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Long Waves Associated With Disturbances Produced in Plasmas

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The disturbances produced in a homogeneous plasma by passing charges is analyzed.

1. Introduction

A source which is effective during a limited period can be described by a function of time or by an equivalent Fourier spectrum. It is well known that the disturbances produced by such a source depend mainly on its highest frequencies shortly after the arrival (determined by a propagation velocity) of the first effect, at $t=t_0$ say, whereas the lower frequencies determine the effects observed later on. The Fourier synthesis of the disturbance, which has to include the contributions of *all* frequencies, will represent the special effect of the low frequencies in an obscured form even when the latter finally dominate. A new representation is then desired; it may be obtained as follows.

The time dependence of the disturbance may be known as a Laplace integral

$$\int_{-\infty+i\delta}^{\infty+i\delta} d\omega \cdot f(\omega) \cdot e^{-i\omega(t-t_0)} \qquad (\delta>0),$$

which is equivalent to a Fourier integral with a path of integration L parallel to the real axis of the ω-plane. This integral can then be reduced in two different ways:

(a) a parallel shift of L to the real axis itself. This results in a conventional Fourier synthesis in terms of real frequencies. In the case of poles on the real axis, these are to be avoided by an indentation just above it. In the limit of an infinitesimal indentation this reduction leads to the sum of a main contribution represented by a Cauchy's principal value, and of additional contributions consisting of half the residues at the poles.

(b) a transformation of L into a closed contour which results by adding to L the upper half of the infinite circle of the ω-plane if $t<t_0$ and its lower half if $t>t_0$. In the first case the integral will vanish, as it should, if $f(\omega)$ has no singularities whatever above L. However, the other contour to be applied for $t>t_0$, encloses the singularities of $f(\omega)$ situated in the half plane Im $\omega<0$ or possibly on the real axis. The integral itself can then be reduced to the residues at the enclosed poles and to contributions along branch lines connected with the other singularities. If the branch lines do not extend up to infinity, we get by contraction of the integration path around the poles and branch lines under consideration contributions connected with discrete frequencies and with frequency bands respectively.

As an example we consider the Laplace integral

$$I=\int_{-\infty+i\delta}^{\infty+i\delta} d\omega \, \frac{e^{-i\omega t}}{(\omega^2-\omega_0^2)}. \tag{1}$$

A shift of the integration path to the real axis here results in:

$$I=PV\int_{-\infty}^{\infty} d\omega \, \frac{e^{-i\omega t}}{(\omega^2-\omega_0^2)}-\frac{\pi}{\omega_0}\sin\,(\omega_0 t) \qquad (PV=\text{principal value}). \tag{2}$$

All frequencies are contained in this integral, though the particular frequency ω_0 has a finite amplitude even when observed in an infinitesimal band around ω_0. On the other hand, transformation to a closed contour as suggested in method (*b*) reveals the vanishing of the integral for $t<0$, while it simply reduces to $(-2\pi/\omega_0)\sin(\omega_0 t)$ for $t>0$. Thus the property of the signal of containing only a single frequency ω_0 right after its start at $t=0$ was hidden in the conventional Fourier representation (2).

Idealized ionospheric disturbances, analyzed with the aid of the above procedure, will be discussed in the next sections. They refer to a single infinitesimal current element, only effective at a special moment, and also to a charge moving with constant speed along a straight line. The latter situation simulates a charged particle emitted by the sun, that travels along a magnetic line of force through the ionosphere. The limiting cases of a zero and of an infinite magnetic field are worked out in detail, the intermediate case of a finite magnetic field is only mentioned briefly. The ionospheric plasma is assumed throughout as homogeneous (constant plasma frequency ω_{pl}) and cold.

2. Disturbance Generated by a Current Element in a Plasma Without Magnetic Field

2.1. Derivation of Laplace Integrals for the Field

Let the current element be situated at the origin, and directed along the z-axis, while effective only at $t=0$. This corresponds, in the case of a normalized momentum $\delta(t)$, to a current-density distribution:

$$\vec{I}=\delta(x)\delta(y)\delta(z)\delta(t)\vec{u}_z, \tag{3}$$

$\vec{u}_z$ being the unit vector in the z-direction.

The effective dielectric constant $\epsilon=1-\omega_{pl}^2/\omega^2$ for a homogeneous plasma at frequency ω then involves the following Maxwell equations in gaussian units:

$$\text{curl }\vec{E}+\frac{1}{c}\frac{\partial\vec{H}}{\partial t}=0,$$

$$\begin{aligned}\text{curl }\frac{\partial\vec{H}}{\partial t}-\frac{1}{c}\left(\frac{\partial^2}{\partial t^2}+\omega_{pl}^2\right)\vec{E}&=\frac{4\pi}{c}\frac{\partial\vec{I}}{\partial t}\\&=\frac{4\pi}{c}\,\delta(x)\delta(y)\delta(z)\delta'(t)\vec{u}_z.\end{aligned} \tag{4}$$

