From
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138 - 220
276 - 351

Chap. III p. 396 C = p. 367

Notes
on
Rankine's Civil Engineering
Part II

for the use of engineering students
University of Michigan

by

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Ann Arbor, Michigan
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Preface.

Rankine's Civil Engineering, while very useful as a manual, is a book full of difficulties for the student, unless he is assisted by copious notes and explanations. To save the time of the young engineer, and to enable him to have his notes in convenient shape and compass, the writer has had the principal portion of such demonstrations and discussions, as he has previously been accustomed to give orally or on the blackboard, prepared by lithography, and he hopes, by this means, to gain in time and accuracy. He has, in some instances, differed from Prof. Rankine in results, and he has added material that he has found useful in past years.

These notes will be further explained and elaborated in the class-room, and supplemented by numerous examples. The order of topics in the text-book is deviated from considerably in the lessons.

A full index to these notes, with references to the corresponding paragraphs of the book, will make any subject easy to be found.

Index.

Errata.

Page 11, line 18, for $3c$ read $3e$.

" 26, equation (8.) should read $r_1 = r_2\sqrt{\left(\frac{f-2p_2+p_1}{f-p_1}\right)}$.

" 27, " (13.) " " $r_1 = r_2\sqrt[3]{\left(\frac{2f+p_1-3p_2}{2(f-p_1)}\right)}$.

" 27, line 12, for $\frac{f+p}{3}$ read $\frac{f+p_1}{3}$.

" 26, " 23, " $\frac{1}{2}(f'+p_1)$ read $\frac{1}{2}(f+p_1)$.

" 27, " 21, " $\frac{1}{3}(2f'+p_1)$ " $\frac{1}{3}(2f+p_1)$.

" 28, Moments of Inertia, to agree with $\frac{x_1 S}{I}$, the coordinates x and y should change places.

" 54, lines 5 and 8, for preceding read Case V.

" 59, line 3, for $\int_0^{x'}$ read $\int_0^{c}$.

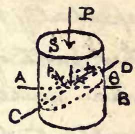
" 65, " 3, " $2wc^2+2w'c^2$ read $2wc^2+w'c^2$.

" 87, last line, for 41 read 42.

Notes to accompany
Rankine's Civil Engineering.

Part II.

Distributed Forces. §108. Internal Stress.— If $p_x = \frac{P}{S}$ is the intensity of stress on and perpendicular to the right section of a short bar or column, the intensity on a section CD, making an angle θ with the former, will be



$\frac{P}{S \sec\theta} = p_x \cos\theta$. Resolve this intensity normally and tangentially to CD.

Normal intensity $= p_n = p_x \cos\theta \cos\theta = p_x \cos^2\theta$;

Tang! intensity $= p_t = p_x \cos\theta \sin\theta$.

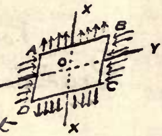
Pass another plane, whose obliquity $\theta' = 90° - \theta$. Then

$p_n' = p_x \cos^2\theta' = p_x \sin^2\theta$; $p_t' = p_t$; $\therefore p_n + p_n' = p_x$.

Hence— On a pair of planes of section at right angles to one another the tangential components or <u>shears</u> are <u>of equal intensity</u>, and the intensities of the normal components are together equal to the intensity of the original normal stress.

<u>Compound</u> <u>Stress</u> is that internal condition of a body which is caused by the combined action of two or more simple stresses in different directions. The investigations which follow are those of compound stress, but will be confined to stresses in or parallel to one plane.

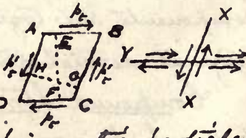
I. Conjugate Stresses.— If the stress on a given plane in a body be in a given direction, the stress on any plane parallel to that direction must be parallel to the first-mentioned plane. For the equal resultant



forces exerted by the other parts of the body on the faces AB and DC of the prismatic particle shown above are directly opposed to one another, their common line of action traversing the axis through O; and they are therefore independently balanced. Therefore the forces exerted by the other parts of the body on the faces AD and BC of this prism must be independently balanced and have their resultant directly opposed; which cannot be unless their direction is parallel to the plane YOY.

A pair of stresses, each acting on a plane parallel to the direction of the other, are said to be conjugate. Their intensities are independent of each other, and they may be of the same or opposite kinds. If they are normal to their planes, and hence at right angles to each other, they are called principal stresses.

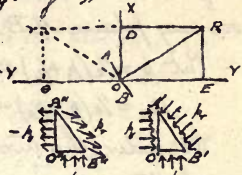
II. Shearing Stress.— If the stresses on a pair of planes are tangential to those planes, those stresses must be of equal intensity. Consider them



as acting along the faces of a small prismatic particle. Taking moments, we must have, for equilibrium, $p_t \cdot AB \cdot EF = p_t' AD \cdot HG$. But the area of the section $AB \cdot EF = AD \cdot HG$. $\therefore p_t = p_t'$.

This construction shows that a shear cannot act alone as a simple stress, but must be combined with a shear of equal intensity on a different plane.

III. If a pair of principal stresses are of equal intensities and of the same kind (p_x and p_y, Fig. 1.), the stress on every plane is normal and of the same intensity. [Example: fluid pressure]



Let the stresses act along X and Y. To find intensity and direction on plane AB:— Take $OD = p_x \cdot O'B'$, and $OE = p_y \, O'A'$, both being positive. Complete the rectangle. Then will

Fig. 2. Fig. 1.

$p_r = \frac{\text{Result. } OR}{\text{Plane } A'B'}$. Since $p_x = p_y$, we have $\frac{OD}{O'B'} = \frac{OE}{O'A'} = \frac{OR}{A'B'}$;

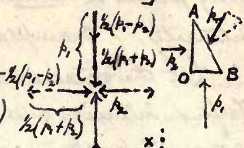
$\therefore p_x = p_y = p_r$; and, because of similarity of triangles, A'O'B' and OER, OR is perpendicular to AB.

IV. If the stresses are of contrary signs (p_x and $-p_y$, Fig. 2.), construct ODre. The resultant Or will be the same in magnitude as OR, and will make the same angle with OX, but on the opposite side; or OX bisects $R\hat{O}r$; hence—

If a pair of principal stresses are of equal intensities, but of opposite kinds, the stress on every plane is of the same intensity, and the angle which its direction makes with the normal to its plane is bisected by the axis of principal stress.

The stress p_r agrees in kind with that one of the principal stresses to which its direction is nearer; and, when it makes angles of 45° with each of the axes, it is shearing or tangential; so that a pull and a thrust of equal intensity, on a pair of planes at right angles to each other, are equivalent to a pair of shearing stresses of the same intensity on a pair of planes at right angles with each other and making angles of 45° with the first pair.

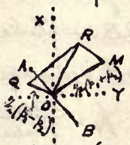
V. If we have two principal stresses (p_1 and p_2) of any intensity, and of the same or opposite kinds,— To find the intensity, direction and kind of stress on any plane AB. [Application of III. and IV.]



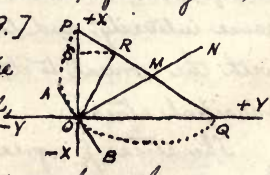
Put $p_1 = \frac{1}{2}(p_1+p_2) + \frac{1}{2}(p_1-p_2)$ and

$p_2 = \frac{1}{2}(p_1+p_2) - \frac{1}{2}(p_1-p_2)$.

Consider the two stresses $+\frac{1}{2}(p_1+p_2)$ on the two axes, and, by III., lay off OM on normal to AB for the intensity and direction of stress on AB due to $\frac{1}{2}(p_1+p_2)$.



Next consider $+\frac{1}{2}(p_1-p_2)$ on the vertical axis, and $-\frac{1}{2}(p_1-p_2)$ on the horizontal axis, and, by IV., lay off OQ, making the same angle with axes as does OM, but in contrary direction, for intensity and direction of stress on AB due to $\pm\frac{1}{2}(p_1-p_2)$. As OM and OQ both act on the same square inch of AB, RO, in the opposite direction to their resultant OR, will give the direction and intensity of stress on AB. Here RO is positive. If it fell on the other side of AB, it would be negative; if along AB, it would be shear. [C.E., pp. 168-9.]



A shorter construction is to strike a semicircle from M on the normal, with a radius $MO = \frac{1}{2}(p_1+p_2)$, and draw MR through the points P and Q where the semicircle cuts the axes. Lay off $MR = \frac{1}{2}(p_1-p_2)$ in the direction of axis of greatest stress, and RO will be the desired stress.

For different planes AB through O, p_1 and p_2 being given, the locus of M is a circle, and of R is an ellipse, with major and minor semidiameters p_1 and p_2. For proof, drop a perpendicular RS, from R on OX.

$\angle MOP = \angle MPO$; $\angle MOQ = \angle MQO$; $QR = p_1$; $RP = p_2$;
then $RS = PR \sin MPO = p_2 \sin \hat{xn}$;

$SO = RQ \sin MQO = p_1 \cos \hat{xn}$. $\therefore$

$$OR = p_r = \sqrt{(\overline{SO}^2 + \overline{RS}^2)} = \sqrt{(p_1^2 \cos^2 \hat{xn} + p_2^2 \sin^2 \hat{xn})} \quad (1.)$$

which is the value of the radius vector of an ellipse, the origin being at the centre. As $\angle NMR = 2\hat{xn}$,

$\sin NOR : \sin OMR = RM : OR = \tfrac{1}{2}(p_1 - p_2) : p$. $\therefore$

$$\sin \hat{nr} = \sin 2\hat{xn}\, \frac{p_1 - p_2}{2p_r}. \quad (2.)$$

For $p_1 = p_2$, we have III and IV.

Example: $p_1 = 100$ lbs.; $p_2 = -50$ lbs.; $\hat{xn} = 30°$; $p = ?$
Construct figure and result $\leftarrow$?. Try other values.

What must be the angle of ↑ plane on which $\underline{p}$ is a shear only, and under what conditions are shearing planes possible?

In that case $\hat{nr} = 90°$; $\sin \hat{nr} = 1$. $\therefore$ from (2.),

$\sin 2\hat{xn} = 2 \sin \hat{xn} \cos \hat{xn} = \dfrac{2p}{p_1 - p_2}$. Substitute from (1.),

$\sin \hat{xn} \cos \hat{xn} (p_1 - p_2) = \sqrt{(p_1^2 \cos^2 \hat{xn} + p_2^2 \sin^2 \hat{xn})}$.

$\sin^2 \hat{xn} \cos^2 \hat{xn} (p_1^2 - 2p_1 p_2 + p_2^2) = p_1^2 \cos^2 \hat{xn} + p_2^2 \sin^2 \hat{xn}$

$p_1^2 \cos^2 \hat{xn} (1 - \sin^2 \hat{xn}) + p_2^2 \sin^2 \hat{xn} (1 - \cos^2 \hat{xn}) + 2p_1 p_2 \sin^2 \hat{xn} \cos^2 \hat{xn} = 0$

$p_1^2 \cos^4 \hat{xn} + 2p_1 p_2 \sin^2 \hat{xn} \cos^2 \hat{xn} + p_2^2 \sin^4 \hat{xn} = 0$

$p_1 \cos^2 \hat{xn} + p_2 \sin^2 \hat{xn} = 0 \therefore \dfrac{\sin \hat{xn}}{\cos \hat{xn}} = \tan \hat{xn} = \sqrt{\left(\dfrac{-p_1}{p_2}\right)}$.

No shearing plane is possible unless p_1 and p_2 differ in sign. There are then two planes of shear.

The same result can be obtained from p. 1.

VI. Given two conjugate stresses, p and p', of the same or opposite signs, to determine the direction, kind and magnitude of the principal stresses, p_1 and p_2.

If we suppose p and p' to be in position, Fig. 2, it is evident that, by definition, they are

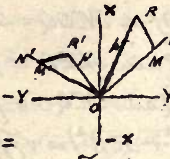


Fig. 2.

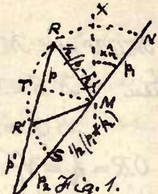


Fig. 1.

equally oblique to the normals to their respective planes; that $OM = OM' = \frac{1}{2}(p_1+p_2)$; and $MR = M'R' = \frac{1}{2}(p_1-p_2)$.

Revolve $OM'R'$ until the normals coincide, as in Fig. 1, and OR' lies in the line OR. If it is of opposite sign, it will lie in the prolongation of RO below O. Then, since RMR' must be an isosceles triangle, bisect RR' at T, drop TM perpendicularly to OR on the normal. Then will $OM + MR = p_1$ and $OM - MR = p_2$. Or, with M as centre, and radius MR, describe a semicircle; when $ON = p_1$, and $OS = p_2$. The axis of p_1 will bisect the angle NMR, giving MX; the axis of p_2 will be at right angles to MX.

Try the construction when p and p' have opposite signs.

VII. Given two stresses p, of any intensity, direction and sign, to determine p_1 and p_2.

The construction of Fig. 3 is evident from Fig. 1. When the normals have been revolved to coincide, the two given stresses will be OR and OR'. By uniting R with R', bisecting at T, dropping perpendicular TM on normal, and describing semicircle about M with radius MR, we have $ON = p_1$; $OS = p_2$; while the direction of $p_1 = MX$, bisecting $\angle RMN$, provided OR was not revolved. Otherwise bisect $\angle R'MN$.

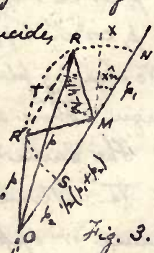


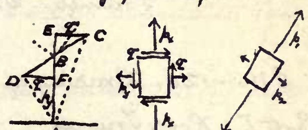
Fig. 3.

All possible values of p will terminate on the semicircle just drawn, and the greatest possible obliquity to the normal will be found by drawing a tangent to the semicircle from O. When the two given stresses have opposite signs, lay one off downwards from O, in the direction of one or the other dotted line. Then can be found the plane on which stress is shear.

The above solution may be considered the general case.

This subject is treated mathematically and at considerable length in Rankine's "Applied Mechanics."

VIII. Deviation of principal stresses by a shearing stress. [C.E., p.169.] Intensities of principal stresses are p_x and p_y; of shear is q.



Lay off $AE = p_x$ on the normal to the horizontal plane, and $EC = q$ at right angles to p_x, as they both act on the same area. Revolve the plane on which p_y acts through 90°, so that normals shall coincide, and lay off $AF = p_y$ and $FD = q$. The resultant stresses on horizontal and vertical planes will be given by the dotted lines AC and AD. Connect extremities C and D by CD. Bisect CD and drop perpendicular on AE, as usual; all of the operation is, in this case, contained in the point B.

$$\text{Then } p_1 = AB + BC = \tfrac{1}{2}(p_x + p_y) + \sqrt{\tfrac{1}{4}(p_x - p_y)^2 + q^2} \quad (3.)$$
$$p_2 = AB - BD = \tfrac{1}{2}(p_x + p_y) - \sqrt{\tfrac{1}{4}(p_x - p_y)^2 + q^2}$$

The new major principal stress p_1 will bisect the angle CBE, and the new minor principal stress p_2 will be at right angles to it, or bisect EBD.

$$\therefore \operatorname{Tan} 2\hat{xn} = \tan CBE = \frac{2q}{p_x - p_y} \quad (4.)$$

See C.E., p. 170, and compare Fig. 134, p. 250. If $p_y = 0$, the point F will fall at A, and we get the construction for the combination of direct pull or thrust from bending moment with the shear at same section. Thus we may obtain the curves of principal stress in a beam.

See, further, pressure of earth, and shaft under torsion, § 174.

Friction. — § 110, p. 171. $f = \tan\varphi$.

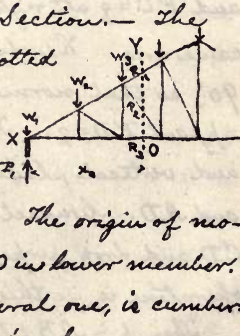
$\sqrt{(1+\tan^2\varphi)} = \sec\varphi = \sqrt{(1+f^2)}$. $\cos\varphi = \frac{1}{\sqrt{(1+f^2)}}$.

$\tan\varphi\cos\varphi = \sin\varphi = \frac{f}{\sqrt{(1+f^2)}}$. (2)

Frames, Chains, etc.

§ 115–120. Bracing of Frames. — See "Graphics," Part I., Roof Trusses.

§ 122. Resistance of Frame at a Section. — The equations of the book, applied at the dotted section of this truss, would be



$R_1\cos i_1 + R_2\cos i_2 + R_3 = 0$,

$R_1\sin i_1 + R_2\sin i_2 = P_1 - W_1 - W_2 - W_3$,

$R_1 n_1 + R_2 n_2 = P_1 x_0 - W_1 x_1 - W_2 x_2 - W_3 x_3$. The origin of moments is taken at the point of section O in lower member.

This method of treatment, as a general one, is cumbersome and unsatisfactory. Modifications, however, are of frequent application.

Suspension Bridge.

§ 125. Chain under Uniform Vertical Load. —

By (2) p. 187, $\tan i = \frac{dy}{dx} = \frac{P}{H}$. $P = px$

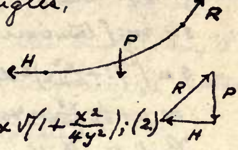
$\therefore \frac{dy}{dx} = \frac{p}{H}x$; $y = \frac{p}{H}\int_0^x x\,dx = \frac{p}{2H}x^2$.

$\therefore x^2 = \frac{2H}{p}y = 4my$. The curve is therefore a parabola.

Then $m = \frac{x^2}{4y} = \frac{H}{2p}$. (3.)

By proportion from similar triangles,

$P:H:R = y : \tfrac{1}{2}x : \sqrt{(y^2 + \tfrac{1}{4}x^2)}$.



As $P = px$, multiply this proportion by $\frac{px}{y}$ and get $P:H:R = px : \frac{px^2}{2y} : px\sqrt{(1+\frac{x^2}{4y^2})}$; (2) in which the second member gives the values of P, H and R.

$\therefore H = \frac{px^2}{2y}$ and $R = px\sqrt{(1+\frac{x^2}{4y^2})}$.

Prob. 1. By (3) p. 188, $m = \frac{x_1^2}{4y_1} = \frac{x_2^2}{4y_2}$; $x_2 = a - x_1$.

$\therefore \frac{x_1^2}{y_1} = \frac{a^2 - 2ax_1 + x_1^2}{y_2}$ or $\frac{x_1}{\sqrt{y_1}} = \frac{a - x_1}{\sqrt{y_2}}$; $x_1 = a\frac{\sqrt{y_1}}{\sqrt{y_1}+\sqrt{y_2}}$ (4.)

x_2 is obtained similarly. $m = \frac{x_1^2}{4y_1} = \frac{a^2}{4(\sqrt{y_1}+\sqrt{y_2})^2}$ (5.)

Prob. 2. By inspection, $\tan i = \frac{2y_1}{x_1}$. Substitute (4.) and obtain (7.).

Prob. 3. By (3.) p. 188, $H = 2pm$. Substitute (5.) and get (9.). By (2.), $H : R_1 = \tfrac{1}{2}x_1 : \sqrt{(y_1^2 + \tfrac{1}{4}x_1^2)}$. $\therefore$

$R_1 = H\sqrt{(\frac{4y_1^2}{x_1^2}+1)}$. (10.) R_2 is similar.

Prob. 4. $ds = \sqrt{(dx^2+dy^2)}$; $s = \int_0^x \sqrt{(1+\frac{dy^2}{dx^2})}\,dx$.

$x^2 = 4my$; $2x\,dx = 4m\,dy$ $\therefore \frac{dy}{dx} = \frac{x}{2m}$.

$s = \int_0^x (1+\frac{x^2}{4m^2})^{\frac{1}{2}}dx = \frac{1}{2m}\int_0^x (4m^2+x^2)^{\frac{1}{2}}dx$. By Courtenay's Calculus, p. 323 and B; or Church's, p. 277; or Olney's (C);

$s = \frac{1}{2m}\cdot\frac{1}{2}(4m^2+x^2)^{\frac{1}{2}}x + \frac{1}{2m}\cdot\frac{1}{2}4m^2\int_0^x (4m^2+x^2)^{-\frac{1}{2}}dx$.

Put $(4m^2+x^2)^{\frac{1}{2}} = z + x$. Then $x = \frac{4m^2-z^2}{2z}$; $dx = -\frac{4m^2+z^2}{2z^2}dz$;

$(4m^2+x^2)^{\frac{1}{2}} = z + \frac{4m^2-z^2}{2z} = \frac{4m^2+z^2}{2z}$. $\therefore$

$\int(4m^2+x^2)^{-\frac{1}{2}}dx = -\int\frac{2z}{4m^2+z^2}\cdot\frac{4m^2+z^2}{2z^2}dz = -\int\frac{dz}{z} = -\log z$ $\therefore$

$s = \frac{x}{4m}(4m^2+x^2)^{\frac{1}{2}} + m[-\log(\sqrt{4m^2+x^2}-x) + \log 2m]$

$= \frac{x}{4m}\sqrt{(4m^2+x^2)} + m\log\frac{\sqrt{(4m^2+x^2)}+x}{2m}$ (A.)

[Or $\int_0^x \frac{dx}{\sqrt{(4m^2+x^2)}} = \int_0^x \frac{(x+\sqrt{4m^2+x^2})dx}{\sqrt{4m^2+x^2}(x+\sqrt{4m^2+x^2})} = \int_0^x \frac{\frac{x}{\sqrt{4m^2+x^2}}dx + dx}{\sqrt{4m^2+x^2}+x} =$ $\log(\sqrt{4m^2+x^2}+x) - \log 2m.$]

By (3.), $2m = \frac{x^2}{2y}$. Substitute in (A.)

$s = \frac{xy}{x^2}\sqrt{(\frac{x^4}{4y^2}+x^2)} + \frac{x^2}{4y}\log\frac{\sqrt{(\frac{x^4}{4y^2}+x^2)}+x}{\frac{x^2}{2y}} = \sqrt{(\frac{x^2}{4}+y^2)} + \frac{x^2}{4y}\log\frac{\sqrt{(\frac{x^2}{4}+y^2)}+y}{\frac{1}{2}x}$ (12.)

$\log_a y = 0.43429\log_e y$; $\log_e y = 2.30258\log_a y$; $e = 2.7182828$.

From (A.) $s = m\{\frac{x}{2m}\cdot\frac{\sqrt{4m^2+x^2}}{2m} + \log(\frac{\sqrt{4m^2+x^2}}{2m}+\frac{x}{2m})\}$;

$\tan i = \frac{dy}{dx} = \frac{x}{2m}$; $\sec i = \sqrt{1+\tan^2 i} = \frac{\sqrt{4m^2+x^2}}{2m}$ $\therefore$

$s = m\{\tan i \sec i + \log(\tan i + \sec i)\}$ (12.)

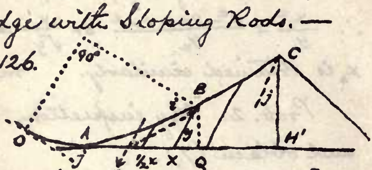
$s = \int_0^x \left(1+\frac{x^2}{4m^2}\right)^{\frac{1}{2}} dx$. Expanding, $s = \int_0^x dx + \int_0^x \frac{1}{2}\frac{x^2}{4m^2}dx + \text{\&c.}$

$= x + \frac{1}{24}\frac{x^3}{m^2} + \text{\&c.} = x + \frac{x^3}{24}\cdot\frac{16y^2}{x^4} + \; = x + \frac{2}{3}\frac{y^2}{x} +$ (13.)

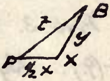
Prob. 5. Differentiate (13.). $ds = \frac{4y}{3x}dy$; $\therefore$

$\frac{d(s_1+s_2)}{dy} = \frac{4}{3}\left(\frac{y_1}{x_1} + \frac{y_2}{x_2}\right)$. (16.)

§126. Suspension Bridge with Sloping Rods. — Note the first paragraph of §126.



The load will be distributed uniformly per foot of a plane perpendicular to the sloping rods; therefore the vertex of this parabola will be at O, where the tangent is perpendicular to the direction BX. OA is not a part of the actual chain.



Olney's Genl Geom. p. 88.

The equation of a parabola referred to oblique coordinates, viz., the tangent at A and a diameter, with origin at A on the curve, is $x^2\cos^2 j = 4my$; $\therefore$ $m = \frac{x^2\cos^2 j}{4y}$ (4.)

In triangle PXB, $\angle PXB = 90° + j$ $\therefore$

$t = \sqrt{(\frac{1}{4}x^2 + y^2 - 2\cdot\frac{1}{2}xy\cos(90°+j))}$. (6.)

$V : P : Q = 1 : \sec j : \tan j = vx : vx\sec j : vx\tan j$. (7.)

$P : H : R = y : \frac{1}{2}x : t$; $\therefore$ $H = \frac{xP}{2y} = \frac{px^2}{2y} = 2\frac{pm}{\cos^2 j} = 2vm\sec^3 j$.

III. When the cable is fastened at A to the horizontal platform, the sloping rods will cause a horizontal thrust $Q = qx$, increasing in amount from A to H'; and the chain will add throughout a thrust H. Hence (10.). The sloping rods ~~may~~ will cause tension in AQ, if there is no horizontal reaction at H'!

From triangle PXB (small sketch), $\angle PXB = 90° + j$; $\angle PBX = 90° - i$.

$\therefore$ $\frac{1}{2}x : t = \sin(90° - i) : \sin(90° + j) = \cos i : \cos j$; $\therefore$

$\cos i = \frac{x}{2t}\cos j$. (11.). At A, $\angle i = \angle j$ as marked.

With origin at O, the length of arc OB, by (12.) p. 190, is

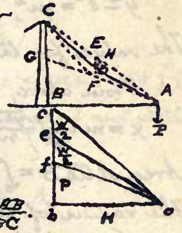
$m\{\tan i \sec i + \log(\tan i + \sec i)\}$. The length from O to A, where $i = j$, is $m\{\tan j \sec j + \log(\tan j + \sec j)\}$. Subtract

$$\text{Arc } AB = m\left\{\tan i \sec i - \tan j \sec j + \log \frac{\tan i + \sec i}{\tan j + \sec j}\right\}. \quad (12.)$$

If the origin is at A, with rectangular coordinates, $BQ = y_1 = y\cos j$; $AQ = x_1 = x + y\sin j$, which values may be substituted in (13.), p. 190,

$$s = x_1 + \frac{2}{3}\frac{y_1^2}{x_1} + = x + y\sin j + \frac{2}{3}\cdot\frac{y^2\cos^2 j}{x + y\sin j} + . \quad (13.)$$

§ 127. Flexible Tie. Backstay of Suspension Bridge.



Continue tangent AF to G. Draw DH perpendicular to AC. As the tangent at D is parallel to AC, and weight on cable below D is $P + \frac{1}{2}W$, make be equal to $P + \frac{1}{2}W$ and draw eo parallel to AC, cutting off $bo = H$. Then

$$H : P + \tfrac{1}{2}W = ob : be = AB : BC; \therefore H = (P + \tfrac{1}{2}W)\frac{AB}{BC}.$$

As H must be constant, fo will be parallel to the tangent at A, or AFG, and co parallel to the tangent at C, or CF; hence fo, co and eo = R_a, R_c and R_d respectively. Hence (1.).

$$DE = \tfrac{1}{2}EF;\ EF = \tfrac{1}{2}CG;\ \therefore DE = \tfrac{1}{4}CG. \quad CG : BC = ef : be$$

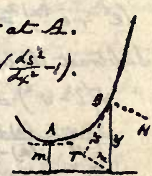
$$= \tfrac{1}{2}W : P + \tfrac{1}{2}W; \quad \therefore \quad CG = BC\frac{\frac{1}{2}W}{P + \frac{1}{2}W};\ DE = \frac{1}{8}BC\frac{W}{P + \frac{1}{2}W} \quad (1.)$$

By (15.) p. 190, $2s_1 = a + \frac{8}{3}\frac{y_1^2}{a}$. $AC = a$; $DH = y_1$. By similar triangles ABC and EDH, $DH = \frac{AB \cdot DE}{AC}$.

$$\overline{ADC} - \overline{AEC} = \frac{8}{3}\frac{y_1^2}{a} = \frac{8}{3}\,\frac{AB^2 \cdot DE^2}{AC^3}.$$

Substitute value of DE and obtain (2.).

§ 128. The Catenary.— [See Weisbach's Mechanics, p. 299]



Weight on $AB = P = ws$. Take origin at present at A.

By (2.), p. 187, $\frac{dy}{dx} = \frac{P}{H} = \frac{ws}{H}$. $\frac{dy}{dx} = \frac{\sqrt{(ds^2 - dx^2)}}{dx} = \sqrt{\left(\frac{ds^2}{dx^2} - 1\right)}$.

Let $H = wm$, to eliminate w. Then

$$\frac{ds^2}{dx^2} - 1 = \frac{s^2}{m^2};\quad \frac{ds}{dx} = \sqrt{\left(\frac{s^2 + m^2}{m^2}\right)};\ dx = \frac{m\,ds}{\sqrt{(s^2 + m^2)}}$$

See integration for parabola, § 125.

$$x = \int_0^s \frac{m\,ds}{\sqrt{(s^2 + m^2)}} = m\log(\sqrt{s^2 + m^2} + s) - m\log m = m\log\left(\frac{s}{m} + \sqrt{\left(1 + \frac{s^2}{m^2}\right)}\right).$$

From this expression find s in terms of x and m. Thus

$$e^{\frac{x}{m}} = \frac{s}{m} + \sqrt{\left(1+\frac{s^2}{m^2}\right)}; \quad \left(e^{\frac{x}{m}} - \frac{s}{m}\right)^2 = 1 + \frac{s^2}{m^2}; \quad e^{\frac{2x}{m}} - 2\frac{s}{m}e^{\frac{x}{m}} + \frac{s^2}{m^2} = 1 + \frac{s^2}{m^2};$$

$$2\frac{s}{m}e^{\frac{x}{m}} = e^{\frac{2x}{m}} - 1; \quad 2\frac{s}{m} = e^{\frac{x}{m}} - e^{-\frac{x}{m}}; \quad s = \frac{m}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right). \quad (1.)$$

Again: $\tan i = \frac{dy}{dx} = \frac{s}{m} = \frac{1}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)$. (1.) Integrate between x & 0:

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) - \frac{m}{2}\cdot 2.$$

If the origin is removed to O, a distance m, $y_1 = y + m$; ∴

$$y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right). \quad (1.)$$

By noting the difference between the values of s and y, we see that

$$y^2 - s^2 = m^2, \text{ or } s = \sqrt{(y^2 - m^2)}. \quad (1.)$$

The values for $\tan i$ and s explain construction of triangle BXT.

From above, $x = m\log\left(\frac{s}{m} + \sqrt{\left(1+\frac{s^2}{m^2}\right)}\right)$; and $s = \sqrt{(y^2 - m^2)}$; ∴

$$x = m\log\left(\frac{\sqrt{y^2-m^2}}{m} + \sqrt{1+\frac{y^2-m^2}{m^2}}\right) = m\log\left(\sqrt{\frac{y^2}{m^2}-1} + \frac{y}{m}\right). \quad (1.)$$

Area $AOXB = \int_0^x y\,dx = \int_0^x \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right)dx = \frac{m^2}{2}\left(e^{\frac{x}{m}} - e^{-\frac{x}{m}}\right)$.

The radius of curvature, $\rho = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}} \div \frac{d^2y}{dx^2}$. $\frac{dy}{dx} = \frac{s}{m} = \sqrt{\frac{y^2}{m^2} - 1}$.

By Olney, p. 52, §104, or Courtenay, pp. 65 and 412,

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{dy}{dx}\cdot\frac{d\frac{dy}{dx}}{dy} = \sqrt{\left(\frac{y^2}{m^2}-1\right)}\frac{\frac{1}{2}\cdot\frac{2y}{m^2}}{\sqrt{\left(\frac{y^2}{m^2}-1\right)}} = \frac{y}{m^2}; \quad ∴$$

$$\rho = \left(1 + \frac{y^2}{m^2} - 1\right)^{3/2} \div \frac{y}{m^2} = \frac{y^3}{m^3} \div \frac{y}{m^2} = \frac{y^2}{m}. \quad \text{Substitute value of } y.$$

The triangles BNX and BXT are similar; ∴ $\rho : y = y : m$; or $\rho = \frac{y^2}{m}$.

IV. (3.) and (4.) may be obtained as follows:—

Let $x_1 + x_2 = h$; $x_1 - x_2 = k$. Then $x_1 = \frac{1}{2}(h+k)$; $x_2 = \frac{1}{2}(h-k)$. From (1.)

$$l = s_1 - s_2 = \frac{m}{2}\left(e^{\frac{h+k}{2m}} - e^{-\frac{h+k}{2m}}\right) - \frac{m}{2}\left(e^{\frac{h-k}{2m}} - e^{-\frac{h-k}{2m}}\right) = \frac{m}{2}\left(e^{\frac{h}{2m}}e^{\frac{k}{2m}} - e^{-\frac{h}{2m}}e^{-\frac{k}{2m}} - e^{\frac{h}{2m}}e^{-\frac{k}{2m}}\right.$$

$$\left. + e^{-\frac{h}{2m}}e^{\frac{k}{2m}}\right) = \frac{m}{2}\left(e^{\frac{h}{2m}} + e^{-\frac{h}{2m}}\right)\left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right) \quad (a.)$$

$$v = y_1 - y_2 = \frac{m}{2}\left(e^{\frac{h+k}{2m}} + e^{-\frac{h+k}{2m}}\right) - \frac{m}{2}\left(e^{\frac{h-k}{2m}} + e^{-\frac{h-k}{2m}}\right) = \frac{m}{2}\left(e^{\frac{h}{2m}} - e^{-\frac{h}{2m}}\right)\left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right) (b)$$

$$l^2 - v^2 = \frac{m^2}{4}\left\{\left(e^{\frac{h}{2m}} + e^{-\frac{h}{2m}}\right)^2 - \left(e^{\frac{h}{2m}} - e^{-\frac{h}{2m}}\right)^2\right\}\left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right)^2. \quad \text{But}$$

$$\left(e^{\frac{h}{2m}} + e^{-\frac{h}{2m}}\right)^2 = e^{\frac{h}{m}} + 2 + e^{-\frac{h}{m}}; \quad \left(e^{\frac{h}{2m}} - e^{-\frac{h}{2m}}\right)^2 = e^{\frac{h}{m}} - 2 + e^{-\frac{h}{m}}.$$

Subtract the latter from the former and the remainder is 4; ∴

$$\sqrt{(l^2 - v^2)} = m\left(e^{\frac{k}{2m}} - e^{-\frac{k}{2m}}\right). \quad (3.)$$

$$\frac{l+v}{l-v} = \frac{(a)+(b)}{(a)-(b)} = \frac{2e^{\frac{h}{2m}}}{2e^{-\frac{h}{2m}}} = e^{\frac{h}{m}}; \quad ∴ h = m\log\frac{l+v}{l-v}.$$

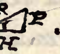
$$x_1 = \tfrac{1}{2}(h+k) = \tfrac{1}{2}\left(m\log\frac{l+v}{l-v} + k\right). \quad (4.)$$

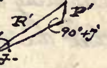
By Courtenay, p. 58, or Olney, p. 72, §132, $e^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \&c.$ ∴

Catenary, $y = \frac{m}{2}\left(e^{\frac{x}{m}} + e^{-\frac{x}{m}}\right) = \frac{m}{2}\left\{\begin{matrix} 1 + \frac{x}{m} + \frac{x^2}{2m^2} + \text{etc} \\ 1 - \frac{x}{m} + \frac{x^2}{2m^2} - \text{etc} \end{matrix}\right\} = m\left(1 + \frac{x^2}{2m^2} + \frac{x^4}{24m^4} + \text{etc}\right)$.

