \frac{2\pi . \alpha . a^3}{r} \times \left\{ U^{(0)} + \frac{a}{r} . U^{(1)} + \frac{a^2}{r^2} . U^{(2)} + \&c. \right\} . (5)$$

If the attracted point is within the surface, we must operate upon the series investigated in No. 8, of which the general term is,

$$- \frac{r^i}{i-2} . \iint \frac{Q^{(i)} . d\mu' . d\varpi'}{R^{i-2}};$$

and if we substitute $a . (1 + \alpha . y')$ for R, and reject the term which is evanescent as before, and likewise all the terms which are above the first order with regard to α; it will become simply,

$$\frac{\alpha . r^i}{a^{i-2}} . \iint y' . d\mu' . d\varpi' . Q^{(i)} = 2\pi . \alpha a^2 . \frac{r^i}{a^i} . U^{(i)}:$$

with regard to the particular term $\iint \log. R . d\mu' . d\varpi' . Q^{(2)}$, we have only to substitute for log. R, its value log. $a + \alpha . y$; and it will become

$$\alpha . r^2 . \iint y' . d\mu' . d\varpi' . Q^{(2)} = 2\pi . \alpha a^2 . \frac{r^2}{a^2} . U^{(2)};$$

also the term $\frac{1}{2} \iint R^2 . d\mu' . d\varpi' . Q^{(0)}$ will become by substitution,

$$\frac{a^2}{2} . \iint d\mu' . d\varpi' . + \alpha a^2 . \int y' . d\mu' . d\varpi' . Q^{(0)} = 2\pi a^2 + 2\pi . \alpha a^2 . U^{(0)}:$$

these things being observed, the value of V relative to a point within the spheroid, will be expressed by this series, viz.

$$V = -\frac{2\pi r^2}{3} + 2\pi . a^2 + 2\pi . \alpha a^2 . \left\{ U^{(0)} + \frac{r}{a} . U^{(1)} + \frac{r^2}{a^2} . U^{(3)} + \frac{r^3}{a^3} . U^{(4)} + \&c. \right\} . \quad (6)$$

The formulas (5) and (6) enable us to compute the attractions of homogeneous spheroids on a point without or within the surface; and, for a point in the surface, we may make use of either series, observing to put $r = a$ in all the terms multiplied by α, and $r = a . (1 + \alpha . y)$ in the rest. When y' is a finite function, the two expressions for V will both stop. It would be easy to deduce from hence the attractions of heterogeneous spheroids; but having nothing new to offer on this head, I shall refer the reader to LAPLACE's work, No. 14, Chap. 2, Liv. 3e.

The two serieses marked (5) and (6) will be found to be entirely equivalent to the formulas (3)* and (4)† which LAPLACE has given in the second chapter of the third book of the *Mécanique Céleste*: for in effect the coefficient of $\alpha . \frac{a^{i+3}}{r^{i+1}}$, in two of the serieses; and the coefficient of $\alpha . \frac{r^i}{a^{i-2}}$, in the other two, are only different expressions of the same integral $\int\int y' . d\mu' . d\varpi' . Q^{(i)}$, the symbol y' being always understood to denote a rational and integral function of three rectangular co-ordinates of a point in the surface of a sphere. In point of result therefore the two methods are one and the same, and the solutions they furnish are both applicable in the same circumstances. Neither of them can be of use, unless the radius of the spheroid be first reduced into such a function as y' is supposed to denote. The one solution can claim no preference

* No. 11. † No. 13.