The complete solution in terms of Laplace integrals reads:

$$\vec{E}=-4\pi\delta(3c)\delta(y)\delta(z)\cos(\omega_{pl}t)U(t)\vec{u}_z$$

$$+\frac{i}{2\pi}\text{ curl curl}\left\{\frac{\vec{u}_z}{r}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega e^{-i\omega t}\frac{\omega e^{i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}}{(\omega^2-\omega_{pl}^2)}\right\}=\vec{E}_1+\vec{E}_2,\text{ say,} \tag{5a}$$

$$\vec{H}=-\frac{\sin\theta}{2\pi c}\frac{d}{dr}\left\{\frac{1}{r}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-w_{pl}^2}}\right\}\vec{u}_\varphi. \tag{5b}$$

The sign of the square root is defined by the property $\sqrt{\omega^2-\omega_{pl}^2}\to\omega$ along the infinity circle, while a crosscut is assumed along the section $-\omega_{pl}<\omega<\omega_{pl}$ of the real axis. Further, $\vec{u}_r$ and $\vec{u}_\varphi$ mark unit vectors in the radial and azimuthal directions corresponding to polar coordinates

$$r,\ \theta,\ \varphi \text{ with } z=r\cos\theta,\ x+iy=\rho e^{i\varphi}=r\sin\theta e^{i\varphi};$$

$U(t)$ represents Heaviside's unit function, while δ may be any positive number.

The solution (5) can be verified as follows. We evaluate curl $E^{(2)}$ with the aid of the vector identity:

$$\text{curl curl curl}=-\text{curl }\nabla^2,$$

as well as the relation:

$$(\nabla^2+k^2)\,\frac{e^{ikr}}{r}=4\pi\delta(x)\delta(y)\delta(x). \tag{6}$$

We thus obtain:

$$\text{curl }\vec{E}^{(2)}=\frac{i}{2\pi c^2}\,\text{curl}\left\{\frac{\vec{u}_z}{r}\int_{-\infty+\delta}^{\infty+i\delta} d\omega e^{-i\omega t}\omega e^{\frac{ir}{c}\sqrt{w^2-w_{pl}^2}}\right\}$$

$$+2i\ \text{curl}\left\{\vec{u}_z\delta(x)\delta(y)\delta(z)\int_{-\infty+i\delta}^{\infty+i\delta} d\omega\,\frac{\omega e^{-i\omega t}}{\omega^2-\omega_{pl}^2}\right\}.$$

The last integral equals $-2\pi iU(t)\cos(\omega_{pl}t)$ from which the second term proves to reduce to $-$curl $E^{(1)}$. Hence we find for the total E field:

$$\text{curl }\vec{E}=\frac{i}{2\pi c^2}\,\text{curl}\left\{\frac{\vec{u}_z}{r}\int_{-\infty+i\delta}^{\infty+i\delta} d\omega e^{-i\omega t}\omega e^{\frac{ir}{c}\sqrt{\omega^2-\omega_{pl}^2}}\right\}.$$

The first Maxwell equation of (4) is then readily verified by applying the further identity

$$\sin\theta\cdot\lambda'(r)\cdot\vec{u}_\varphi=\vec{u}_z\times\overrightarrow{\text{grad}}\,\lambda(r)=-\text{curl}\{\vec{u}_z\cdot\lambda(r)\} \tag{7}$$

to the magnetic field H.

The other Maxwell equation can be checked, without working out the curl curl operator occurring in curl $\partial\vec{H}/\partial t$ when (5b) is replaced by a curl equivalent to it according to (7). In the further elementary derivation the application of the operator $\partial^2/\partial t^2+\omega_{pl}^2$ to $\vec{E}^{(2)}$ amounts to a cancellation of the denominator of the integrand.

The first contribution $E^{(1)}$ of the electric field is recognized as a nonpropagating field with a frequency equal to the plasma frequency; it can not move away from the source. The main contribution $\vec{E}^{(2)}$, as well as $\vec{H}$, is composed of spherical waves of complex frequencies.

2.2. Frequency Spectrum After the Start of the Field

An application of method (a) of the first section shows how all real frequencies contribute to the final field, but the explicit form of this spectrum is not very instructive. The method (b), however, directly shows the predominant role of the lower frequencies after the signal has started at the point of observation. In fact, the approximation

$$e^{-i\omega(t-r/c)}$$

for the exponential factor in the integrand of both integrals of (5) involves that the integration path can be closed along the upper or lower half of the infinite circle when $t<\frac{r}{c}$ or $t>\frac{r}{c}$ respectively. In the first case the only singularities of the integrands, situated at $\omega=\omega_{pl}$ and $\omega=-\omega_{pl}$, are outside the contour and the integrals vanish. In the second case the contour can be contracted to a loop L around the part $-\omega_{pl}<\omega<\omega_{pl}$ of the real axis which was introduced as the branch line fixing the sign of $\sqrt{\omega^2-\omega_{pl}^2}$; the ends of this line constitute the only other singularities of the integrand, viz, its two poles at ω_{pl} and $-\omega_{pl}$. Hence the main contribution $\vec{E}^{(2)}$, e.g., can be represented at all times by:

$$\vec{E}^{(2)}=\frac{i}{2\pi}\text{ curl curl}\left\{\frac{\vec{u}_z U\left(t-\frac{r}{c}\right)}{r}\int_L d\omega e^{-i\omega t}\frac{\omega e^{i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}}{(\omega^2-\omega_{pl}^2)}\right\}.$$