Parabola, $y = \frac{x^2}{4m_1} = \frac{x^2}{2m} = m\frac{x^2}{2m^2}$. $y_1 = y + m = m\left(1 + \frac{x^2}{2m^2}\right)$.

The remaining values are similarly obtained.

§ 130. Transformation of Frames. — The original coordinates are rectangular, and P is vertical,  $\therefore R = \sqrt{(P^2 + H^2)}$. (3.)

Next, the load P and the vertical coordinates  being unchanged, make $x' = ax$, when $H' = aH$ for equilibrium; and, if x' is inclined at angle j to horizon, we get. $R' = \sqrt{(P'^2 + H'^2 \pm 2P'H' \sin j)}$; (5.) and

$P' : H' : R'$ = sines of opposite angles. (6.)

The intensity of P on plane x' will not be p, as $x' = ax$, $\therefore p' = \frac{p}{a}$ (7.)

§ 131. Transformed Catenary. — The values for p and w OABX are obtained by substitution from p. 196. Then

$$\frac{dy}{dx} = \frac{P}{H} = \frac{s}{m}; \; H = \frac{m}{s}P = \frac{m}{s}(ms)w = wm^2. \;(1.)$$

$$\sqrt{(P^2 + H^2)} = \sqrt{(w^2m^2s^2 + w^2m^4)} = wm\sqrt{(s^2 + m^2)} = wmy. \;(1.)$$

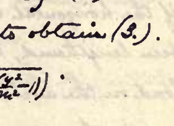
By (2.) $\frac{y'}{y} = \frac{y_0}{m}$. Multiply (1.) by this equation to obtain (3.).

By (1.) p. 196, $\frac{x}{m} = \log\left(\frac{y}{m} + \sqrt{\left(\frac{y^2}{m^2} - 1\right)}\right)$ or $m = \frac{x}{\log\left(\frac{y}{m} + \sqrt{\left(\frac{y^2}{m^2} - 1\right)}\right)}$.

By (2.) $\frac{y}{m} = \frac{y'}{y_0}$; $\therefore$ (4.).

Graphical constructions for the catenary and transformed catenary will be found in Greene's Graphics, Part III., Arches.

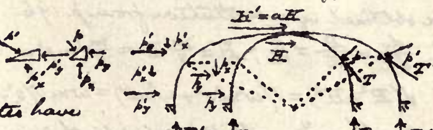
§ 133. Circular Rib. — By § 150, $T = pr$.



By § 108, if $p_x = p_y$, the pressure on any plane will be normal to that plane, and of the same intensity; $\therefore p = p_x = p_y$. As T is constant, the reaction at the abutment is vertical and equals $pr = P$; the thrust at the crown is $H = pr$; and the horizontal component of the thrust in the rib at any point diminishes from H at the crown to zero at the springing points, being neutralized by the successive applications of the normal intensities p. It is well to remember that the horizontal thrust of an arch is constant only so long as the external forces are vertical. The stress diagram in the sketch may also show these points.

§ 134. Elliptic Rib.— Imagine that the semi-circle of the previous section is changed to a semi-ellipse by lengthening or shortening one set of rectangular coordinates, and that the external forces (not intensities) are changed in the same proportion. Then, by the principle of parallel projection, or transformation, § 130, the new rib will be in equilibrium under the new set of forces.

If the circular rib, its load and its reactions are drawn on a sheet of rubber, and the sheet is evenly stretched, parallel to one or the other of the coordinate axes, the same effect will be produced.



It will then appear, if the horizontal coordinates have been lengthened, that the total vertical load on the arch, being parallel to the unchanged coordinates, remains the same, and the reaction, still vertical, is

$$P' = P = pr = p_x r. \qquad (2.)$$

As P' is equal to the load on the half arch, the vertical intensity of load must be less, since it is distributed over a greater area, and hence $p'_x = \frac{pr}{ar} = \frac{p}{a} = \frac{p_x}{a}$. (4.) The thrust at the crown of the new rib must have the relation

$$H' = aH = apr = ap_y r; \quad (2.)$$

and, since this horizontal thrust is the sum of the uniform horizontal intensities p'_y on the same plane of height r, we have

$$p'_y = \frac{aH}{r} = ap = ap_y. \quad (4.)$$

The new intensities are therefore changed, and, in place of

$p_x = p_y = p$, we have $p'_x = \frac{p}{a}$; $p'_y = ap$; (4.) Whence

$$\frac{p'_y}{p'_x} = a^2, \text{ or } a = \sqrt{\left(\frac{p'_y}{p'_x}\right)}. \quad (5.)$$

The intensity of pressure on any element, ds, of the elliptic rib will not be normal, except at crown and springing, but will point towards the centre, and will vary in magnitude from point to point. The thrust T' at any point in the rib, being transformed from T in the circular rib which was tangential or perpendicular to the radius, will still be tangential. If two radii at right angles in the circle are projected in the ellipse, they will become conjugate semi-diameters, and the thrusts T' at their extremities will be proportional to the conjugate semi-diameters to which they are parallel. ∴ (3.) The intensities at the extremities of such diameters, being found by dividing the total pressure on a quadrant or projected quadrant by the area of the plane on on which it acts, will follow (6.) and (7.).

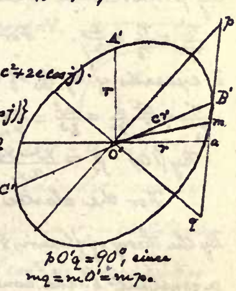
§ 135. Distorted Elliptic Rib.— If the sheet of rubber, referred to before, is stretched obliquely, we get the present case. Oblique coordinates take the place of rectangular ones.

Rankine's construction of axis. $A+B = pq = 2\,O'm$; $A-B = B'a$; $\angle B'O'a = j$.

$B'a = \sqrt{(r^2+c^2r^2-2cr\cdot r\cos j)} = r\sqrt{(1+c^2-2c\cos j)}$.

By Davies' Legendre, B. IV, Prop. XIV, or Olney's Elem. of Geom., §673, p. 262,

in $\Delta\, B'O'a$, $O'a^2 + O'B'^2 = 2\,O'm^2 + 2(\tfrac{1}{2}B'a)^2$; ∴



$2\,O'm^2 = r^2 + c^2r^2 - 2\,\dfrac{r^2+c^2r^2-2cr^2\cos j}{4}$; $2\,O'm = r\sqrt{(1+c^2+2c\cos j)}$.

$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B) = \tfrac{r}{2}\{\sqrt{(1+c^2+2c\cos j)} + \sqrt{(1+c^2-2c\cos j)}\}$

$B = \quad " \quad - \quad " \quad = \tfrac{r}{2}\{\sqrt{\quad "} \quad - \sqrt{\quad "} \quad \}$

Olney's Gen. Geom., p 86, Ex. 10. $A' = \dfrac{AB}{\sqrt{(B^2\cos^2\alpha + A^2\sin^2\alpha)}}$

$c^2r^2 = \dfrac{A^2B^2}{B^2(1-\sin^2\alpha)+A^2\sin^2\alpha}$ ∴ $\sin\alpha = \sin B'O'p =$

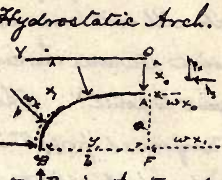
$\sqrt{\left(\dfrac{\frac{A^2B^2}{c^2r^2} - B^2}{A^2-B^2}\right)} = \dfrac{B}{cr}\sqrt{\left(\dfrac{A^2-c^2r^2}{A^2-B^2}\right)}$ (6.)

$pO'q = 90°$, since $mq = mO' = mp$.

By Olney's Gen. Geom., §318, e, p. 194, and a, p. 193, $A'B'\sin(\alpha'-\alpha) = AB$, and $A'^2+B'^2 = A^2+B^2$. (1.) $A'O'p = \alpha$ and $C'O'p = \alpha'$; ∴ $C'O'A' = \alpha'-\alpha$; $\sin(\alpha'-\alpha) = \sin A'O'B' = \sin(90°-j) = \cos j$; ∴ $A'B'\cos j = AB$. Double this equation, add to and subtract from (1.). $\sqrt{(A'^2+B'^2+2A'B'\cos j)} = A+B$; $A' = r$; $B' = cr$.

$\sqrt{(A'^2+B'^2-2A'B'\cos j)} = A-B$. Half sum and half diff.^{e} give (5.)

§ 136. Rib under Normal Pressure. Hydrostatic Arch.



The thrust in the rib must be constant; since the external pressure (fluid) is normal at all points, and a normal force cannot change a tangential one, as it has no component in the tangent. Substitute the plane AF for the right half of the arch: FAB will be in equilibrium under fluid pressure on its exterior and the upward reaction P_1 at B. Total horizontal pressure on AF is

$w\cdot\frac{1}{2}(x_1+x_0)(x_1-x_0) = w\frac{x_1^2-x_0^2}{2}$. The total hor. pressure on the half arch must balance it; and, as thrust in rib is constant,

$$H = \int_{x_0}^{x_1} p\,dx = \int_{x_0}^{x_1} wx\,dx = w\frac{x_1^2-x_0^2}{2} = T = P_1. \quad (4.)\quad (7.)$$

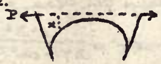
The osculating circle may be substituted at any point, since the pressure is normal, ∴ $T = wx\rho = wx_0\rho_0 = wx_1\rho_1$. (5.) (3.)

Problem. p. 211. — A hydrostatic arch approaches nearly a semi-elliptic arch of the same height, and having its max and min. radii of curvature in the same proportion. So draw an elliptic arch having the same rise $\underline{a}$, and r_0' and r_1' at the crown and springing proportioned to each other as those of the hydrostatic arch, i.e. $\frac{r_1'}{r_0'} = \frac{r_1}{r_0} = \frac{x_0}{x_1}$. For the ellipse, $r_0' = \frac{b^2}{a}$; $r_1' = \frac{a^2}{b}$. The semi-axis b exceeds y_1 by $\frac{1}{30}$ of radius of curvature at A. So $b = y_1 + \frac{y_1^2}{30a}$. (11.)

$$\frac{r_1'}{r_0'} = \frac{a^2}{b}\cdot\frac{a}{b^2} = \frac{a^3}{b^3} = \frac{x_0}{x_1} = \frac{x_0}{x_0+a};\quad x_0 = \frac{a^3(x_0+a)}{b^3} = \frac{a^4}{b^3-a^3} = a\frac{a^3}{b^3-a^3}. \quad (11.)$$

By (10.) $\rho_1 = \frac{x_1^2-x_0^2}{2x_1} = \frac{(x_0+a)^2-x_0^2}{2(x_0+a)} = \frac{2ax_0+a^2}{2(x_0+a)} = \frac{2(ax_0+a^2)-a^2}{2(x_0+a)} = a - \frac{a^2}{2(x_0+a)}$.

For the elastic curve, the bending moment $M = Px$.



By the theory of flexure, § 169, $\frac{1}{r} = \frac{M}{EI} = \frac{Px}{EI}$. ∴ $xr = \frac{EI}{P}$, a constant by the conditions of the case. (3.) is satisfied.

§ 137. Geostatic Arch. This arch is derived from the preceding, as the elliptic was derived from the circular arch.

$p_x' = \frac{p_x}{c}$; $p_y' = cp_y$; $dx' = dx$; $dy' = c\,dy$. $P_x' = \int p_x'\,dy' = P_x$.

$P_y' = \int p_y'\,dx' = cP_y$. The load or pressure at crown and springing being normal (it is not so at other points), let ρ_0 and ρ_1' be radii of geostatic curve at crown and springing. Then $H' = p_y' \rho_0' = \frac{p_y}{c}\rho_0'$. In the hydrostatic arch $H = p_y \rho_0$ ∴ $H' = cH = cp_y\rho_0$. ∴

$$\frac{p_y}{c}\rho_0' = cp_y\rho_0, \quad \text{or} \quad \rho_0' = c^2\rho_0.$$

So $P_1' = p_x'\rho_1' = cp_x\rho_1' = P_1$. But $P_1 = p_x\rho_1$ ∴ $p_x\rho_1 = cp_x\rho_1'$ ∴ $\rho_1' = \frac{\rho_1}{c}$

§ 138. *Linear Rib of any Figure.*— If, under the given loads or vertical external forces, the thrust in the rib does not coincide with the tangent, horizontal forces must be supplied. Such forces may come from the backing of a masonry arch. The geometrical construction explains the equations (1.) to (4.) (5.) is an expression for the point of application of the resultant of parallel forces. $P_1 \cot i_1$ is the horizontal thrust at springing.

This section may be useful also in finding the ring tension (or compression) in a thin dome.

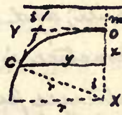
Ex. 1, p. 216. $p_y = 0$; H is constant.

Ex. 2, p. 217. $x_1 = r$; $H_1 = P_1 \cot i_1 = 0$; $x_H = \frac{\frac{1}{2}px_1^2}{pr} = \frac{pr^2}{2pr} = \frac{1}{2}r$. (7.)

Ex. 3, " $p_y = wx$. See (7.) p. 210.

Ex. 4 " Change coordinates from x and y to i and r.

$x = r(1-\cos i)$; $dx = r\sin i\,di$. $y = r\sin i$; $dy = r\cos i\,di$.



The vertical load between O and C, $P_x = \int p_x dy$

$= w\int(mr+x)dy = wr^2\int(m+1-\cos i)\cos i\,di$

$= wr^2[(1+m)\sin i + \frac{1}{2}(\cos i \sin i) - \frac{1}{2}i]$*

Introduce this value in (4.)

$$p_y = -\frac{d(P_x \cot i)}{dx} = \frac{1}{r\sin i}\cdot\frac{d(P_x\cot i)}{di}$$

$$= -\frac{wr}{\sin i}\cdot\frac{d}{di}\left[(1+m)\cos i - \tfrac{1}{2}\cos^2 i - \frac{i\cos i}{2\sin i}\right]$$

$$= wr\left(1+m-\cos i - \frac{i-\cos i\sin i}{2\sin^3 i}\right) \quad (9.)$$

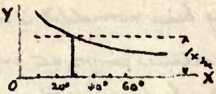
From (1.), $H_0 = P_x \cot i_0 = wr^2[(1+m)\cos i_0 - \frac{\cos^2 i_0}{2} - \frac{i_0 \cot i_0}{2}]$ (11.). $x_H = \frac{\int_{x_0}^{x_1} x p_y dx}{H_0}$

$$= \frac{r^3}{H_0}\int_{i_0}^{90°} p_y \sin i(1-\cos i)\,di. \quad (12.)$$

*$\int u\,dv = uv - \int v\,du$. $\cos i\,di = dv$; $\cos i = u$; $\sin i = v$; $-\sin i\,di = du$. $\int\cos^2 i\,di = \cos i\sin i + \int\sin^2 i\,di = \cos i \sin i + \int(1-\cos^2 i)\,di = \cos i\sin i + i - \int\cos^2 i\,di$. ∴ $2\int\cos^2 i\,di = \cos i\sin i + i$.

$T_{\text{at crown}} = mwr^2$.

Graphical solution of (10) $p_y = 0$.



$1 + m = \cos i_0 + \frac{i_0 - \cos i_0 \sin i_0}{2 \sin^2 i_0}$. Assume four or five values of i, say 20°, 30° .... 60°. Compute the respective values of the second member, plot on rectangular coordinates and draw curve. Plot $1+m$ and find intersection.

Ex. 5, p. 218. $H_0 = H_1$; $x = r(1 - \cos i)$; $dx = r \sin i \, di$. See (5) & (12.)

Section V.

Strength of Materials.

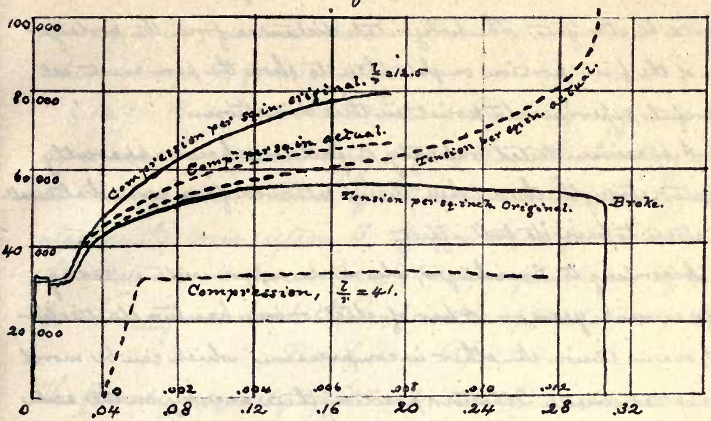
§ 142. For proof strength use the term elastic limit which is explained later.

The safe load which a structure will bear depends not only on the proportion which the safe load has to the breaking load, but on the number of times the load is to be applied and released and also upon the suddenness of the application.

§ 143. The term factor of safety has fallen into disrepute, and is not used by the best authorities. The proper practice is to definitely specify the allowable working stresses per square inch of section, having regard to the kind and range of stress and the frequency and suddenness of action of the applied forces.

§ 144. Testing Machines. See Van Nostrand's Science Series, No. 74. The test load, for a piece which is afterwards to be used, should not exceed the elastic limit, and is generally limited to double the allowed working stress.

Some testing machines have automatic recording attachments which give graphical representations of the stresses and accompany-



ing strains. Such diagrams are illustrated by the one above, where vertical ordinates measure stress in pounds, and horizontal distances show elongation or shortening in ~~decimals~~ percentages of an inch. The specimens represented above were of ductile wrought iron. The dotted curves are corrected from the full curves, for the change in cross-section. The lowest diagram, with more exaggerated horizontal scale, belongs to a slender column, as will be explained later.

The elastic limit is the point (and hence the stress) where curvature in the lines more or less sensibly begins, or where strain ceases to be proportional to stress. [Elastic limit marks that load at which either permanent set can first be detected, or where increments of stress and strain cease to be proportional to each other, the earliest certain and continued indication that way being taken to mark that limit. Yield point marks that load at which rapid and considerable yield first takes place under a steady load. Yield point is the point called elastic limit in ordinary testing done without very accurate means for measuring small extensions. The original elastic limit is far below yield point in many specimens. C. A. Marshall.]

If there were no permanent set, it is probable that the strain would continue to be proportional to the stress, or that these lines would be straight. The horizontal distances from the prolongation of the first portion ought then to show the permanent set. Careful experiments point in this direction.

A specimen tested rapidly to failure shows apparently greater strength than when time is allowed for a somewhat smaller stress to have its full effect.

According to Bauschinger, who made experiments extending over several years,— A bar of steel or iron has two elastic limits, one in tension the other in compression, which can be moved by applied stresses. Whatever position these occupy on the scale of loads, the range between them is nearly a constant quantity. By alternately stressing a bar in tension and compression just beyond the elastic limits, these after a certain number of repetitions occupied positions equally distant from the point of zero load, and the limits thus obtained are called by Bauschinger the natural elastic limits of the bar. It was then noted that the stress corresponding to these limits sensibly coincided with that found by Wohler as the limiting stress to which a bar could be subjected to alternate tension and compression. It would thus appear that a bar will bear an indefinite number of repetitions of stress, provided the range of stress does not exceed the elastic range mentioned above.

§ 146. Modulus of Elasticity.— Let P = force applied in pounds to the ends of a bar whose cross-section = S sq. inches; let l = length in inches between two points where λ, the elongation or shortening is measured. Then the modulus of elasticity

$$E = \frac{\text{intensity of stress}}{\text{strain}} = \frac{P}{S}\cdot\frac{l}{\lambda}. \qquad \text{Hence}$$

$\lambda = \dfrac{Pl}{ES}$, = change of length in a distance l; a more convenient formula than (2.) p. 227. E is the same for tension and compression. It is the tangent of the angle at the origin in previous sketch. E enters into all problems where change of form is involved.

§ 149. Bars under Tension. — In designing it is better to use the safe working stress than the breaking stress, as Rankine does. Then, if the tension is uniformly distributed on the cross-section S; if P is the safe load and f the maximum safe working stress in pounds per sq. inch, for the way in which the load is to be applied (see p. 18)

$$P = fS. \quad (1.)$$

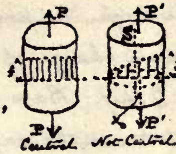
This formula is equally applicable to a short strut or column centrally loaded.

If the stress is not uniformly distributed on cross-section S, the variation may be due to lack of uniformity in the material, when those fibres which offer the most resistance to stretching or shortening will be subject to the greatest stress. To prevent injurious stress in such a case, f must be taken at a smaller value, in determining P or S by the above formula.

If the variation is due to the fact that the point of application of P does not coincide with the centre of figure of S, the load P′ that can be imposed without causing a stress greater than f on the most stressed fibre or particle is less than P, and depends upon the distance of the point of application from the centre of S.

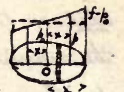
As, for stresses within the elastic limit, strain is proportional to stress, a uniformly varying stress is always assumed in the direction of deviation of load. Divide the stress on each particle into two parts, the mean stress and the variable part.

$P'x_0$ = moment of P′, to be balanced by moments of the variable portion of stress about the axis of the piece.



Let p = intensity of stress at distance x from axis of piece, measured in the direction of x_0 and x_1, which latter is distance to edge where stress is f.

p_0 = mean intensity of stress, found at centre.



$p - p_0 = p'$ represents the variable portion. Then

$P' = p_0 S$; $p' : f - p_0 = x : x_1$ or $p' = \frac{f-p_0}{x_1} x$.

If z = variable width of section perpendicular to x, the moment of the variable portion of stress about the axis O is

$$M = \int_{-x_1}^{x_1} p' x z\,dx = \frac{f-p_0}{x_1}\int_{-x_1}^{x_1} x^2 z\,dx = \frac{f-p_0}{x_1} I$$

where I represents the moment of inertia of the cross-section.

But $p_0 = \frac{P'}{S}$; $\therefore M = \left(f - \frac{P'}{S}\right)\frac{I}{x_1} = P' x_0$ as above. $\therefore$

$$P' = \frac{fS}{1 + \frac{x_0 x_1 S}{I}} = \frac{fS}{1 + \frac{x_0 x_1}{r^2}} \qquad (2.)$$

Also $f = \frac{P'}{S}\left(1 + \frac{x_0 x_1}{r^2}\right)$, which gives the max. stress under a given load when x_0 is known.

Illustrated by a bar subjected to a force not centrally applied, or to a force and a moment due to lack of homogeneity. In either case the bar is bent, and is weaker than when $P = fS$, in the ratio of p_0 to f.

As the above demonstration applies also to compression, compare (2.) p. 233, and (5.) and (7.) p. 163.

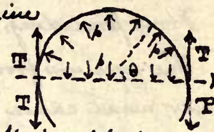
The deviation x_0, although comparatively small, will have a decided effect in increasing f for a given P', or *vice versa*; and thus may be explained some considerable variations in the strength of apparently similar pieces. For this reason, among others, working stresses are reduced below what otherwise might be used.

$$E = \frac{\text{intensity of Stress}}{\text{Strain of unit length}} = \frac{P}{S}\cdot\frac{x}{\Delta x}. \qquad (2.) \quad \frac{P}{S} = f'.$$

$$\text{Work} = \tfrac{1}{2} P \cdot \Delta x = \frac{f'S}{2}\cdot\frac{f'x}{E}. \qquad (3)$$

Work done is simply — Mean force, $\tfrac{1}{2}P$, multiplied by strain of whole bar or total length through which force moves.

§ 150. Boilers and Pipes. — Imagine the cylinder to be cut by a diametral plane, and the equilibrium of this



half cylinder to be considered. It is evident that, for unity of distance along the cylinder, the total pressure on the diameter, $2pr$, must balance the pressure on the semicircumference in a direction perpendicular to the diameter. But the pressure $2pr$ must cause a tension T in the material at each end; and hence

$$T = pr.$$

As all points of the circle are similarly situated, the tension in the ring at all points is pr; and, if t is the thickness, and f the safe working intensity of stress, assumed to be uniform, we have

$ft = pr$; or required net thickness $t = \frac{pr}{f}$. (1.)

In a boiler or similar cylinder additional allowance must be made for rivets, as will be seen later.

Another proof: — Small force on arc $ds = p\,ds$; Vertical component $= p\,ds \sin\theta = p\,dx$. $\int_{-r}^{+r} p\,dx = 2pr$ to be resisted by stress in material at ends of diameter.

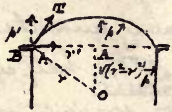
This investigation applies only to thin cylinders.

§ 151. Spherical Shells. — The pressure on a right section $= \pi r'^2 p$. This pressure causes tension in the direction of the axis of the cylinder, and, dividing by the circumference × thickness, $= 2\pi r' t$, we have stress per sq. inch $= \frac{pr'}{2t}$, which is only one-half the amount of f in § 150, for same r. Hence a boiler is twice as strong against rupture circumferentially as longitudinally. Hence, also, the longitudinal seams are often double riveted.

See also statement in text.

From preceding, stress on flange bolts $= \pi r'^2 p$. (1.)

The pressure from below on the dome or cover must cause a pull BT along the tangent to dome at the flange. This



pull has a vertical component, as in previous paragraph, $p' = \frac{1}{2} p r'$, and a horizontal component, acting inwardly, of intensity which we will call h. But

$$h : p' = OA : AB, \quad \text{or} \quad h = \tfrac{1}{2} p r' \frac{\sqrt{r^2 - r'^2}}{r'}.$$

As h is a normal stress, like p of § 150, but of opposite sign, it causes a compression in the flange at all sections of $h r' = \frac{1}{2} p r' \sqrt{r^2 - r'^2}$. (2.)

§ 152. Thick Hollow Cylinder.— If the walls of a hollow cylinder or sphere are comparatively thick, it will not be sufficiently accurate to assume that the stress in any section is uniformly distributed throughout it. If the material were perfectly rigid, the external or internal pressure would be resisted by the immediate layer against which the pressure was exerted, and the remainder of the material would be useless. As, however, the substance of which the wall is composed yields under the force applied, the pressure is transmitted through the material at a decreasing rate, each layer resisting or neutralizing a portion of the normal pressure which extends or compresses it.

More intense pressure on inside:— Let the sketch represent the right section of a cylinder, such as that of an hydraulic press. Let r_1 and r_2 be the internal and external radii in inches; p_1 and p_2 the internal and external intensities of normal pressure in pounds per sq. inch, p_1 be-

ing the greater; and p the normal intensity of pressure on any ring whose radius is r.

The intensity of tension caused in a thin layer of radius r and thickness dr will be denoted by t, and will be due to that portion of p which is resisted by the layer and not transmitted to the next exterior layer.

As the stress in a thin ring of radius r, under any intensity of pressure p, is pr, the entire tension on a section from r_1 to r must be $p_1r_1 - pr$, and may also be expressed by $\int_r^{r_1} t\,dr$. Differentiate this equality $p_1r_1 - pr = \int_r^{r_1} t\,dr$, obtaining

$$-d(pr) = t\,dr, \text{ or } \quad p\,dr + r\,dp + t\,dr = 0. \quad (1.)$$

The fibres or layers between r_1 and r, being compressed, will be diminished in thickness. The compression of a piece a unit in thickness by a force of intensity p will be $\frac{p}{E}$, and of one dr thick will be $\frac{p\,dr}{E}$. The total diminution of of thickness between r_1 and r, from what it was at first, will therefore be $\frac{1}{E}\int_r^{r_1} p\,dr$.

But the annular fibre or ring whose radius is r and length $2\pi r$ has been elongated $\frac{t}{E}$ per inch of length. Its length will now be $2\pi r(1+\frac{t}{E})$ and its radius $r(1+\frac{t}{E})$. The internal radius must similarly have become $r_1(1+\frac{f}{E})$, where f is the value of t for radius r_1. The thickness $r-r_1$ has now become $r(1+\frac{t}{E}) - r_1(1+\frac{f}{E})$, and, by subtracting this value from $r-r_1$ we get diminution of thickness, to equate with previous expression,

$$r_1\frac{f}{E} - r\frac{t}{E} = \frac{1}{E}\int_r^{r_1} p\,dr. \qquad \text{Differentiating we get}$$

$$-d(tr) = p\,dr, \quad \text{or} \quad t\,dr + r\,dt + p\,dr = 0. \quad (2.)$$

Add (1.) and (2.), and multiply by r

$$2(t+p)r\,dr + r^2(dt+dp) = 0. \qquad \text{Integrate—}$$

$$r^2(t+p) = \text{Constant}, \therefore = r_1^2(f+p_1) = r_2^2(f'+p_2). \quad (3.)$$

Subtract (1.) from (2.); $dt - dp = 0$. Integrate —

$$t - p = \text{Constant}, \therefore = f - p_1 = f' - p_2. \quad (4.)$$

From (3.) and (4.) by addition and subtraction,

$$t = \frac{f-p_1}{2} + \frac{r_1^2}{r^2}\cdot\frac{f+p_1}{2}\ ; \quad p = -\frac{f-p_1}{2} + \frac{r_1^2}{r^2}\cdot\frac{f+p_1}{2}. \quad (5.)$$

If the internal radius is given, the external radius may be found by eliminating f' from (3.) and (4.), or

$$r_2 = r_1\sqrt{\left(\frac{f+p_1}{f-p_1+2p_2}\right)} \quad (6.)$$

If p_2 is atmospheric pressure, it may be neglected when p_1 is large. $\therefore$ we get (2.) § 152.

As r_2 becomes infinite when the denominator of (6) is zero, no thickness will suffice to bring f within the safe stress if p_1 exceeds $f + 2p_2$. These formulae do not apply to bursting pressures, nor to those which bring f above the elastic limit, for E will not then be constant.

More intense pressure on outside:— In this case the direction or sign of p will be reversed, acting from without inwards. So, from the preceding equations, without independent analysis, and noting that t is now thrust, we have

$$r^2(t-p) = r_1^2(f-p_1) = r_2^2(f'-p_2) \text{ and } t+p = f+p_1 = f'+p_2.$$

Using the outer radius and pressure in place of the inner, we get

$$t = \frac{f'+p_2}{2} + \frac{r_2^2}{r^2}\cdot\frac{f'-p_2}{2}\ ; \quad p = \frac{f'+p_2}{2} - \frac{r_2^2}{r^2}\cdot\frac{f'-p_2}{2}. \quad (7.)$$

$$r_1 = r_2\sqrt{\left(\frac{f'-p_1}{f'-2p_2+p_1}\right)}. \quad (8.)$$

In this case, p_2, the external pressure, must be less than $\tfrac{1}{2}(f'+p_1)$ in order that the internal radius shall not be zero.

§ 153. Thick Hollow Sphere:— Greater pressure on inside.— Let the previous sketch represent a meridian section of a sphere. Suppose f, t, etc., to be perpendicular to the plane of the paper. The entire normal pressure on the circle of radius r_1 will be $p_1\pi r_1^2$, and the tension in the ring between radii r_1 and r will be $\pi(p_1 r_1^2 - pr^2)$.

Any ring of radius r and thickness dr will carry

$2\pi r t\,dr$, and hence is derived the first equation

$$\pi(p_1 r_1^2 - pr^2) = 2\pi\int_r^{r_1} r t\,dr, \text{ or } -d(pr^2) = 2rt\,dr. \therefore$$

$$r^2 dp + 2pr\,dr + 2rt\,dr = 0.$$

The second equation will be the same as obtained for the cylinder,

$$-d(tr) = p\,dr; \therefore r\,dt + t\,dr + p\,dr = 0.$$

Strike out the common factor r from first equation, multiply the second by 2, and subtract. $2r\,dt - r\,dp = 0$, or

$$2\,dt - dp = 0; \therefore 2t - p = \text{Constant} = 2f - p_1 = 2f' - p_2. \quad (9.)$$

Again, add together, multiply by r^2 and obtain

$$r^3(dp + dt) + 3r^2 dr(p + t) = 0; \therefore$$

$$r^3(p+t) = \text{Constant} = r_1^3(p_1 + f) = r_2^3(p_2 + f'). \quad (10.).$$ From (9.) & (10.)

$$t = \frac{2f - p_1}{3} + \frac{r_1^3}{r^3}\cdot\frac{f + p_1}{3}; \quad p = -\frac{2f - p_1}{3} + 2\frac{r_1^3}{r^3}\cdot\frac{f + p_1}{3}. \quad (11.)$$

$$r_2 = r_1\sqrt[3]{\left(\frac{2(f + p_1)}{2f - p_1 + 3p_2}\right)}. \quad (12.)$$

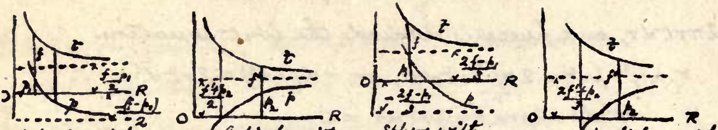
For a finite value of r_2, p_1 must be less than $2f + 3p_2$. If p_2 is atmospheric pressure it may be neglected and we get (2.) § 153. These formulae are not applicable to bursting pressures for the reason given before.

Greater pressure on outside:— Here again, reverse sign of p and obtain $t = \frac{2f' + p_2}{3} + \frac{r_2^3}{r^3}\cdot\frac{f' - p_2}{3}$; $p = \frac{2f' + p_2}{3} - 2\frac{r_2^3}{r^3}\cdot\frac{f' - p_2}{3}$.

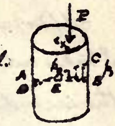
$$r_1 = r_2\sqrt[3]{\left(\frac{2f' + p_1 - 3p_2}{2(f' - p_1)}\right)}. \quad (13.)$$

That r_1 should be greater than zero, requires that $p_2 < \frac{1}{3}(2f' + p_1)$.

Curves may be drawn to represent the variation of p and t in the four preceding cases. They are all hyperbolic, and, if r is laid off from the centre O on the horizontal axis, each curve will have the vertical axis for one asymptote, and for the other a line parallel to the horizontal axis at the distance indicated by the first term in each value of t or p. The four accompanying sketches show the various curves; the values of f and f' which correspond to the given values of p_1 and p_2 are found at the extremities of the abscissae which represent r_1 and r_2.



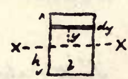
Cylinder with internal pressure. Cylinder with external pressure. Sphere with internal pressure. Sphere with external pressure.



§157. Resistance to Compression:– Load not Central. P is imposed at x_0 from the centre of section AB of the column. CB = p_1 = maximum intensity of compression. Ordinate at E, the middle, = p_0 the mean intensity. Total load P = volume of cylinder or prism of base AB and upper surface bounded by the plane through CD, the portion below plane AB being tension and negative. As $p_0 S = P$; as p_1 must not be more than the safe stress f; and as, if the load were central and uniformly distributed, the column might safely carry $p_1 S$;– it is weaker now in the ratio $\frac{p_0}{p_1}$ as stated in the text. See Notes, pp. 21, 22.

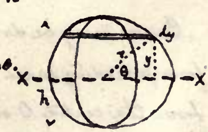
Notice that $p_1 = CB = p_0 + p_0 \frac{x_0 x_1 S}{I}$ (1.)

Values of $\frac{x_1 S}{I}$. h is to be taken in direction of x_0.



I. If the column is rectangular in section, I for axis thro' centre of gravity and parallel to b is, $I = \int_{-\frac{1}{2}h}^{+\frac{1}{2}h} b y^2 dy$
$= \Big/_{-\frac{1}{2}h}^{+\frac{1}{2}h} \frac{1}{3} b y^3 = \frac{bh^3}{24} + \frac{bh^3}{24} = \frac{1}{12} bh^3$. $\therefore \frac{x_1 S}{I} = \frac{1}{2}h \cdot bh \cdot \frac{12}{bh^3} = \frac{6}{h}$.