to the other, except in deducing the same conclusion with greater clearness and expressing it with greater simplicity, and in a form better fitted to fulfil the views of the analyst. In these respects it can hardly be denied that the procedure delivered in the preceding pages has some advantages above that of the author of the *Mécanique Céleste.* The analysis here given is direct; and it exhibits the several coefficients in separate and independent expressions derived immediately from the radius of the spheroid. On the other hand LAPLACE's investigation is indirect; and the coefficients are found successively by decomposing the radius of the spheroid into a series of parts which follow a known law. If we now compare the two methods with respect to the grounds on which the investigations are founded we shall not find the same agreement between them. In this paper it is admitted as a necessary hypothesis, that the radius of the spheroid must be a rational and integral function of three co-ordinates of a point in the surface of a sphere: and, in consequence, the result of the analysis is limited to spheroids of that description. LAPLACE, grounding his investigation on a property which, according to his demonstration, belongs to all spheroids that differ little from spheres, seems to prove that the radius of such a spheroid cannot be an arbitrary expression, and in this inference it is necessarily implied that the radius must be such a function as we have supposed it to be.* What in the one

* Méc. Cél. Liv. 3e, No. 11. In No. 11, by substitution in his fundamental theorem, LAPLACE obtains this formula

$$4\alpha\pi \,.\, a^2 y = \frac{U^{(0)}}{a} + \frac{3.U^{(1)}}{a^2} + \frac{5.U^{(2)}}{a^3} + \&c.:$$

of this he remarks, a few lines below; "Cette expression de y n'est donc point solution is assumed as a necessary hypothesis without which the investigation will not succeed, in the other, is derived as a necessary consequence of a more general supposition. Here then the two methods are so much at variance, that if one be rigorous and exact, the other cannot be exculpated from the charge of erroneous or insufficient reasoning. This contradiction between the preceding analysis and the procedure of LAPLACE is entirely consonant to the conclusions obtained in my former paper alluded to in the beginning of this discourse; and the origin of it is to be sought for in the error I there pointed out in the investigation of that geometer. It cannot be denied that an error of calculation does exist in the demonstration of the theorem on which that author's method is grounded: his reasoning is therefore imperfect and inconclusive; and the inferences he has drawn from it cannot be supported in opposition to a rigorous analysis.

10. The same procedure which has been applied to approximate to the attractions of spheroids differing little from spheres, may likewise be employed to find accurate expressions in serieses of the attractive forces of any spheroid, provided the radius of it be such a function as the analysis requires. In both cases the research turns upon the same sort of integrals. Resume the general term of the series for the attractive force on a point without the surface, viz.

"arbitraire, mais elle derive du developpement en serie, des attractions des sphe-"roides."

In this formula it is necessarily implied, that y is a rational and integral function of three rectangular co-ordinates of a sphere; because all the terms $\frac{U'^{(0)}}{a}$, $\frac{U^{(1)}}{a^2}$ &c. are necessarily such functions.

$$B^{(i)} = \frac{1}{i+3} \cdot \iint R^{i+3} \cdot d\mu' \cdot d\varpi' \cdot Q^{(i)};$$

suppose R to be a function of μ' only, without ϖ'; then, as before,

$$B^{(i)} = \frac{2\pi \cdot C^{(i)}}{i+3} \cdot \frac{\int (1-\mu'^2)^i \cdot \frac{d^i \cdot R^{i+3}}{d\mu'^i} \cdot d\mu'}{2.4.6 \ldots\ldots\ldots\ldots 2i}:$$

but if R be a function of the most general kind, then it must be reduced to this form, viz.

$$R^{i+3} = M^{(0)} + (1-\mu'^2)^{\frac{1}{2}} \cdot M^{(1)} \cdot \cos. \varpi' + \&c.$$
$$+ (1-\mu'^2)^{\frac{1}{2}} \cdot N^{(1)} \cdot \sin. \varpi' + \&c.:$$

and the several parts that $B^{(i)}$ will consist of must be separately computed, as in the analogous case already considered. The same process will apply when the attracted point is within the surface.