The contraction of L involves the sum of the complete residues at the poles, and of the limit for $\epsilon\rightarrow+0$ of an integral along the interval $-\omega_{pl}+\epsilon<\omega<\omega_{pl}-\epsilon$ of the real axis; along the latter we have to take into account the new values $i\sqrt{\omega_{pl}^2-\omega^2}$ and $-i\sqrt{\omega_{pl}^2-\omega^2}$ of $\sqrt{\omega^2-\omega_{pl}^2}$ just above and below the branch line. Both residues can be added together to a term $-2\pi i \cos(\omega_{pl}t)$. The remaining integral can be reduced to an integration over the positive interval $0<\omega<\omega_{pl}-\epsilon$ only, which, however, involves an integral yet to converge for $\epsilon=0$. The residues are missing in the corresponding integral for $\vec{H}$. We thus arrive at the following final expressions without an ϵ limit:

$$\vec{E}^{(2)}=\text{curl curl}\left[\vec{u}_z\frac{U\left(t-\frac{r}{c}\right)}{r}\left\{\cos(\omega_{pl}t)+\frac{2}{\pi}\int_0^{\omega_{pl}}d\omega\cdot\sin(\omega t)\frac{\omega\sinh\left(\frac{r}{c}\sqrt{\omega_{pl}^2-\omega^2}\right)}{(\omega_{pl}^2-\omega^2)}\right\}\right] \tag{8a}$$

$$\vec{H}=\frac{2\sin\theta}{\pi c}\cdot\frac{d}{dr}\left\{\frac{U\left(t-\frac{r}{c}\right)}{r}\int_0^{\omega_{pl}}d\omega\cos(\omega t)\cdot\sinh\left(\frac{r}{c}\sqrt{\omega_{pl}^2-\omega^2}\right)\right\}\cdot\vec{u}_\varphi. \tag{8b}$$

For $t>\frac{r}{c}$ we infer a dependence on the lower frequencies $\omega\leq\omega_{pl}$ only.

2.3. The Field Expressed in Terms of Bessel Functions

The integrals in (5) are connected with Bessel functions. This can be shown with the aid of the operational relation [Van der Pol and Bremmer, 1955]

$$J_0(\sqrt{t^2-\tau^2})\cdot U(t-\tau)\doteqdot\frac{q}{\sqrt{q^2+1}}e^{-\tau\sqrt{q^2+1}}\qquad(\text{Re } q>0), \tag{9}$$

in which we use the nomenclature $h(t)\doteqdot f(q)$ for

$$f(q)=q\int_{-\infty}^{\infty}dt e^{-qt}h(t).$$

We replace t by $\omega_{pl}t$ [which requires the replacement of q by q/ω_{pl}] and also the nonoperational variable τ by $\omega_{pl}\tau$. We then obtain the following extension of (q):

$$J_0(\omega_{pl}\sqrt{t^2-\tau^2})\cdot U(t-\tau)\doteqdot\frac{q}{\sqrt{q^2+\omega_{pl}^2}}e^{-\tau\sqrt{q^2+\omega_{pl}^2}}\qquad(\text{Re } q>0). \tag{10}$$

We next apply to both sides the operator (likewise nonoperational) $\int_{r/c}^{\infty}d\tau$, which leads to:

$$U\left(t-\frac{r}{c}\right)\int_{r/c}^{t}d\tau J_0(\omega_{pl}\sqrt{t^2-\tau^2})\doteqdot\frac{qe^{-\frac{r}{c}\sqrt{q^2+\omega_{pl}^2}}}{(q^2+\omega_{pl}^2)}\qquad(\text{Re } q>0). \tag{11}$$

The inversion integral of this new operational relation reads:

$$U\left(t-\frac{r}{c}\right)\int_{r/c}^{t}d\tau J_0(\omega_{pl}\sqrt{t^2-\tau^2})=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}dq\frac{e^{qt-\frac{r}{c}\sqrt{q^2+\omega_{pl}^2}}}{(q^2+\omega_{pl}^2)}$$

$$=\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\frac{e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}}{(\omega_{pl}^2-\omega^2)}. \tag{12}$$

The derivative of this relation with respect to t yields at once a new expression for the $E^{(2)}$ contribution of (5). We find:

$$\vec{E}^{(2)}=\frac{\partial}{\partial t}\,\text{curl curl}\left\{\vec{u}_z\,\frac{U\left(t-\frac{r}{c}\right)}{r}\int_{r/c}^{t}d\tau\cdot J_0(\omega_{pl}\sqrt{t^2-\tau^2})\right\}. \tag{13a}$$

A corresponding reduction for the magnetic field represented by (5b) is arrived at by first deriving from (12):

$$\left(\frac{d^2}{dt^2}+\omega_{pl}^2\right)\left\{U\left(t-\frac{r}{c}\right)\int_{r/c}^{t}d\tau\cdot J_0(\omega_{pl}\sqrt{t^2-\tau^2})\right\}=\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\cdot e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}. \tag{14}$$