II. For a square section make $b = h$.



IV. For a circular section, $y = r\sin\theta$, $\frac{x}{2} = r\cos\theta$.

$I_x = \int_{-r}^{+r}\int_{-r\cos\theta}^{+r\cos\theta} y^2 dy\, dx$ or $\int_0^r \int_0^{2\pi} r^3 \sin^2\theta\, dr\, d\theta$.

$I_y = \iint r^3 \cos^2\theta\, dr\, d\theta$. $I_x + I_y = I_p = \int_0^r\int_0^{2\pi} r^3(\sin^2\theta + \cos^2\theta)\, dr\, d\theta$

$= \int_0^r \int_0^{2\pi} r^3 dr\, d\theta = \frac{2\pi r^4}{4}$. As $I_x = I_y$ $\therefore I_x = \frac{\pi r^4}{4} = \frac{\pi d^4}{64} = \frac{\pi h^4}{64}$;

$\frac{x_1 S}{I} = \frac{h}{2} \cdot \frac{\pi h^2}{4} \cdot \frac{64}{\pi h^4} = \frac{8}{h}$.

III. As the value of x in the ellipse is to x in the circle as b to h, and the moment of strip $x dy$ varies as the breadth alone, we have, for ellipse, $I = \frac{\pi h^4}{64} \cdot \frac{b}{h} = \frac{\pi b h^3}{64}$. $\frac{x_1 S}{I} = \frac{h}{2} \cdot \frac{\pi bh}{4} \cdot \frac{64}{\pi b h^3} = \frac{8}{h}$.

V. The moments of inertia of hollow symmetrical sections are found by taking I for the cavity from I for the entire section including the cavity; ∴

$$\frac{x_1 S}{I} = \frac{h}{2}(hb - h'b')\frac{12}{bh^3 - b'h'^3} = \frac{6h(hb - h'b')}{h^3 b - h'^3 b'}$$

VI. Make $b = h$; $6h\,\frac{h^2 - h'^2}{h^4 - h'^4} = \frac{6h}{h^2 + h'^2}$.

VII. $\frac{h}{2}\cdot\frac{\pi}{4}(h^2 - h'^2)\frac{64}{\pi(h^4 - h'^4)} = 8h\,\frac{h^2 - h'^2}{h^4 - h'^4} = \frac{8h}{h^2 + h'^2}$.

The sketch of the column on p. 28 shows AD, the <u>least</u> pressure, as tension. If $AD = 0$, $p_1 = 2p_0$ and by (1.) above, $x_0 = \frac{I}{x_1 S}$, or the reciprocal of the values above. Thus a deviation from the centre of 1/8 the diameter makes the intensity of pressure on the edge of a solid cylinder twice the mean intensity.

157, II. Crushing by Shearing.— The action of granular substances under compression. We have seen that, if $p = \frac{P}{S}$ is the intensity of thrust on the right section of a short column, on a plane making an angle θ with the right section the

Normal intensity $= p\cos^2\theta$; Tangl intensity $= p\sin\theta\cos\theta$.

If q = coefficient of frictional resistance of the material to sliding, the intensity of frictional <u>resistance to sliding</u> along the oblique plane will be $qp_n = qp\cos^2\theta$, and the shearing stress, tending to produce fracture along this plane will be $p_\theta = p(\cos\theta\sin\theta - q\cos^2\theta)$. (1.)

Fracture, if possible, will take place by shearing along that plane for which p_θ is maximum, or $\frac{dp_\theta}{d\theta} = 0$.

$\frac{dp_\theta}{d\theta} = p(\cos^2\theta - \sin^2\theta + 2q\sin\theta\cos\theta) = 0$. Solve for θ.

$\sin^2\theta - 2q\sin\theta\cos\theta + q^2\cos^2\theta = \cos^2\theta + q^2\cos^2\theta$ ∴

$$\tan\theta = q + \sqrt{(1+q^2)} \quad (2.)$$

If q were zero, θ max. would be $45°$. Therefore θ max is always greater than $45°$. As θ may be negative as well as positive, fracture tends to form pyramids or cones.

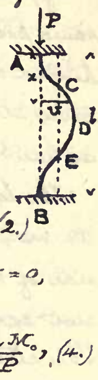
If, in cast-iron, $q = 0.3$, $\theta_{max} = \tan^{-1} 1.344 = 53°21'$.

If this deviation of the plane of fracture from $45°$ is wholly due to a resistance analogous to friction, it follows that p_s must be the shearing strength of the material, and that a crushed, granular column of moderate length, which gives way by shearing, will indicate only a crushing resistance compatible with this shearing strength; while its crushing strength in short blocks will be far higher. Thus, if the shearing strength of a certain cast-iron is 28,000 lbs. per sq. inch, and $q = 0.3$, we have, from (1.) $28,000 = p(.802 \times .597 - .3 \times .597^2) = .372\,p$ or $p = 75,000$ lbs. But, in short blocks of cast-iron, an average value of 112,000 lbs. crushing strength has been found.

§158. Resistance of long Columns.— A column, strut or other piece subjected to pressure is shortened by the compression. As there is not perfect homogeneousness in any material, the longitudinal fibres or elements will yield in different degrees, so that there is apt to be an incipient, although generally imperceptible, curvature of the strut, and the action line of the applied pressure will no longer traverse the axis of the piece. Equilibrium will only take place when, at each cross-section, the moment of resistance against lateral flexure equals the bending moment of the external force, the load.

The flexure usually occurs in a direction parallel to the least diameter of the strut, or perpendicular to that axis in a plane section which gives the least moment of resistance. By the application of longitudinal pressure to a slender rod, its flexure may be made very apparent. The *form* of the column formula ought to resemble (2.) § 157.

If the column is fixed in direction at its ends, by being securely fastened, or by having broad, well-bedded bases, it will act in flexure much like a beam fixed at the ends; a couple or bending moment, which may be represented by M_0, will thus be introduced at each end. Let P = applied external force; v = any deflection ordinate, measured at right angles to the action-line of P, from the original axis of the column to any point in the axis when bent; x = distance along axis from one end to any ordinate v.



Then at any section the bending moment is

$M = M_0 - Pv$, and, if the flexure is very small, we may write, as for a beam,

$$\frac{d^2v}{dx^2} = \frac{M}{EI} = \frac{1}{EI}(M_0 - Pv). \quad (1.) \qquad \text{(See notes to § 169.)}$$

If $M = 0$, the ordinate at pt. of contraflexure, $v_c = \frac{M_0}{P}$. (2.)

Multiply (1.) by dv and integrate. As $\frac{dv}{dx} = 0$ when $v = 0$,

$$\frac{1}{2}\left(\frac{dv}{dx}\right)^2 = \frac{1}{EI}\left(M_0 v - \tfrac{1}{2}Pv^2 + (C=0)\right) \quad (3.)$$

When $\frac{dv}{dx} = 0$, at D, the middle of length, we have $v_m = \frac{2M_0}{P}$, (4.) which is double (2.) and is max. ordinate.

Taking the square root of (3.) we get

$$\frac{dv}{dx} = \sqrt{\left(\frac{P}{EI}\right)}\sqrt{\left(\frac{2M_0}{P}v - v^2\right)} \quad \text{or} \quad dx = \sqrt{\left(\frac{EI}{P}\right)}\frac{dv}{\sqrt{\left(\frac{2M_0}{P}v - v^2\right)}}.$$

Integrate. [See Courtenay's Calculus, p. 256, or Olney's do. pp. 106-7.]

$$x = \sqrt{\left(\frac{EI}{P}\right)}\cdot\left(\text{versin}^{-1}\frac{P}{M_0}v\right) + (C = 0, \text{ as } v = 0 \text{ when } x = 0). \quad \therefore$$

$$v = \frac{M_0}{P}\text{versin}\left(x\sqrt{\frac{P}{EI}}\right) = \frac{M_0}{P}\left(1 - \cos x\sqrt{\frac{P}{EI}}\right). \quad \text{As } 1 - \cos\alpha = 2\sin^2\frac{\alpha}{2},$$

$$v = \frac{2M_0}{P}\sin^2\left(\tfrac{1}{2}x\sqrt{\frac{P}{EI}}\right). \quad (5.)$$

If in (5.), for the point D, we make $x = \frac{1}{2}l$, where $AB = l$, we get v_m to equate with (4.) $\therefore$

$$\frac{2M_0}{P} = \frac{2M_0}{P}\sin^2\left(\frac{l}{4}\sqrt{\frac{P}{EI}}\right), \quad \text{or} \quad 1 = \sin^2\left(\frac{l}{4}\sqrt{\frac{P}{EI}}\right).$$

Since $\sin^{-1} 1 = \frac{\pi}{2}$, $\quad \frac{1}{4}l = \frac{1}{2}\pi\sqrt{\frac{EI}{P}}$; (6.) or $P = \frac{4\pi^2 EI}{l^2}$. (7.)

Therefore $v = \frac{2M_0}{P} \cdot \sin^2 \pi \frac{x}{l}$ is the equation of the curve.

To find the points of contraflexure, substitute (2.) in (5.).

$\frac{M_0}{P} = \frac{2M_0}{P} \sin^2\left(\frac{1}{2} \times \sqrt{\frac{P}{EI}}\right)$, or $\sin\left(\frac{1}{2} \times \sqrt{\frac{P}{EI}}\right) = \frac{1}{\sqrt{2}} = \sin 45^\circ = \sin \frac{\pi}{4}$;

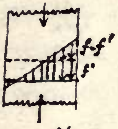
$\therefore \frac{x}{2}\sqrt{\frac{P}{EI}} = \frac{\pi}{4}$, or $x = \frac{\pi}{2}\sqrt{\frac{EI}{P}} = (\text{by } (6.)) = \frac{1}{4} l$ and $\frac{3}{4} l$.

Hence pts. of contraflexure divide column at C and E into quarters.

Therefore a column hinged or free to turn at its ends, and of length $CE = \frac{1}{2} l$, will have the same stress at D from bending as does a column of length AB, $= l$, which is fixed at its ends.

If, in actual cases, C is considered to be practically in the same position as before bending, a column fixed at one end and hinged at the other, of length $CB = \frac{3}{4} l$, will also have the same stress; and the max. deflection will be at ⅓ its length, as has been verified by experiment.

The load which a column will safely carry is determined by the maximum stress on the outside fibre on the concave side of the strut at D, or at A and B, which stress must not exceed f, the safe working stress for the material. This stress may be separated into two parts, the uniform compressive stress f' from P, or $f' = \frac{P}{S}$, where S = cross-section, and $f - f'$, the maximum compression on the outside fibre from bending moment. [On the tension side we might write $f + f'$].



The available strength of the column to resist the bending moment will therefore be in the ratio of $f - f'$ to f, or $1 - \frac{f'}{f}$, and we may get the load the column will safely carry by multiplying (7.) by this ratio, or

$$P = \frac{4\pi^2 EI}{l^2}\left(1 - \frac{f'}{f}\right) = \frac{4\pi^2 EI}{l^2}\left(1 - \frac{P}{fS}\right). \quad \text{Solving for } P;$$

$$P = \frac{\frac{4\pi^2 EI}{l^2}}{\frac{4\pi^2 EI}{l^2 fS} + 1} = \frac{fS}{1 + \frac{l^2 fS}{4\pi^2 EI}} = \frac{fS}{1 + a\frac{l^2}{r^2}}, \quad (4.)$$

where $r^2 = \frac{I}{S}$, or what is known as the square of the radius of gyration, and $a = \frac{f}{4\pi^2 E}$, a quantity dependent on the material.

As seen above, for l write $2l$ when the strut is hinged at the ends, in order to obtain the equivalent length of column fixed at ends; and, for a strut fixed at one end and hinged at the other, write $4/3\,l$ for l, for the same reason. Therefore, we should, theoretically, use $\underline{a}$ for a column fixed at ends and of length l; $4\underline{a}$ for a column hinged at both ends and of length l; and $16/9\,\underline{a}$ for a column hinged at one end and fixed at the other, of length l. Actual tests appear to show that a column bearing on a pin at its ends is not hinged or perfectly free to turn; hence the multipliers of $\underline{a}$ more commonly used, instead of 1, 16/9 and 4, are 1, 4/3 and 2.

Some regard struts as neither perfectly fixed nor perfectly hinged, and use one value of $\underline{a}$ for all. The moment of friction on a pin is certainly considerable.

For further details, see Iron Columns, § 366.

[In the case of very slender columns, when the resistance of the material to tension is decidedly less than its resistance to compression, as is the case with cast-iron, the convex side may be the weaker; $f'+f$ will be the maximum tension and $P = \frac{4\pi^2 EI}{l^2}\left(\frac{f'}{f}+1\right) = \frac{fS}{a\frac{l^2}{r^2}-1}$.]

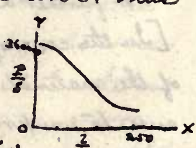
If a column is very short, the formula practically becomes $P = fS$, and some engineers use this form for iron and steel built struts up to a limit of $l = 60$ or $80\,r$.

On the other hand, in the formula, for very slender columns the second term of the denominator overpowers the first, and we practically come back to (7.), known as Euler's.

In that case the moment of flexure far outweighs, and permits the neglect of, the direct thrust. Euler's formula still has its advocates.

The breaking strengths of different columns ~~different~~ columns, for various lengths and values of S and r, have been plotted, with ordinates $\frac{P}{S} = f$ and abscissae $\frac{l}{r}$, and formulae derived to agree with mean values. These formulae could hardly be expected to agree with the one deduced in this section; for this one only applies below the elastic limit, for working, not breaking loads. As, however, the actual strength in compression, for iron and steel, when used in long struts, is little, if any, above the elastic limit, (as may be seen from the lowest curve of the diagram on a preceding page), the formulae do not disagree very widely.

The equation of a straight line may be substituted for Gordon's or Rankine's formula, and the same results practically obtained within working limits of $\frac{l}{h}$ or $\frac{l}{r}$. The formula gives a rather flat, reversed curve. Thus a straight line through $\frac{l}{r} = 0$, $\frac{P}{S} = f$, and the point of contraflexure may be found as follows:— If $y = \frac{f}{1+ax^2}$; where $y = \frac{P}{S}$ and $x = \frac{l}{r}$;



$$\frac{dy}{dx} = \frac{-2afx}{(1+ax^2)^2}; \quad \frac{d^2y}{dx^2} = \frac{-2af(1+ax^2)+8a^2fx^2}{(1+ax^2)^3} = 0$$

$$= \frac{6a^2fx^2-2af}{(1+ax^2)^3}. \quad \therefore 3ax^2 = 1; \; x = \frac{1}{\sqrt{3a}}; \; y = 3/4 f;$$

the coordinates of point of contraflexure.

Then the equation of the straight line well be

$$y = f - 1/4\sqrt{3a} f x.$$

If $f = 36000$, $a = \frac{f}{4\pi^2 E}$ and $E = 27,000,000$, we get, for iron,

$$\frac{P}{S} = 36000 - 90\frac{l}{r} \text{ for fixed ends, and}$$

$$36000 - 180\frac{l}{r} \text{ for hinged ends.}$$

A straight line, tangent at point of contraflexure, will have for its equation, $y = \frac{9}{8}f - \frac{9}{8}f\sqrt{3a}\,x$, giving

$$\frac{P}{S} = 40500 - 135\frac{l}{r} \text{ for fixed ends, and}$$

$$40500 - 270\frac{l}{r} \text{ for hinged ends.}$$

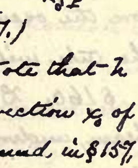
Closer agreement within working limits can be obtained by a straight line through $\frac{l}{r} = 0$, $\frac{P}{S} = 37000$, and the point of contraflexure, resulting in

$$\frac{P}{S} = 37000 - 100\frac{l}{r} \text{ for fixed ends, and}$$

$$37000 - 200\frac{l}{r} \text{ for hinged ends.}$$

Such modifications of Rankine's formula are convenient for calculation and are coming into use.

Page 238. Long Column with load applied eccentrically.— From (4.), §158, $f = \frac{P}{S}\left(1 + a\frac{l^2}{r^2}\right) = p_0 + p_0 a\frac{l^2}{r^2}$; and from (2.), §157, $f = \frac{P}{S}\left(1 + \frac{x_0 x_1 S}{I}\right) = p_0 + p_0\frac{x_0 x_1}{r^2}$. If then, a column is long, and also the line of action of P deviates from the original axis of the strut by a distance x_0, we must add the two expressions, for stress from the moment of flexure and from Px_0, (not doubling p_0), and get



$$f = p_0\left(1 + a\frac{l^2}{r^2} + \frac{x_0 x_1}{r^2}\right). \quad \therefore P = \frac{fS}{1 + a\frac{l^2}{r^2} + \frac{x_0 x_1}{r^2}} \quad (7.)$$

Therefore add to the divisor of (4) $\frac{x_0 h}{2r^2}$. Note that h and r must in this case be measured in the direction x_0 of deviation. Substitute values of r^2 previously found in §157, and obtain $\frac{6x_0}{h}$, etc., as given in text.

That the moment Px_0, although small, has a decided weakening effect on the column is proved by experiment, and its presence may explain some anomalies in tests. See "Experiments on Strength of wrought-iron Struts," by Jas. Christie, Trans. Am. Soc. C.E., Vol. XIII., Apl. 1884, p. 112.

The resisting power, or the proper dimensions of a strut, carrying also a transverse load, is involved in some doubt. Theoretically, the formula would be deduced as follows:—

From the formula for the moment of resistance of a beam, p. 252, $M = \frac{f'I}{y_1}$, we have the stress on the extreme or outside fibre from the bending moment, $f' = \frac{My_1}{I}$. Hence, if M is the moment of the transverse load at the point of maximum strut deflection, we may write

$$f = p_0\left(1 + a\frac{l^2}{r^2}\right) + \frac{My_1}{I}, \text{ and } P = \frac{\left(f - \frac{My_1}{I}\right)S}{1 + a\frac{l^2}{r^2}} \text{ or}$$

$$P = \frac{fS - \frac{My_1}{r^2}}{1 + a\frac{l^2}{r^2}} \qquad (8.)$$

for the safe load on the column when the maximum fibre-stress does not exceed f.

When a tie acts as a beam also, we have

$$f = p_0 + \frac{My_1}{I}\,; \quad P = fS - \frac{My_1}{r^2}. \qquad (9.)$$

with which compare (1.) § 340.

Strut-beams are not economical, and should be avoided.

§ 159. Resistance to Collapsing.— Radius of curvature of an ellipse at end of minor axis b is $\frac{a^2}{b}$; therefore the osculating circle, which determines the strength of the tube will have a diameter $d = \frac{2a^2}{b}$.

§ 160. Beams.— [Substitute for pp. 241-3]. If a beam, under the action of transverse or perpendicular forces, and supported in any manner, is cut by an imaginary plane of section at right angles to the beam, we shall have, on one side of the plane of section, the resultant of all the external ~~applied~~ forces, that is, applied forces and reactions, which act upon the portion of the beam on that side; and, on the other side of the section, another resultant of the external force on the second side. Since equilibrium exists, these two resultants must be equal and opposite to one another. Either one is called the shearing force, and together they tend to move the two parts of the beam in opposite directions along the plane of section, giving rise to a resisting stress in the fibres of the beam at the section to prevent the movement. From the nature of the movement, this stress is called a shearing stress, and is positive on one side of the section and negative on the other.

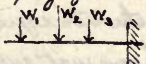
It is proved, in Mechanics, that a force at any distance from a point is equivalent, so far as its action in regard to that point is concerned, to the same force at the point and a moment equal to the product of the force by its perpendicular distance from the point. The resultants referred to above will, therefore, when reference is had to a point in the transverse section, be equivalent to the two equal and opposite forces at the section, or the shear just explained; — and, secondly, —

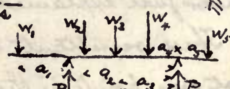
These resultants have equal and opposite moments about any point in the section; and, as they tend to bend the beam, either one is termed the bending moment at the section. These two equal moments neutralize one another by means of the moment of resistance, in the given transverse section of the beam, due to the resistances to extension and compression of the fibres at the section. Each such resistance, in a direction perpendicular to the plane of section, is to be multiplied by its lever arm from any convenient point in the section, and the sum of these moments makes up the moment of resistance.

Since the beam is curved by the elongation of the fibres on one side, and the compression of those on the other, the bending moments are positive or negative, according to the curvature induced in the beam. Rankine calls a positive bending moment that which tends to make a beam concave on the upper side; and a positive shear that which acts upward on the left side of a given section. The distinction is arbitrary.

Since the external forces are all transverse, their only effects at the plane of section must be those due to shear and bending moment. For the three conditions of equilibrium, that the sum of the X forces, the sum of the Y forces, and the sum of the moments, each equals zero, are thus satisfied.

Supporting Forces or Reactions.— Denote them by P_1, P_2, etc., positive when acting upwards. They need not generally be found when a beam is simply fixed by one end alone. For a beam supported at two points, we have, by moments about one of the points of support,





$$P_1 l = \Sigma Wa, \text{ or } P_1 = \frac{\Sigma Wa}{l};$$

where l is distance between points of support, and a is the arm of W about point where P_2 acts, + on the left, − on the right.

For a distributed load, weighing w per foot,

$$P_1 l = \int w\,da \cdot a,$$ between limits over which w extends.

The cases where the reactions are more than two will be examined later, as will also the cases where the beam is not simply placed on its supports.

The shear F, positive when acting upwards on the left of any section distant x' from the left-hand end, is the sum of the vertical forces on the left of the section; or, in a beam fixed at right-hand end only,

$$-F = \Sigma W; \text{ or } -F = \int_0^{x'} w\,dx, \text{ for distributed load. (1.)}$$

In a beam supported at two points

$$F = P_1 - \Sigma W; \text{ or } F = P_1 - \int_0^{x} w\,dx, \text{ for dist}^{d} \text{ load (2.)}$$

But, for sections to left of P_1, P_1 disappears.

The bending moment M, positive when right-handed on the left of any section, or tending to make the beam concave on its upper side, will be at any section distant x' from the left-hand end, the sum of the moments on the left of the section. Beam fixed at one end,

$$-M = \Sigma Wx; \text{ or } -M = \int_0^{x'} wx\,dx, \text{ for distributed load (3.)}$$

Beam supported at two points,

$$M = P_1 x - \Sigma Wx; \text{ or } M = P_1 x - \int_0^{x} wx\,dx, \text{ for dist}^{d} \text{ load; (4.)}$$

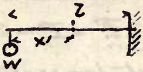
$\underline{x}$ being, in all cases, the arm of P_1, W or $w\,dx$ about the <u>section</u> in question, and the integration covering only the loaded portion. If P_1 is not to left of section, drop term $P_1 x$, and thus obtain (3.).

Notice that the shear is always $\frac{dM}{dx}$. Hence M is maximum where F is zero or changes sign. The <u>intensity</u> of the load may also be considered as the differential coefficient of the shear, which latter, therefore, has maximum values where the external forces change in sign.

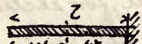
If the origin of coordinates be taken at any point in the length of the beam, and $-w$ be the intensity of the load, either constant or variable, we may write

$$F_x = -\int w\,dx = F_0 - wx; \text{ if } w \text{ is constant.}$$
$$M_x = -\iint w\,dx^2 = \int (F_0 - wx)\,dx = M_0 + F_0 x - \frac{wx^2}{2};$$

where F_0 and M_0 are values of F and M at the origin, or the constants of integration. A general expression for the bending moment at any point of any beam has therefore the form $A + Bx + Cx^2$. What do these letters represent?



§ 161. Applications.— I. $F_{x'} = -W$, and is constant; $M_{x'} = -Wx'$; $M_{max} = -Wl$; $m = 1$.

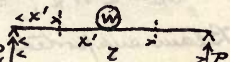


II. $F_{x'} = -wx'$; $M_{x'} = -\int_0^{x'} wx\,dx = -\frac{wx'^2}{2}$;

$F_{max} = -wl$; $M_{max} = -\frac{wl^2}{2} = -\frac{(wl)l}{2}$. $\therefore m = -\frac{1}{2}$.

In these two cases the <u>load</u> and the <u>shear</u> will each change sign at the wall by the introduction and addition of the upward reaction; $\therefore F_{max}$ and M_{max} at wall.

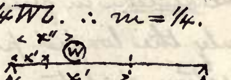
III. Add I. and II.



IV. $P_1 = \frac{1}{2}W$. $F_{x'} = \frac{1}{2}W$ on left of weight, and $= \frac{1}{2}W - W = -\frac{1}{2}W$ on right of weight.

$M_{x'} = \frac{1}{2}Wx'$ on left and $= \frac{1}{2}Wx' - W(x' - \frac{1}{2}l) = \frac{1}{2}W(l - x')$ on right.

As F changes sign at distance $\frac{1}{2}l$, $M_{max} = \frac{1}{4}Wl$. $\therefore m = \frac{1}{4}$.



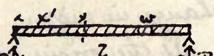
V. $P_1 = W\frac{l - x''}{l}$. $P_2 = W - P_1 = W\frac{x''}{l}$.

$F_{x'} = W\frac{l-x''}{l}$ on left and $= W\frac{l-x''}{l} - W = -\frac{Wx''}{l}$ on right of weight. $M_{x'} = W\frac{l-x''}{l}x'$ on left and $=$

$W\frac{l-x''}{l}x' - W(x' - x'') = W\frac{l-x'}{l}x''$ on right of weight.

M_{max} (where F changes sign) $= W\frac{l-x''}{l}x''$. $\therefore m = \frac{x''(l-x'')}{l^2}$.

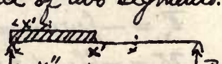
The bending moment may often be conveniently found by taking moments on the right of a section.



VI. $P_1 = \frac{1}{2}wl = P_2$. $F_{x'} = \frac{1}{2}wl - wx'$ $= w(\frac{1}{2}l - x')$. $F_{max} = \frac{1}{2}wl$. $F = 0$ when $x' = \frac{1}{2}l$.

$M_{x'} = \frac{1}{2}wl \cdot x' - \frac{1}{2}wx'^2 = \frac{1}{2}wx'(l - x')$. $M_{max} = \frac{1}{8}wl^2$. $\therefore m = \frac{1}{8}$.

The bending moment varies as the product of two segments.

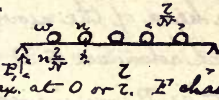


VII. $P_1 = \int_{l-x''}^{l} \frac{w\,a\,da}{l} = \frac{wl^2}{2l} - \frac{w(l-x'')^2}{2l}$ $= w(x'' - \frac{x''^2}{2l})$; more easily found by using centre of gravity of load. $F_{x'} = w(x'' - \frac{x''^2}{2l} - x')$ on the loaded portion, and will diminish until $x' = x''$; when $F_{x'} = -\frac{wx''^2}{2l}$, and remains constant over the unloaded portion. $F = 0$ when $x' = x'' - \frac{x''^2}{2l}$. $M_{x'} = w(x'' - \frac{x''^2}{2l})x' - w\frac{x'^2}{2}$ up to the end of loaded portion; thence to the end of the beam

$$M_{x'} = w(x'' - \frac{x''^2}{2l})x' - \int_{x'-x''}^{x'} wx\,dx = \frac{wx''^2}{2l}(l - x') = P_2(l - x').$$

$$M_{max} = \frac{w}{2}\left(x'' - \frac{x''^2}{2l}\right)^2. \quad \therefore m = \frac{x''}{2l}\left(1 - \frac{x''}{2l}\right)^2.$$

VIII. $W = (N-1)w$. $P_1 = P_2 = \frac{1}{2}(N-1)w$.



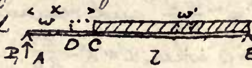
$F_n = (\frac{N-1}{2} - n)w$. Origin being at P_1, F max. at 0 or l. F changes sign when $n > \frac{1}{2}(N-1)$. $M_n = \frac{1}{2}(N-1)w \cdot n\frac{l}{N} - nw \cdot \frac{1}{2}(n-1)\frac{l}{N} = \frac{n(N-n)}{2N} lw$. This value varies as the product of the two segments n and $N-n$. When N is even, M_{max} for $n = \frac{1}{2}N$. $\therefore M_{max} = \frac{Nlw}{8}$. When N is odd, M_{max} for $n = \frac{N-1}{2}$ or $\frac{N+1}{2}$. $\therefore M_{max} = \frac{N-1}{2} \cdot \frac{N+1}{2} \cdot \frac{lw}{2N} = \frac{N^2-1}{8N} lw$. Divide by $(N-1)wl$ to find m. Value of $\frac{1}{8}$ is wrong for given load, but is right for a load of Nw, counting $\frac{1}{2}w$ at each point of support. In that case the value $\frac{N+1}{8N}$ is wrong.

Notice that M_{max} for odd number of spaces or panels is less than M_{max} for even number of spaces by the factor $\frac{1}{N^2}$; so that $\frac{1}{8}Nwl$ diminished to $(1 - \frac{1}{N^2})\frac{N}{8}wl$ gives the max. bending moment in case of odd number of spaces, a convenient matter for use in trusses with odd number of panels. Howe and Pratt trusses come under this case.

Notice also and remember the values of m in Cases I., II., IV. and VI. The coefficients 1, ½, ¼ and ⅛ are handy to use.

In ordinary problems special formulas are of doubtful value. The student will do well to accustom himself to calculate shears and bending moments from numerical examples, by the general principles of pp. 38, 39.

IX. The greatest shear at any section C of a beam loaded uniformly, over a portion of its length occurs when the longer segment alone is covered with the moving load. As the



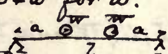
load advances from B, the shear at C (from moving load only) will be equal to P_1 at A, and will increase until the head of the moving load reaches C. If the moving load advances to D, the shear at C will be $P_1 - w'.CD$; but, as P_1 was increased by a portion only of $w'.CD$, the shear at C will now be less than when the moving load covered BC. While this position of load gives the absolute maximum shear at any section C, there will be a smaller maximum of the opposite sign when the shorter segment only is loaded. This latter value is now sometimes used in determining minimum stresses on web members of trusses, as will be seen later.

The greatest bending moment at any section occurs with a full load; for every increment of load increases the bending moment at all points. This statement is limited here to beams and trusses simply supported at their two ends. [For better diagrams, see "Graphics," Bridges]

P_1 (for load of sketch), $= \frac{1}{2}wl + w'(l-x)\frac{l-x}{2l}$.

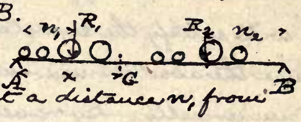
$F_x = P_1 - wx = w\left(\frac{l}{2}-x\right) + \frac{w'(l-x)^2}{2l}$. Max. bending moment is the same as in Case VI., when we write $w+w'$ for w.

X. $P_1 = W$. $F = W$, from 0 to a; $F = 0$ in middle portion; and $F = -W$ on right. $M = Wx$ on left. $M = P_1x - W(x-a) = Wa$ in middle portion, $\therefore$ constant, as might be expected where F is zero. Case of single track R.R. floor beam.



To find the position of a given system of loads to give

Maximum Moment at any point C, distant x from the left abutment of a beam AB.



Let R_1 be the resultant of all loads to the left of C and acting at a distance n_1 from left abutment; let R_2 be the resultant of all loads to the right of C and acting at a distance n_2 from right abutment.

Then at C, $M_c = R_2 \frac{n_2 x}{l} + R_1 \frac{n_1(l-x)}{l}$.

If now the entire system of loads be moved a distance Δ to the left, $M_c' = R_2 \frac{(n_2+\Delta)x}{l} + R_1 \frac{(n_1-\Delta)(l-x)}{l}$.

and M_c' is greater than M_c, if $R_2 x > R_1(l-x)$, or if $\frac{R_2}{l-x} > \frac{R_1}{x}$.

But $\frac{R_2}{l-x}$ is the <u>average</u> <u>load</u> per running foot to the right of C, and $\frac{R_1}{x}$ is the average load per ft. to the left of C.

Hence the rule:— If the <u>average</u> <u>load</u> per running foot to the right of the section is greater than the average on the left, then M at the section will be increased by moving the system of loads to the left, and <u>vice versa</u>. For the maximum moment at any section, some load must be at that section.

Suppose that, when W_n is just to right of section, the average load on the right is larger than that on the left; then move W_n just to left of section, and if the average load on the right is still greater than that on the left, move the next load W_{n+1} up until it is just to the left of the section, and repeat the trial, regardless whether, during the operation, new loads come upon the beam to the right, or leave it to the left. For, when W_n is just to the left of the section, we move to the left, because the average load to the right is greater than that to the left; and if, in moving the loads to the left, additional loads come upon the beam to the right, or leave it to the left, then the average load to the right is so much larger still, and we must continue to move to left.

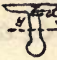
It sometimes happens that two or more different loadings will satisfy the condition just explained, and to determine the absolute max. moment each must be worked out numerically. By remembering, however, that, to produce the max. moment at any point, the heaviest loads must be as near that point as possible, the cases in which trial is necessary will be greatly reduced in number.

Position for Maximum Shear.— For the same point C in the beam, as the load advances from the right, the shear at C will increase until the first wheel reaches C. If it passes C, the shear suddenly diminishes by W_1, but will then increase until W_2 reaches C. Let R be the sum of all loads on beam when W_1 is at C. The shear at C = P_1, the left-hand reaction, or $\frac{Rn}{l}$, where n is the distance from centre of gravity of loads R to right abutment.

If the train moves a distance a_1 to the left, so that W_2 is just to the right of C, the shear will be $R\frac{n+a_1}{l} - W_1$ plus a small quantity d, which is the increase in the left-hand reaction due to any additional loads which may have come on the beam during this advance of the train. The shear will therefore be increased by moving up W_2, if $R\frac{a_1}{l} + d > W_1$, or (as d can often be neglected) if $R\frac{a_1}{l} > W_1$. It is only necessary to consider d when these two terms are nearly equal. Similarly, W_3 should be moved up to C, if

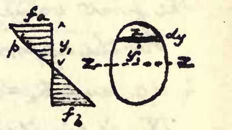
$R'\frac{a_2}{l} > W_2$; R' being the sum of loads on span when W_2 is at C. [Geo. F. Swain, Trans. Am. Soc. C.E., 1887-II. p. 22.

§162. Moment of Resistance.— As the external forces are transverse to the beam, they have no longitudinal components; hence, for equilibrium, the longitudinal compression and tension of the moment of resistance must be equal, or their sum be zero; $\therefore \int pz\,dy = a\int yz\,dy = 0$.



But, in finding the centre of gravity of any section, the distance y_0 from the origin to centre of gravity $= \frac{\int yz\,dy}{S}$. If $\int yz\,dy$ in any case equals zero, then y_0 must be zero, and the origin lies in the centre of gravity of the section.

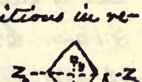
The formula for the moment of resistance of a beam, $M = \frac{fI}{y_1} = nfbh^2$, is applicable only when f does not exceed the stress at the elastic limit. Above that point, the assumption of a uniformly varying stress is not likely to be true. Hence, also, the substitution of breaking weights, obtained by experiments carried to rupture of beams, in the above formula, results in values of f, the modulus of rupture, agreeing with neither the tensile nor the compressive strength of the material, and of no special value. Use in practice, for f, the <u>safe working strength</u> in tension or compression, whichever will give $\frac{f}{y_1}$ the less value.



$p : y = f : y_1$; $p = \frac{f}{y_1}y$.

$$M = \int_{-y_1}^{+y_1} pz\,dy \cdot y = \frac{f}{y_1}\int_{-y_1}^{+y_1} y^2 z\,dy = f\frac{I}{y_1}.$$

It is useful to recall two or three propositions in regard to values of I for plane sections.