11. To complete the plan of this discourse, it remains that we apply the theory laid down in it to the case of the ellipsoid. Let the semi-axes be k, k', k'', the first being the least of all the three; and let x, y, z, respectively parallel to the same axes, be three co-ordinates of a point in the surface: then will the equation of the solid be

$$\frac{x^2}{k^2} + \frac{y^2}{k'^2} + \frac{z^2}{k''^2} = 1:$$

put $x = R\mu'$; $y = R \cdot \sqrt{1-\mu'^2} \cdot \cos. \varpi'$; and $z = R \cdot \sqrt{1-\mu'^2} \cdot \sin. \varpi'$; then by substitution,

$$R^2 \left\{ \frac{\mu'^2}{k^2} + \frac{(1-\mu'^2)\cos.^2 \varpi'}{k'^2} + \frac{(1-\mu'^2)\sin.^2 \varpi'}{k''^2} \right\} = 1:$$

farther, let $e = \frac{k^2}{k'^2}$; $f = \frac{k^2}{k''^2}$; and $s = \mu'^2 + e(1-\mu'^2)\cos.^2 \varpi' + f \cdot (1-\mu'^2) \cdot \sin.^2 \varpi'$; then $R = \frac{k}{\sqrt{s}}$: and if this value of R, or the radius of the ellipsoid, be substituted in the general term of the series for the attractive force on a point within the spheroid (No. 8), that term will become

$$-\frac{1}{i-2}\cdot\frac{r^i}{k^{i-2}}\cdot\iint s^{\frac{i-2}{2}}\,.\,d\mu'\,.\,d\varpi'\,.\,Q^{(i)}.$$

In the first place I say that all the terms in which i is odd are evanescent. For $s=\frac{e+f}{2}+\left(1-\frac{e+f}{2}\right).\mu'^2+\frac{e-f}{2}.(1-\mu'^2)$ cos. $2\varpi'$; whence it follows that $s^{\frac{i-2}{2}}$ may be expanded into a series of this form, viz. $A^{(0)}+A^{(1)}$. cos. $2\varpi'+A^{(2)}$. cos. $4\varpi'$ ... $A^{(n)}$. cos. $2n\varpi'$ .. &c.; of which the general term is $A^{(n)}$. cos. $2n\varpi'$, and if we combine this quantity with the expansion of $Q^{(i)}$ (No. 5), there will result one term, and only one, independent of sines and cosines, viz.

$$\beta^{(2n)}.(1-\mu^2)^{\frac{2n}{2}}.\frac{d^{2n}C^{(i)}}{d\mu^{2n}}\iint(1-\mu'^2)^{\frac{2n}{2}}.\frac{d^{2n}C'^{(i)}}{d\mu'^{2n}}.A^{(n)}.d\mu'.d\varpi';$$

all the other terms, produced by the multiplication, contain sines or cosines of variable angles; on which account they vanish when they are integrated with regard to $d\varpi'$ between the required limits: since s contains no other power of μ' but μ'^2, it is plain that every coefficient of the developement of $s^{\frac{i-2}{2}}$, as $A^{(n)}$, will be an even function of μ', or will contain only even powers of that quantity: and, because i is odd, therefore $C^{(i)}$, and all its fluxions of the even orders, will be odd functions of μ': upon the whole then the quantity under the double sign of integration will be an odd function of μ'; or it will be an assemblage of the odd powers of that quantity: therefore the integral, between the limits $\mu'=1$ and $\mu'=-1$, is equal to nothing (No. 4). Therefore all the terms are evanescent when i is odd.

Again I say that all the terms are evanescent when i is even, except when it is $= 2$. For in this case $s^{\frac{i-2}{2}}$ will be an integer power, and it will contain a finite number of terms which may be generally represented thus, viz.

$$(1-\mu'^2)^{\frac{2n}{2}} . M^{(n)} . \cos. 2n\varpi';$$

$M^{(n)}$ being a rational and integral function of μ': and this quantity when combined with the developement of $Q^{(i)}$, will produce one term, and only one, clear of sines and cosines, viz.