We next apply the operator $-\partial/\partial\tau$ to (10) while taking $\tau=\frac{r}{c}$, and also the operator $(d^2/dt^2+\omega_{pl}^2)\doteqdot(q^2+\omega_{pl}^2)$ to (11). In both cases we get the same operational "image" in the right-hand side which involves an identity for Bessel functions according to which (14) can be replaced by:

$$-c\frac{\partial}{\partial r}\left\{J_0\left(\omega_{pl}\sqrt{t^2-\frac{r^2}{c^2}}\right)\cdot U\left(t-\frac{r}{c}\right)\right\}=\frac{1}{2\pi}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\cdot e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}. \tag{15}$$

Hence the magnetic field (5b) can be expressed as follows in terms of Bessel functions, without any integral:

$$\vec{H}=\sin\theta\cdot\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{J_0\left(\omega_{pl}\sqrt{t^2-\frac{r^2}{c^2}}\right)\cdot U\left(t-\frac{r}{c}\right)\right\}\right]\cdot\vec{u}_\phi. \tag{13b}$$

3. Disturbance Generated by a Charge Moving Along the z-Axis Through a Plasma Without Magnetic Field

3.1. Derivation of Laplace Integrals for the Field

We assume a unit charge moving with a constant velocity v along the z-axis according to the equation $z=vt$. The corresponding current-density distribution is given by:

$$\vec{I}=\delta(x)\delta(y)\delta\left(t-\frac{z}{v}\right)\vec{u}_z.$$

The connection with the nonmoving current element of the previous section is shown by the equivalent representation:

$$\vec{I}=v\int_{-\infty}^{\infty}dt_0\cdot\delta(x)\delta(y)\delta(z-vt_0)\delta(t-t_0)\cdot\vec{u}_z.$$

Hence this new current distribution is obtained from the former (3) by first substituting $z-vt_0$ for z, and $t-t_0$ for t, and by applying next the operator $v\int_{-\infty}^{\infty}dt_0$. In view of the linear dependence of the field vectors $\vec{E}$ and $\vec{H}$ on the sources, the fields for the present situation may be derived by applying the mentioned substitutions and operator to all results arrived at in section 2.

We first consider the electric field. In the case of (5a) the mentioned procedure results in the following expression if the t_0 integration is evaluated in the first contribution $E^{(1)}$, and if the order of the integrations with respect to ω and t_0 is inverted in the main contribution $E^{(2)}$:

$$\vec{E}=-4\pi\delta(x)\delta(y)\cos\left\{\omega_{pl}\left(t-\frac{z}{v}\right)\right\}\cdot U\left(t-\frac{z}{v}\right)\cdot\vec{u}_z$$

$$+\frac{iv}{2\pi}\,\text{curl curl}\left\{\vec{u}_z\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\,\frac{e^{-i\omega t}\omega}{\omega^2-\omega_l}\int_{-\infty}^{\infty}dt_0\,\frac{e^{\omega t_0+\frac{i}{c}\sqrt{\omega^2-\omega_{pl}^2}\sqrt{\rho^2+(z-vt_0)^2}}}{\sqrt{\rho^2+(z-vt_0)^2}}\right\}.$$

The substitution

$$t_0=\frac{z}{v}-i\,\frac{\rho}{v}\sin\left\{\chi+i\ \text{arc}\sinh\frac{\sqrt{\omega^2-\omega_{pl}^2}}{\sqrt{\left(\frac{c^2}{v^2}-1\right)\omega^2+\omega_{pl}^2}}\right\}$$

reduces the t_0-integral to a Sommerfeld integral for a Hankel function $H_0^{(1)}(iz)=\frac{2}{\pi i}K_0(z)$ which is given explicitly by

$$\frac{2}{v}e^{i\omega\frac{z}{v}}\cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\},$$

provided that the modulus of the argument of the square root is smaller than $\pi/2$. This involves the following final expression for the electric field:

$$\vec{E}=-4\pi\delta(x)\delta(y)\cos\left\{\omega_{pl}\left(t-\frac{z}{v}\right)\right\}\cdot U\left(t-\frac{z}{v}\right)\cdot\vec{u}_z$$

$$+\frac{i}{\pi}\ \text{curl curl}\left[\vec{u}_z\int_{-\infty+i\delta}^{\infty+i\delta}d\omega e^{-i\omega\left(t-\frac{z}{v}\right)}\frac{\omega\cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}}{(\omega^2-\omega_{pl}^2)}\right].\qquad(16a)$$

The corresponding reduction of the magnetic field can be carried through by first transforming the formula (5a) for this field in the case of a nonmoving current element into

$$\vec{H}=\frac{1}{2\pi c}\cdot\text{curl}\left\{\frac{\vec{u}_z}{r}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\cdot e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}{}^2}}\right\},$$

which is equivalent to the original expression in view of (7). The resulting expression for the magnetic field of the moving charge then becomes:

$$\vec{H}=-\frac{\vec{u}_\phi}{\pi c}\cdot\frac{\partial}{\partial\rho}\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\cdot e^{-i\omega\left(t-\frac{z}{v}\right)}\cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\},\qquad(16b)$$

in the derivation of which we have applied a vector identity similar to (7), viz:

$$f(z)\cdot g'(\rho)\cdot\vec{u}_\phi=-\text{curl}\ \{\vec{u}_z\cdot f(z)\cdot g(\rho)\}.$$