Moment of inertia about an axis parallel to one through the centre of gravity and distant y' from it.

$$I = \iint (y+y')^2\,dy\,dz = \iint y^2\,dy\,dz + 2y'\iint y\,dy\,dz + \iint y'^2\,dy\,dz.$$

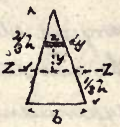
In the second term of second member $\iint y\,dy\,dz$ is the moment of area about axis through centre of gravity, and is therefore zero. $\iint y'^2\,dy\,dz = y'^2\iint dy\,dz = y'^2 \times \text{Area}$. $\therefore$

$$I = I' + y'^2 A' = (r^2 + y'^2)\,A'. \qquad (7.)$$

If a surface has two axes of symmetry not at right angles to each other, its moment of inertia is the same about all axes passing through its centre of gravity and lying in it. The same thing is true if the moments of inertia about four axes at right angles to each other, in pairs, are equal.

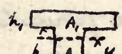
Any two sections which have the same value of I do not have the same moment of resistance unless y_1 is also the same in both cases.

§163. Examples.— For I. to VI., see Notes, §157.



VII. $h : b = \frac{2}{3}h - y : z$; $\therefore z = \frac{b}{h}(\frac{2}{3}h - y)$

$$I = \int_{-\frac{1}{3}h}^{+\frac{2}{3}h} y^2 z\, dy = \frac{b}{h}\int_{-\frac{1}{3}h}^{+\frac{2}{3}h} (\tfrac{2}{3}h - y) y^2 dy = \frac{1}{36} b h^3.$$



VIII. By moments around base,

$$y_1 = \frac{A_1(h_2 + \frac{1}{2}h_1) + A_2 \frac{1}{2} h_2}{A_1 + A_2} = \frac{2A_1 h_2 + A_1 h_1 + A_2 h_2}{2A}$$

$$= \frac{A_1(h_1 + h_2) + A_2(h_1 + h_2) + A_1 h_2 - A_2 h_1}{2A} = \frac{h}{2} + \frac{A_1 h_2 - A_2 h_1}{2A} \quad (1.)$$

By (7.), $I = \frac{A_1 h_1^2}{12} + \frac{A_2 h_2^2}{12} + A_2(y_1 - \frac{h_2}{2})^2 + A_1(h_2 + \frac{h_1}{2} - y_1)^2$

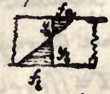
$$= \frac{A_1 h_1^2 + A_2 h_2^2}{12} + \frac{A_1 A_2 (h_1 + h_2)^2}{4A}. \quad (1.)$$

The first value is generally sufficiently convenient.

For the approximate values, it will suffice to make $h_1 = 0$, and $h_2 = h = h'$.

IX. is similar and of little value.

§164. Cross-Section of Equal Strength.— If a material will resist tension better than it will compression, or *vice versa*, we may form the section



so that the neutral axis lies nearer the weaker side; and, if we so design the cross-section that the maximum allowable stress of each kind occurs at the opposite edges, top and bottom, at the same time, we have a *cross-section of equal strength*. Then, by similar triangles, $f_a : f_b = y_a : y_b = f_a + f_b : y_a + y_b$; and such is the relationship to be satisfied.

Such cross-sections are, however, but little used, unless perhaps in cast-iron. In wrought-iron and steel, the <u>safe</u> <u>working</u> <u>intensity</u> of stress for tension and for compression are not very different, even if the ultimate or breaking strengths may be ; and hence symmetrical sections are commonly employed for beams. Such sections also give stiffer beams.

Omit pp. 257-8.

§165. Beams of Uniform Strength.— The moment of resistance at any section may be written $M = nfbh^2$. If, then, bh^2 be varied at successive sections as the value of the external bending moment varies, the beam will be equally strong at all sections, and there will be no waste of material, for a given form of cross-section, provided we do not waste in fashioning it.

The eight cases of the text may be tabulated as follows:—

	$M = nfbh^2$.	bh^2 prop.l to	h^2 constant. b varies as	b constant. h^2 varies as
I.	$-Wx'$	x'	x', plan triang.r	x', elev.n parabolic.
II.	$-\frac{wx'^2}{2}$	x'^2	x'^2 " parabolic	x'^2 or $h \propto x'$, elev. triang.r
III.	$W\frac{x'(l-x'')}{l}$	x'	x' } " triang.r	x' } elev.n parabolic.
	$W\frac{x''(l-x')}{l}$	$l-x'$	$l-x'$	$l-x'$
IV.	$\frac{wx'(l-x')}{2}$	$x'(l-x')$	$x'(l-x')$ " parabolic	$x'(l-x')$ " circular or elliptic.

For sketches see the text.

§167. Allowance for Weight of Beam.— For simplicity, omit the symbols for factors of safety in the first analysis, and introduce them in the results, if desired. As the depth of a beam or truss is fixed by considerations of stiffness and economy of material, the breadth of the beam, or the cross-sections of the chords of a truss, should be increased until it is able to carry safely its own weight in addition to the external load. It is supposed that the beam is sufficiently heavy to make its own weight of consequence.

Having a beam of weight B' and breadth b', proportioned so as to just carry safely a gross or entire load W', we have $\frac{\text{Wt. of beam}}{\text{Entire Load}} = \frac{B'}{W'}$. Then $\frac{\text{Entire load}}{\text{External load}} = \frac{W'}{W'-B'}$.

The weight of the beam (proportional to $b'hl$), the gross load it can carry ($W' = \frac{nfb'h^2}{ml}$), and consequently the net load, all increase directly as the breadth b', when h and l are unchanged. If then we desire to increase the net load from $W'-B'$ to W', we must increase the breadth in this ratio, or the new breadth b will give $\frac{b}{b'} = \frac{W'}{W'-B'}$, or $b = b'\frac{W'}{W'-B'}$; (1.)
and, as the weight, net load and gross load are increased in the same ratio, $B = B'\frac{W'}{W'-B'}$; (2.) $W = \frac{W'^2}{W'-B'}$. (3.)

If we write $b = pl$ and $h = ql$, the wt. of a beam may be written as $kw'(bhl) = kw'pql^3$, and hence varies as the cube of any dimension, for similar beams. Again,

$mWl = nfbh^2$; $\therefore$ $W = \frac{nfbh^2}{ml} = \frac{nfpq^2}{m}\cdot\frac{l^3}{l}$. $\therefore$

The load varies as the square of any dimension. Hence a limiting length, when a beam can carry only its own weight, is possible.

By (6.) p. 253, $\frac{1}{8}Wl = nfbh^2$; $\therefore W = \frac{8nfhA}{l}$. (7.)

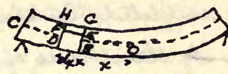
$\frac{W}{B} = \frac{8nfh}{kw'l^2}$. (9.) At the limiting length, $B = W$ or

$1 = \frac{8nfh}{kw'L^2}$; $\therefore$ $L = \frac{8nfh}{kw'l}$, (10.) if $\frac{h}{l}$ is fixed or constant.

Then, for other spans, $W : B :: L : l$, or $L : l : L-l = W : B : W'$. (11.)

This formula is useful for computing the approximate weight of one beam or truss from that of another similar known one. §323.

§168. Distribution of Shearing Stress in the Cross-Section of Beams, Pins, etc. — From the principles of internal stress, §108, II., or Notes, p. 1, the intensities of shear on a pair of planes at right angles must be equal. Whatever can be proved true in regard to the horizontal shear in a beam must then be true of the vertical shear. If the sketch represents a bent beam, the existence of shear on planes parallel to EF is shown by



the tendency of the layers to slide by one another upon flexure.

Let the cross-section be constant. If the moment of flexure at section H differs from that at G, there must be a corresponding difference in the direct stresses on faces HF and GE of the solid HFEG, which difference constitutes a horizontal force, to be resisted by a shearing stress on plane EF, equal and opposite to that horizontal force. That amount, divided by the area EF on which it acts, gives the intensity of horizontal shear; this intensity must increase from H to the neutral axis, and then decrease to other edge. Intensity of vertical shear on plane HD must follow the same law of distribution. Pins, keys and rivets, which do not fit tightly, and hence are exposed to bending, have also max. intensity of shear at centre of cross-section. See § 361.

$OB = x$; $OC = c$; $BE = y$; $BG = y_1$. Dotted line OBDC is neutral axis.

$BD = EF$ sensibly $= dx$. Breadth of beam at any point $= z$, and at neutral axis $= z_0$. Intensity of horizontal stress at E $= p$, of shearing stress at E $= q$, and max. intensity of shear at B $= q_0$.

By (2A.) p. 252, $p = \frac{My}{I}$. Stress on plane GE $= \frac{M}{I}\int_y^{y_1} yz\,dy$. (a.)

By (8.) p. 243; $M = \int F\,dx$. Therefore the difference between M at B and M at D must be $F\,dx$, where F = shear at section D.

The horizontal force on FE is the excess of (a.) for GE over its value for HF, or $\frac{F\,dx}{I}\int_y^{y_1} yz\,dy$. Divide by the area $z\,dx$ of FE over which it acts and

$$q = \frac{F}{Iz}\int_y^{y_1} yz\,dy; \quad \therefore \quad q_0 = \frac{F}{Iz_0}\int_0^{y_1} yz\,dy. \quad (2.)$$

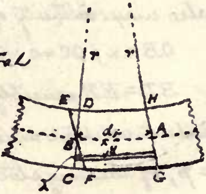
(3.) is obtained by proportion from (1.) and (2.)

I. Rectangle: $\frac{A}{Iz_0}\int_0^{y_1} yz\,dy = \frac{12}{bh^3}\cdot\frac{bh}{b}\cdot b\int_0^{\frac{1}{2}h} y\,dy = \frac{12}{h^2}\cdot\frac{h^2}{8} = 3/2.$

II. Circle: $\frac{4}{\pi r^4}\cdot\frac{\pi r^2}{2r}\int_0^r \sqrt{r^2-y^2}\,2y\,dy = \frac{4}{\pi r^4}\cdot\frac{\pi r^2}{2r}\cdot\frac{2}{3}r^3 = 4/3.$

§ 169. Flexure of Beams.— After the following analysis is completed, Rankine's method of treatment may be read.

As the stresses of tension and compression which make up the moment of resistance at any section of a beam cause elongation and shortening of the longitudinal fibres or layers, a curvature of the beam will result, the curve depending upon the material, upon the magnitude and distribution of the load, the span of the beam and method of support, and upon the form and variation of the cross-sections. It is convenient to be able to determine the amount of <u>deflection</u>, or displacement from its original position, of any beam carrying a given load; and the investigation of the forces which act on beams supported in any other than the simpler ways requires the use of equations involving expressions for the deflection and the inclination or slope of the tangent to the neutral axis at any point.



If, through the points A and B on the neutral axis of a bent beam and distant <u>dx</u> from one another, we draw normals to the curve of the neutral axis, the distance from AB to their intersection will be the radius of curvature for that portion of the curve. If we pass through B a plane EF parallel to GH, the distances from CD to EF will represent the elongations and compressions of the respective fibres which were <u>dx</u> long before flexure. Cross-sections plane before flexure are considered as plane after flexure, since deformation is proportional to stress within the elastic limit, and stress has already been assumed as uniformly varying. Experiment bears out the assumption.

Let r = radius of curve of neutral axis; y = distance from neutral axis to any fibre. This fibre, of length dx will be changed in length an amount λ. By similar triangles,

$r : dx = y : \lambda$, or $r = \frac{y}{\lambda}\,dx$. By Notes to §146, $\lambda = \frac{p\,dx}{E}$.

$\therefore\ r = \frac{yE}{p} = \frac{y_1 E}{f}$, if f = stress on extreme fibre, distant y_1.

As, by (24.) p. 252, $M = \frac{fI}{y_1}$, we have the

$$\text{Curvature} = \frac{1}{r} = \frac{M}{EI}. \quad (1.)$$

If the curve of the neutral axis is referred to rectangular coordinates, x and v, the latter being vertical, we have, from Calculus, $r = \frac{ds^3}{dx\,d^2v}$. For slight curvature put $ds = dx$. Then

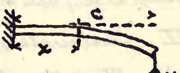
$$\frac{1}{r} = \frac{d^2v}{dx^2} = \frac{M}{EI}.$$

The first integral, $\frac{dv}{dx}$, will give the inclination or slope of the tangent to the curve at any point x, and the second integral will give v, the deflection or vertical ordinate from the axis of x.

If $\frac{M}{I}$ is constant, the beam will bend in a circular arc.

Applications.

I. Beam fixed at one end; single load at the other; length = c; origin at fixed end; horizontal distance of section = x. $M = W(c-x)$.



$$\frac{d^2v}{dx^2} = \frac{W}{EI}(c-x). \qquad \tan i_x = \frac{dv}{dx} = \frac{W}{EI}\int_0^x (c-x)\,dx = \frac{W}{EI}(cx - \tfrac{1}{2}x^2).$$

$$\tan i_c = \frac{W}{EI}(c^2 - \tfrac{1}{2}c^2) = \frac{Wc^2}{2EI}.$$

$$v_x = \frac{W}{EI}(\tfrac{1}{2}cx^2 - \tfrac{1}{6}x^3). \qquad v_c = \frac{Wc^3}{3EI}. \qquad \text{See II. p. 274.}$$

To determine the max. allowable deflection of a given beam, substitute in v_c the value of W in terms of f. Thus, max. bending moment $= Wc = \frac{fI}{y_1}$ the moment of resistance; $\therefore\ W = \frac{fI}{y_1 c}$, and

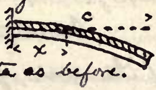
$v_1 = \frac{1}{3}\frac{fc^2}{y_1 E}$. See "Proof Deflection," II. p. 274.

The ease with which cases are worked depends much upon point taken for origin, as affecting the integration.

Notice that, for a given weight, the max. bending moment

varies as the span c; the max. slope as c^2; and the max. deflection as c^3. The slope and deflection also vary inversely as I, or inversely as the breadth and cube of the depth. The proof deflection, v_1, varies as c^2, and inversely as $2y_1$, or as the depth. See pp. 271 and 273.
The same will be found true of the following cases.

II. Beam fixed at one end; uniform load w per foot over whole length; other data as before.



$W = wc$; $\mathcal{M} = w\frac{(c-x)^2}{2}$; $\frac{d^2v}{dx^2} = \frac{w}{2EI}(c^2 - 2cx + x^2)$

$\frac{dv}{dx} = \frac{w}{2EI}\Big[_0^x(c^2x - cx^2 + 1/3\,x^3) = \tan i_x$.

$\tan i_0 = \frac{w}{2EI}(c^3 - c^3 + 1/3\,c^3) = \frac{wc^3}{6EI} = \frac{Wc^2}{6EI}$.

$v_x = \frac{w}{2EI}\Big[_0^x(1/2\,c^2x^2 - 1/3\,cx^3 + 1/12\,x^4)$

$v_0 = \frac{w}{2EI}(1/2\,c^4 - 1/3\,c^4 + 1/12\,c^4) = \frac{wc^4}{8EI} = \frac{Wc^3}{8EI}$. III. p. 274

Again, for max. allowable deflection, $1/2\,wc^2 = \frac{fI}{y_1}$ ∴

$v_1 = \frac{1}{4}\cdot\frac{fc^2}{y_1E}$. III. p. 274.

If the above bending moments had been written with the negative sign, all results would have had signs reversed.

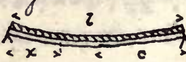
III. Combination of Single and Uniform Load.

$\mathcal{M}_{0\text{ or }max} = Wc + 1/2\,wc^2$. $i_0 = \frac{1}{EI}(1/2\,Wc^2 + 1/6\,wc^3)$.

$v_0 = \frac{1}{EI}(1/3\,Wc^3 + 1/8\,wc^4) = \frac{c^3}{3EI}(W + 3/8\,wc)$.

Note, in the last expression, the relative deflections due to a load at the end and the same load distributed along the beam; and compare with the relative maximum bending moments.

IV. Beam of length l; supported at both ends; uniform load of w per foot; origin at left pt. of support.



$\mathcal{M} = \frac{w}{2}(lx - x^2)$; $\frac{d^2v}{dx^2} = \frac{w}{2EI}(lx - x^2)$; $\frac{dv}{dx} = \frac{w}{2EI}(1/2\,lx^2 - 1/3\,x^3 + C)$;

$\frac{dv}{dx} = 0$ when $x = 1/2\,l$; ∴ $C = \frac{l^3}{24} - \frac{l^3}{8} = -1/12\,l^3$. ∴

$\frac{dv}{dx} = \frac{w}{2EI}(1/2\,lx^2 - 1/3\,x^3 - 1/12\,l^3)$. Why not integrate between x+0.

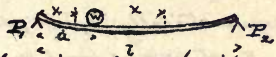
$\tan i_{max}$ (when $x = 0$ or l) $= \pm\frac{wl^3}{24EI} = \frac{wc^3}{3EI} = \frac{Wc^2}{6EI}$; if $c = 1/2\,l$ and $W = w.2c$.

$v_x = \frac{w}{2EI}\left(\frac{lx^3}{6} - \frac{x^4}{12} - \frac{l^3x}{12} + C'\right)$. $v = 0$ when $x = 0$; ∴ $C' = 0$.

v_{max} (when $x=\frac{1}{2}l$) $= \frac{w}{2EI}\left(\frac{l^4}{48}-\frac{l^4}{192}-\frac{l^4}{24}\right) = -\frac{5}{384}\frac{wl^4}{EI} = \frac{5}{24}\frac{wc^4}{EI} = \frac{5}{12}\frac{Wc^3}{EI}$.

Since $\frac{wl^2}{8}=\frac{fI}{y_1}$; $v_1 = \frac{5}{48}\frac{fl^2}{Ey_1} = \frac{5}{12}\frac{fc^2}{Ey_1}$. V. p. 274.

V. Beam of length l; supported at both ends; load W at distance $\underline{a}$ from left; origin at left.



As the load is eccentric, equations must be written for each side.

On left of weight.

$P_1 = W\frac{l-a}{l}$; $M = W\frac{l-a}{l}x$.

$$\frac{d^2v}{dx^2} = \frac{W}{EI}\cdot\frac{l-a}{l}x$$

$$\frac{dv}{dx}=\frac{W}{EIl}\left(\frac{(l-a)x^2}{2}+C\right)$$

$$v = \frac{W}{EIl}\left(\frac{(l-a)x^3}{6}+Cx+C''\right).$$

When $x=0$, $v=0$; $\therefore C''=0$.

On right of weight.

$P_2=\frac{Wa}{l}$; $M=\frac{Wa}{l}(l-x)$.

$$\frac{d^2v}{dx^2}=\frac{W}{EI}\cdot\frac{a}{l}(l-x)$$

$$\frac{dv}{dx}=\frac{W}{EIl}\left(alx-\frac{ax^2}{2}+C'\right)$$

$$v=\frac{W}{EIl}\left(\frac{alx^2}{2}-\frac{ax^3}{6}+C'x+C'''\right).$$

When $x=l$, $v=0$; $\therefore C'''=-\frac{al^3}{2}+\frac{al^3}{6}-C'l$
$= -\frac{al^3}{3}-C'l$.

When $x=a$, $\frac{dv}{dx}$ on left $= \frac{dv}{dx}$ on right; and v_a on left $= v_a$ on right.

$\left[\frac{dv}{dx}\right]$ gives $\frac{1}{2}(l-a)a^2+C = a^2l-\frac{1}{2}a^3+C'$; or $C=C'+\frac{1}{2}a^2l$.

$[v]$ " $\frac{1}{6}a^3l-\frac{1}{6}a^4+Ca = \frac{1}{2}a^3l-\frac{1}{6}a^4+C'a+C'''$; or $C=\frac{1}{3}a^2l+C'+\frac{C'''}{a}$.

$\therefore C'''=\frac{1}{6}a^3l$; $C'=-\frac{1}{6}a^3-\frac{1}{3}al^2$; $C=\frac{1}{2}a^2l-\frac{1}{6}a^3-\frac{1}{3}al^2$. Substitute above.

$$\frac{dv}{dx}=\frac{W}{EIl}\left[\tfrac{1}{2}(l-a)x^2+\tfrac{1}{2}a^2l-\tfrac{1}{6}a^3-\tfrac{1}{3}al^2\right]$$

$x=0$, gives $\tan i_{max}$

$$v=\frac{W}{EIl}\left[\tfrac{1}{6}(l-a)x^3+\tfrac{1}{2}a^2lx-\tfrac{1}{6}a^3x-\tfrac{1}{3}al^2x\right]$$

$$\frac{dv}{dx}=\frac{W}{EIl}\left(alx-\tfrac{1}{2}ax^2-\tfrac{1}{6}a^3-\tfrac{1}{3}al^2\right)$$

$x=l$, gives $\tan i_{max}$

$$v=\frac{W}{EIl}\left(\tfrac{1}{2}alx^2-\tfrac{1}{6}ax^3-\tfrac{1}{6}a^3x-\tfrac{1}{3}al^2x+\tfrac{1}{6}a^3l\right)$$

For v_{max} make $\frac{dv}{dx}=0$ on right, to find value of x. We thus get

$alx-\frac{1}{2}ax^2=\frac{1}{6}a^3+\frac{1}{3}al^2$. $a=0$ is a minimum value.

$x^2-2lx+l^2=l^2-\frac{1}{3}a^2-\frac{2}{3}l^2$; $l-x=\sqrt{\left(\frac{l^2-a^2}{3}\right)}$, the dist. from right.

$x=l-\sqrt{\left(\frac{l^2-a^2}{3}\right)}$. Substitute in v on right, for max. deflection.

It will be noticed that, when the weight is eccentric, the point of max. deflection does not agree with point of max. bending moment.

VI. Beam as above; $a=\frac{1}{2}l$, or weight in middle.

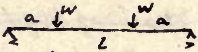
$\frac{d^2v}{dx^2}=\frac{W}{2EI}x$; $\frac{dv}{dx}=\frac{W}{2EI}\left(\frac{1}{2}x^2+C\right)$; $\frac{dv}{dx}=0$ when $x=\frac{1}{2}l$, $\therefore C=-\frac{1}{8}l^2$.

$\frac{dv}{dx}=\frac{W}{2EI}\left(\frac{1}{2}x^2-\frac{1}{8}l^2\right)$; $x=0$, $\frac{dv}{dx}=\frac{Wl^2}{16EI}$; $v=\frac{W}{2EI}\left(\frac{1}{6}x^3-\frac{1}{8}l^2x+(C'=0)\right)$

since $v=0$ when $x=0$. When $x=\frac{1}{2}l$, $v_{max}=\frac{W}{2EI}\left(\frac{1}{48}l^3-\frac{1}{16}l^3\right)=-\frac{Wl^3}{48EI}$.

Since $\frac{Wl}{4}=\frac{fI}{y_1}$; $v_1=\frac{fl^2}{12Ey_1}=\frac{fc^2}{3Ey_1}$. IV. p. 274.

Notice the numerical coefficients of v_{max} in Cases I, II, VI and IV. They are $1/3$, $1/8$, $1/48$ and $\frac{5}{8\times48}$. M_{max} varies as 1, $1/2$, $1/4$ and $1/8\ Wl$.

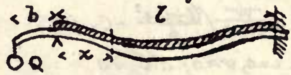
VII. Beam with two equal weights, 

symmetrically placed, distant a from either end.

v_{max} at $x = \frac12 l$; found by making $x=\frac12 l$ in preceding value of v on right, and doubling; $\therefore v_{max} = \frac{2W}{EIl}\left(\frac18 al^3 - \frac1{48}al^3 - \frac1{12}a^3l - \frac16 al^3 + \frac16 a^3l\right)$ $= -\frac{Wa}{24EI}(3l^2-4a^2)$. The deflection under a weight will be found by adding the values of v_a and v_{l-a} of the preceding case.

VIII. Beam of length l, fixed at both ends, uniformly loaded with w per foot over whole length.

Origin at left point of support.



The reactions and end moments are now unknown. The beam may be considered either as built in at its ends (as at right), or as having a couple Qb applied at each point of support (as at left), of a magnitude just sufficient to keep the tangent horizontal. The reaction at either end will then be $\frac12 wl + Q$.

$M = (\frac12 wl+Q)x - \frac12 wx^2 - Q(b+x) = \frac12 wlx - \frac12 wx^2 - Qb.$ Compare IV.

A constant subtractive or negative moment Qb is felt over the whole span.

$\frac{d^2v}{dx^2} = \frac{1}{EI}(\frac12 wlx - \frac12 wx^2 - Qb)$.

$\frac{dv}{dx} = \frac{1}{EI}(\frac14 wlx^2 - \frac16 wx^3 - Qbx + C)$. $\frac{dv}{dx} = 0$ when $x=0$; $\therefore C=0$.

$v = \frac{1}{EI}(\frac1{12} wlx^3 - \frac1{24}wx^4 - \frac12 Qbx^2 + C')$. $v=0$ when $x=0$; $\therefore C'=0$.

$\frac{dv}{dx}=0$ for $x=l$; $\therefore \frac14 wl^3 - \frac16 wl^3 = Qbl$; or $-Qb = -\frac1{12}wl^2$, the negative moment at each point of support. $\therefore M$ at middle $= \frac1{24}wl^2$.

Substituting value of Qb, we get $\frac{d^2v}{dx^2} = \frac{w}{2EI}(lx - x^2 - \frac16 l^2)$;

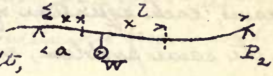
$\frac{dv}{dx} = \frac{w}{2EI}(\frac12 lx^2 - \frac13 x^3 - \frac16 l^2x)$; $v = \frac{w}{2EI}(\frac16 lx^3 - \frac1{12}x^4 - \frac1{12}l^2x^2)$.

The slope, $\frac{dv}{dx}$, is maximum at points of contraflexure, where M or the second diff. coefficient is zero; $\therefore x^2 - lx = -\frac16 l^2$, and $x = \frac12 l \pm \frac12 l\sqrt{\frac13}$. The second term, $\frac12 l\sqrt{\frac13}$, is therefore the distance from the middle each way to points of contraflexure.

What would M_{max} be in Case IV, for span of $l\sqrt{\frac13}$? $\frac1{24}wl^2$.

If $x = \tfrac12 l$, $v_{max} = \frac{w}{2EI}\left(\tfrac1{48}l^4 - \tfrac1{192}l^4 - \tfrac1{48}l^4\right) = \frac{wl^4}{384EI}$; only 1/5 the value of Case IV.

IX. Beam fixed at both ends, single weight W in middle. By a similar analysis to the preceding it may be found that the negative bending moment at ends is $-\frac{Wl}{8}$; that the max. positive moment at middle is $\frac{Wl}{8}$; that the points of contraflexure are $\tfrac14 l$ from each end; and that the max. deflection is $\frac{Wl^3}{192EI}$.



X. Beam of span l, fixed at left, supported at right, weight W at distance $\underline{a}$ from left. Origin at left. Reaction at right is P_2.

On left of weight.

$M = P_2(l-x) - W(a-x)$

$\frac{d^2v}{dx^2} = \frac{1}{EI}[P_2(l-x) - W(a-x)]$

$\frac{dv}{dx} = \frac{1}{EI}[P_2 lx - \tfrac12 P_2x^2 - Wax + W\frac{x^2}{2} + C)$

$\frac{dv}{dx} = 0$ when $x = 0$; $\therefore C = 0$

$v = \frac{1}{EI}(\tfrac12 P_2 lx^2 - \tfrac16 P_2x^3 - \tfrac12 Wax^2 + \tfrac16 Wx^3 + C')$

$v = 0$ when $x = 0$; $\therefore C' = 0$.

On right of weight.

$M = P_2(l-x)$.

$\frac{d^2v}{dx^2} = \frac{1}{EI}P_2(l-x)$

$\frac{dv}{dx} = \frac{1}{EI}(P_2 lx - \tfrac12 P_2x^2 + C'')$

$v = \frac{1}{EI}(\tfrac12 P_2 lx^2 - \tfrac16 P_2x^3 + C''x + C''')$

If $x = a$, $\frac{dv}{dx}$ on left $= \frac{dv}{dx}$ on right; $\therefore C'' = -Wa^2 + \tfrac12 Wa^2 = -\tfrac12 Wa^2$.

" " " v on left $= v$ on right; $\therefore C''' = -\tfrac12 Wa^3 + \tfrac16 Wa^3 + \tfrac12 Wa^3 = \tfrac16 Wa^3$.

If $x = l$, v on right $= 0$; $\therefore \tfrac12 P_2 l^3 - \tfrac16 P_2 l^3 - \tfrac12 Wa^2 l + \tfrac16 Wa^3 = 0$ or

$$P_2 = Wa^2(\tfrac12 l - \tfrac16 a)\frac{3}{l^3} = \frac{Wa^2}{2l^3}(3l-a).$$

Substitute this value of P_2 in above equations and obtain

$\frac{d^2v}{dx^2} = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(l-x) - (a-x)\right]$ $\quad\frac{d^2v}{dx^2} = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(l-x)\right]$

$\frac{dv}{dx} = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(lx - \tfrac12 x^2) - ax + \tfrac12 x^2\right]$ $\quad\frac{dv}{dx} = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(lx - \tfrac12 x^2) - \tfrac12 a^2\right]$

$v = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(\tfrac12 lx^2 - \tfrac16 x^3) - \tfrac12 ax^2 + \tfrac16 x^3\right]$ $\quad v = \frac{W}{EI}\left[\frac{a^2}{2l^3}(3l-a)(\tfrac12 lx^2 - \tfrac16 x^3) - \tfrac12 a^2x + \tfrac16 a^3\right]$

M_{max} by inspection, when $x = 0$, or $x = a$.

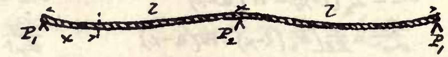
$x = 0$, $M = W\frac{a^2}{2l^3}(3l^2 - al) - Wa$. $\quad x = a$, $M = W\frac{a^2}{2l^3}(3l-a)(l-a)$.

Point of contraflexure where $\frac{d^2v}{dx^2} = 0$, or $\frac{a^2}{2l^3}(3l-a)(l-x) - (a-x) = 0$.

Max. deflection where $\frac{dv}{dx}=0$, on right or left $\frac{l}{2}$, according to value of $\underline{a}$.

Various problems are possible, as to values of $\underline{a}$ which shall make $\frac{d^2v}{dx^2}$, $\frac{dv}{dx}$ or v a maximum, with position of corresponding points of contraflexure and maximum deflection. They are more curious than useful.

In solving the more intricate problems in flexure of beams, each equation of condition can be used but once in the same problem, and as many unknown quantities can be determined as there are independent equations of condition. The reactions and end moments are usually unknown, and must be found through such flexure equations as have just been used.

XI. Beam of span l; 

fixed or horizontal at P_2, supported at P_1, uniform load of $\underline{w}$ per foot. Origin at P_1.

$\frac{d^2v}{dx^2}=\frac{1}{EI}(P_1x-\frac{1}{2}wx^2)$; $\quad \frac{dv}{dx}=\frac{1}{EI}(\frac{1}{2}P_1x^2-\frac{1}{6}wx^3+C)$.

$\frac{dv}{dx}=0$, when $x=l$; ∴ $C=\frac{1}{6}wl^3-\frac{1}{2}P_1l^2$. ∴

$\frac{dv}{dx}=\frac{1}{EI}(\frac{1}{2}P_1x^2-\frac{1}{6}wx^3+\frac{1}{6}wl^3-\frac{1}{2}P_1l^2)$.

$v=\frac{1}{EI}(\frac{1}{6}P_1x^3-\frac{1}{24}wx^4+\frac{1}{6}wl^3x-\frac{1}{2}P_1l^2x+C')$.

$v=0$ when $x=0$; ∴ $C'=0$. If $x=l$, $v=0$; ∴

$P_1=(\frac{1}{24}-\frac{1}{6})wl^4\div(\frac{1}{6}-\frac{1}{2})l^3=\frac{3}{8}wl$. Substitute, and

$\frac{d^2v}{dx^2}=\frac{w}{EI}(\frac{3}{8}lx-\frac{1}{2}x^2)$

$\frac{dv}{dx}=\frac{w}{EI}(\frac{3}{16}lx^2-\frac{1}{6}x^3+\frac{1}{6}l^3-\frac{3}{16}l^3)=\frac{w}{EI}(\frac{3}{16}lx^2-\frac{1}{6}x^3-\frac{1}{48}l^3)$

$v=\frac{w}{EI}(\frac{1}{16}lx^3-\frac{1}{24}x^4-\frac{1}{48}l^3x)$.

For v_{max}, $\frac{dv}{dx}=0$, or $\frac{1}{6}x^3-\frac{3}{16}lx^2=-\frac{1}{48}l^3$, or $x^3-\frac{9}{8}lx^2=-\frac{1}{8}l^3$.

As a minimum value of v, or $v=0$, occurs for $x=l$, divide $8x^3-9lx^2+l^3=0$ by $x-l=0$, and obtain $8x^2-lx-l^2=0$ or

$x=l\frac{1\pm\sqrt{33}}{16}=0.4215l$. Then $v_{max}=0.0054\frac{wl^4}{EI}$.

To find points of max. M, put $\frac{d^3v}{dx^3}=0$. $\frac{3}{8}l-x=0$; $x=\frac{3}{8}l$.

Also, by inspection, $\mathcal{M}_{max}$ when $x = l$.

For $x = 3/8\,l$, $\mathcal{M} = (9/64 - 9/128)wl^2 = 9/128\,wl^2$. For $x = l$, $\mathcal{M} = (3/8 - \frac{1}{2})wl^2 = -1/8\,wl^2$.

For point of contraflexure, $\mathcal{M} = 0$, or $3/8\,lx - \frac{1}{2}x^2 = 0$; $x = 3/4\,l$.

It will be seen that a beam of two equal spans $\underline{l}$, uniformly loaded, has end reactions of $3/8\,wl$, and middle reaction of $2 \times 5/8\,wl = 5/4\,wl$; that points of contraflexure divide the span at $1/4\,l$ from the middle pier; and that the bending moment at the middle of this space of $3/4\,l$ is, as above, $1/4 \times 3/4 \times 1/8\,wl^2 = 9/128\,wl^2$.

The general discussion of a continuous beam will come under § 178.

§ 169. Deflection of Beams. — To explain the book.

Since we may write $f'_a + f'_b : f' = h : y_1$; $\frac{1}{r_0} = \frac{f'}{Ey_1}$ will become

$$\frac{1}{r_0} = \frac{f'_a + f'_b}{Eh}. \quad (2A.) \qquad \text{Since } \mathcal{M}_0 = \frac{f' I_0}{y_1};$$

$$\frac{1}{r_0} = \frac{\mathcal{M}_0}{EI_0} = \frac{f'}{Ey_1} = \frac{f'}{Enh}. \quad \text{Curvature varies inversely as } h.$$

At any other cross-section, multiplying and dividing by $\frac{\mathcal{M}_0}{I_0}$,

$$\frac{1}{r} = \frac{\mathcal{M}}{EI} = \frac{\mathcal{M}_0}{I_0} \cdot \frac{I_0}{\mathcal{M}_0} \cdot \frac{\mathcal{M}}{EI} = \frac{f'}{Enh} \cdot \frac{\mathcal{M} I_0}{I \mathcal{M}_0}; \text{ the latter a numerical factor.}$$

$$\frac{dv}{dx} = \tan i = \int_0^{x} \frac{dx}{r} = \frac{f'}{Enh} \int_0^{x'} \frac{\mathcal{M} I_0}{I \mathcal{M}_0}\,dx. \quad (5.)$$

$$\tan i_1 = \frac{f'}{Enh} \int_0^c \frac{\mathcal{M} I_0}{I \mathcal{M}_0}\,dx = \frac{f}{Enh}\,m''c. \quad (7.) \quad \therefore$$

Slope varies directly as $\underline{c}$, inversely as $\underline{h}$.

$$v = \int_0^x \frac{dv}{dx}\,dx = \frac{f'}{Enh} \int_0^{x'}\!\!\int_0^{x'} \frac{\mathcal{M} I_0}{I \mathcal{M}_0}\,dx^2. \quad (10.)$$

$$v_1 = \frac{f'}{Enh} \int_0^c\!\!\int_0^{x'} \frac{\mathcal{M} I_0}{I \mathcal{M}_0}\,dx^2 = \frac{f'}{Enh}\,n''c^2. \quad (13.) \quad \therefore$$

Deflection varies directly as $\underline{c}^2$, inversely as $\underline{h}$.