$$(1-\mu^2)^{\frac{2n}{2}} . \frac{d^{2n}C^{(i)}}{d\mu^{2n}} . \int\int \beta^{(2n)} . (1-\mu'^2)^{\frac{2n}{2}} . \frac{d^{2n}C'^{(i)}}{d\mu'^{2n}} . M^{(n)} . d\mu' . d\varpi':$$

now since μ'^4 is the greatest power in s, the greatest power in $s^{\frac{i-2}{2}}$ will be μ'^{i-2}; therefore $(1-\mu'^2)^{2n} . M^{(n)}$. cannot contain any power of μ' greater than $i-2$, nor $M^{(n)}$ any greater than $i-2n-2$, which number the dimensions of $M^{(n)}$ cannot pass: but $i-2n$, greater than $i-2n-2$, denotes the dimensions of $\frac{d^{2n}C'^{(i)}}{d\mu'^{2n}}$: therefore, by a property of this sort of integrals already demonstrated (No. 4), the preceding quantity is evanescent. Therefore all those terms of the series are evanescent in which i is an even number; but from this the case of $i = 2$, when the term assumes a particular form, must be excepted.

If now we reject all the terms that have been proved to be evanescent, we shall have, for a point within or in the surface of the ellipsoid,

$$V = -\frac{2\pi . r^2}{3} + \frac{k^2}{2} . \iint \frac{d\mu' . d\varpi'}{s} - \frac{r^2}{2} . \iint \log. s . d\mu' . d\varpi' Q^{(2)}:$$

in the last term I have written $-\frac{1}{2}\log. s$ for $\log. \frac{k}{\sqrt{s}} = \log. k - \frac{1}{2}\log. s$; because $\int Q^{(2)} . d\mu' . d\varpi' = 0$.

Before we pursue the investigation farther, we shall stop to demonstrate a property of the attractions of a shell of homogeneous matter bounded by the surfaces of two ellipsoids, similar to one another and similarly placed, on a point within the shell. If we suppose k to denote the axis of the greater ellipsoid, and put h for the corresponding axis of the smaller one; then the value of V relatively to the latter solid will be found merely by changing k into h in the last expression; because s contains no quantities but such as are common to the two solids: therefore the value of V, relatively to the shell of matter included between the two surfaces, will be equal to

$$\frac{k^2 - h^2}{2} . \iint \frac{d\mu' . d\varpi'}{s};$$

a quantity which is independent on the position of the attracted point: therefore the differential coefficients of V for any coordinates of the attracted point are evanescent; and consequently so are the attractive forces parallel to the co-ordinates (No. 6). Therefore a material point within such a shell is attracted equally in opposite directions.

Let us now investigate the value of

$$\iint \frac{d\mu' d\varpi'}{s}:$$

put $p = e + (1-e) . \mu'^2$; $q = f + (1-f) . \mu'^2$; then $s = p . \cos.^2\varpi' + q . \sin.^2\varpi'$: assume $\sqrt{\frac{q}{p}} . \frac{\sin. \varpi'}{\cos. \varpi'} = \frac{\sin. u}{\cos. u}$; then $\frac{d\varpi'}{s} = \frac{du}{\sqrt{p . q}}$; therefore by restoring the values of p and q, we get

$$\iint \frac{d\mu' \,.\, d\varpi'}{s} = \iint \frac{d\mu' \,.\, du}{\sqrt{\left\{e+(1-e)\,.\,\mu'^2\right\}\,.\,\left\{f+(1-f)\,.\,\mu'^2\right\}}}:$$

let $\frac{1-e}{e} = \frac{k'^2-k^2}{k^2} = \lambda^2$; $\frac{1-f}{f} = \frac{k''^2-k^2}{k^2} = \lambda'^2$; and

$F = \int \frac{dx}{\sqrt{(1+\lambda^2 x^2)\,.\,(1+\lambda'^2 x^2)}}$ between the limits $x = o$ and $x = 1$: then observing that the preceding integrals increase as much from $\mu' = -1$ to $\mu' = o$, as they do from $\mu' = o$ to $\mu' = 1$; and likewise that the limits of u are from $u = o$ to $u = 2\pi$; we shall get

$$\frac{k^2}{2} \,.\, \iint \frac{d\mu' \,.\, d\varpi'}{s} = \frac{2\pi \,.\, k^2}{\sqrt{e\,.\,f}} \,.\, F.$$