3.2 Discussion of the Frequency Spectrum

The integrals in (16) cannot be transformed into contour integrals, in contrast with the corresponding Laplace integrals in (5) for a nonmoving current element. Mathematically this follows from the exponential behavior at infinity of the part

$$e^{-i\omega\left(t-\frac{z}{v}\right)}\cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}$$

of the integrands. In view of the asymptotic approximation of K_0 the total exponential factor of the integrands can then be represented for $|\omega|\rightarrow\infty$ by

$$e^{-i\omega\left(t-\frac{z}{v}\right)-\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\,\omega}.$$

For $v<c$ this involves an infinite increase of the integral along those parts of the infinite circle of the ω-plane at which

$$\tan\,(\arg\omega)>\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\Big/\left(t-\frac{z}{v}\right).$$

Both the upper and lower half of the infinite circle therefore contain a sector in which the integrand increases exponentially. The physical reason for the impossibility of obtaining contour integrals which are different for large and small t is the lack of a starting time before which the signal vanishes. The situation would be quite different in the opposite case $v>c$ where the disturbances propagating with velocity c move behind the source; this latter situation is realized for Cerenkov radiation.

We shall further restrict ourselves to the case $v<c$. We then can only apply method (a) of the procedures mentioned in the introduction, which amounts to a shift of the integration path of (16) to the real axis, or to determining the limit for $\delta\rightarrow 0$. In (16a) we then have to take, in view of the poles at $\omega=\pm\omega_{pl}$, the sum of Cauchy's principal value (V.P.) connected with the latter, and of half the associated residues. Moreover, the path of integration can finally be reduced to the positive half of the real axis. In (16b) the poles are missing, and it is unnecessary to consider a principal value. The final results become:

$$\vec{E}=-4\pi\delta(x)\delta(y)\cos\left\{\omega_{pl}\left(t-\frac{z}{v}\right)\right\}\cdot U\left(t-\frac{z}{v}\right)\vec{u}_z$$

$$+\operatorname{curl}\operatorname{curl}\left[\vec{u}_z\left\{\frac{2}{\pi}\text{V.P.}\int_0^\infty d\omega\cdot\sin\left\{\omega\left(t-\frac{z}{v}\right)\right\}\frac{\omega K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}}{\omega^2-\omega_{pl}^2}\right.\right.$$

$$\left.\left.+\cos\left\{\omega_{pl}\left(t-\frac{z}{v}\right)\right\}\cdot K_0\left(\frac{\omega_{pl}}{v}\rho\right)\right\}\right],$$

$$\vec{H}=-\frac{2}{\pi c}\vec{u}_\phi\cdot\frac{\partial}{\partial\rho}\int_0^\infty d\omega\cdot\cos\left\{\omega\left(t-\frac{z}{v}\right)\right\}\cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}. \tag{17}$$

Obviously, the first term of $\vec{E}$ represents a contribution left behind along the trajectory which has been covered by the moving charge up to its position at the moment of observation. All contributions only depend on ρ and $t-z/v$, which indicates that the field moves along with the same velocity as the charge. The spectral density of the finite contributions depends first of all on the monotonically decreasing K_0 function. This decrease determines the predominance of the lower frequencies. As a matter of fact, only those frequencies contribute considerably for which the argument of K_0 does not exceed the order of magnitude of unity. Working out the corresponding condition in the case $v<<c$ we infer that the field is mainly restricted to a cylinder of radius c/ω_{pl} around the trajectory of the charge (z-axis) while the cutoff frequency decreases rapidly inside this cylinder when moving away from its axis. Only very low frequencies may yet penetrate a little outside the mentioned cylinder, up to distances roughly of the order of v/ω.

3.3. The Field Expressed in Terms of Elementary Functions

In the case of a nonmoving current element the field could be reduced to Bessel functions (see section 2.3). For a moving charge the magnetic field can even be connected with a simple algebraic function by substituting in (16b), for $\delta\rightarrow 0$, the expression

$$K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}=\frac{1}{2}\int_{-\infty}^{\infty}d\phi\cdot e^{-\frac{\omega_{pl}}{c}\rho\cosh\phi+i\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\,\omega\sinh\phi}.$$

This relation is again connected with the representation of Hankel functions by a Sommerfeld integral.

After the substitution we invert the order of integration with respect to ω and φ. The integral over ω then amounts to a delta function according to which we find:

$$\int_{-\infty}^{\infty} d\omega \cdot e^{-i\omega\left(t-\frac{z}{v}\right)} \cdot K_0\left\{\rho\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\omega^2+\frac{\omega_{pl}^2}{c^2}}\right\}$$

$$=\pi\int_{-\infty}^{\infty} d\phi \cdot e^{-\frac{\rho}{c}\omega_{pl}\cdot\cosh\phi} \cdot \delta\left(t-\frac{z}{v}-\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sinh\phi\right).$$

This integral can be worked out most conveniently by introducing the argument of the delta function as a new integration variable, and applying next the sifting property of integrals containing such a function. The formula (16b) thus finally proves to be equivalent to the representation:

$$\vec{H}=-\frac{\vec{u}_\phi}{c}\cdot\frac{\partial}{\partial\rho}\frac{e^{-\frac{\omega_{pl}}{\sqrt{\frac{c^2}{v^2}-1}}\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\rho^2+\left(t-\frac{z}{v}\right)^2}}}{\sqrt{\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\rho^2+\left(t-\frac{z}{v}\right)^2}}.$$

The exponential factor, which is missing in the absence of a plasma, indicates that the latter compresses the field in space and in time.