Beams under any load. —

$$\frac{1}{r} = \frac{\mathcal{M}}{EI} = \frac{\mathcal{M}}{En'bh^3}. \quad \therefore$$ Curvature, under equal bending moments varies inversely as $\underline{b}$ and $\underline{h}^3$. $\tan i' = \int_0^{x'} \frac{dx}{r} = \frac{1}{E} \int_0^{x'} \frac{\mathcal{M}}{I}\,dx$.

$$\tan i'_1 = \frac{1}{E} \int_0^c \frac{\mathcal{M}}{I}\,dx = \frac{\mathcal{M}_0}{EI_0} \int_0^c \frac{\mathcal{M} I_0}{\mathcal{M}_0 I}\,dx = \frac{mWl}{En'bh^3}\,m''c = \frac{mm''}{n'} \cdot \frac{Wlc}{Ebh^3} = \frac{m'''}{n'} \cdot \frac{Wlc}{Ebh^3}. \quad (6)$$

Under equal loads, therefore, slopes vary directly as $\underline{c}^2$, or lc, inversely as $\underline{b}$ and $\underline{h}^3$.

$$v' = \frac{1}{E} \int_0^{x'}\!\!\int_0^{x'} \frac{\mathcal{M}}{I}\,dx^2. \qquad v'_1 = \frac{1}{E} \int_0^c\!\!\int_0^{x'} \frac{\mathcal{M}}{I}\,dx^2 = \frac{\mathcal{M}_0}{EI_0} \int_0^c\!\!\int_0^{x'} \frac{\mathcal{M} I_0}{I \mathcal{M}_0}\,dx^2 = \frac{mWl}{En'bh^3}\,n''c^2$$

$$= \frac{mn''}{n'} \cdot \frac{Wlc^2}{Ebh^3} = \frac{m'''}{n'} = \frac{Wlc^2}{Ebh^3}. \quad (12)$$ Under equal loads, deflections vary directly as c^3, or lc^2, and inversely as $\underline{b}$ and $\underline{h}^3$.

Beam fixed at one end, $l = c$; Beam supported at both ends, $l = 2c$.

$$\left.\begin{array}{l}\text{Beam fixed one end, } m''' = m m'' \\ \text{" supported both ends, } m''' = 2 m m''\end{array}\right\}(8.) \qquad \left.\begin{array}{l} n''' = m n'' \\ n''' = 2 m n''\end{array}\right\}(14.)$$

Beam having cross-sections of equal strength:— By (1.), p. 256,

$f_a + f_b : f_a : f_b = h : y_a : y_b$. $\therefore \frac{1}{r_0} = \frac{f'}{Ey} = \frac{f_a + f_b}{Eh}$. (2 A.)

$\tan i_1 = \frac{m'' f' c}{E m' h} = \frac{m''(f_a' + f_b')c}{Eh}$. (7 A.) $v_1 = \frac{n'' f' c^2}{E m' h} = \frac{n''(f_a' + f_b')c^2}{Eh}$. (13 A.)

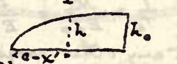
p. 274. A. See previous examples.

" B. Uniform Strength and Uniform Depth.—

$M = n f b h^2$, and varies as bh^2; $I = n' b h^3$, and varies as bh^3; $\therefore \frac{M}{I}$ varies as $\frac{1}{h}$. h is constant; $\therefore \frac{M}{I}$ is constant. $\frac{1}{r} =$ constant. $\frac{M I_0}{I M_0} = \frac{h}{h} = 1$.

$\therefore \int_0^c dx = c$; $m'' = 1$. $\int_0^c \int_0^{x'} dx^2 = \frac{c^2}{2}$; $n'' = 1/2$. Multiply by m.

C. Uniform Strength and Uniform Breadth.— $\frac{M}{I}$ varies as $\frac{1}{h}$; $\frac{I_0}{M_0}$ varies as $\frac{h_0}{1}$; $\therefore \frac{M I_0}{I M_0} = \frac{h_0}{h}$.



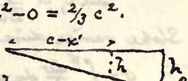
X. Fixed at one end, loaded at other. $h_0^2 : h^2 = c : c - x$.

$\frac{M I_0}{I M_0} = \sqrt{\frac{c}{c-x'}}$; $\int_0^{x'} \sqrt{\frac{c}{c-x'}}\, dx = \int_0^{x'} c^{1/2}(c-x')^{-1/2} dx = \Big[_0^{x'} -2c^{1/2}(c-x)^{1/2} = -2c^{1/2}(c-x)^{1/2} + 2c$.

When $x' = c$; $\int_0^c \frac{M I_0}{I M_0} dx = 0 + 2c$; $\therefore m'' = 2$.

$\int_0^c (-2c^{1/2}(c-x')^{1/2} + 2c)\, dx = \Big[_0^c \left(\frac{4}{3} c^{1/2}(c-x)^{3/2} + 2cx\right) = 0 + 2c^2 - \frac{4}{3}c^2 - 0 = \frac{2}{3} c^2$.

$\therefore n'' = 2/3$. $m = 1$; $\therefore m''' = 2$; $n''' = 2/3$.



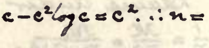
XI. Fixed at one end, uniformly loaded. $\frac{h_0}{h} = \frac{c}{c-x'}$.

$\int_0^{x'} \frac{c}{c-x'} dx = -c\Big[_0^{x'} \log(c-x') = -c \log(c-x') + c \log c$. $c \int_0^c \frac{dx}{c-x'} = -c \log 0 + c \log c$

$= \infty c$; $\therefore m'' = \infty$. $-c \int_0^c \{\log(c-x')^* - \log c\} dx = -c \Big[_0^c \{x' \log(c-x') - x'$

$- c \log(c-x) - x' \log c\} = -c^2 \log 0 + c^2 + c^2 \log 0 + c^2 \log c - c^2 \log c = c^2$. $\therefore n = 1$.

$m = 1/2$; $n''' = 1/2$.

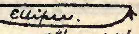


XII. Supported at both ends, loaded in the middle.

$\frac{h_0}{h} = \frac{\sqrt{c}}{\sqrt{c-x}}$. Same as X. $m = 1/2$; $\therefore m''' = 1$; $n''' = 1/3$.

---

* $\log(c-x') = u$; $dx = dv$; $x = v$; $-\frac{dx}{c-x'} = du$. $\therefore \int \log(c-x')\, dx = x' \log(c-x')$ $+ \int \frac{x'}{c-x'} dx$: $\frac{x'}{c-x'}$ by division $= -1 + \frac{c'}{c-x'}$. $\therefore \int \frac{x'}{c-x'} dx = -\int dx + c \int \frac{dx}{c-x'}$ $= -x' - c \log(c-x')$.



XIII. Supported at both ends, uniformly loaded.— $\frac{h_0}{h} = \frac{c}{\sqrt{c^2-x'^2}}$. $c\int_0^{x} \frac{dx}{\sqrt{c^2-x'^2}} = c\int_0^{x'} \frac{\frac{dx}{c}}{\sqrt{(1-\frac{x'^2}{c^2})}} = c\Big[_0^{x'} \sin^{-1}\frac{x'}{c}$
$= c\sin^{-1}\frac{x'}{c}$. $c\Big[_0^{c}\sin^{-1}\frac{x'}{c} = c\sin^{-1}\frac{c}{c} = \frac{\pi}{2}c = 1.5708c$. $\therefore m'' = 1.5708$.

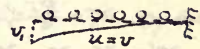
$\int u dv = uv - \int v du$; $\therefore c\int_0^c \sin^{-1}\frac{x'}{c}dx = c\Big[_0^c x'\sin^{-1}\frac{x'}{c} - c\int_0^{x'} x' \frac{\frac{dx}{c}}{\sqrt{(1-\frac{x'^2}{c^2})}} =$
$c\Big[_0^c x'\sin^{-1}\frac{x'}{c} - c\int_0^c (c^2-x'^2)^{-\frac{1}{2}} x' dx = c\Big[_0^c\{x'\sin^{-1}\frac{x'}{c} + (c^2-x'^2)^{\frac{1}{2}}\} = c^2 \tfrac{1}{2}\pi - c^2$
$= (\tfrac{1}{2}\pi - 1)c^2$. $\therefore n'' = \tfrac{1}{2}\pi - 1 = 0.5708$. $m = \tfrac{1}{4}$; $\therefore m''' = \frac{1.5708}{4}$; $n''' = \frac{0.5708}{4}$.

See note, top of p. 275.

§170. A similar expression, applying to the case of any given load W which produces f, may be deduced from (12.) p. 273.

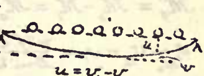
$$v_1' = \frac{n'''Wc^3}{En'bh^3} = \frac{mWln''c^2}{En'bh^3} = \frac{nfbh^2n''c^2}{En'bh^3} = \frac{nn''}{n'}\cdot\frac{f}{E}\cdot\frac{c^2}{h};\ \therefore \frac{v_1'}{l} = \frac{nn''}{4n'}\cdot\frac{f}{E}\cdot\frac{l}{h}.$$

Compare (1.)



§173. Resilience or Spring of a Beam.—

Resilience or work done $= \tfrac{1}{2}Wv_1 = \tfrac{1}{2}\int uw dx$.

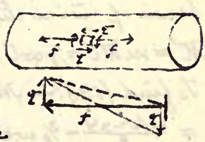


Beam supported at both ends, load in middle,

$mWl = nf'bh^2$. $W = \frac{nf'bh^2}{ml}$. By (13.) p. 273, $\frac{Wv_1}{2} = \tfrac{1}{2}\cdot\frac{nf'bh^2}{ml}\cdot\frac{n''f'c^2}{En'h} =$
$\tfrac{1}{2}\frac{nn''}{mn'}\cdot\frac{f'^2}{E}\cdot\frac{bh^2c^2}{lh} = \frac{4}{2}\cdot\frac{nn''}{n'}\cdot\frac{f'^2}{E}\cdot\frac{bhc^2}{2c} = \frac{nn''}{m'}\cdot\frac{f'^2}{E}\cdot bhc$.

(4.) is similar, for a beam fixed at one end and loaded at the other. Compare top p. 280.

§174. Effect of Twisting on a Beam.— Combination of bending moment and moment of torsion.



Let the stress on extreme fibre of beam from bending moment $= f$. Let the stress on same extreme point of section from the torsional moment $= q$, a shearing stress. Referring to §108, and Notes, p. 7, VIII, deviation of principal stresses by a shearing stress, construct as above, p_y being zero. One stress is the resultant of f and q, both acting on the same plane; the other is q.

Then $p_1 = \tfrac{1}{2}f + \sqrt{(\tfrac{1}{4}f^2 + q^2)}$ (a.)

Since $M = \frac{fI_x}{y_1}$, (2 A.) p. 252, and the moment of torsion $T = \frac{qI_p}{y_1}$, where I_x = rectangular and I_p = polar moment of inertia of cross-section, which latter value may be approximately written $= 2 I_x$ (exactly true when the beam is square or circular), — we may multiply (a) by $\frac{I_x}{y_1}$, and write

$$\mathcal{M}_1 = \frac{\mathcal{M}}{2} + \sqrt{\left(\frac{\mathcal{M}^2}{4} + \frac{T^2}{4}\right)}. \quad (2.)$$

As the section on which the new principal stress acts is not the same as the original right-section, (2.) involves another small inaccuracy.

§176. Beam Fixed at both Ends. — See Case IX. p. 54, Notes. The results on pp. 283-6 may be found by the following notes:—

Ex. I. By (6.) p. 271, $i'_1 = \frac{m''' W c^2}{EI}$. But $i'_1 = \int_0^c \frac{\mathcal{M}}{EI} dx$, and, for a constant moment of flexure, $i'_1 = \frac{\mathcal{M}}{EI}\int_0^c dx = \frac{\mathcal{M}}{EI} c$; $\therefore$ for a constant moment, $\mathcal{M}_1 = \frac{EI}{c} i'_1 = \frac{EI}{c} \cdot \frac{2 m m'' W c^2}{EI} = m'' 2 m W c = m'' \mathcal{M}_0$. (1.)

$$\mathcal{M}'_0 = \mathcal{M}_0 - \mathcal{M}_1 = \mathcal{M}_0 - m'' \mathcal{M}_0 = (1 - m'') \mathcal{M}_0. \quad (2.)$$

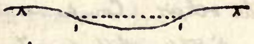
$v_{1_{\mathcal{M}_0}} - v_{\mathcal{M}_1} = v_1 = \frac{n'' f' c^2}{E m' h} - \frac{1}{2} \frac{f' c^2}{m' h}$, since $n'' = 1/2$ for constant $\mathcal{M}$.

By (2 A.) p. 252, $\frac{f'}{m' h} = \frac{\mathcal{M}}{I}$; $\therefore$ $v_1 = \frac{n'' \mathcal{M}_0 c^2}{EI} - \frac{1}{2} \frac{\mathcal{M}_1 c^2}{EI} = \frac{n'' \mathcal{M}_1 c^2}{m'' EI} - \frac{1}{2} \frac{\mathcal{M}_1 c^2}{EI}$

$$= \left(\frac{n''}{m''} - \frac{1}{2}\right) \frac{\mathcal{M}_1 c^2}{EI} = \left(n'' - \frac{m''}{2}\right) \frac{\mathcal{M}_0 c^2}{EI}. \quad (4.)$$

Ex. III. By Case V., p. 274, $\frac{\mathcal{M}}{\mathcal{M}_0} = 1 - \frac{x'^2}{c^2}$ or $\mathcal{M} = \left(1 - \frac{x^2}{c^2}\right)\mathcal{M}_0 = \left(1 - \frac{x'^2}{c^2}\right)\frac{3}{2}\mathcal{M}_1$. At the point of contraflexure, $\mathcal{M}$ must equal $\mathcal{M}_1$, or $3/2\left(1 - \frac{x'^2}{c^2}\right) = 1$; $\therefore$ $1 - \frac{x'^2}{c^2} = \frac{2}{3}$.

$x'^2 = c^2 - 2/3 c^2$; $x' = \frac{c}{\sqrt{3}}$. (9.)



Ex. IV. $v_1 = \frac{n'' f' c^2}{E m' h} = \frac{1}{2} \cdot \frac{f' c^2}{E m' h}$. Deflection varies as c^2; $\therefore$ a beam of twice the span has four times the deflection.

$$\mathcal{M}'_0 = m W l' = 1/8\, w l'^2 = 1/8\, w c^2 = \frac{2wc \cdot c}{16} = \frac{Wc}{16} = \frac{Wl}{32}. \quad (11.)$$

To find (13.):— $\mathcal{M} - \mathcal{M}_1 = \mathcal{M} - (\mathcal{M}_0 - \mathcal{M}'_0) = \frac{w(c^2 - x^2)}{2} - \left(\frac{wc^2}{2} - \frac{wc^2}{8}\right)$

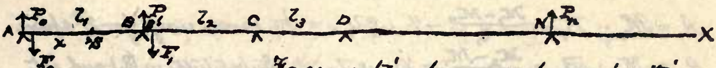
$= \frac{w(c^2 - x^2)}{2} - 3/8\, w c^2$. $\mathcal{M} = n f b h^2$; h is constant; $\therefore$ b varies as $\mathcal{M}$.

$\frac{b}{b_1} = \frac{\mathcal{M} - \mathcal{M}_1}{\mathcal{M}_1} = \frac{1/2 (c^2 - x^2) - 3/8 c^2}{3/8 c^2}$. $b = 1/3\left(\frac{4(c^2 - x^2)}{c^2} - 3\right) b_1 = 1/3\left(1 - \frac{4x^2}{c^2}\right) b_1$. (13.)

§177. By comparing this section with IX., p. 54 and XI., p. 56, of Notes, the statement of this section is seen to be inaccurate.

§178. Continuous Girders. — The following discussion may precede or be substituted for this section.

Clapeyron's Formula for Continuous Loading. To find Reactions, Shears and Bending Moments in a horizontal continuous beam,



Forces acting downwards are positive.

loaded with w_1, w_2, w_3, etc., loads per running foot over the successive spans l_1, l_2, l_3, etc., as sketched. P_0, P_1, etc., denote reactions; M_0, M_1, M_2, etc., are the unknown moments at points of support A, B, C, &c. Origin at A.

Consider the condition of equilibrium of the first span AB, or l_1. If we take moments on the left of a section S, distant x from the origin A, their sum, or the bending moment at S, will always be a function of the second degree (See p. 40, Notes), in which the term in x^2 will be $\frac{1}{2} w_1 x^2$. Therefore, using A_1 and B_1 for literal coefficients, we write

$$M = EI\frac{d^2v}{dx^2} = A_1 + B_1 x + \tfrac{1}{2} w_1 x^2. \quad (1.)$$

Let $i_0, i_1, i_2 \dots i_n$ = tangent of inclination of neutral axis at A, B, C ..... N. Integrate (1.) and obtain between limits 0 and x,

$$EI\left(\frac{dv}{dx} - i_0\right) = A_1 x + \tfrac{1}{2} B_1 x^2 + \tfrac{1}{6} w_1 x^3. \quad (2.) \qquad \text{When } x = l_1,$$

$$EI(i_1 - i_0) = A_1 l_1 + \tfrac{1}{2} B_1 l_1^2 + \tfrac{1}{6} w_1 l_1^3. \quad (3.)$$

Integrating (2.), and determining constant by $v = 0$ when $x = 0$,

$$EI(v - i_0 x) = \tfrac{1}{2} A_1 x^2 + \tfrac{1}{6} B_1 x^3 + \tfrac{1}{24} w_1 x^4. \quad (4.)$$

Make $x = l_1$; then $v_1 = 0$; and we get

$$-EI\, i_0 l_1 = \tfrac{1}{2} A_1 l_1^2 + \tfrac{1}{6} B_1 l_1^3 + \tfrac{1}{24} w_1 l_1^4, \text{ or}$$

$$-EI\, i_0 = \tfrac{1}{2} A_1 l_1 + \tfrac{1}{6} B_1 l_1^2 + \tfrac{1}{24} w_1 l_1^3. \quad (5.)$$

Eliminate i_0 by subtracting (5.) from (3.).

$$EI\, i_1 = \tfrac{1}{2} A_1 l_1 + \tfrac{1}{3} B_1 l_1^2 + \tfrac{1}{8} w_1 l_1^3. \quad (6.)$$

If the origin be taken at B in place of A, we shall find for the second span an equation like (5.), or

$$-EI\, i_1 = \tfrac{1}{2} A_2 l_2 + \tfrac{1}{6} B_2 l_2^2 + \tfrac{1}{24} w_2 l_2^3. \quad (7.) \qquad \text{Add (6.) and (7.).}$$

$$0 = \tfrac{1}{2} A_1 l_1 + \tfrac{1}{2} A_2 l_2 + \tfrac{1}{3} B_1 l_1^2 + \tfrac{1}{6} B_2 l_2^2 + \tfrac{1}{8} w_1 l_1^3 + \tfrac{1}{24} w_2 l_2^3. \quad (8.)$$

To determine A_1, A_2, B_1 and B_2.

(1.) must reduce to M_0 for $x = 0$, and to M_1 for $x = l_1$; $\therefore$

$A_1 = M_0$; $B_1 = \frac{M_1 - M_0}{l_1} - \frac{1}{2} w_1 l_1$. In same way, for 2^d span, $A_2 = M_1$; $B_2 = \frac{M_2 - M_1}{l_2} - \frac{1}{2} w_2 l_2$. Substitute in (8.) and get

$$0 = \tfrac{1}{2} M_0 l_1 + \tfrac{1}{2} M_1 l_2 + \tfrac{1}{3}(M_1 - M_0) l_1 - \tfrac{1}{6} w_1 l_1^3 + \tfrac{1}{6}(M_2 - M_1) l_2$$
$$- \tfrac{1}{12} w_2 l_2^3 + \tfrac{1}{8} w_1 l_1^3 + \tfrac{1}{24} w_2 l_2^3; \quad \text{or}$$

$$M_0 l_1 + 2 M_1 (l_1 + l_2) + M_2 l_2 = \tfrac{1}{4}(w_1 l_1^3 + w_2 l_2^3) \quad (9.)$$

which is the desired formula.

If the two spans are equal and have the same load, we have

$$M_0 + 4 M_1 + M_2 = \tfrac{1}{2} w l^2. \quad (10.)$$

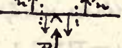
If we have $\underline{n}$ spans, we shall have $n-1$ equations between $n+1$ quantities $M_0, M_1, \ldots M_n$. But if the beam is simply placed on the points of support, the extremities being free, $M_0 = 0$, $M_n = 0$, and there remain $n-1$ equations to determine $M_1 \ldots M_{n-1}$.

As the shear is the first differential coefficient of the bending moment, (p. 39, Notes) we have $\frac{dM}{dx} = F$ on left of section.

From (1.) $F = B_1 + w_1 x$, and we shall have a similar equation for each span. As $\underline{w}$ is positive, $+F$ acts downwards.

It will be seen that A_1, A_2 &c., are pier moments; that B_1, B_2 &c., are shears at $\underline{\text{right}}$ of piers. (Compare Notes, p. 40.)

The reaction at any pier is equal and opposite to the sum of the shears on each side of the pier, that on the left of the pier having its sign reversed, to give it on the right of the section.



Application: — 2 spans, 3 supports, equidistant. $w_1 = w_2 = w$. $M_0 = 0$; $M_2 = 0$. (10.) becomes $4 M_1 = \frac{1}{2} w l^2$; or $M_1 = \frac{1}{8} w l^2$. $\therefore$ $A_1 = 0$; $A_2 = \frac{1}{8} w l^2$; $B_1 = \frac{1}{8} wl - \frac{1}{2} wl = -\frac{3}{8} wl$; $B_2 = -\frac{1}{8} wl - \frac{1}{2} wl = -\frac{5}{8} wl$.

(5.) gives $i_0 = -\frac{1}{EI}(0 - \frac{3}{48} wl^3 + \frac{1}{24} wl^3) = + \frac{wl^3}{48EI}$; (6.) gives $i_1 = \frac{1}{EI}(0 - \frac{1}{8} wl^3 + \frac{1}{8} wl^3) = 0$; and the analogous equation for the 2^d span is

$$i_2 = \frac{1}{EI}(\tfrac{1}{16} wl^3 - \tfrac{5}{24} wl^3 + \tfrac{1}{8} wl^3) = -\frac{wl^3}{48EI}. \quad \text{Compare } i_0 \text{ and } i_2.$$

(11.) gives, for $x = 0$, $F_0 = -\frac{3}{8} wl$, and, for $x = l$, $F_1' = \frac{5}{8} wl$.

A similar equation for 2^d span gives $F_1 = -\frac{5}{8} wl$; $F_2' = \frac{3}{8} wl$.

Then $P_0 = -\frac{3}{8} wl$; $P_1 = -F_1' + F_1 = -\frac{5}{4} wl$; $P_2 = -F_2' = -\frac{3}{8} wl$.

Equation (2.) gives $EI\frac{dv}{dx} = \frac{1}{48} wl^3 - \frac{3}{16} wlx^2 + \frac{1}{6} wx^3$;

and (4.) gives $EI \cdot v = \frac{1}{48} wl^3 x - \frac{1}{16} wlx^3 + \frac{1}{24} wx^4$. (11.)

These equations determine the slope and deflection at each point. Putting $\frac{dv}{dx} = 0$, we obtain an equation containing the root $x = l$, already known. Taking out that root, we have remaining an equation of the second degree, which gives $x = 0.4215\,l$, the point of maximum deflection, which can be substituted in (11.).

From (1.), $M = -\frac{3}{8} wlx + \frac{1}{2} wx^2$. If $M = 0$, $-\frac{3}{8} l + \frac{1}{2} x = 0$; or $x = \frac{3}{4} l$, the point of contraflexure. Differentiate M, and $F = -\frac{3}{8} wl + wx$. If $F = 0$, M_{max} at $\frac{3}{8} l$.

$M_{max} = -\frac{3}{8} wl \cdot \frac{3}{8} l + \frac{1}{2} w \cdot \frac{9}{64} l^2 = -\frac{9}{128} wl^2$. (Compare XI., p. 56.)

If we have 5 equal spans with $\underline{w}$ per foot, on each, we have $M_0 + 4M_1 + M_2 = \frac{1}{2} wl^2 = M_1 + 4M_2 + M_3 = M_2 + 4M_3 + M_4 = M_3 + 4M_4 + M_5$. $M_0 = 0$; $M_5 = 0$. Hence $M_1 = \frac{4}{38} wl^2 = M_4$; $M_2 = \frac{3}{38} wl^2 = M_3$.
$F_0 = \frac{15}{38} wl$; $F_1' = \frac{20}{38} wl$; $F_2 = \frac{17}{38} wl$; $F_3 = \frac{18}{38} wl$; $F_4 = \frac{23}{38} wl$; $F_5 = 0$.
Then $P_0 = \frac{15}{38} wl = P_5$; $P_1 = \frac{43}{38} wl = P_4$; $P_2 = \frac{37}{38} wl = P_3$. Their sum must equal $5wl$.

It has been found that the numerical coefficients for moments and reactions at points of support, where all spans are equal, and the load is uniformly distributed over the whole length, may be tabulated easily for reference and use. Thus the values just obtained for five equal spans, for the pier moments and reactions, can be selected from the lines marked V.

The rule for writing either table is as follows:— For an even number of spans, the numbers in any horizontal line are obtained by multiplying the fraction above, in any diagonal row, both numerator and denominator, by two, and adding numerator and denominator of the fraction preceding that. Thus $\frac{2 \times 1 + 1}{2 \times 10 + 18} = \frac{3}{38}$. For an odd number of spans, add

Pier Moments.	Reactions.
I. $\frac{0}{2}$ $\frac{0}{2}$ I.	I. $\frac{1}{2}wl$. $\frac{1}{2}wl$. I.
II. 0 $\frac{1}{8}wl^2$ 0 II.	II. $\frac{3}{8}wl$. $\frac{10}{8}wl$. $\frac{3}{8}wl$. II.
III. 0 $\frac{1}{10}wl^2$ $\frac{1}{10}wl^2$ 0 III.	III. $\frac{4}{10}wl$. $\frac{11}{10}wl$. $\frac{11}{10}wl$. $\frac{4}{10}wl$. III.
IV. 0 $\frac{3}{28}wl^2$: $\frac{2}{28}wl^2$: $\frac{3}{28}$ 0 IV.	IV. $\frac{11}{28}wl$. $\frac{32}{28}wl$. $\frac{26}{28}wl$. $\frac{32}{28}$ $\frac{11}{28}$. IV.
V. 0 $\frac{4}{38}wl^2$: $\frac{3}{38}$ $\frac{3}{38}$ $\frac{4}{38}$ 0 V.	V. $\frac{15}{38}wl$. $\frac{43}{38}$ $\frac{37}{38}$ $\frac{37}{38}$ $\frac{43}{38}$ $\frac{15}{38}$ V.
&c. &c. &c.	&c. &c. &c.

the two preceding fractions in the same diagonal column, numerator to numerator and denominator to denominator; thus $\frac{1+3}{10+28} = \frac{4}{38}$. By remembering the values for a 2 span beam, all the others may be written; but it should be noted that, in the table of reactions, for four spans, the above rule gives $^{25}/_{28}wl$ at middle point; $^{26}/_{28}wl$ is correct. $\left(\frac{11+32+26+32+11}{28} = 4wl\right.$

§178. Continuous Girder.— By comparing the expressions in the text, the integrations of which are only indicated, with those just obtained in the preceding pages of notes, the trend of this investigation can be seen. The special case, on pp. 289 to 292, can be solved independently; but it is of little value, as the results are only true of the beam or truss of equal spans (except end ones), with full travelling load on alternate spans. Some of the equations are obtained as follows:— p. 290,

$$i = \int_0^{x'} \frac{M}{EI}\,dx = \frac{1}{EI}\left\{\frac{w-w'}{6}c^2x' - \frac{w}{6}x'^3\right\}$$
$$v = \int_{x'}^{c} i\,dx = \frac{1}{EI}\left[\left(\frac{w-w'}{12}c^2x'^2 - \frac{w}{24}x'^4\right)\right]_{x'}^{c}$$
$$= \frac{1}{EI}\left\{\frac{w-2w'}{24}c^4 - \frac{w-w'}{12}c^2x'^2 + \frac{w}{24}x'^4\right\}$$

$$i = \frac{1}{EI}\left\{\frac{w+2w'}{6}c^2x - \frac{w+w'}{6}x^3\right\}$$
$$v = \frac{1}{EI}\left[\left(\frac{w+2w'}{12}c^2x^2 - \frac{w+w'}{24}x^4\right)\right]_{x}$$
$$= \frac{1}{EI}\left\{\frac{w+3w'}{24}c^4 - \frac{w+2w'}{12}c^2x^2 + \frac{w+w'}{24}x^4\right\}$$

For (8.), make $x = 0$ in above values of M.

For (9.), make $x = 0$ in above values of v. To introduce f' for proof load, $\frac{1}{I} = \frac{f'}{M_0 n h} = \frac{6f'}{(w+2w')c^2 n h}$. $\therefore$ (9 A.).

For point of contraflexure, lightly loaded span,

$M = w\frac{c^2-x^2}{2} - \frac{2w+w'}{6}c^2 = 0$; $3wc^2 - 3wx^2 = 2wc^2 + w'c^2$;

$3wx^2 = (w-w')c^2$; $\therefore x = \pm c\sqrt{\left(\frac{w-w'}{3w}\right)}$. (10.)

For point of contraflexure, heavily loaded span,

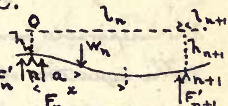
$\mathcal{M} = (w+w')\frac{c^2-x^2}{2} - \frac{2w+w'}{6}c^2 = 0;\ \ 3wc^2 - 3wx^2 + 3w'c^2 - 3w'x^2 =$

$2wc^2 + 2w'c^2;\ \ 3(w+w')x^2 = (w+2w')c^2;\ \therefore\ x = \pm c\sqrt{\left(\frac{w+2w'}{3(w+w')}\right)}.$ (10.)

The part beyond C at the end must not be longer than the length required to cause $\mathcal{M}_0$ at A, or $\frac{w+w'}{2}(l'-c)^2 \leq \frac{w+2w'}{6}c^2$; $\left(\frac{l'-c}{c}\right)^2 \leq \frac{w+2w'}{3(w+w')}$; $\therefore\ l'-c \leq c\sqrt{\left(\frac{w+2w'}{3(w+w')}\right)}$, or $l' \leq c\left(1+\sqrt{\left(\frac{w+2w'}{3(w+w')}\right)}\right)$ (12.)

A more general investigation will now be given, the resulting equations from which are of great practical value.

Three Moment Theorem for Single Load.



O is the origin; the supports are at distances $\underline{h}$ below the axis of x; single weight W distant $\underline{a}$ from O. The moment at section x, beyond the weight, is

$$\mathcal{M} = \mathcal{M}_n - F_n x + W_n(x-a). \quad (1.) \qquad \text{(Compare previous discussion.)}$$

If $x = l_n$, $\mathcal{M} = \mathcal{M}_{n+1}$, and we get from (1.)

$$F_n = \frac{\mathcal{M}_n - \mathcal{M}_{n+1}}{l_n} + \frac{W_n}{l_n}(l_n - a). \quad (1a.)$$

For an unloaded span, $W = 0$, and $F_m = \frac{\mathcal{M}_m - \mathcal{M}_{m+1}}{l_m}$.

For the shear on the $\underline{\text{right}}$ of a section just to the left of the $\underline{\text{right}}$ support of loaded span,

$F'_{n+1} = W_n - F_n = \frac{\mathcal{M}_{n+1} - \mathcal{M}_n}{l_n} + \frac{W_n a}{l_n}$. For unloaded span, $W = 0$, and

$F'_m = \frac{\mathcal{M}_m - \mathcal{M}_{m-1}}{l_{m-1}}$.

As F'_m is the shear at left of support $\underline{m}$, and F_m is the shear at right of same, — the former on the right and the latter on the left of the section, — the reaction is

$P_m = F'_m + F_m$. As before,

$$\int_0^x EI\frac{d^2v}{dx^2}dx = \int_0^x \mathcal{M}dx = \int_0^x \mathcal{M}_n dx - \int_0^x F_n x\,dx + W_n\int_a^x (x-a)dx. \quad (2.)$$

Note that the integral of last term is between limits $\underline{a}$ and $\underline{x}$ only.

$$EI\left(\frac{dv}{dx} - i_n\right) = \mathcal{M}_n x - \tfrac{1}{2}F_n x^2 + \tfrac{1}{2}W_n(x-a)^2. \quad (3.)$$

Since the origin is at a distance h_n above the support $\underline{n}$, the constant for the next integration is h_n.

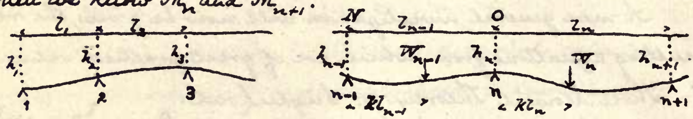
$$EI(v - i_n x - h_n) = \tfrac{1}{2}\mathcal{M}_n x^2 - \tfrac{1}{6}F_n x^3 + \tfrac{1}{6}W_n(x-a)^3. \quad (4.)$$

which is the general equation of the elastic curve, the term in W disappearing for values of $\underline{x}$ less than $\underline{a}$.

If $x = l_n$, $v = h_{n+1}$. If we put $\underline{a} = kl_n$ and insert also the value of F'_n from (1a.), we find for i_n

$$i_n = \frac{h_{n+1} - h_n}{l_n} - \frac{1}{6EI}\left[2M_n l_n + M_{n+1} l_n - W_n l_n^2 (2k - 3k^2 + k^3)\right]. \quad (5.)$$

The equation of the curve is therefore completely determined when we know M_n and M_{n+1}.



k is inseparable from l; kl_{n-1} and kl_n are different k's.

The equation of the elastic line, between W_n and the $n+1$ st. point of support, is given by (4.), and the tangent of its angle with axis of abscissas by (3.). If we substitute in (3.) for F_n its value from (1a.), and for i_n its value from (5.), and make $x = l_n$; then $\frac{dv}{dx} = i_{n+1}$, the tangent at $n+1$, and we have

$$i_{n+1} = \frac{h_{n+1} - h_n}{l_n} + \frac{1}{6EI}\left[M_n l_n + 2M_{n+1} l_n - W_n l_n^2 (k - k^3)\right].$$

Remove origin from O to N, and derive expression for i_n by diminishing indices.

$$i_n = \frac{h_n - h_{n-1}}{l_{n-1}} + \frac{1}{6EI}\left[M_{n-1} l_{n-1} + 2M_n l_{n-1} - W_{n-1} l_{n-1}^2 (k - k^3)\right]$$

Equate with (5.), and transpose.

$$\begin{aligned} M_{n-1} l_{n-1} + 2M_n (l_{n-1} + l_n) + M_{n+1} l_n = -6EI\left[\frac{h_n - h_{n-1}}{l_{n-1}} + \frac{h_n - h_{n+1}}{l_n}\right] \\ + W_{n-1} l_{n-1}^2 (k - k^3) + W_n l_n^2 (2k - 3k^2 + k^3) \end{aligned}$$

which is the most general form of the Three Moment Theorem for a girder of constant cross-section. End moments usually zero.