It remains to find the value of

$$\frac{r^2}{2} \,.\, \iint \log.\, s \,.\, d\mu' \,.\, d\varpi' \,.\, Q^{(2)}.$$

Taking the value of $Q^{(2)}$ in terms of γ (No. 3), we have $r^2 Q^{(2)} = r^2 \,.\, (\frac{3}{2}\gamma^2 - \frac{1}{2})$: let a, b, c, denote the co-ordinates of the attracted point; then $a = r\,.\,\mu$; $b = r\,.\,\sqrt{1-\mu^2}\,.\,\cos.\,\varpi$; $c = r\,.\,\sqrt{1-\mu^2}\,.\,\sin.\,\varpi$; therefore

$$r\,.\,\gamma = a\,.\,\mu' + b\,.\,\sqrt{1-\mu'^2}\,.\,\cos.\,\varpi' + c\,.\,\sqrt{1-\mu'^2}\,.\,\sin.\,\varpi':$$

consequently

$$\begin{aligned} r^2\,.\,Q^{(2)} = {} & a^2\left(\tfrac{3}{2}\mu'^2 - \tfrac{1}{2}\right) + b^2\,.\left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\} \\ & + c^2\,.\left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\} \\ & + 3ab\,.\,\mu'\sqrt{1-\mu'^2}\,.\,\cos.\,\varpi' + 3ac\,.\,\mu'\sqrt{1-\mu'^2}\,.\,\sin.\,\varpi' \\ & + 3bc\,.\,(1-\mu'^2)\,.\,\cos.\,\varpi'\sin.\,\varpi': \end{aligned}$$

but log. s may be reduced into a series of this form, viz.

$$A^{(0)} + A^{(1)}\,.\,\cos.\,2\varpi' + A^{(2)}\,.\,\cos.\,4\varpi' + \&c.$$

and we may neglect all such parts of $Q^{(2)}$ as multiplied by this series would produce only quantities containing sines and cosines: on this account, we may make

$$r^2 . Q^{(2)} = a^2 . (\tfrac{3}{2}\mu'^2 - \tfrac{1}{2}) + b^2 . \left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\}$$
$$+ c^2 . \left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\}:$$

therefore,

$$\tfrac{r^2}{2} . \iint \log. s . d\mu' . d\varpi' . Q^{(2)} = \tfrac{a^2}{2} . \iint \log. s . d\mu' . d\varpi' . (\tfrac{3}{2}\mu'^2 - \tfrac{1}{2})$$
$$+ \tfrac{b^2}{2} . \iint \log. s . d\mu' . d\varpi' . \left\{\tfrac{3}{2}(1-\mu'^2)\cos.^2\varpi' - \tfrac{1}{2}\right\}$$
$$+ \tfrac{c^2}{2} . \iint \log. s . d\mu' . d\varpi' . \left\{\tfrac{3}{2}(1-\mu'^2)\sin.^2\varpi' - \tfrac{1}{2}\right\}.$$

Let the term multiplied by $\frac{a^2}{2}$ be integrated by parts with respect to $d\mu'$, then $\int \log. s . d\mu' . \left(\frac{3\mu'^2}{2} - \frac{1}{2}\right) = \log. s \times \frac{\mu'^3 - \mu'}{2}$ $- \int \frac{\mu'^3 - \mu'}{2} . \frac{\left(\frac{ds}{d\mu'}\right)}{s} . d\mu'$: but $- \frac{\mu'^3 - \mu'}{2} . \left(\frac{ds}{d\mu'}\right) = \mu'^2 - \mu'^2 . s$: therefore, observing that the term without the sign of integration vanishes both when $\mu' = -1$ and $\mu' = 1$; the value of the coefficient of $\frac{a^2}{2}$ will be equal to