4. Disturbance Generated by a Current Element Along a Line of Force of an Infinite Magnetic Field Embedded in a Plasma

4.1. Derivation of the Laplace Integrals for the Field

According to the Appleton-Hartree formula the external magnetic field only enters into the refractive index of the plasma in the ratio ω/ω_H (ω_H=electron gyrofrequency). Hence, the limiting case of an infinite magnetic field is related to that for very low frequencies and therefore is instructive for these frequencies.

Let the homogeneous external field be in the z direction, and the current element of normalized momentum at the origin. Its current-density distribution is then given, once again, by (3). The dielectric constant of the medium under consideration is given by the tensor:

$$\left\|\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1-\frac{\omega_{pl}^2}{\omega^2}\end{matrix}\right\|.$$

This tensor involves the Maxwell equations:

$$\text{curl } \vec{E}+\frac{1}{c}\frac{\partial\vec{H}}{\partial t}=0, \tag{18a}$$

$$\text{curl}\frac{\partial\vec{H}}{\partial t}-\frac{1}{c}\frac{\partial^2\vec{E}}{\partial t^2}-\frac{\omega_{pl}^2}{c}\vec{u}_z E_z=\frac{4\pi}{c}\frac{\partial\vec{I}}{\partial t}=\frac{4\pi}{c}\vec{u}_z\delta(x)\delta(y)\delta(z)\delta'(t). \tag{18b}$$

The solution corresponding to (5) in the case without magnetic field reads:

$$\vec{E}=\frac{i}{2\pi}\left(\overrightarrow{\text{grad}}\frac{\partial}{\partial z}-\frac{\vec{u}_z}{c^2}\frac{\partial^2}{\partial t^2}\right)\int_{-\infty+i\delta}^{\infty+i\delta} d\omega\,\frac{e^{-i\omega t+\frac{i}{c}\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}}}{\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}},$$

$$\vec{H}=-\frac{i}{2\pi c}\sin\theta\vec{v}_\varphi\frac{\partial^2}{\partial r\partial t}\int_{-\infty+i\delta}^{\infty+i\delta} d\omega\,\frac{e^{-i\omega t+\frac{i}{c}\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}}}{\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}}. \tag{19}$$

The first equation (19) is verified readily, when representing the $\vec{H}$ field as a curl vector, applying (7). In order to check the second equation of (19), the curl curl operator then occurring in its first term may be split into curl curl$=-\nabla^2+$grad div. The Laplace operator ∇^2 operating on the quantity

$$\frac{e^{\frac{i}{c}\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}}}{\sqrt{\omega^2r^2-\omega_{pl}^2\rho^2}}$$

should next be transformed with the aid of the relation:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1}{\alpha}\frac{\partial^2}{\partial z^2}+k^2\right)\frac{e^{ik\sqrt{x^2+y^2+\alpha z^2}}}{\sqrt{x^2+y^2+\alpha z^2}}=-\frac{4\pi}{\sqrt{\alpha}}\,\delta(x)\delta(y)\delta(z)\qquad\left(\arg\sqrt{\alpha}=\frac{1}{2}\arg\alpha\right)\tag{20}$$

for

$$\alpha=\omega^2/(\omega^2-\omega_{pl}^2)\text{ and }k=\sqrt{\omega^2-\omega_{pl}^2}/c.$$

The term with delta functions then leads to the right-hand member of the second equation (18) whereas the other terms prove to cancel. Thus the correctness of (19) depends first of all on the validity of (20), a relation which can be proved, e.g., by integrating it (applying the method leading to Gauss' integral theorem) over a volume element around the origin; this relation constitutes an extension, for $\alpha\neq1$, of the property (6) for the point-source solution of the three-dimensional Helmholtz equation.

4.2. Frequency Spectrum After the Start of the Field

The derivation is completely similar to that in the case of the absence of a magnetic field, used when deriving (8) from (5). The situation is even simpler since the branch point of

$$(\omega_{pl}^2\rho^2-\omega^2r^2)^{1/2}\text{ at }\omega=\omega_{pl}\rho/r$$

constitutes the only singularity, as the pole is now missing even for the electric field. The results can be represented by:

$$\vec{E}=\frac{2}{\pi}\left(\overrightarrow{\text{grad}}\,\frac{\partial}{\partial z}-\frac{\vec{u}_z}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{U\left(t-\frac{r}{c}\right)\int_0^{\omega_{pl}\sin\theta}d\omega\cos(\omega t)\,\frac{\cosh\left(\frac{1}{c}\sqrt{\omega_{pl}^2\rho^2-\omega^2r^2}\right)}{\sqrt{\omega_{pl}^2\rho^2-\omega^2r^2}}\right\},$$

$$\vec{H}=-\frac{2\sin\theta}{\pi c}\,\vec{u}_\varphi\,\frac{\partial^2}{\partial r\partial t}\left\{U\left(t-\frac{r}{c}\right)\int_0^{\omega_{pl}\sin\theta}d\omega\cos(\omega t)\,\frac{\cosh\left(\frac{1}{c}\sqrt{\omega_{pl}^2\rho^2-\omega^2r^2}\right)}{\sqrt{\omega_{pl}^2\rho^2-\omega^2r^2}}\right\}.\tag{21}$$

Just as in the case without magnetic field, described by (8), the field starts at the moment $t=r/c$, but the cutoff frequency now equals $\omega_{pl}\sin\theta$ instead of ω_{pl}. The amplitude is maximal near this cutoff. This implies a concentration of the lowest frequencies to directions very near to that of the magnetic line of force passing through the radiating source. Here we infer a similarity to "whistler" propagation.