When supports are on a level, $h_1 = h_2$ etc., and term in EI disappears.

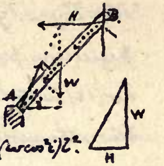
Any reaction $P_n = F'_n + F_n$; ∴

$$P_n = \frac{M_n - M_{n-1}}{l_{n-1}} + \frac{M_n - M_{n+1}}{l_n} + W_n (1 - k) + W_{n-1} k.$$

See application in "Graphics," Bridge Trusses, Chap. VIII.

§ 179. Sloping Beam with an Abutment. — $W = w.l\cos i$ ∴ $w = \frac{W}{l\cos i}$

The longitudinal component of the pressure at A must balance the



sum of the longitudinal components of H and W; $\therefore$

$$H\cos i + W\sin i = \frac{W\cos^2 i}{2\sin i} + W\sin i = \frac{W\cos^2 i + 2W\sin^2 i}{2\sin i}$$

$$= \frac{W}{2\sin i} + \frac{W\sin^2 i}{2\sin i} = \frac{W}{2}\left(\frac{1}{\sin i} + \sin i\right). \quad (3.)$$

From (4.), transverse intensity of load $= \frac{wl\cos^2 i}{2}$; $\therefore M = m(w\cos^2 i)l^2$.

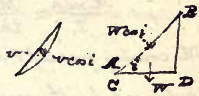
$$p' + p'' = \frac{W}{2A}\left(\frac{1}{\sin i} + \sin i\right) + \frac{Mn'h}{I} = \frac{W}{2A}\left(\frac{1}{\sin i} + \sin i\right) + \frac{W\cos i\, l}{12} \cdot \frac{n'h}{I}. \quad (7A.)$$

If A be made a factor, the last term will contain $\frac{n'hA}{I}$; put this $= \frac{1}{qh}$;

then $q = \frac{I}{n'h^2 A} = \frac{n'bh^3}{n'h^2 A} = \frac{nbh}{A}$.

Rectangle: $q = \frac{bh^3}{12} \cdot \frac{2}{h^2 \cdot bh} = \frac{1}{6}$. Ellipse: $q = \frac{\pi bh^3}{64} \cdot \frac{2}{h^2} \cdot \frac{4}{\pi bh} = \frac{1}{8}$.

Deflection of AB : Defl. of $CD = W'l^3 : Wl^3\cos^3 i$



$$= W\cos i\, l^3 : Wl^3\cos^3 i = 1 : \cos^2 i \quad (9.)$$

$$\frac{\text{Vert. compt. defl. of } AB}{\text{Perpendicular " " "}} = \frac{v\cos i}{v} = \cos i.$$

$$\frac{\text{Vert. comp. defl. of } AB}{\text{Defl. } CD} = \frac{\cos i}{\cos^2 i} = \frac{1}{\cos i}. \quad (10.)$$

Deflection varies inversely as breadth.

§ 180. *Arched Rib under Vertical Loads.*— By (1.), p. 214,

$$H = P\frac{dx}{dy};\ \therefore H\frac{dy}{dx} = P = \int w_1\, dx; \quad w_1 = H\frac{d^2y}{dx^2}.$$

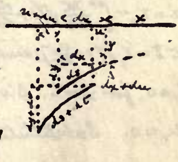
The vertical component of the shearing force at any point C is due to that part of the load which produces bending (the part whose intensity is w_1 produces no shear, just as w_1 hanging on chain causes no shear), and, by p. 242, $F = F_0 - \int_0^x (w - w_1)\,dx = F_0 - \int_0^x w\,dx + \int_0^x H\frac{d^2y}{dx^2}\,dx$

$$= F_0 - \int_0^x w\,dx + H\left(\frac{dy}{dx} - \frac{dy_0}{dx_0}\right). \quad (2.) \qquad (3.) \text{ follows } (2.)$$

$$i = i_0 - \int_0^x \frac{M}{EI}\,ds = i_0 - \int_0^x \frac{M}{EI}\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}\,dx. \quad (4.)$$

Note the expressions— Alteration of curvature and alteration of slope.

The thrust along the rib, due to the part $\int w_1\,dx$ of the load, is equal, p. 214, to $H \cdot$ sec. inclination; and, if A = cross-section of rib, $\frac{H\,\text{sec. inclin.}}{A}$ = intensity of thrust along the rib at C. Let dt be the shortening of ds. Then $dt : ds = \frac{H\,\text{sec. incl.}}{A} : E$; $\therefore \frac{dt}{ds} = -\frac{H\frac{ds}{dx}}{EA}$. (6.)



$$du = \frac{ds}{dx}\,dt - \frac{dy}{dx}\,dv,\ (7.) = -\frac{ds}{dx} \cdot \frac{H\frac{ds}{dx}}{EA}\,ds - \frac{dy}{dx}\,i\,dx$$

$$= -\frac{H}{EA}\left(1 + \frac{dy^2}{dx^2}\right)\left(1 + \frac{dy^2}{dx^2}\right)^{1/2} dx - i\frac{dy}{dx}\,dx$$

$$= -\frac{H}{EA}\left(1 + \frac{dy^2}{dx^2}\right)^{3/2} dx - i\frac{dy}{dx}\,dx. \quad (7A.)$$

$$H\frac{ds}{dx} \div A = f'; \quad \frac{Mn'h}{I} = f''; \quad f' \pm f'' = p_1. \quad (12.)$$

$\mathcal{M} = Hy; \therefore y = \frac{\mathcal{M}}{H}$. (13.) $\mathcal{M} = Tp; \therefore p = \frac{\mathcal{M}}{H\frac{ds}{dx}}$. (14.)

Deflection, p. 302; see top p. 276.

For (18.) introduce (4 A.) in (8 A.), substituting (17.). $\int_0^l i_0 \frac{dy}{dx} dx = i_0(y_1 - y_0) = 0$.

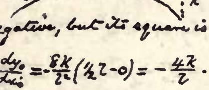
Since the vertical deflection is the same at every point as in a uniform, straight, horizontal beam, with cross-section A_1, and under the same moments as act on the arch, the stress on the outside fibres of the arched rib due to the bending part of the load must be the same as it would be in such a horizontal beam; that is

$$p = \frac{\mathcal{M} m^i h}{I_1} = \frac{\mathcal{M}}{2hA_1}; \quad \therefore p_1 = \frac{1}{A_1}\left\{ H \pm \frac{\mathcal{M}}{2h} \right\} \quad (12\,A.)$$

$\mathcal{M}$ coming from $\int F\,dx$ (F being the vertical component of shearing force), is balanced by the moment of resistance on vertical section.

The change of temperature will change the horizontal span and a term should be added for this change in u_1 of (8 A.). If t = greatest deviation of temperature from assumed standard, and e = coefficient of expansion, the horizontal expansion or contraction due to change of temperature $= \int_0^l te\,dx = tel$; $\therefore$

$$u_1 = -\frac{H}{EA_1}\int_0^l \left(1 + \frac{dy^2}{dx^2}\right)dx \pm tel - \int_0^l i \frac{dy}{dx}\,dx. \quad (18\,A'.) \quad [\text{See p. } 539.]$$



Page 304. $y : k = (\frac{1}{2}l - x)^2 : (\frac{1}{2}l)^2$; $\therefore$

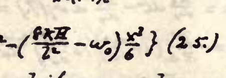
$y = \frac{4k}{l^2}(\frac{1}{2}l - x)^2$. If $x > \frac{1}{2}l$, $\frac{1}{2}l - x$ will be negative, but its square is +.

$dy = \frac{2 \cdot 4k}{l^2}(\frac{1}{2}l - x)(-dx)$; $\therefore \frac{dy}{dx} = -\frac{8k}{l^2}(\frac{1}{2}l - x)$. $\frac{dy_0}{dx_0} = -\frac{8k}{l^2}(\frac{1}{2}l - 0) = -\frac{4k}{l}$.

$\int_0^l \left(1 + \frac{dy^2}{dx^2}\right)dx = \int_0^l \left[1 + \frac{64k^2}{l^4}(\frac{1}{2}l - x)^2 dx = l - \frac{64k^2}{3l^4}\left[_0^l (\frac{1}{2}l - x)^3 = l - \frac{64k^2}{3l^4}(\frac{1}{2}l - l)^3\right.\right.$
$+ \frac{64k^2}{3l^4}(\frac{1}{2}l - 0)^3 = l + \frac{64k^2}{3l^4}\left(\frac{l^3}{8} + \frac{l^3}{8}\right) = l + \frac{16k^2}{3l}$. $d\frac{dy}{dx} = \frac{8k}{l^2}$. (21.)

$\int u\,dv = uv - \int v\,du$. Let $u = \frac{dy}{dx}$; $v = v$. $\int_0^l \frac{dy}{dx}\,dv = \frac{dy}{dx}(\overset{0}{v_1} - \overset{0}{v_0}) - \int_0^l v\frac{d^2y}{dx^2}\,dx$
$= -\frac{8k}{l^2}\int_0^l v\,dx$. (22.)

From (2.), $F = F_0 - w_0x + H\left(-\frac{8k}{l^2}(\frac{1}{2}l - x) + \frac{4k}{l}\right) = F_0 - w_0x + \frac{8k}{l^2}Hx$. (23.)



(24.) follows easily.

$i = i_0 - \int_0^x \frac{\mathcal{M}}{EI}\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}dx$. $i_0 = 0$.

Put $I = I_1\sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$; and $i = \frac{1}{2m^i h^2 EA_1}\left\{-\mathcal{M}_0x - \frac{1}{2}F_0x^2 - \left(\frac{8kH}{l^2} - w_0\right)\frac{x^3}{6}\right\}$ (25.)

$i_1 = 0$, for $x = l$. Divide by l. $\{x - (1-r)l\}$, when $x = l$, becomes rl.

$u_1 = 0$. Substitute value of v and integrate.

$$-\left(l+\frac{16k^2}{3l}\right)\frac{H}{EA_1}+\frac{8k}{l^2}\int_0^l v\,dx=0=-\frac{4m'k^2}{8kl}\left(l+\frac{16k^2}{3l}\right)H+\frac{1}{l^3}\int_0^l\Big\{-M_0\frac{x^2}{2}-F_0\frac{x^3}{6}$$
$$-\left(\frac{8kH}{l^2}-w_0\right)\frac{x^4}{24}+\frac{w}{24}[x-(1-r)l]^4\Big\}dx=-\frac{4m'k^2}{8k}\left(1+\frac{16k^2}{3l^2}\right)H-\tfrac{1}{6}M_0-\tfrac{1}{24}F_0l$$
$$-\frac{8kH}{120}+\frac{w_0l^2}{120}+\frac{wr^5l^2}{120}. \quad (29.)$$

Eliminations between (27.), (28.) and (29.) are not difficult. There result (31.), (32.) and (39.). From (24.) get (33.).

Differentiate (34.) in respect to r, and put $=0$.

$$\frac{2}{3q^2}(12r-24r^2+12r^3)-\frac{\frac{2}{3q^2}-\frac{1}{k}}{1+B}(30r^2-60r^3+30r^4)=0. \qquad r=0.$$

$$12-24r+12r^2-\frac{1-\frac{3q^2}{2k}}{1+B}(30r-60r^2+30r^3)=0.$$

$$12(r^2-2r+1)-\frac{1-\frac{3q^2}{2k}}{1+B}30r(r^2-2r+1)=0. \qquad r^2-2r+1=0$$
$$r=1.$$

$$30r=12\frac{1+B}{1-\frac{3q^2}{2k}}; \qquad r_1=\frac{2}{5}\frac{1+B}{1-\frac{3q^2}{2k}}. \quad (36.)$$

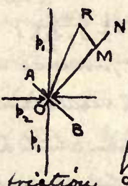
The other problems are similar.

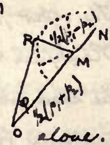
Rankine's results are hardly applicable to practical cases, as the load from which he deduces maximum stresses extends continuously from one abutment. A load over the central portion, or over both flanks at once, should be considered. The general equations, however, are instructive, as is the investigation. They are limited to the parabola, but will answer for flat circular segments.

Chapter II.
Earthwork.
Section I.

§183. Theory of the Stability and Pressure of Loose Earth.—

I. As the plane in a mass of earth, on which the stress is to be found, may have any inclination, the line RO will make a greater or less angle with the normal to its plane (Notes p. 6); but the angle NOR, or $\hat{rr}$ in earth, cannot exceed φ, by the condition that equilibrium is due to friction alone.

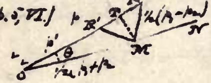




MR may therefore make various angles with OM, and angle ROM will be the greatest when RO is tangent to the circle whose centre is M and radius $MR = \frac{1}{2}(p_1-p_2)$. Then $\angle MOR = \varphi$, and $ORM = 90°$;

$$\therefore \sin\varphi \geq \frac{RM}{MO} \geq \frac{p_1-p_2}{p_1+p_2};\ p_1\sin\varphi + p_2\sin\varphi \geq p_1 - p_2 \text{ or } \frac{p_2}{p_1} \geq \frac{1-\sin\varphi}{1+\sin\varphi}. \quad (1.A)$$

II. $OM = \frac{1}{2}(p_1+p_2) = \frac{OT}{\cos\theta} = \frac{p+p'}{2\cos\theta}$. (Notes, p. 5, VI.)



$$MR = \sqrt{(OM^2 + OR^2 - 2\,OM.OR\cos\theta)}$$

$$\tfrac{1}{2}(p_1-p_2) = \sqrt{\left(\frac{(p+p')^2}{4\cos^2\theta} + p^2 - \frac{2(p+p')p\cos\theta}{2\cos\theta}\right)} = \sqrt{\left(\frac{(p+p')^2}{4\cos^2\theta} - pp'\right)}.$$

$$\frac{MR}{OM} = \frac{p_1-p_2}{p_1+p_2} = \frac{\sqrt{\left(\frac{(p+p')^2}{4\cos^2\theta} - pp'\right)}}{\frac{p+p'}{2\cos\theta}} \leq \sin\varphi;$$ square and divide.

$$1 - \frac{4\cos^2\theta\, pp'}{(p+p')^2} \leq \sin^2\varphi. \quad \therefore 1-\sin^2\varphi = \cos^2\varphi \leq \frac{4\cos^2\theta\, pp'}{(p+p')^2} \text{ or}$$

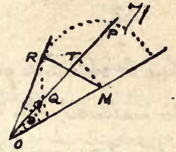
$$\frac{(p+p')^2}{4pp'} \leq \frac{\cos^2\theta}{\cos^2\varphi}.$$ By composition and division,

$$\frac{(p+p')^2 - 4pp'}{4pp'} = \frac{(p-p')^2}{4pp'} \leq \frac{\cos^2\theta - \cos^2\varphi}{\cos^2\varphi}. \quad \therefore \frac{(p-p')^2}{(p+p')^2} \leq \frac{\cos^2\theta - \cos^2\varphi}{\cos^2\theta}$$

Extract the square root; then, by composition and division,

$$\frac{p}{p'} \leq \frac{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}}. \quad (2.)$$ For other limits reverse the fraction.

To prove Rankine's geometrical construction, which is really a combination of the two sketches above:—



$OR^2 = OQ \cdot OP$ [segments of secant and tangent]

$= (OT - TQ)(OT + TQ)$, MT being drawn perpendicular to OP. Then $TQ^2 = OT^2 - OR^2$.

$$\frac{OP}{OQ} = \frac{OT + TQ}{OT - TQ} = \frac{OT + \sqrt{(OT^2 - OR^2)}}{OT - \sqrt{(OT^2 - OR^2)}}.$$

Divide numerator and denominator by OM. $\frac{OT}{OM} = \cos\theta$; $\frac{OR}{OM} = \cos\varphi$; $\therefore$

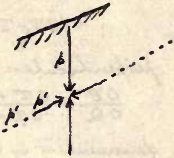
$$\frac{OP}{OQ} = \frac{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}{\cos\theta - \sqrt{(\cos^2\theta - \cos^2\varphi)}} \quad . \quad (2.)$$

III. Mr. Moseley's principle of least resistance is, that

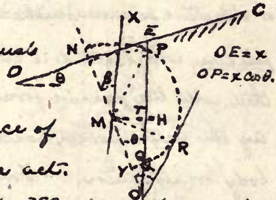
If the forces which balance each other in or upon a given body or structure be distinguished into two systems, called respectively <u>active</u> and <u>passive</u>, which stand to each other in the relation of cause and effect, then will the passive forces be the least which are capable of balancing the active forces, consistently with the physical condition of the body or structure. For the passive forces, being caused by the application of the active forces to the body, will not increase after the active forces have been balanced by them, and will therefore not increase beyond the <u>least</u> amount capable of balancing the active forces.

In a mass of earth, loaded with its own weight only, the gravitation of the earth causes the vertical pressure; the vertical pressure causes a tendency to spread laterally; and the tendency to spread causes the conjugate pressure. Therefore the conjugate pressure is the least which is consistent with the conditions of stability. The <u>third</u> pressure, being perpendicular to the plane of p and p', must be a principal pressure, and, being a passive force, must have the least intensity consistent with stability, and must, therefore, be equal to the least pressure in the plane of p and p'.

If, in the sketch above, $OP = p$ is given, as well as the angles θ and φ, and it is desired to find $OQ = p'$, it can be done by finding the centre M of a circle which shall pass through P and be tangent to OR. Two circles are possible; one is sketched above, the other lies entirely above P. The value OQ' which would thus be obtained gives with p the reverse of (2.) If p is vertical, for instance, the smaller value of p' is the least intensity of resistance which will keep the earth from slipping down and to the left; the larger value of p' is the greatest intensity of pressure which can be brought to bear consistent with the given value of p as a resistance against motion of the mass upward. Equilibrium is assured for all values of p' between these two limits; hence (2.) and (2 A.).



Page 322. To find p_1 and p_2, which equals p'' also as shown above. Sketch is like the one on preceding page; CD is surface of ground; O is point at which pressures act.



$$p_1 = ON = OM + MN = OM + MR = OM\left(1 + \frac{MR}{OM}\right) = OM(1 + \sin\varphi);$$

$$p_2 = OY = OM - MY = OM - MR = \qquad OM(1 - \sin\varphi).$$

$$p = OP = OT + TP = OT + TQ = (\text{page } 71), \qquad OM(\cos\theta + \sqrt{\cos^2\theta - \cos^2\varphi}).$$

$$\therefore\ OM = \frac{p = wx\cos\theta}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}\ ;\ \text{and}\ p_1 = \frac{wx\cos\theta\,(1 + \sin\varphi)}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}} \qquad (13.)$$

$$p_2 = p'' = wx\cos\theta\,\frac{1 - \sin\varphi}{\cos\theta + \sqrt{(\cos^2\theta - \cos^2\varphi)}}\ . \qquad (10.)$$

MX, the direction of p_1, bisects PMN. (Notes, p. 6).

$NMX = \beta$; $NMP = 2\beta$; $NOP = \theta$; $\therefore\ MPO = 2\beta - \theta$.

In triangle MPO, $\frac{\sin(2\beta - \theta)}{\sin\theta} = \frac{p_1 + p_2}{p_1 - p_2} = \frac{1}{\sin\varphi}$, by (1.). $\therefore$

$$\sin(2\beta - \theta) = \frac{\sin\theta}{\sin\varphi};\ \therefore\ 2\beta = \theta + \sin^{-1}\frac{\sin\theta}{\sin\varphi},\ \text{and}\ \beta = \tfrac{1}{2}\left(\theta + \sin^{-1}\frac{\sin\theta}{\sin\varphi}\right).$$

$$\angle XMH = \psi = 90^\circ + \theta - \beta = \tfrac{1}{2}\left(180^\circ + \theta - \sin^{-1}\frac{\sin\theta}{\sin\varphi}\right). \qquad (14.)$$

The axis of greatest pressure lies in the acute angle between the vertical and the direction of greatest declivity of the earth's surface

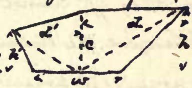
IV. p. 323. (17.) comes from (2 A.). Then $P' = XD \cdot \frac{OX\cos\theta}{2} = \frac{wx^2}{2}\cos\theta\,\frac{p'}{p}$. (18.)

For graphical treatment of this matter see "Graphics," Part II., Bridge Trusses, Ed. 1890, Appendix.

Section II.

The methods of calculating earthwork and of staking out given in the text may well be superseded by those given in American books. See Henck, Shunk, Searles, Wellington, &c.

§186. The prismoidal formula may be put in following shape: c_1 and c_2 = two successive centre heights, distant l apart; d_1, d_1', d_2, d_2', = distances out to heights h_1, h_1', h_2, h_2'.



$$\tfrac{1}{2}c_1(d_1+d_1')+\tfrac{1}{4}w(h_1+h_1')+\tfrac{1}{2}c_2(d_2+d_2')+\tfrac{1}{4}w(h_2+h_2')+4\Big[\frac{c_1+c_2}{4}\Big(\frac{d_1+d_2}{2}+\frac{d_1'+d_2'}{2}\Big)+\frac{w}{4}\Big(\frac{h_1+h_2}{2}+\frac{h_1'+h_2'}{2}\Big)\Big]$$, or, sum of end areas + 4 times middle area.

$$=(c_1+\tfrac{1}{2}c_2)(d_1+d_1')+(c_2+\tfrac{1}{2}c_1)(d_2+\tfrac{1}{2}d_2')+\tfrac{3}{4}w(h_1+h_1'+h_2+h_2').$$ Multiply by length and divide by 6×27 to find cubic yards. Or mult. by $.0061\times$ length.

Section III.

§194. Classification of Earthwork. — N. Y., W. S. & B. Ry. Specification.

Earth = soft clay, sand, gravel, loam, decomposed rock, stones and boulders of less than one c. ft. capacity.

Hardpan = quicksand, tough, indurated clay or cemented gravel which needs blasting, cannot be ploughed with fewer than four horses or requires two pickers to three shovelers.

Hard Clay is between earth and hardpan; requires one picker to two shovelers and is with difficulty ploughed with two horses.

Loose Rock = masses of stone between one c. ft. and one c. yd.; also slate, shale and soft, friable sandstone; also rock in strata not exceeding 8 inches thick, which can be economically removed without blasting.

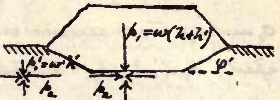
Solid Rock = rock in ledge or masses of more than one c. yd. which can be best removed by blasting. Special allowance for frozen earth.

Fifty cubic yards of earth, a good day's work for a 2-horse wheeled scraper. 80 yards have been moved in ten hours.

§196. Prevention of Slips.— Consolidation of Earthwork.— See Proc. Inst. C.E. 1874 - Eng. News, Vol. IV, 1877, p. 257, and elsewhere.

§198. Overhaul.— See Eng. News. Mar. 14, 1891, and elsewhere.

§204. Embankments on Soft Ground.— By (3) and (3A) p. 320, $\frac{p_2}{p_1}$ cannot be less than $\frac{1-\sin\varphi'}{1+\sin\varphi'}$.



Therefore the embankment must cause a horizontal pressure under its base of

$$p_2 = p_1 \frac{1-\sin\varphi'}{1+\sin\varphi'} = w(h+h') \frac{1-\sin\varphi'}{1+\sin\varphi'}.$$

This horizontal pressure causes a vertical pressure p_1' outside the base; and p_2 may equal, but, for equilibrium, must not exceed $p_1' \frac{1+\sin\varphi'}{1-\sin\varphi'} = w'h' \frac{1+\sin\varphi'}{1-\sin\varphi'}$, since $p_1' = w'h'$. Equating these two values of p_2, we have

$$w(h+h') \frac{1-\sin\varphi'}{1+\sin\varphi'} = w'h' \frac{1+\sin\varphi'}{1-\sin\varphi'}, \text{ or } (h+h')\, w k' = \frac{w'h'}{k'};$$

$$h'(w' - wk'^2) = hwk'^2; \quad \therefore \quad h' = \frac{hwk'^2}{w' - wk'^2}. \qquad (1.)$$

If φ' is small, k' will come nearer unity, and, w being greater than w', the value of h' becomes large, even impossible, and the embankment sinks, while the sides rise. The smaller <u>w</u> and <u>h</u>, the better.

§207. Blasting.— Chamber in limestone made for blasting. Copper tube 1⅛" diam. put in drill-hole and fitted to its mouth with hemp. Into this tube was inserted a ½" rubber tube. Muriatic acid was run into drill-hole, allowed to stand, and then a new supply was added, forcing out the old between copper and rubber tubes, to be caught for reuse. One quart of acid made cavity of about 2 c. ft. For cavity of one cubic metre (1⅓ c. yds), 20 qts. acid were used; time required, 15 hours.

Rackarock:— a solid, mainly potassium chlorate in fine powder and given a reddish tint by some coloring matter,— and an oily liquid, having the strong, bitter-almond smell characteristic of nitro-benzol.

Chapter III.
Masonry.
Section III.

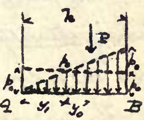
§230. Concrete. — At Forth bridge: concrete of 1 c. yd. from 27 c. ft. broken trap, 7 c. ft. sand, 5½ c. ft. cement.

Voids in stone, gravel and sand can be found by placing materials in a box separately and measuring quantity of water required to fill to level of material. Then proportion to fill voids, with a little excess of cement.

Not a bad rule is — 1 part cement, 2 sand, 3 coarse gravel, 4 broken stone. See Gillmore on "Limes, Mortars & Cements"; also Trans. Am. Soc. C. E. Nov. 1885, report of Com. on uniform tests of Cements; Behavior of Cement Mortars; also Baker's "Masonry Construction."

Section IV.

§236. Rock Foundations. — For incompressible or rock foundations, as there is to be no tension on the base, the pressure at time of greatest allowable deviation must be zero at A, and increase uniformly to B. The same variation will be true if the pressure is oblique, instead of normal as sketched.

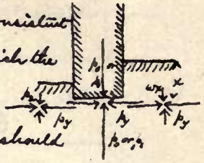


$P = p_o S$; $M = \frac{fI}{y_1} = Py_o$; $\therefore y_o = \frac{fI}{Py_1}$.

When stress at A is 0, $f = p_o$ and $y_o = \frac{p_o I}{P y_1} = \frac{I}{S y_1}$. $\frac{I}{S} = r^2$ and $\frac{r^2}{y_1} = 2h$.

The pressure at B will be $2p_o$, which must not exceed safe working pressure on masonry or foundation.

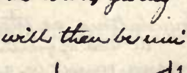
§237. Earth Foundations. — In an earth foundation, as in §204, the horizontal pressure, at the limit, will be the least which is consistent with the vertical intensity due to the weight of the structure. If that horizontal pressure does not exceed the greatest intensity consistent with the weight of the earth above the plane on which the structure rests, this earth will not be forced up and stability will barely be assured. A greater depth should be reached for safety. Then



$$\frac{p_0}{p_y} = \frac{1+\sin\varphi}{1-\sin\varphi}; \; p_0 = p_y\frac{1+\sin\varphi}{1-\sin\varphi}; \; \frac{wx}{p_y} \geqq \frac{1-\sin\varphi}{1+\sin\varphi}; \; wx \geqq p_y\frac{1-\sin\varphi}{1+\sin\varphi}; \; \therefore$$

$$\frac{wx}{p_0} \geqq \left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)^2. \qquad (1.)$$

As the same horizontal pressure must be kept up under all parts of the wall where the load gives a vertical intensity of p_0, we must have, under adjacent openings and on the cellar side of walls, at least $p_2 \geqq wx$ or $\frac{wx}{p_2} \leqq 1$. (3.) Hence, footing courses should be a sufficient depth below cellar bottom, or settlement will ensue. If pressure on foundation is ~~varying~~ not uniform, p_1 being the greatest and p_2 the least intensity, p_2 must not be less than wx, or that part pressing with p_2 may be forced up. After beginning to yield it may offer increased resistance, but even small settlement may cause cracks. Chicago foundations are now usually isolated piers with carefully calculated, central loads giving same intensity on foundations. Settlement will then be uniform on a compressible soil.



The expression in § 236 now becomes, since

$$f = p_0 - p_2; \quad y_0 = \frac{(p_0-p_2)I}{Py_1} = \frac{p_0-p_2}{p_0}\cdot\frac{I}{Sy_1} = q\,h\,\frac{p_0-p_2}{p_0} = .\,\delta. \; (5.) \text{ Also}$$

$$p_0 = \tfrac{1}{2}(p_1+p_2) = \tfrac{1}{2}(p_1+wx) \leqq \tfrac{1}{2}wx\left\{\frac{(1+\sin\varphi)^2}{(1-\sin\varphi)^2}+1\right\} \leqq \tfrac{1}{2}wx\,\frac{(1+\sin\varphi)^2+(1-\sin\varphi)^2}{(1-\sin\varphi)^2}$$

$$\leqq wx\,\frac{1+\sin^2\varphi}{(1-\sin\varphi)^2}; \quad \therefore \; \frac{wx}{p_0} \geqq \frac{(1-\sin\varphi)^2}{1+\sin^2\varphi}. \quad (6.)$$

$$\delta = q\,h\,\frac{p_0-p_2}{p_0} = q\,h\,\frac{p_1+p_2-2p_2}{p_1+p_2} = q\,h\,\frac{p_1-p_2}{p_1+p_2} = q\,h\,\frac{wx\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 - wx}{wx\left(\frac{1+\sin\varphi}{1-\sin\varphi}\right)^2 + wx}$$

$$= q\,h\,\frac{(1+\sin\varphi)^2-(1-\sin\varphi)^2}{(1+\sin\varphi)^2+(1-\sin\varphi)^2} = q\,h\,\frac{4\sin\varphi}{2(1+\sin^2\varphi)} = q\,h\,\frac{2\sin\varphi}{1+\sin^2\varphi}. \quad (7.)$$

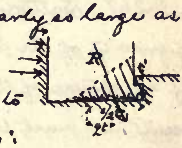
§ 239. Same demonstration as for (1.) p. 343.

See Johnson's Cyclopedia, "Foundations;" also Engg. + Bldg. Record, Mar. 30, 1889, or Jan. 31, Feb. 21, 28, 1871, or Eng. News, Feb. 14-21, 1891.

§ 262. Local custom, in measuring brick or stone masonry, will govern, unless specifications are distinct to the contrary. Under this head is included doubling corners and measuring openings as solid wall. A perch of masonry is in some places $16\frac{1}{2}$ c. ft., and in others $24\frac{3}{4}$ c. ft.

Section VII.

§ 263. Stability of Masonry. — When any yielding or movement is objectionable, q should have the value given, as $q = \frac{1}{6}$ for a rectangular base; but, where the foundation is more or less compressible, and slight yielding will do no harm, the pressure may be considered as distributed over a portion of the base, the remainder not being in actual pressure-contact with the foundation. Then q may be fixed by the requirement that f' shall not be greater than the safe pressure on the material under foundation. By the graphical construction for a retaining wall with inclined or battered back (See "Bridges; Appendix"), q is not nearly as large as by Rankine's construction.

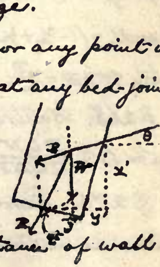


Distance from front edge of greatest pressure to centre of pressure (action line of R), $= (\frac{1}{2} - q)t$. ∴

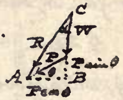
Whole pressed area $= 3(\frac{1}{2} - q)tb$. Total pressure, $R = \frac{f'}{2}(\frac{3}{2} - 3q)bt$.

$f' = \frac{2R}{(\frac{3}{2} - 3q)bt}$, and $q = \frac{1}{2} - \frac{2R}{3f'bt}$. (1.) The last term is the fraction of distance from the front edge.

The moments of W and P must balance for any point in the resultant R, either where it cuts the base or at any bed-joint, W being the weight above the joint, and P the external pressure above the same. The coordinates of point of application of P from centre of resistance are x' and y'. Then the moment of resistance of wall



$M = W(q \pm r)t\cos j$. (2.) Overturning moment $= P(x'\cos\theta - y'\sin\theta)$; $\therefore$ for stability, (5.).

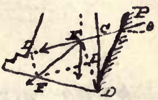


$\tan\alpha = \dfrac{AB}{BC} = \dfrac{P\cos\theta}{W + P\sin\theta}$; $\therefore$ (6.)

§ 264. Vertical-Faced Buttress.— From (2.) comes (4.). Then $nwh_0b_0t_0 = P\left(\dfrac{\cos\theta}{\tan\varphi} - \sin\theta\right) = P\dfrac{\cos\theta\cos\varphi - \sin\theta\sin\varphi}{\sin\varphi} = P\dfrac{\cos(\varphi+\theta)}{\sin\varphi}$.

$$h_0 = \frac{\cos(\varphi+\theta)}{\sin\varphi}\cdot\frac{P}{wbt} = \frac{\cos(\varphi+\theta)}{\sin\varphi}\cdot\frac{P}{wb\sqrt{\left(\frac{P\cos\theta}{qwb}\right)}} = \frac{\cos(\varphi+\theta)}{\sin\varphi}\sqrt{\left(\frac{qP}{wb\cos\theta}\right)} = \frac{\cos(\varphi+\theta)}{\sin\varphi\cos\theta}qt.$$

§ 265. Stability of Retaining Walls.—



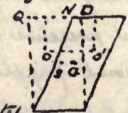
$CD = \frac{1}{3}x$; $DK = \frac{1}{3}x\cos\theta$; $DF = (q+\frac{1}{2})t$; $\angle IFD = \theta + j$; $DI = (q+\frac{1}{2})t\sin(\theta+j)$. Arm of $P = LF = DK - DI$ $= \frac{1}{3}x\cos\theta - (q+\frac{1}{2})t\sin(\theta+j)$. r is changed to q'.

Rankine's assumption of a vertical plane at CD, in place of FD, alters the direction and intensity of the earth pressure, and hence the value of q will be affected, as derived from existing structures. [See graphical construction, Greene's "Bridges," Ed. 1890. Appendix.

§ 266. Upright, Rectangular Retaining Wall.— $\dfrac{t}{x} = \sqrt{a} = \sqrt{\left\{\dfrac{w'}{6qw}\cdot\dfrac{1-\sin\varphi}{1+\sin\varphi}\right\}} = \sqrt{\dfrac{w'}{6qw}}\sqrt{\left\{\dfrac{(1-\sin\varphi)(1+\sin\varphi)}{(1+\sin\varphi)^2}\right\}} = \sqrt{\dfrac{w'}{6qw}}\cdot\dfrac{\cos\varphi}{1+\sin\varphi}$. By (12.) p. 41, $\tan\frac{1}{2}A = \dfrac{\sin A}{1+\cos A} = \dfrac{\cos(90^\circ - A)}{1+\sin(90^\circ - A)}$; $\therefore \dfrac{\cos\varphi}{1+\sin\varphi} = \tan\frac{1}{2}(90^\circ - \varphi)$; $\therefore \dfrac{t}{x} = \sqrt{\dfrac{w'}{6qw}}\tan\dfrac{90^\circ - \varphi}{2}$.

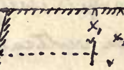
§ 267. Battering-Faced Retaining Wall.— Since the centre of gravity of the triangle is, by hypothesis, vertically over the centre of resistance of the base,— upon removal of the triangle, the new resultant must traverse the same centre of resistance, but will be more inclined to the vertical. Thickness at top $= t - QN$ $= t - 3(\frac{1}{2} - q)t = (3q - \frac{1}{2})t$. Weight of wall is diminished in the ratio of $\frac{1}{2}QN$ to t, or $(\frac{3}{4} - \frac{3}{2}q)$ to 1. Therefore the remaining weight of wall $= (1 - \frac{3}{4} + \frac{3}{2}q)W = (\frac{1}{4} + \frac{3}{2}q)W$. For a horizontal topped bank, $\tan WAR = \dfrac{P}{W}$. The diminution of W increases $\tan WAR$.