$$\iint \frac{\mu'^2 . d\mu' . d\varpi'}{s} - \iint \mu'^2 . d\mu' . d\varpi':$$

and because $\frac{d\varpi'}{s} = \frac{du}{\sqrt{p . q}}$; therefore the first term of the quantity sought will be equal to

$$\pi . a^2 . \int \frac{\mu'^2 . d\mu'}{\sqrt{\left\{e + (1-e) . \mu'^2\right\} . \left\{f + (1-f)\mu'^2\right\}}} - \frac{2\pi . a^2}{3}:$$

which is equal to

$$\frac{2\pi . a^2}{\sqrt{ef}} . \int \frac{x^2 . dx}{\sqrt{(1+\lambda^2 x^2) . (1+\lambda'^2 x^2)}} - \frac{2\pi . a^2}{3},$$

the fluent here being taken from $x = o$ to $x = 1$. Seeking to express this value by means of the integral F, I have found in general, $\int \frac{x^2 . dx}{\sqrt{(1+\lambda^2 x^2) . (1+\lambda'^2 x^2)}} = \frac{x^3}{\sqrt{(1+\lambda^2 x^2) . (1+\lambda'^2 x^2)}} + \frac{1}{\lambda} . \left(\frac{dF}{d\lambda}\right) + \frac{1}{\lambda'} . \left(\frac{dF}{d\lambda'}\right)$;

therefore, making $x = 1$, the first term will become,

$$\frac{2\pi . a^2}{\sqrt{ef}} . \left\{\frac{1}{\sqrt{(1+\lambda^2) . (1+\lambda'^2)}} + \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) + \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right)\right\} - \frac{2\pi . a^2}{3}.$$

With regard to the term containing b^2, it may be changed into an equivalent expression similar to the first term we have just been considering: for if, at entering on this investigation, we had substituted in the equation of the solid, $x = R . \sqrt{1-\mu'^2} . \cos.^2 \varpi'$; $y = R' . \mu'$; $z = R . \sqrt{1-\mu'^2} . \sin.^2 \varpi'$; which substitutions are entirely arbitrary; we should have found $s = e\mu'^2 + (1-\mu'^2) \cos.^2 \varpi' + f \times (1-\mu'^2) \sin.^2 \varpi'$; and the term we are seeking, multiplied by b^2, would have been changed into

$$\frac{b^2}{2} . \iint \log. s . d\mu' . d\varpi' . \left(\tfrac{3}{2} \mu'^2 - \tfrac{1}{2}\right):$$

and hence by proceeding as before, we derive this value of that term

$$\frac{b^2}{2} . \iint \frac{e . \mu'^2 . d\mu' . d\varpi'}{s} - \frac{b^2}{2} . \iint \mu'^2 . d\mu' . d\varpi':$$

and if we put $p = 1 + (e-1) . \mu'^2$; $q = f + (e-f) . \mu'^2$; also $\sqrt{\frac{q}{p}} . \frac{\sin. \varpi'}{\cos. \varpi'} = \frac{\sin. u}{\cos. u}$: then, $s = p . \cos.^2 \varpi' + q . \sin.^2 \varpi'$; $\frac{d\varpi'}{s} = \frac{du}{\sqrt{pq}}$; consequently, by substitution, and integrating with regard to u, and confining the integration with regard to μ' between the limits $\mu' = o$ and $\mu' = 1$; we shall get,

$$2\pi . b^2 . \int \frac{e . \mu'^2 . d\mu'}{\sqrt{\left\{1+(e-1) . \mu'^2\right\} . \left\{f+(e-f) . \mu'^2\right\}}} - \frac{2\pi . b^2}{3}.$$