4.3. The Field Expressed in Terms of Bessel Functions

The expressions corresponding to (13) in the absence of the magnetic field are obtained here with the aid of the relation:

$$\int_{-\infty+i\delta}^{\infty+i\delta}d\omega\,\frac{e^{-i\omega t+i\frac{r}{c}\sqrt{\omega^2-\omega_{pl}^2}}}{\sqrt{\omega^2-\omega_{pl}^2}}=-2\pi iJ_0\left(\omega_{pl}\sqrt{t^2-\frac{r^2}{c^2}}\right)U\left(t-\frac{r}{c}\right),$$

of which (15) constitutes the derivative with respect to r. Applying this relation for ω_{pl} replaced by $\omega_{pl}\sin\theta$, we obtain from (21):

$$\vec{E}=\left(\overrightarrow{\text{grad}}\,\frac{\partial}{\partial z}-\frac{\vec{u}_2}{c^2}\frac{\partial^2}{\partial t^2}\right)\left\{U\left(t-\frac{r}{c}\right)\frac{J_0\left(\omega_{pl}\sin\theta\sqrt{t^2-\frac{r^2}{c^2}}\right)}{r}\right\},$$

$$\vec{H}=-\frac{\sin\theta}{c}\vec{u}_\varphi\frac{\partial^2}{\partial r\partial t}\left\{U\left(t-\frac{r}{c}\right)\frac{J_0\left(\omega_{pl}\sin\theta\sqrt{t^2-\frac{r^2}{c^2}}\right)}{r}\right\}.$$

5. Disturbance Generated by a Charge Moving Along a Line of Force of an Infinite Magnetic Field Embedded in a Plasma

5.1. Discussion of the Frequency Spectrum

Here, again, the field for a moving charge can be deduced from that of the nonmoving current element, just as discussed in section 3.1 in the absence of a magnetic field. Starting from (19) we now have to evaluate the following integral over t_0:

$$L=\int_{-\infty}^{\infty}dt_0\,\frac{e^{i\omega t_0+\frac{i}{c}\sqrt{(\omega^2-\omega_{pl}^2)\rho^2+\omega^2(z-vt_0)^2}}}{\sqrt{(\omega^2-\omega_{pl}^2)\rho^2+\omega^2(z-vt_0)^2}}.$$

Again, this integral could be reduced with the aid of Sommerfeld integrals to Bessel and Hankel functions. However, it is impossible to get a single representation for all ω-values of the integration path of (19) which, therefore, has to be split into different sections. This splitting is simplest in the limit $\delta\rightarrow 0$ and can then be reduced to the positive part of the real ω-axis. The final results read as follows in terms of Neumann and K functions with positive arguments:

$$E=2\left(\overrightarrow{\text{grad}}\,\frac{\partial}{\partial}z-\frac{\vec{u}_z}{c^2}\frac{\partial^2}{\partial t^2}\right)\left[-\frac{1}{2}\int_0^{\omega_{pl}}d\omega\,\frac{\sin\left\{\omega\left(t-\frac{z}{v}\right)\right\}}{\omega}\,Y_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega_{pl}^2-\omega^2}\right)\right.$$
$$\left.+\frac{1}{\pi}\int_{\omega_{pl}}^{\infty}d\omega\,\frac{\sin\left\{\omega\left(t-\frac{z}{v}\right)\right\}}{\omega}\,K_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega^2-\omega_{pl}^2}\right)\right],$$

$$\vec{H}=\frac{2}{c}\,\text{curl}\left[\vec{u}_z\left\{-\frac{1}{2}\int_0^{\omega_{pl}}d\omega\cos\left\{\omega\left(t-\frac{z}{v}\right)\right\}Y_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega_{pl}^2-\omega^2}\right)\right.\right.$$
$$\left.\left.+\frac{1}{\pi}\int_{\omega_{pl}}^{\infty}d\omega\cos\left\{\omega\left(t-\frac{z}{v}\right)\right\}K_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega^2-\omega_{pl}^2}\right)\right\}\right]\cdot \tag{22}$$

These expressions can be verified most conveniently by showing that they do satisfy the Maxwell equations (18), with the right-hand side of (18b) replaced by

$$\frac{4\pi}{c}\vec{u}_z\delta(x)\delta(y)\delta'\left(t-\frac{z}{v}\right)\cdot \tag{23}$$

The proof can be carried through, apart from the application of elementary rules of vector analysis, with the aid of the relations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\lambda^2\right)Y_0(\lambda\rho)=4\delta(x)\delta(y),$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\lambda^2\right)K_0(\lambda\rho)=-2\pi\delta(x)\delta(y),$$

$$\int_0^\infty d\omega\omega\sin\left\{\omega\left(t-\frac{z}{v}\right)\right\}=-\pi\delta'\left(t-\frac{z}{v}\right).$$

Once again, the predominant role of the lower frequencies in (22) is obvious. In fact, the amplitudes of the frequencies $\omega<\omega_{pl}$ depend numerically on an oscillating Neumann function, those of the higher frequencies $\omega>\omega_{pl}$ on the exponentially decaying K_0 junction.