§ 268. Battering Walls of Uniform Thickness.— By VIII. § 104, $Gg = OO'\dfrac{EQN}{EQDF}$; $\therefore Gg = t\dfrac{EQN}{tx} = \dfrac{EQN}{x}$. (1.)

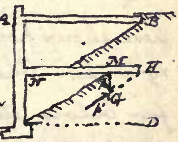


§ 270. Surcharged Wall.— Make the thickness intermediate between that for horizontal bank at top of wall and that for extended slope at angle φ.

§272. Land Ties for Retaining Walls.— The earth at the middle of plate, having vertical pressure on the horizontal plane of $p_1 = w'\frac{x_1+x_2}{2}$, causes a conjugate horizontal pressure against the vertical plate of least intensity $p_2 = w'\frac{x_1+x_2}{2}\left(\frac{1-\sin\varphi}{1+\sin\varphi}\right)$. But the plate may press horizontally with an intensity $p_2' = w'\frac{x_1+x_2}{2}\cdot\frac{1+\sin\varphi}{1-\sin\varphi}$, before the earth will be forced away in front. As p_2 first acts on both sides of the plate, the difference between these mean intensities is available for holding the wall. Multiplying the difference by the area x_2-x_1 of the plate, we have $H = w'\frac{(x_2^2-x_1^2)}{2}\left(\frac{1+\sin\varphi}{1-\sin\varphi}-\frac{1-\sin\varphi}{1+\sin\varphi}\right)$ $= w'\frac{x_2^2-x_1^2}{2}\cdot\frac{4\sin\varphi}{\cos^2\varphi}$. (1.) Such ties are used in wharf fronts, etc. The anchorage must be in solid ground.



§274. Relieving Arches.— The vertical intensity of pressure on plane GH at point G = $w \cdot MG\cos\varphi$. The conjugate pressure $p' = p$ when $\theta = \varphi$, and, as pressure is uniform throughout the conjugate plane, the pressure at H will be the same, in direction GH. Horizontal component of this pressure at H = $MG\, w\cos^2\varphi$. But the horizontal thrust at $H = wx\frac{1-\sin\varphi}{1+\sin\varphi}$, where $x = BH$. $\therefore MG\, w\cos^2\varphi = wx\frac{1-\sin\varphi}{1+\sin\varphi}$; $l = NM + MH$ $= h\cot\varphi + MG\cot\varphi = h\cot\varphi + \frac{x\cot\varphi}{\cos^2\varphi}\cdot\frac{1-\sin\varphi}{1+\sin\varphi} = \cot\varphi\left(h + \frac{x}{(1+\sin\varphi)^2}\right)$
The above theory is approximate only.



Section VIII.

§281. Linear Rib applied to Real Arch.— We may apply equal and opposite couples at two points of a curve of equilibrium without disturbing its equilibrium; but the position of the curve will be shifted. The curve of equilibrium is the locus of the resultant force at every section; and, by Mechanics, the combination of a force and a couple, in one plane, is the same force moved perpendicularly to itself an amount $= \frac{M}{F}$. The distance at crown will be vertically $\frac{M}{H}$, and, at other points, perpendicularly $\frac{M}{\text{Thrust}}$. The application of two equal and opposite couples makes their sum zero, and equilibrium of the structure as a whole is unchanged.

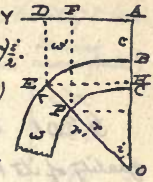
§ 283. Hydrostatic Arch.— Note. By (12.), p. 211, $\rho_0 = a + \frac{a^2}{2x_0}$, and $x_0 = \frac{a^4}{b^3 - a^3}$. Then $\rho_0 = a + \frac{a^2(b^3-a^3)}{2a^4} = a + \frac{a^2b^3 - a^5}{2a^4} = \frac{a}{2}\left(1 + \frac{b^3}{a^3}\right)$.

§ 284. Geostatic Arch.— By (11.), p. 211, $x_0 = \frac{a^4}{b^3-a^3}$; $b^3 x_0 = a^3(x_0 + a)$; $\therefore\ b = a\left(\frac{x_0+a}{x_0}\right)^{\frac{1}{3}}$. By the same reasoning by which we obtained (11.), p. 211, $b = y_1 + \frac{y_1^2}{30a}$, we may write, and with more exactness, $y_1 = b - \frac{b^2}{30a}$. (1.)

§ 285. Stability of any Proposed Arch.— Rankine here shows how he applies the methods of § 138 to a masonry arch. The horizontal pressures from without, called for below the point of rupture, are to be supplied by the backing. For that part of the arch between the point of rupture and the crown, where by the method of § 138, tension from without would be required, he substitutes some other curve for the linear rib like the intrados of the stone arch, but does not trace it. He draws its tangents at the two points referred to, and these must intersect on the vertical through the centre of gravity of the load on that portion. The points of application of the thrust at crown and at point of rupture are not known but assumed, and, that the thrust shall be horizontal at the crown, requires a symmetrical load. No provision is therefore made for moving load. The writer considers this treatment too crude and approximate, and thinks that the comparative insignificance of the moving load on a masonry arch is what has permitted such treatment to be used.

§ 286. Circular Arch. — Analysis from § 138.

Vertical load from crown to any point P:-



$DABE = DAHE + HOE - BOE$. $BCPE = BOE - COP = (r'^2 - r^2)\frac{i}{2}$.

$DAHE = EH \times AH = r'\sin i\{r'(1-\cos i) + c\} = r'^2(\sin i - \sin i\cos i)$

$+ cr'\sin i$. $HOE = \frac{r'\cos i \cdot r'\sin i}{2} = \frac{1}{2}r'^2\sin i\cos i$.

$BOE = \frac{1}{2}r'^2 i$. $\frac{dy}{dx} = \cot i = \frac{r'\cos i}{r'\sin i} = \frac{\cos i}{\sin i}$.

$$P_x = w'\{r'^2(\sin i - \sin i\cos i) + cr'\sin i + \tfrac{1}{2}r'^2\sin i\cos i - \tfrac{1}{2}r'^2 i\} + w(r'^2 - r^2)\tfrac{1}{2}i,$$

$$= w'r'^2(\sin i - \tfrac{1}{2}\sin i\cos i + \tfrac{c}{r'}\sin i - \tfrac{1}{2}i) + w(r'^2 - r^2)\tfrac{1}{2}i.$$

$$H = P_x\frac{dy}{dx} = w'r'^2\left(\cos i - \tfrac{1}{2}\cos^2 i + \tfrac{c}{r'}\cos i - \frac{i\cos i}{2\sin i}\right) + w(r'^2 - r^2)\frac{i\cos i}{2\sin i},$$

$$= w'r'^2\left\{\left(1+\tfrac{c}{r'}\right)\cos i - \tfrac{1}{2}\cos^2 i - \frac{i\cos i}{2\sin i}\right\} + w(r'^2 - r^2)\frac{i\cos i}{2\sin i}. \quad (2.)$$

$p_y = -\frac{dH}{dx} = -\frac{dH}{r'\sin i}$. Divide by $r'\sin i$ and change signs.*

$$p_y = w'r'\left\{\left(1+\tfrac{c}{r'}\right) - \cos i + \frac{\cos i}{2\sin^2 i} - \frac{i}{2\sin^3 i}\right\} + \frac{w}{r'}(r'^2 - r^2)\left\{\frac{i}{2\sin^3 i} - \frac{\cos i}{2\sin^2 i}\right\} = 0$$

Multiply by r' and divide by w

$$r'^2\left\{\frac{w'}{w}(1-\cos i) + \frac{w'}{w}\frac{c}{r'} + \frac{w'\cos i}{2w\sin^2 i} - \frac{w'i}{2w\sin^3 i}\right\} + \frac{r'^2 i}{2\sin^3 i} - \frac{r'^2\cos i}{2\sin^2 i} - \frac{r^2 i}{2\sin^3 i} + \frac{r^2\cos i}{2\sin^2 i},$$

$$= r'^2\left\{\frac{w'}{w}(1-\cos i) + \left(1-\frac{w'}{w}\right)\left(\frac{i - \cos i\sin i}{2\sin^3 i}\right)\right\} + \frac{w'}{w}cr' - \frac{i - \cos i\sin i}{2\sin^3 i}r^2. \quad (1.)$$

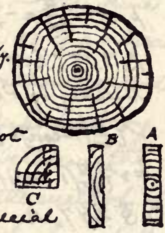
§ 297, A. The tunnel is considered as elliptic, to obtain (2.) and (3.). Then, quadrant of ellipse $= \frac{\pi}{4}a'b' = 0.8\,a'b'$; and radius of curvature at end of b is $\frac{a'^2}{b}$.

*$d\left(\frac{i\cos i}{2\sin i}\right) = \frac{\cos i\,di}{2\sin i} + i\left(\frac{-2\sin^2 i - 2\cos^2 i}{4\sin^2 i}\right)di = \frac{\cos i}{2\sin i}di - \frac{i}{2\sin^2 i}di.$

Chapter IV.
— Carpentry —
Section I.

§ 301. Good Timber: — All timber must be of best quality of its kind, and sawed true to dimension, out of <u>wind</u>, and free from wind shakes, large or loose knots, decay or wormholes, or other defects which will impair its strength or durability. Sometimes specifications read, "little or no sap."

§ 309. Seasoning. — Timber piled to season is sometimes painted on ends to hinder checking.



Timber in drying contracts little or none radially. Hence the rings must crack. A board like A, sawed through the middle of the log, does not shrink nor warp nearly so much as B, if at all. Lumber may thus be selected for special work. Quarter-sawed lumber is worked similarly, as shown by sketch C. The medullary rays are also exposed, giving a handsome appearance to some woods.

§ 311. Preservation of Timber: — Tannin is used with success with chloride of zinc, to keep the zinc salt in the pores of the wood. All these preservatives may be looked on as poisons, which kill or prevent from growing the fungi or germs which cause the decay. Paint keeps them, and the dampness which is necessary for their life, out. Unseasoned timber should not be painted. Dry air and ventilation make the life of timber very extended. Immersion under water preserves.

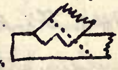
§ 312. Strength of Timber. — Bridge Specifications, Colorado Midland Ry. Working stresses per square inch, struts and braces, compression only, Flat-Ends, $\frac{1000}{1+\frac{l^2}{250h^2}}$; One Pin End, $\frac{1000}{1+\frac{l^2}{190h^2}}$; Two Pin Ends, $\frac{1000}{1+\frac{l^2}{125h^2}}$; in which l = length in inches between bearings and h = width of member in inches in direction of greatest liability to bend.

Kind of Wood.	Transverse Stress, outer fibre	Tension	Shear with grain.	Bearing with grain.	Bearing across grain.
White Oak	1200	1100	135	1000	300
Long leaf So. Pine	1200	1000	100	1000	250
Oregon Pine or Fir	1200	1400	150	900	190
White Pine, Eastn.	1000	600	85	800	150
Spruce	800	700	85	800	150
Colorado Yellow Pine.	800.	900	85	800	150.

Bearing = Compression. For roofs, where liberal allowance is made for snow and wind, these stresses may be increased 50%.

§ 314. Lengthening Ties. — Keys should be of hard wood, if resistance to shearing is to equal bearing resistance and tenacity. A packing block, to remain tight in spite of shrinkage, should have its grain in same direction as that of stick in which it is driven.

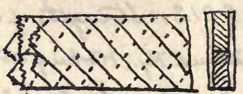
§ 320. Strut and Tie Joints. — With the joints shown, if shrinkage takes place, the bearing is eccentric, and the pieces are weakened, as explained in these Notes pp. 21, 22. Notching as in this sketch is preferable. A bolt on the dotted line is superior to a bridle.



§ 323. Nails. — American nails are different. Cut Nails 3 penny — 1¼" long, 460 in a lb.; 4 penny — 1½" long, 380 in lb.;
6 " 2" " 160 " "; 8 " 2½" " 92 " ";
10 " 3" " 60 " "; 20 " 4" " 24 " ";
40 " 5" " 14 " "; 60 " 6" " 10 " ".

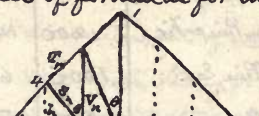
Wire Nails, same lengths, but lighter.

§ 331. Built Beams.— A very serviceable beam may be made by



nailing boards at 45°. on both sides of two planks, as above, setting them close and in opposite directions on the two sides. The beam may be made wider by more layers; and old, seasoned floor joists and boards have been made to span quite an opening in this way.

§ 339. Roof-Trusses.— III. A set of formulas for this truss may be used. Let N = whole number of panels in truss, here 6. Let $\underline{n}$ = any panel from abutment



towards middle, calling 3–5 two panels; w = panel wt. on roof.

$V_n = \frac{1}{2}(n-1)w$ = stress in n^{th} vertical; (to be doubled for middle)

$S_n = \frac{1}{2}(n-1)w \sec\theta$ = comp.ⁿ in brace in n^{th} panel. l = span

By moments, $T_n \cdot$ perpendicular from $5 = \frac{1}{2}(N-1)wn\frac{l}{N} - (n-1)w\frac{n}{2}\frac{l}{N}$.

$T_n = \frac{1}{2}(N-n)w \operatorname{cosec} i$ = comp.ⁿ in n^{th} rafter, since $\frac{nl}{Np} = \operatorname{cosec} i$.

$H_n \cdot$ ht. to $4 = \frac{1}{2}(N-1)w(n-1)\frac{l}{N} - (n-1)w\frac{n-2}{2}\frac{l}{N}$, or

$H_n = \frac{1}{2}(N-n+1)w \cot i$ = tension in n^{th} main tie.

H_n begins with $n = 2$. Stress diagrams are better and easier.

The way of deducing these formulas will be explained under the Howe Truss, § 343.

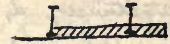
IV. Hammer-beam Truss.— See Graphics, "Roof Trusses."

§ 340. Tie-Beams.— $f'' = \frac{6M}{bh^2}$; $f''' = \frac{H}{bh}$; each represents maximum stress from its own cause. $f' = f'' + f'''$, and must not exceed safe intensity. $\therefore f' = \frac{H}{bh} + \frac{6M}{bh^2}$. (1.)

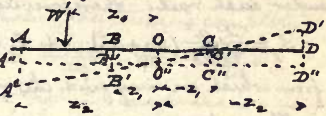
If h is assumed, $b = \left(\frac{H}{h} + \frac{6M}{h^2}\right) \div f'$; (2.) which gives same result as found by calculating a breadth for bending moment and an additional breadth to resist H. This formula should <u>not</u> <u>be</u> <u>used</u> for strut-beams and bent struts.

See Notes, p. 35.

§ 341. Bridge Trusses. I. Load Unequal Transversely. The trusses of a bridge with roadway and



two outside sidewalks will each have maximum stresses when opposite sidewalk alone is empty, and minimum stresses when opposite sidewalk is alone loaded. The floor beam will have max + M for load on roadway alone, and min. + M for load on sidewalks only.



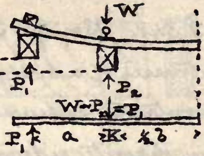
Case given in text. Beam AD, resting on four trusses at A, B, C and D. Under load W' the trusses deflect, the beam assuming the position, greatly exaggerated, of A'D', and remaining sensibly straight, as stated in the text. The mean deflection is OO'' at middle. The truss at A is depressed to A', an amount = AA'' + A''A'. As the four trusses are alike, their shares of W' will be proportional to deflections, and OO'' = AA'' must be the amount caused by an equal amount of W' on each, or $\frac{1}{4}W'$. A''A' will measure the remainder carried at A, which we may denote by az_2, as A''A' varies as distance from O. Then, by moments about O,

$$W'z_0 = (\tfrac{1}{4}W' + az_2)z_2 + (\tfrac{1}{4}W' + az_1)z_1 + (\tfrac{1}{4}W' - az_1)(-z_1) + (\tfrac{1}{4}W' - az_2)(-z_2)$$

$$= 2a(z_1^2 + z_2^2); \quad \therefore a = \frac{W'z_0}{2(z_1^2 + z_2^2)}.$$

Hence (1). If a load comes on both tracks at once, each girder will carry $\frac{1}{2}W'$. If then we so space the girders that $\frac{1}{4}W' + az_2$, the load on A, shall not exceed $\frac{1}{2}W'$, we have secured economy of construction. $\therefore \frac{z_0 z_2}{2(z_1^2 + z_2^2)} = \frac{1}{4}$ or $z_1^2 = 2z_0z_2 - z_2^2$.

The Portion of Load on Outside or Safety Stringers of a Bridge Floor. — Supposing the ties and stringers to have uniform bearings, the distribution of a load may be determined approximately. Let W be the load on tie at each rail in centre of span. W is divided into two parts, P_1 acting on the safety stringer and P_2 on the track stringer. The span of the stringers being l, under these loads the stringers will deflect



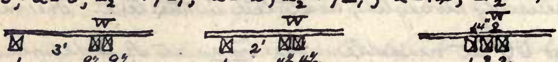
Safety Stringer; $d_1 = \frac{P_1 l^3}{48EI_1}$,
Track Stringer; $d_2 = \frac{P_2 l^3}{48EI_2}$; [See VI. p. 53]
where I_1 and I_2 are respective moments of inertia.

The difference between these deflections is the deflection d_3 of the cross-tie at point K, considered as a beam loaded with equal weights P_1 under each rail. Therefore, using $\underline{a}$ and $\underline{b}$ as marked, (VII. p. 54)

$d_3 = \frac{P_1 a^2}{6EI_3}(2a+3b)$. As $d_3 = d_2 - d_1$; $\frac{P_1 a^2}{I_3}(2a+3b) = \frac{P_2 l^3}{8I_2} - \frac{P_1 l^3}{8I_1}$;

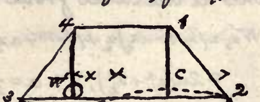
from which we may find the ratio of P_1 to P_2.

Thus, if $l = 15' = 180''$; track stringers $= 2-8''\times16''$, ($I_2 = 5460$); side-stringer $= 1-8''\times16''$, ($I_1 = 2730$); ties $= 6''\times8''$ flat, ($I_3 = 144$); we have, for $b = 60''$; $a = 3'$, $P_2 = 19P_1$; $a = 2'$, $P_2 = 9P_1$; $a = 14''$, $P_2 = 4P_1$.



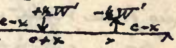
The last result shows the effect of having the rail over the middle of the inner two of three stringers:– the outside stringer then carries but 2/10 W. It is also clear that a side-stringer over 2½ ft. or 3 ft. out does little to relieve the track stringers, and hence that the latter should then be calculated as if the side-stringers were absent. [G. F. Swain, R. R. Gazette, Jan. 27, 1888.

II. Load Unequal Longitudinally.– An equal pull on rods 4-6 and 1-5 is required for equilibrium of the truss. For a



small deformation, 1 or 5 must rise as much as 4 or 6 sinks, for the system of bars 3412 turns about 3 and 2 as centres. Then the upward pull at 5 on the $\underline{\text{beam}}$ must equal the portion of W' carried by the beam at 6, and not supported by 4-6. But as, for equilibrium of truss, pull on 1-5 = pull on 4-6, ½W' must rest on beam at 6, and ½W' be carried by 4-6. The beam 3-2 will then be loaded with +½W' at 6 and –½W' at 5, with point of contraflexure, by symmetry, in the middle. Hence

$\mathfrak{M}$ at $6 = \frac{1}{2} W' \frac{x(c-x)}{c}$; (3.) $\mathfrak{M}$ at $5 = -\frac{1}{2} W' \frac{x(c-x)}{c}$.

Or we may take moments on beam of span $2c$, 

Reaction at left $= \frac{1}{2} W' \frac{c+x}{2c} - \frac{1}{2} W' \frac{c-x}{2c} = \frac{W'x}{2c}$; $\mathfrak{M} = \frac{W'x}{2c}(c-x)$.

As total reaction of bridge at left must be $\frac{W'(c+x)}{2c}$, we have, as above, $\frac{W'}{2}$ through the end brace, and $\frac{W'x}{2c}$ through the beam. And at 2, $\frac{W'}{2} - \frac{W'x}{2c} = \frac{W'(c-x)}{2c}$.

If braces are put in the middle panel, the truss is complete, and the shear in that panel will be $\frac{W'(c+x)}{2c} - W' = -\frac{W'(c-x)}{2c}$, equal and opposite to reaction at right. Compression in brace 4-5 given by multiplying by its length, $\sqrt{4x^2+k^2}$, and dividing by k.

Look

§ 343. Howe Truss.— Before studying this analysis, take § 377, and Notes.

Howe Truss, Fig. 1.

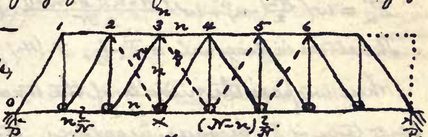


Fig. 1.

Pratt Truss, Fig. 2.

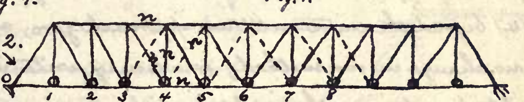


Fig. 2.

Number on the chord joints, as in respective figures, and note which pieces are indicated by letter $\underline{n}$. Verticals alone are affected by moving the load to top.

Span $= l$; height $= k$; whole no. of panels $= N$; any panel or joint $= n$. Bending moments and chord stresses are maximum for full load, $w+w'$ on each joint. Neglecting weights at abutments.

$P_1 = \frac{1}{2}(N-1)(w+w')$. Taking moments of external forces about and on the left of n^{th} joint, as in Case VIII., p. 247 (which see),

$$\mathfrak{M}_{max} = \tfrac{1}{2}(N-1)(w+w')\frac{l}{N}n - (n-1)(w+w')\frac{l}{N}\cdot\frac{n}{2} = n(N-n)\frac{l}{2N}(w+w') \quad (1.)$$

Note that $\mathfrak{M}$ and hence chord stress varies as product of two segments.

$$H_n = \frac{\mathfrak{M}}{k} = n(N-n)\frac{l}{2Nk}(w+w'). \quad (1.)$$

Minimum stresses will be found by using $\underline{w}$ only.

Stresses in diagonals and verticals are maximum for partial rolling load (See Case IX. p. 41, Notes.).

In Fig. 1, P_1 for $w = \frac{N-1}{2}w$; by moments, rolling load extending from right up to and including n^{th} joint, P_1 for $w' = (N-n)w'\,\frac{N-n+1}{2N}$. The shear in panel which contains n^{th} diagonal, $F_n = w\left[\frac{N-1}{2}-(n-1)\right] + w'\frac{(N-n)(N-n+1)}{2N} = w\left(\frac{N+1}{2}-n\right) + w'\frac{(N-n)(N-n+1)}{2N}$; (2.) which is also the stress in the n^{th} vertical. For the middle vertical $\underline{w+w'}$ is to be used, if greater than $F_{\frac{N}{2}}$.

If the load is on the top chord of the Howe Truss, or on the bottom chord of the Pratt Truss, Fig. 2, write, in the formula for the verticals, $n+1$ for n, as the vertical and diagonal in stress which connect two loaded joints have the same shear, or

$F'_n = w\left(\frac{N-1}{2}-n\right) + w'\frac{(N-n)(N-n-1)}{2N}$. (2'.) The book is wrong here.

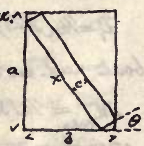
The stress in diagonals $= F_n\frac{s}{k}$; ∴ (4.)

When we pass the middle of the truss, we have $n > \frac{N+1}{2}$, and the first term of F_n becomes negative. We may compute values of (4.) beyond the centre until we approach zero, and we thus make no change in our method; or we may write $N-n$ for n and obtain values on the right for a load advancing from the left, thus getting the stresses in the counterbraces. Thus

$$T_n = F_{N-n}\frac{s}{k} = \frac{s}{k}\left[w\left(\frac{N+1}{2}-(N-n)\right) + w'\frac{(N-N+n)(N+1-N+n)}{2N}\right] = -\frac{s}{k}w\left(\frac{N-1}{2}-n\right)$$
$$+w'\frac{s}{k}\frac{n(n+1)}{2N}. \quad (5.)$$

If minimum as well as maximum stresses are desired, the values of F_n and $F_n\frac{s}{k}$ should be computed up to $n = N$.

Length of Howe Truss Brace.— Length of brace $= x$.



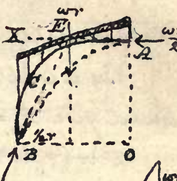
$x\sin\theta + c\cos\theta = b$; $x\cos\theta + c\sin\theta = a$.

$$x = \frac{b - c\cos\theta}{\sin\theta} = \frac{a - c\sin\theta}{\cos\theta}.$$

$b\cos\theta - a\sin\theta = c(\cos^2\theta - \sin^2\theta) = c\cos 2\theta$.

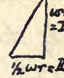
Assume a value of θ, and correct by approximation, in last equation. Then find x and dimensions of angle-block.

§ 345. Timber Semi-circular Arch.— As the rib is supposed to be uniformly loaded horizontally, and hinged at A



and B, the curve of equilibrium must be a parabola, shown by the dotted curve, drawn through those points and tangent to thrusts at crown and springing.

$(P = wr) : H = r : \frac{1}{2} r$; $\therefore H = \frac{1}{2} wr$. (1.)



The bending moment at any point $= H(y - y')$; where y' = vertical ordinate to circle from tangent at A, and y = do. do. to parabola from tangent at vertex A. M_{max} where $y - y'$ is max.

Parabola:- $x^2 = 4my$; when $x = r$, $y = r \therefore r^2 = 4mr$, $4m = r$; $\therefore x^2 = ry$.

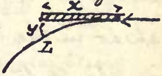
Circle:- $x^2 + (r - y')^2 = r^2$ or $r - y' = \sqrt{(r^2 - x^2)}$; $\therefore y' = r - \sqrt{(r^2 - x^2)}$.

$$y - y' = \frac{x^2}{r} - r + \sqrt{r^2 - x^2}. \quad \frac{d(y-y')}{dx} = \frac{2x}{r} + \tfrac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x) = 0. \text{ One value, } x = 0.$$

$$\frac{2}{r} - \frac{1}{\sqrt{r^2 - x^2}} = 0; \quad \frac{1}{r^2 - x^2} = \frac{4}{r^2}; \quad r^2 = 4(r^2 - x^2) \quad \therefore x = r\sqrt{\tfrac{3}{4}} = .866r = \begin{cases} \sin 60° \\ \cos 30° \end{cases}$$

$$y - y' = \tfrac{3}{4}\frac{r^2}{r} - r + \sqrt{(r^2 - \tfrac{3}{4}r^2)} = -\frac{r}{4} \pm \frac{r}{2} = \tfrac{1}{4}r. \quad M_{max} = H(y - y')_{max} = \frac{wr}{2} \cdot \frac{r}{4} = \frac{wr^2}{8} \ (2)$$

or same as that of a straight beam of span r, similarly loaded, but at a different place, i.e. 30° from B.



M_{max} may also be obtained as follows:-

$$M \text{ at } L = -Hy' + \frac{wx^2}{2} = -\tfrac{1}{2}wry' + \tfrac{1}{2}w(2ry' - y'^2), \text{ to be a max.}$$

$$\frac{d}{dy}(-ry' + 2ry' - y'^2) = 0 = r - 2y'; \ \therefore y' = \tfrac{1}{2}r, \text{ or at } 30° \text{ from } B.$$

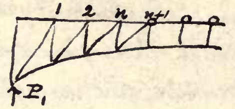
$$M_{max} = -\frac{wr^2}{4} + \frac{w}{2}\left(r^2 - \frac{r^2}{4}\right) = \frac{wr^2}{8}.$$

At C, 30° from springing; load from A to C $= wr.\sqrt{3/4} = W$. Tangent of circle at that point is parallel to that of the curve of equilibrium, the parabola; $\therefore$ Thrust at C $= \sqrt{(W^2 + H^2)} = wr\sqrt{(\tfrac{1}{2}^2 + \tfrac{3}{4})}$ $= wr = 2H$. Rib at C must resist this thrust and M_{max}.

Read

§ 346. Timber Spandril.— Rankine's reasoning appears to be that, since the parabola would be in equilibrium under a full, uniform load, and there would be no shear nor stress in the braces at any panel,— a brace at any point, when the span is loaded from one end to that panel, must supply the shearing resistance which would be given by <u>the</u> <u>load</u> <u>which</u> <u>is</u> <u>not</u> <u>on</u> <u>the</u> <u>truss</u>, and that thus equilibrium would be produced in the rib at that panel.

Thus a load on $\underline{n}$ joints from the left would give $P_2 = \frac{nw'(n+1)}{2N} = F$ and



$T = \frac{w'S}{k} \cdot \frac{n(n+1)}{2N}$. (1.)

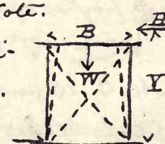
If, then, this resistance is given, Rankine considers that equilibrium will be assured. He apparently applies the same principle to the bowstring girder; § 347 and § 377.

But the horizontal thrust, and hence the part of the shear carried by the curved member, at any point, vary with the amount and position of the load on the frame; and it is wrong to use $H_{max} = \frac{wl^2}{8k}$ for a partial load. We must find the amount of vertical force carried by bow in any panel, when the load extends to that panel, and the total shear in the panel minus (or plus) this vertical force will be the shear in the brace. Then, in true analysis of bowstring girder, (See Notes on § 379) we get, for diagonal ties, $\frac{(N-n)n}{2N} \cdot \frac{w's}{Y_n}$.

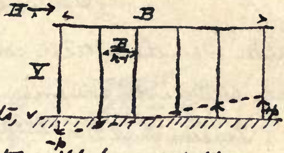
The braced arch above is treated in Graphics, Part III, "Arches".

§ 347. Bowstring Girder.— See preceding Note.

§ 348. Timber Piers.— Posts either vertical or inclined, but without auxiliary bracing. The external couple, being HY, is balanced by a couple at base of posts. $PB = HY \therefore P = \frac{HY}{B}$ and vertical force at foot of a post is $\frac{W}{2} \pm \frac{HY}{B}$. Horizontal force at each foot is $\frac{1}{2}H$. If then $B = \frac{2HY}{W}$, the pressure will be zero at foot of windward post. No account is here taken of the bending moments experienced by the pieces.



If diagonals, shown by the broken lines, are introduced, the reactions are unchanged, but the amount of force, or rather its distribution, in the posts is altered. With strut diagonal also the case is unlike that of tie diagonal.



If the pier or trestle-bent has several posts transversely, the moment of resistance at feet of posts, to balance HY overturning moment, will be as follows:-

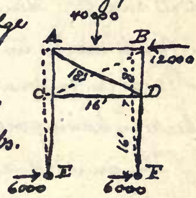
Let the resistance of outside posts $= \pm p$. Arm $= B$.

Resistance of next posts to outside $= \pm p\frac{n-3}{n-1}$. Arm $= B\frac{n-3}{n-1}$.

$\therefore HY = pB + p\frac{n-3}{n-1}\cdot B\frac{n-3}{n-1} + p\frac{n-5}{n-1}\cdot B\frac{n-5}{n-1}$ &c.

$= pB\,\frac{(n-1)^2+(n-3)^2+(n-5)^2\ \&c.}{(n-1)^2}$; $\therefore\ p = \frac{HY(n-1)^2}{B\{(n-1)^2+(n-3)^2\ \&c\}}$

$P = \frac{W}{n} \pm p$. (3.) & (4.) If P is made zero, we have the minimum breadth of base, to avoid tension in any post.



Trestle with partial bracing, or Bridge Portal. To make the analysis plain, this example is assumed. Load of 40,000 lbs. on top at middle; horizontal force of 12000 lbs. applied at B; diagonals to be ties; no bending moments at bases of posts. The two posts, being of the same section, may be expected to carry equal bending moments and shears; hence 6000 lbs. horizontal force will be resisted at E and F. Diagonal AD will be in action, and the posts will tend to deflect in the dotted lines.

M at middle of $AB = \frac{40000\times16}{4} = 160,000$ ft. lbs. Shear in AB $= \pm 20000$ lbs. $=$ vertical force at A and B, resisted by ground.

Taking moments about E or F, for the horizontal force of 12000 lbs., $\frac{12000\times24}{16}$ = compression at E, tension at F of 18000 lbs.

$CE = 20000 + 18000 = 38000$ compression; $DF = 20000 - 18000 = 2000$ comp.

Or, by moments for both external forces about F, $\frac{40000\times8 + 12000\times24}{16}$ $= 38000$ lbs. compⁿ. at E as before.

Shear in DF and CE, each 6000 lbs.

M_{max} in posts at D and C, $= 6000\times16 = 96000$ ft. lbs.

Since AD will be in action, compression in $BD = 20000$ lbs, load at B. Then the vertical component in AD must be the difference between the compressions in BD and DE, or 20000 − 2000. = 18000 lbs.

∴ Tension in AD = $\frac{18000\times18}{8}$ = 40500 lbs.

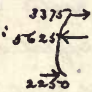
AC will then carry 20000 + 18000 = 38000 lbs. comp.ⁿ; check on CE.

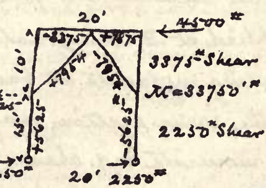
Horizontal component in AD = $\frac{18000\times16}{8}$ = 36000 lbs., of which one-half, or 18000 lbs. springs the post at D, and the other half compresses DC and springs the post at C.

Shear in BD and AC must be enough, when applied at B and A, to give the previously calculated max. M of 96000 ft. lbs at D and C. Hence, shear in BD and AC = $\frac{96000}{8}$ = 12000 lbs.

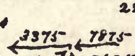
∴ Compression in AB = 36000 from AD − 12000, or 24000 lbs.; or 12000 lbs. external force at B + 12000 lbs. shear in BD = 24000 lbs. Check by having resultant of 12000 lbs. on any hor.^{l} section.

The annexed example shows another arrangement of braces. The posts are sprung thus:





Equilibrium at the middle of top member gives

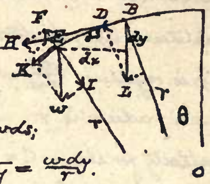


Fixing posts at the bottom introduces a moment there.

§ 349. Centres for Arches.—(1.) The load on unity of arc being w, its normal component EI is $w \cos\theta$, which is the max. intensity of load on the lagging. The weight next above, at B, a distance ds, may be resolved normally and tangentially. The latter component will intersect the normal component of the weight at E, and can now be resolved into a component EK, parallel to the tangent here, and one, EF, opposed to EI. Therefore the normal pressure at E is now EI − EF.

Load on $ds = w\,ds$; tangential comp.^{t} at B $= w\,ds\frac{dy}{ds} = w\,dy$.

Force EF : $w\,dy = ds : r$, or $EF = \frac{w\,dy\,ds}{r}$;



∴ Intensity $= \frac{EF}{ds} = \frac{w\,dy}{r}$. ∴ $p = w\cos\theta - \frac{1}{r}\int_{y_0}^{y} w\,dy.$ (1.)

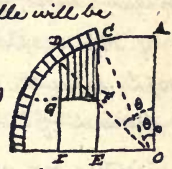
Or, from similar triangles, OEB and FEH,

$\frac{FE}{HE} = \frac{EB}{EO}$, or $FE = \frac{HE\cdot EB}{EO} = \frac{BD\cdot EB}{EO}$. $BL = w\,ds$;

$BD = \frac{w\,ds\,dy}{ds} = w\,dy$; $FE = \frac{w\,dy\,ds}{r}$ Intensity $= \frac{w\,dy}{r}$.

p = normal intensity; $p\,ds$ = normal pressure on ds; $p\,ds\frac{dx}{ds} = p\,dx$ = vertical component on ds. ∴ (2.)