If we make $\mu' = \frac{x}{\sqrt{e+(1-e)x^2}}$;* the integral in the last expression will be transformed into

$$\frac{1}{\sqrt{ef}} \cdot \int \frac{x^2 \,.\, dx^2}{\left\{1+\lambda^2 x^2\right\}^{\frac{3}{2}} \cdot \left\{1+\lambda'^2 x^2\right\}^{\frac{1}{2}}} = -\frac{1}{\sqrt{ef}} \cdot \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right):$$

therefore the value of this term is

$$-\frac{2\pi \,.\, b^2}{\sqrt{ef}} \cdot \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) - \frac{2\pi \,.\, b^2}{3}.$$

And, by proceeding in a manner entirely analogous, it may be shewn that the remaining term multiplied by c^2, is equal to

$$-\frac{2\pi \,.\, c^2}{\sqrt{ef}} \cdot \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right) - \frac{2\pi \,.\, c^2}{3}.$$

If M denote the mass of the ellipsoid, then $M = \frac{4\pi}{3} \,.\, kk'k'' = \frac{4\pi \,.\, k^3}{3 \cdot \sqrt{ef}}$; and $\frac{2\pi}{\sqrt{ef}} = \frac{3M}{2k^3}$: therefore by collecting all the parts of V, into one sum, we have

$$\left.\begin{aligned} V = \frac{3M}{2k} \,.\, F - \frac{3M \,.\, a^2}{2k^3} \cdot \left\{\frac{1}{\sqrt{(1+\lambda^2) \,.\, (1+\lambda'^2)}} + \frac{1}{\lambda}\left(\frac{dF}{d\lambda'}\right) + \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right)\right\} \\ + \frac{3M \,.\, b^2}{2k^3} \cdot \frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) + \frac{3M \,.\, c^2}{2k^3} \cdot \frac{1}{\lambda'}\left(\frac{dF}{d\lambda'}\right). \\ \frac{k'^2 - k^2}{k^2} = \lambda^2; \quad \frac{k''^2 - k^2}{k^2} = \lambda'^2. \\ F = \int \frac{dx}{\sqrt{(1+\lambda^2 x^2) \,.\, (1+\lambda'^2 x^2)}} \ (\text{from } x = 0, \text{ to } x = 1). \end{aligned}\right\}$$

The case of an oblate elliptical spheroid of revolution corresponds to the supposition of $k' = k''$ or $\lambda = \lambda'$: but in taking the partial fluxions of F we must attend to the peculiarity that takes place when $\lambda = \lambda'$: for in general $dF = \left(\frac{dF}{d\lambda}\right) d\lambda + \left(\frac{dF}{d\lambda'}\right) d\lambda'$; and hence when $\lambda = \lambda'$, $dF = 2\left(\frac{dF}{d\lambda}\right) . d\lambda$: now when $\lambda = \lambda'$, $F = \frac{1}{\lambda}$. arc. tan. λ; consequently $\frac{1}{\lambda}\left(\frac{dF}{d\lambda}\right) = \frac{1}{\lambda'}\left(\frac{dF}{d\lambda}\right)$

* Méc. Cél. Liv. 3e, No. 3.

$= -\frac{1}{2\lambda^3}$ (arc. tan. $\lambda - \frac{\lambda}{1+\lambda^2}$): in this case then we shall have

$$V = \frac{3M}{2k} \cdot \frac{1}{\lambda} \cdot \text{arc. tan. } \lambda - \frac{3M \cdot a^2}{2k^3} \cdot \frac{1}{\lambda^3} \cdot \left\{\lambda - \text{arc. tan. } \lambda\right\}$$
$$- \frac{3M}{2k^3} \cdot (b^2 + c^2) \cdot \frac{1}{2\lambda^3} \cdot \left\{\text{arc. tan. } \lambda - \frac{\lambda}{1+\lambda^2}\right\}.$$

If this value of V be substituted in the equation of the surface of a homogeneous fluid mass which is in equilibrium by the joint effect of the attractions of its molecules and a rotatory motion;* it will be proved that the oblate spheroid satisfies the conditions of equilibrium, and the relation between the velocity of rotation and the eccentricity of the spheroid will likewise be determined.

* Méc. Cél. Liv. 3e, No. 23 et 24.