We mention also the following alternative representations equivalent to (22):

$$\vec{E}=\frac{i}{\pi}\left(\overrightarrow{\text{grad}}\,\frac{\partial}{\partial z}-\frac{\vec{u}_z}{c^2}\frac{\partial^2}{\partial t^2}\right)\int_{-\infty}^{\infty}d\omega\,\frac{e^{-i\omega\left(t-\frac{z}{v}\right)}}{\omega}\,K_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega^2-\omega_{pl}^2}\right), \tag{24a}$$

$$\vec{H}=\frac{1}{\pi c}\,\text{curl}\left\{\vec{u}_z\int_{-\infty}^{\infty}d\omega e^{-i\omega\left(t-\frac{z}{v}\right)}\,K_0\left(\rho\sqrt{\frac{1}{v^2}-\frac{1}{c^2}}\sqrt{\omega^2-\omega_{pl}^2}\right)\right\}. \tag{24b}$$

The square root $\sqrt{\omega^2-\omega_{pl}^2}$ is to be considered here as positive for $\omega^2>\omega_{pl}^2$, whereas in the section $-\omega_{pl}<\omega<\omega_{pl}$ we have to take the average of the values of the K function just above and just below this crosscut of the square root. The average follows from the identity:

$$K_0(\sqrt{-z})=\frac{1}{2}K_0\left(e^{\frac{i\pi}{2}}\sqrt{z}\right)+\frac{1}{2}K_0\left(e^{\frac{-i\pi}{2}}\sqrt{z}\right)=-\frac{\pi}{2}Y_0(\sqrt{z}).$$

The denominator in (24a) vanishes when performing the differentiations with respect to z and t.

5.2. The Field Expressed in Terms of Elementary Functions

Both the electric and the magnetic field can be reduced to elementary functions by applying further special formulas connected with Sommerfeld's integrals. The final expressions obtained, viz:

$$\vec{E}=-\left(\overrightarrow{\text{grad}}+\frac{v}{c^2}\vec{u}_z\frac{\partial}{\partial t}\right)\frac{\cos\left\{\frac{\omega_{pl}}{v}\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}\right\}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}},$$

$$\vec{H}=\frac{v}{c}\,\text{curl}\left[\vec{u}_z\,\frac{\cos\left\{\frac{\omega_{pl}}{v}\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}\right\}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}}\right], \tag{25}$$

can also be verified straightforwardly by substitution into the Maxwellian equations (18), the right-hand side of (18b) again being replaced by (23). The proof depends on the relation

$$\left\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2}{\partial z^2}+\omega_{pl}^2\left(\frac{1}{v^2}-\frac{1}{c^2}\right)\right\}\frac{\cos\left\{\frac{\omega_{pl}}{v}\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}\right\}}{\sqrt{\left(1-\frac{v^2}{c^2}\right)\rho^2+(z-vt)^2}}$$

$$=-\frac{4\pi}{v}\,\delta(x)\delta(y)\delta\left(t-\frac{z}{v}\right),$$

which results from the general relation (20).

According to (25) the plasma induces oscillations in the field existing otherwise in free space, which is in contrast with the exponential influence of the plasma in the case of a zero magnetic field (described in section 3.3).

6. General Conclusions

The described situations, referring either to a zero or to an infinite magnetic field, show that the field disturbances produced by a current element which is only effective at a special moment can, after their start at the point of observation, be decomposed into a spectrum containing exclusively low frequencies. The corresponding disturbances produced by a charge moving along a rectilinear path (assumed as a line of force in the case of the infinite magnetic field) comprise predominantly low frequencies, though theoretically all frequencies occur. The limitation to low frequencies is particularly pronounced at large distances from the path of the moving charge. These general results are expected to hold also for a finite magnetic field. According to preliminary calculations the spectral density then becomes infinite at frequencies for which the phase velocity equals that of the moving charge; therefore these frequencies should be observable first of all. This is in accordance with Gallet's [1959] theory concerning exospheric VLF emissions; in fact, the frequencies radiated from a charged particle moving through the exosphere are derived there from the condition expressing the equality of the two velocities mentioned. The general problem of a charge moving through a finite static magnetic field has also been approached, using a different analysis in the case of a spiral motion, by Pakhomov, Aleksin, and Stepanov [1962].

7. References

Gallet, R. M. (1959), The very-low-frequency emissions generated in the earth's exosphere, Proc. IRE **47,** 211–231.

Pakhomov, V. I., V. F. Aleksin, and K. N. Stepanov (1962), The radiation from an electron moving in a spiral in a magnetoactive plasma, Soviet Physics—Tech. Phys. **6,** 856–866.

Van der Pol, B., and H. Bremmer (1955), Operational calculus, 318, formula (13), Cambridge.

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