If the centre forms a girder, since the arch is supposed to go up on both sides together, the moment at the middle will be



$Pc - \int_{x_0}^{x_1} p\,dx\cdot x.$ (3.) By (1.) $p = w\cos\theta - \frac{1}{r}\int_{r(1-\cos\theta_0)}^{r(1-\cos\theta)} w\,dy$

$= w\cos\theta - \frac{w}{r}\{r(1-\cos\theta) - r(1-\cos\theta_0)\} = 2w\cos\theta - w\cos\theta_0$ (5.)

$y = r - r\cos\theta$; $\cos\theta = \frac{r-y}{r}$; $\cos\theta_0 = \frac{r-y_0}{r}$. Substitute in (5.)

$p = w\,\frac{2r-2y-r+y_0}{r} = w\,\frac{r-2y+y_0}{r}$ (5.). For (5A.) see (1A.).

$P = \int_{r\sin\theta_0}^{r\sin\theta} p\,dx = w\int_{r\sin\theta_0}^{r\sin\theta}(2\cos\theta - \cos\theta_0)\,dx$. $x = r\sin\theta$; $dx = r\cos\theta\,d\theta$; ∴

$P = wr\int_{\theta_0}^{\theta}(2\cos^2\theta - \cos\theta\cos\theta_0)\,d\theta = wr\Big[_{\theta_0}^{\theta}(\cos\theta\sin\theta + \theta - \sin\theta\cos\theta_0)$

$= wr\{\theta - \theta_0 - \sin\theta(\cos\theta_0 - \cos\theta)\} = w\{s - s_0 - \frac{x}{r}(y - y_0)\}$ (6.)

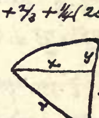
$\int\cos^2\theta\,d\theta = \frac{\cos\theta\sin\theta + \theta}{2}$

$M = P_1c - \int_0^{x_1} p\,x\,dx = P_1c - \int_0^{\theta_1} w(2\cos\theta - 1)\,r\sin\theta\,r\cos\theta\,d\theta = P_1c - \int_0^{\theta} wr^2$

$(2\cos^2\theta\sin\theta - \sin\theta\cos\theta)\,d\theta = P_1c - wr^2\Big[_0^{\theta_1} -\frac{2}{3}\cos^3\theta - wr^2\int_0^{\theta_1}\frac{1}{2}\sin2\theta\,d\theta$

$= P_1c - wr^2\left(-\Big[_0^{\theta_1}\frac{2}{3}\cos^3\theta + \frac{1}{4}\Big[_0^{\theta_1}\cos2\theta\right) = P_1c - wr^2\left(-\frac{2}{3}\cos^3\theta_1 + \frac{2}{3} + \frac{1}{4}(2\cos^2\theta_1 - 1)\right.$

$\left.-\frac{1}{4}\right) = P_1c - wr^2\left(\frac{1}{6} + \frac{1}{2}\cos^2\theta_1 - \frac{2}{3}\cos^3\theta_1\right).$ (7.)



$x^2 = y(2r-y) = 2ry - y^2$; $r = \frac{x^2+y^2}{2y} = \frac{y + \frac{x^2}{y}}{2}$. (8.)

Formula for parabolic arc, $s = \int_0^x\left(1 + \frac{x^2}{4m^2}\right)^{1/2}dx$. [Notes, p10, (13)

$= \int_0^x dx + \frac{1}{2}\int_0^x\frac{x^2}{4m^2}dx - \frac{1}{4}\cdot\frac{1}{2}\int_0^x\frac{x^4}{16m^4}dx = x + \frac{x^3}{24m^2} - \frac{x^5}{640m^4} + c. = x + \frac{2}{3}\frac{y^2}{x} - \frac{2}{5}\frac{y^4}{x^3} + c.$ (8.)

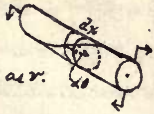
For the graphical method, see (5.). $p = 2w\cos\theta - w\cos\theta_0$.

If $AO = 2w$, and $AOD = \theta$; $DI = 2w\cos\theta$, $EF = \frac{1}{2}CE = w\cos\theta_0$;

$DG = DI - EF = 2w\cos\theta - w\cos\theta_0 = p.$ (5.)

When arch ring reaches crown, E comes at middle of AO, and the horizontal line through that point strikes the rib at 30° from springing, — hence, remark, middle p. 486.

§ 349 A. Resistance of Timber to Torsion. — If $d\theta$ is the small angle at centre that radius revolves in passing longitudinally a distance dx, distortion $= \frac{r\,d\theta}{dx}$; and, as the pitch of this helix is regular, $\frac{d\theta}{dx} = \frac{\theta}{x}$. Let p = intensity of shear at the point whose radius is r. Then the modulus of elasticity for shear, $C = \frac{p}{\text{distortion}}$. $\therefore$



$\frac{d\theta}{dx} = \frac{p}{Cr} = \frac{f}{Cr_1}$, since intensity of shear varies as r.

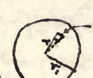
$\frac{d\theta}{dx} = \frac{\theta}{x} = \frac{f}{Cr_1}$; $\theta = \frac{fx}{Cr_1} = \frac{2fl}{Cd}$.

$M' = \frac{\pi}{16} f d^3$ for round shaft; $f = \frac{16 M'}{\pi d^3}$, $\therefore \theta = \frac{32 M' l}{\pi C d^4} = \frac{M' l}{0.1 C d^4}$.

For square shaft, if $M' = \frac{f}{r_1} I_p = \frac{f}{\sqrt{\frac{1}{2}}\,h} \cdot \frac{h^4}{6} = 0.236 f h^3$; $\theta = \frac{M' l}{0.118 C h^4}$;

if $M' = .208 f h^3$, $\theta = \frac{M' l}{0.104 C h^4}$. The old edition has $M' = 0.280 f h^3$; hence the 0.1405 of text.

Moment of Torsion. (Also § 353, p. 502.) —



In twisting a bar we cause one section to rotate by the next one, and develop a shearing stress that varies with the motion, or directly as the distance from the centre. The shearing stresses make up the moment of resistance to torsion. Let f' = intensity of shear at circumference; intensity at any other point $= \frac{f'}{r_1} r$.

Stress on a particle $= \frac{f'}{r_1} r \cdot r\,dr\,d\theta$; lever arm $= r$, hence moment

$M' = \int_0^{r_1}\int_0^{2\pi} \frac{f'}{r_1} r^3\,dr\,d\theta = \frac{f' I_p}{r_1}$, where I_p = polar moment of inertia.

$= \frac{2\pi f'}{r_1} \cdot \frac{r_1^4}{4} = \frac{\pi f' r_1^3}{2} = \frac{\pi f' h^3}{16} = 0.196 f' h^3$ for cylinders.

$I_p = \frac{h^4}{6}$ for square bars, and $M' = \frac{f'}{\sqrt{\frac{1}{2}}\,h} \cdot \frac{h^4}{6} = 0.236 f' h^3$.

The assumption in this last approximate value is that the shear varies as r along all radii, notwithstanding the differing lengths of distances to perimeter. The error is not important.

Chapter V.
Metallic Structures.
Section I.

§ 352. Cast Iron.— "Steel and Iron," by Greenwood, p.66; "Gray cast-iron is more fluid when melted than white iron, but it requires a much higher temperature for its fusion." Gray iron melts at 1600° to 1700° C.; white iron at 1400° to 1500° C. Also Crookes & Röhrig, pp. 268, 290.

§ 353, p. 502. — See Notes, p. 94.

§ 356. Steel.— Modern structural and other steel contains but very little carbon, and differs from wrought iron more in name than in reality. Wrought iron is made by puddling, and the process of rolling from the puddled ball rolls out the slag, oxides, and impurities in the spongy mass; and these, by hindering the complete welding of the adjacent particles, produce that appearance of fibre, which is considered a characteristic of wrought iron. It is rather an accident of the process of manufacture and does not, in itself, give strength. Lack of so good lateral adhesion explains the less degree of strength in wrought iron across the grain.

Steel, on the other hand, cast in an ingot, and then rolled, is homogeneous and free from impurities, and hence does not develop fibre in rolling. Therefore the strength of steel bars, of the least carbon percentage, ought to exceed that of wrought iron bars.

Mild steel, so called, may not exceed 0.12 of one per cent. of carbon, and will not harden and temper, but welds; it is iron under another name, denoting a better article; but not what has been commonly understood by steel. Hard steel, structurally speaking, used for compression members, may not exceed 0.36 of one per ct. of carbon. Compare these per cents. with Rankine's 0.5 to 1.5 per cent.

Open-hearth steel is more uniform than Bessemer metal, and has more ductility for a given elastic limit and ultimate strength. Carbon exists in steel in two principal shapes, hardening and non-hardening: Hardening carbon is the form found in steel which has been heated to a high red heat and quenched in with water. Non-hardening carbon is found in steel which has been heated to a red heat and slowly cooled. [John Coffin. See R.R. Gazette, Dec. 23, 1887.] This view of the matter brings steel into harmony with cast-iron, as to two forms of carbon, combined and graphitic, and makes it more nearly fit its intermediate position between wrought iron, with no carbon, and cast-iron, with the maximum amount.

§ 357. Strength of Wrought Iron and Steel.– It is the opinion of some engineers that neither the chemical constitution nor the mechanical processes of manufacture should be specified, in calling for a certain grade of iron or steel, but only breaking strength, elastic limit, elongation and reduction of area. Hence, carbon percentage of steel and double refining of iron will not be mentioned.

The following specifications agree very well with present practice.

All wrought iron must be tough, fibrous and uniform in character. It shall have a limit of elasticity of not less than 26000 lbs. per sq. in. Finished bars must be thoroughly welded during rolling, and be free from injurious seams, blisters, buckles, cinder spots or imperfect edges.

For all tension members the muck bars shall be rolled into flats, and again cut, piled and rolled into finished sizes. They shall stand the following tests:– Full-sized pieces of flat, round or square iron, not over 4½ sq. in. in sectional area, shall have an ultimate strength of 50000 lbs. per sq. in., and stretch 12½ per cent. in their whole length.

Bars of larger section than 4½ sq. in., when tested in usual way, will be allowed a reduction of 1000 lbs. for each additional sq. inch, down to a minimum of 46000 lbs. per sq. inch.

When tested in specimens, of uniform section of at least ½ sq. in. in length of 10 inches, taken from tension members rolled to a section not more than 4½ sq. inches, the iron shall show ultimate strength of 52000 lbs. and stretch 18% in distance of 8 inches. Specimens from bars larger than 4½ sq. inches will be allowed a reduction of 500 lbs. for each additional sq. inch of section, down to minimum of 50000 lbs.

The same sized specimen taken from angle and other shaped iron shall have an ultimate strength of 50000 lbs. and elongate 15%. in 8 inches. The same sized specimen from plate iron shall have 50000 lbs. ult. str. and 15%. elongation in 8 inches.

All iron for tension members must bend cold for about 90°, to a curve of diameter not over twice the thickness of piece, without cracking. At least one sample in three must bend 180° to this curve without cracking. When nicked on one side and bent from a blow with a sledge, the fracture must be nearly all fibrous, showing but few crystalline specks. Specimens from angle, plate and shaped iron must stand bending cold through 90°, to a curve of diameter not over three times the thickness of piece, without cracking. When nicked and bent, its fracture must be mostly fibrous.

Steel for rivets and eyebars shall contain not more than 25/100 of one per cent. of carbon, and less than 1/10 of one per cent. of phosphorus. A sample bar ¾ inch in diameter shall bend 180°, and be set back upon itself without showing crack or flaw; when tested in a lever machine it shall have an elastic limit of not less than 40000 lbs, and an ultimate strength of not less than 70000 lbs. per sq. inch; it shall elongate at least 18% in a length of 8 inches, and shall show a reduction of area of at least 45% at the point of fracture. In full-sized bars this steel shall have an elastic limit of at least 35000 lbs., and an ultimate strength of at least 65.000 lbs. per sq. inch; it shall elongate 10% before breaking, and for strains less than 30000 lbs. per sq. inch shall show a modulus of elasticity between 28 000 000 and 30 000 000 lbs.

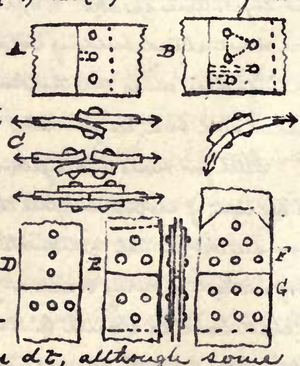
Steel used in compression members shall not contain more than 1/10 of one per cent. of phosphorus. A sample bar 3/4 inch in diameter shall bend 180° around its own diameter without showing crack or flaw, and, when tested in tension, it shall have an ultimate strength of not less than 80 000 lbs. and an elastic limit of not less than 50000 lbs. per sq. inch, shall elongate at least 15% in 8 inches, and show a reduction of area of at least 30% at point of fracture.

It shall be incapable of tempering.

Examples.	Elastic limit.	Ult. Strength.	Elong.n in 8 in.	Red.n Area.	Carbon.	Manganese.	Phosphorus.
Not Annealed	53140#	83680#	20.75%	37.78%	.27	.83	.067
	51190#	84440#	18.75%	33.23%	.26	.77	.067

Section II.

§360. Rivets.— Formulas (1.), (2.) and (3.) are not very serviceable. If d = diam. of rivet, p = pitch, or distance from centre to centre in a row, t = thickness of plate, and w = distance from centre of hole to edge of plate; the shearing area of a rivet in single shear is $\frac{\pi d^2}{4}$; in double shear it is twice as much. The bearing area, where the compression is exerted, is usually taken as dt, although some



use the semicircumference in place of the diameter. The tension area, between two holes, is $(p-d)t$. The shearing area of the plate, for one rivet, is $2wt$. If more than one row of rivets is needed, the rows are staggered, as at B, and the dotted diagonal should not be less than the pitch. The tendency of a lap joint to cause an uneven distribution of stress by reason of bending, and the same tendency when a single cover strip is used, is shown at C. The case is not so bad in a boiler. Rivet-holes, in good practice, are allowed for as if 1/8 inch more in diameter than the rivet.

In splicing ties, D is a bad arrangement, the upper plan failing to distribute the stress across the tie, and the lower plan wasting the section by excessive cutting away. The rivets at E are well distributed across the breadth, and weaken the tie by but one hole, as only two-thirds the stress has to pass the section reduced by two holes. The covers, however, will be weakened by two holes, and hence their combined thickness, if two are employed, should exceed that of the tie. F, similarly is better than G, and the tie is again weakened by one hole.

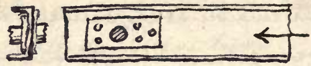
In compression, as the rivets should fit the holes tightly, the holes are not deducted.

It is not usual to compute rivets for bending, although such treatment has been advocated.

Heads may be conical, as in boiler work , or button shaped as in bridge and structural work. They are often flattened, or countersunk , when the heads would otherwise be in the way.

Rivets should not be subject to tension.

§361. Pins. — See Notes, pp. 49, 50. In pin-jointed structures, the pins are subjected to compression on their cylin-



drical surfaces, to shear on the cross-section, and to bending moment. The compression of the pinhole is reduced to the proper intensity, if necessary, by riveting reinforcing plates to the sides of the members, as seen above, with a sufficient number of rivets to transmit the proper proportion of the force.

The shear at any section of the pin is found from the given forces in the pieces connected. As the pin will not probably fit the hole tightly (a difference of diameter of 1/50 of an inch being usually permitted), the max. intensity of shear, by §168, will be 4/3 the mean, and this is usually allowed for in specifications by reducing the unit stress for shear by one-quarter, e.g. from 10000 lbs. to 7500 lbs. per sq. inch, and thus finding the pin section directly.

Bearing area is also figured as if projected on the diameter with, e.g., 15000 lbs. in place of 10000 lbs. per sq. inch of semicircumference of hole.

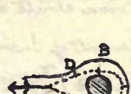
At a joint where several pieces are assembled, the moment of resistance, required to resist the maximum bending moment on the pin caused by the forces in those pieces, will generally determine the diameter of the pin. In computing the bending moments, the centre of each bearing is considered the point of application of the force. This assumption is likely to give a result somewhat in excess of the truth, as any yielding tends to diminish slightly the arm of each force. It will be convenient to decompose the applied forces into components in two planes at right angles, and finally, from the component moments, to find the maximum bending moment, which will occur at one of the places where the shear is zero or changes sign. Some computers use the shears to find bending moments, by (8.) p. 243.

Graphics offers a very convenient solution when the location of point of absolute max. bending moment is doubtful, also when the pin acts as a continuous beam of two spans.

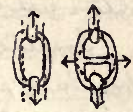
In computing a bridge pin from data on the strain sheet, care must be taken not to combine web stresses from a partial load with chord stresses from a full load. The nut on a pin, being subject to little force, may be made shallow. The arrangement of pieces on a pin may have much to do with the magnitude of the bending moment and size of pin.

§362. Bolts. For economy, long bolts should be upset or enlarged at the screw end, to give, according to good practice, a net area, at bottom of thread, some 15% in excess of that of the bolt. Nuts must be deep enough not to shear off or strip the thread, and heads thick enough to resist shear on the cylindrical surface of diameter equal to that of the bolt. The head and nut must have sufficient bearing area. Washers increase this area on yielding material, and for soft wood are large.

Section III.



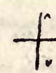
§364. Ties.—III. Eyes for tie-bars are usually given a section on each side of the pin of 75% of the body of the bar, making the excess in the section BC 50%. The places of greatest intensity of tensile stress are at the edges marked at D, B and A. The curve of equilibrium, or locus of resultant force at each section, probably has a shape like the dotted line. Hence eyes made by turning a bar round a pin and welding should be made long in the loop. 



IV. The equilibrium polygon for a studded link is like the dotted rhombus; reduce the transverse force to zero, by removing the stud, and the rhombus becomes two parallel lines. The bending moments for the same shaped link are likely to be less when the stud is used.

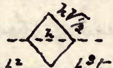
§ 365. Cast-Iron Struts. — For h^2 of the formula may be substituted r^2, as in Notes, p. 32. Then columns of the same sectional area may be compared by r^2. The method of finding r^2 will be explained in next article, as applied to thin sections. Moment of inertia of ring of thickness t and radius r about its polar axis is $2\pi r t \cdot r^2$. As $I_x + I_y = I_p$, since $x^2+y^2=r^2$;

$I_x = \frac{1}{2} I_p = \pi r^3 t = \frac{\pi h^3 t}{8}$; $r^2 = \frac{I}{S} = \frac{\pi h^3 t}{8 \cdot \pi h t} = \frac{h^2}{8}$.



I for cross, about horizontal axis, $= \frac{1}{3} h t \cdot \frac{h^2}{4} = \frac{h^3 t}{12}$;

$r^2 = \frac{h^3 t}{12} \div 2ht = \frac{h^2}{24}$; ∴ $24 : 8 = 3 : 1$.



I for hollow square, about diagonal of length $h = \frac{1}{3} 4ht\sqrt{\frac{1}{2}} \cdot \frac{h^2}{4} = \frac{h^3 t}{3\sqrt{2}}$.

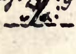
$r^2 = \frac{h^3 t}{3\sqrt{2}} \div 4h\sqrt{\frac{1}{2}} = \frac{h^2}{12}$; ∴ $12 : 8 = 3 : 2$.

§ 366. Wrought-Iron Struts. — Rankine's first statement may be derived as follows:— $\frac{80000\,S}{1+\frac{l^2}{400h^2}} = \frac{36000\,S}{1+\frac{l^2}{3000h^2}}$; $36 + \frac{36l^2}{400h^2} = 80 + \frac{8l^2}{300h^2}$;

$38l^2 = 26400h^2$; $l = 26.3\,h$, for same sections, equal strength.

Cast-iron struts and columns are not regarded with favor.

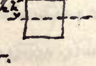
Moment of Inertia of a thin bar, length L, thickness t, about an axis through its extremity.



$\Sigma m r^2 = \int_0^L t\,dl \cdot l^2 \sin^2\alpha = \frac{1}{3} t L^3 \sin^2\alpha = \frac{1}{3} t L\, y^2$, or one-third of the bar multiplied by the square of perpendicular distance to extreme end.

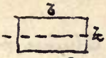
I. I for solid rectangle $= \frac{bh^3}{12}$ (Notes, p. 28); $r^2 = \frac{bh^3}{12} \div bh = \frac{h^2}{12}$.

II. Sides parallel to axis, $I = 2ht\frac{h^2}{4}$; do. perpend. to axis, $I = \frac{2ht}{3} \cdot \frac{h^2}{4}$.

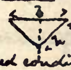


∴ $I = \frac{2h^3t}{3}$. $r^2 = \frac{2h^3t}{3} \div 4ht = \frac{h^2}{6}$. Omit t hereafter.

III. $I=\frac{2h}{3}\cdot\frac{h^2}{4}+2b\cdot\frac{h^2}{4}=(h+3b)\frac{h^2}{6}$. $r^2=\frac{h+3b}{h+b}\cdot\frac{h^2}{12}$.

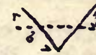


IV. About vertical axis, $I=\frac13 b\cdot\frac{b^2}{4}+\frac{2h}{3}\cdot\frac{b^2}{4}=(2h+b)\frac{b^2}{12}$.



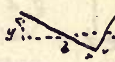
$r^2=\frac{2h+b}{2h+b}\cdot\frac{b^2}{12}=\frac{b^2}{12}$. Moment about vert. axis by imposed condition.

V. (See Notes, p. 28, IV.) $r^2=\frac{\pi r^4}{4}\div\pi r^2=\frac{r^2}{4}=\frac{h^2}{16}$.



VI. $I_x=\frac12 I_p=\frac12 2\pi r\cdot r^2=\pi r^3$. $r^2=\pi r^3\div 2\pi r=\frac{r^2}{2}=\frac{h^2}{8}$.

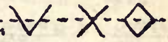
VII. $I=\frac23 b\cdot\frac{b^2}{8}=\frac{b^3}{12}$. $r^2=\frac{b^3}{12}\div 2b=\frac{b^2}{24}$.



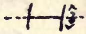
VIII. $y:\frac12 b=h:\sqrt{b^2+h^2}$; $\therefore y^2=\frac{b^2h^2}{4(b^2+h^2)}$.

$I=\frac13(b+h)\frac{b^2h^2}{4(b^2+h^2)}$; $r^2=I\div(b+h)=\frac{b^2h^2}{12(b^2+h^2)}$.

IX. Same as VII. Slide one arm across the other



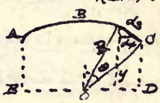
X. $I=\frac{A}{3}\cdot\frac{b^2}{4}$; $r^2=I\div A+B=\frac{b^2}{12}\cdot\frac{A}{A+B}$.



XI. I' about base $=\frac13 Ah^2$; $y'=\frac{\frac12 Ah+B\cdot 0}{A+B}$; $I=I'-y'^2S$

$=\frac13 Ah^2-\frac{A^2h^2}{4(A+B)}$. $r^2=I\div(A+B)=h^2\left\{\frac{A}{3(A+B)}-\frac{A^2}{4(A+B)^2}\right\}=h^2\left\{\frac{A}{12(A+B)}+\frac{AB}{4(A+B)^2}\right\}$.

Omit XII and XIII.



XIV. Length of arc $=2R\theta$. $I'=\int y^2 ds$ about ED.

By similar triangles, $ds:dx=R:y$; $\therefore y\,ds=R\,dx$.

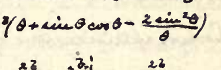
$I'=R\int y\,dx=R\cdot$ area $ABCDE=R^3(\theta+\sin\theta\cos\theta)$.

Distance of c. of g. of arc from O, $y_1=\frac{\int y\,ds}{\int ds}=\frac{R\int dx}{\int ds}=\frac{R\cdot\text{chord }AC}{\text{arc }ABC}=\frac{R\sin\theta}{\theta}$

$I=I'-y_1^2A=R^3(\theta+\sin\theta\cos\theta)-\frac{R^2\sin^2\theta}{\theta^2}2R\theta=R^3\left(\theta+\sin\theta\cos\theta-\frac{2\sin^2\theta}{\theta}\right)$

$r^2=I\div 2R\theta=R^2\left(\frac12+\frac{\sin\theta\cos\theta}{2\theta}-\frac{\sin^2\theta}{\theta^2}\right)$.

Compare the transpositions of this sketch



$I=\frac{4\cdot\frac12 a}{3}\cdot\frac{a^2}{4}+4b\cdot\frac{a^2}{4}=(a+6b)\frac{a^2}{6}$. $r^2=\frac{a^2}{12}\left(\frac{a+6b}{a+2b}\right)$. Also middle form, about vertical axis, distance apart being d:—

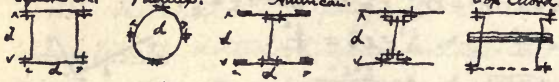
$I=2a\cdot\frac{d^2}{4}+\frac{4(b+\frac12 d)^3}{3}-\frac43\cdot\frac{d^3}{8}=(a+2b)\frac{d^2}{2}+\frac{4b+6d}{3}b^2$. $r^2=I\div(2a+4b)$

$=\frac{d^2}{4}+\frac{2b+3d}{3(a+2b)}b^2$. To find value of d to make I's the same, solve $d^2(\ \cdot+2b)+d\cdot(2b)^2=\frac13 a^3+2a^2b-\frac83 b^3$.

As the shear is the first differential coefficient of the bending moment, and as, by IV. p.3, and §168, the shears are equivalent to a pull and thrust at 45° with the axis, differentiating (5.) p. 31 ought to yield a formula for finding size of lacing bars on a compression member.

Column or Strut Formulas have undergone very material modifications from time to time. Formerly, values like the following were much in use. For sections represented, where $\frac{l}{d}=H$, the ultimate stress, or $\frac{P}{S}=\frac{f}{1+a\frac{l^2}{d^2}}$, was supposed to be represented by three Formulas.

Square Col.	Phoenix.	American.		Top Chord.
$\frac{38500}{1+\frac{H^2}{5820}}$	$\frac{42500}{1+\frac{H^2}{4500}}$	$\frac{36500}{1+\frac{H^2}{3750}}$	$\frac{36500}{1+\frac{H^2}{2700}}$	for flat ends.
$\frac{38500}{1+\frac{H^2}{3000}}$	$\frac{40000}{1+\frac{H^2}{2250}}$	$\frac{36500}{1+\frac{H^2}{2250}}$	$\frac{36500}{1+\frac{H^2}{1500}}$	for one pin end.
$\frac{37800}{1+\frac{H^2}{1900}}$	$\frac{36600}{1+\frac{H^2}{1500}}$	$\frac{36500}{1+\frac{H^2}{1750}}$	$\frac{36500}{1+\frac{H^2}{1200}}$	for two pin ends.



For top-chord sections, the pin being so placed that the moment of inertia is, as near as practicable, equal on both sides, use the 1st column.

[There seems to be no sound reason for prescribing this position of the pin; and it is now placed in the axis passing through the centre of gravity.] The allowable unit stress in compression was then obtained by dividing each of the above expressions by $4+\frac{5H}{100}$, thus making the allowable unit stress smaller as the length of the strut increased.

Cooper's specifications of 1884 gave $\frac{P}{S}=\frac{8000}{1+\frac{l^2}{40000r^2}}$ for square ends, changing 40000 to 30000 and 20000 for one and for two pin ends. No member was to have an unsupported length of more than 45 times its least width.

Jos. M. Wilson specifies [1885] For pieces subject to one kind of stress only (all compression or all tension), permissible unit

$$\text{stress } \underline{a} = \left\{\begin{array}{l} 7500 \text{ for double rolled iron in tension} \\ 7000 \text{ plate and shape " " "} \\ 6500 \text{ rolled iron in compression} \end{array}\right\}\left(1+\frac{\text{min. stress}}{\text{max. stress}}\right).$$

For pieces subject to stresses acting in opposite directions:

$$a=\left\{\begin{array}{l} 7500 \text{ do. do} \\ 7000 \text{ " "} \\ 6500 \text{ " "} \end{array}\right\}\left(1-\frac{\text{max. stress of lesser kind}}{\text{2 max. stress of greater kind}}\right).$$

The permissible stress $\underline{a}$ for members in compression is to be reduced by the following formulas; allowable stress = $\frac{a}{1+\frac{l^2}{36000r^2}}$ for both ends fixed; 24000 r^2 and 18000 r^2 are substituted in case of one and two pin ends.

C. C. Schneider, [in 1886] proposed for a design for Harlem arch:— Strain to be used regardless of sign. Permissible unit stress
$a = 10000\left(1+\frac{1}{2}\frac{\text{min. stress}}{\text{max. stress}}\right)$ for wrought iron; 12000 do. do. for steel; for members subject to one kind of stress only.
$a = 10000\left(1-\frac{1}{2}\frac{\text{min. stress}}{\text{max. stress}}\right)$ for wrought iron; 12000 do. do. for steel: for members subject to alternate tension and compression.
For compression members, allowed unit stress, $b = \dfrac{a}{1+\frac{a}{8E}\cdot\frac{l^2}{r^2}}$.

J. A. L. Waddell, [1887] for highway bridges, live load 100# per sq. ft. specifies

Flat Ends	$9000-30\frac{l}{r}$	for iron	$12000-45\frac{l}{r}$	for steel.
One Pin "	$9000-35$ "		$12000-53$ "	
Two " "	$9000-40$ "		$12000-60$ "	

Pencoyd Iron Works, C. C. Schneider, Engineer, [1887] gives $\frac{P}{S} = \dfrac{14000}{1+\frac{l^2}{15000r^2}}$ for wrought iron and $\frac{P}{S} = \dfrac{18000}{1+\frac{l^2}{10000r^2}}$ for steel; and no compression member to have a length exceeding 120 times its least radius of gyration. But in these formulas the effects of impact and vibration shall be considered and added to the maximum strains resulting from engine and train loads, [which explains the apparently large numbers used]. The effect of impact is to be found by — Effect of impact = calculated max. live load stress $(0.7+\frac{5}{L})$ where L = number of panels loaded × panel length [to produce the live load stress in question].

Colorado Midland Ry. Specifications — Main Posts, Chords and Struts:
Flat ends ... $(7500-25\frac{l}{r})(1+\frac{\text{min. stress}}{\text{max. stress}}$ in member.).
One pin end $(7500-30\frac{l}{r})($ " " $)$. Lateral and other
Two pin ends $(7500-35\frac{l}{r})($ " " $)$. struts, subject to maximum stresses as calculated, $11000-45\frac{l}{r}$.

Cooper's specifications for 1888 read — $\frac{P}{S}$ or unit stress for

Chord segments	$8000-30\frac{l}{r}$	for live load strains	
	$16000-60\frac{l}{r}$	" dead "	"
All posts	$7000-40\frac{l}{r}$	" live "	"
	$14000-80\frac{l}{r}$	" dead "	"
Lateral struts	$10500-60\frac{l}{r}$	" wind strains	
	$9000-50\frac{l}{r}$	" assumed initial strains.	

with the same restriction as to length as on previous page.

No distinction is here made between pin and fixed ends, all being practically treated as if not fixed. The unit stress is reduced for pieces whose loads are more rapidly or frequently imposed, thus taking account of impact in a different way. The dead and live loads give the ratio 2 as suggested by Rankine.

A comparative reference by the student to the latter part of the notes on §158 will not be amiss.

Alternating Stresses.— For members subject to a reversal of stress, good practice requires an increase of section. A specification of some years ago read — Section = $\frac{\text{Max. tension}}{10000}$ + $\frac{\text{Max. compression}}{\text{1/4 column Strength}}$, in case a larger section is given than by the usual formula for columns.

Pencoyd specifications say — Members subject to alternate strains of tension and compression shall be so proportioned that the total sectional area is equal to the sum of areas required for each strain.

Colorado Midland Ry. Specifications — For compressive stress alone, use formula for posts. For the greater kind of stress use $7000(1 - \frac{\text{max. lesser stress}}{\text{2 max. greater stress}})$.

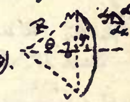
J. A. L. Waddell says: In any portion of a bridge in which the stresses of tension and compression alternate, the sectional area required is to be determined by dealing with the part thus affected — first for the calculated max. tension, then for the calculated max. compression, employing for intensities of working stresses the — Intensity that would be used were there no reversion of stress $\times (1 - \frac{1}{2}\, \frac{\text{smaller stress}}{\text{larger stress}})$, and adopting the greater of the two areas thus found.

Cooper specifies:— Members subject to alternate strains of tension and compression shall be proportioned to resist each kind of strain. Both of the strains shall, however, be considered as increased by an amount equal to 8/10 of the least of the two strains, for determining the sectional areas. See also pp. 104, 105.

Compression members, consisting of two or more pieces of shape iron connected by lacing bars, should have connection plates at ends of such length that the distance between extreme rivets shall be at least equal to the transverse distance between rows.

§ 369. Rolled Beams.— Working stress, bridge work, 7500 lbs. to 8000 lbs. Table, top of p. 527, not used. Formula (2.) uses the whole section to resist bending, and not the flanges alone. Moment of web, approx. $= \frac{f A_w h'}{6}$; of flanges $= f A_f h'$.

§ 370 Built Beams.— Upper member, stress 6000# to 7000# per sq. inch gross section; lower member, 7000# to 8000# per sq. inch net section. The gross section of the two flanges is therefore practically the same. As on p. 103,



$I = \int y^2 dS = R \int y\, dx = R \cdot \text{area segment} = R^3(\theta - \sin\theta\cos\theta)$.

$b = 2R \sin\theta$; $\therefore R^2 = \frac{b^2}{4\sin^2\theta}$; $S = 2R\theta$. $\therefore$

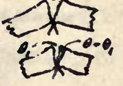
$r^2 = \frac{I}{S} = R^2\left(\frac{1}{2} - \frac{\sin\theta\cos\theta}{2\theta}\right) = R^2\left(\frac{1}{2} - \frac{\sin 2\theta}{4\theta}\right) = \frac{b^2}{8\sin^2\theta}\left(1 - \frac{\sin 2\theta}{2\theta}\right)$.

5000 is modified by values of I., II., III. and VI. of § 366, p. 523. Table on p. 529 not used.

IV. Vertical Web.— To be stiffened at intervals as stated in text. Practical examples appear to show that stiffeners are not needed so soon as the formula would require. It may be suggested that, by reason of this restraint offered at right angles by the tensile stress in the web, a diagonal strip of the web does not flex freely as a column of length s, but buckles in a series of waves, so that the distance from crest to crest should be used. This length would depend somewhat on thickness of plate. The resistance of the web will then be higher than that found by (1.).

§ 372. Erection of Continuous Girder.— Alternate spans being loaded, as in § 178, (7.) and (8.) give (1.) and (2.) here.

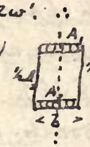
Find an angle θ, which would be the angle to be closed for a moment at the pier of $\frac{w+w'}{16} l^2$;



by the proportion $\theta_1 : \theta = \frac{w+w'}{16} : \frac{2w+w'}{24} = 3(w+w') : 4w+2w'$. Then

$\theta - \theta_1 : \theta = 4w+2w' - 3(w+w') : 4w+2w' = w - w' : 4w+2w'$. $\therefore$

$\theta - \theta_1$, the angle of filling piece $= \frac{w-w'}{4w+2w'}\,\theta$. (5.)

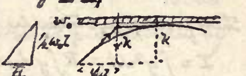


§ 373. Wind against Tubular Girder.—

$M = \frac{fI}{y_1}$; $M_1 = \frac{p_1 I}{\frac{1}{2}b}$. $I = \frac{A_1 b^2}{12} + \frac{A_3 b^2}{12} + \frac{A_2 b^2}{4}$.

$p_1 = \frac{M_1 \frac{1}{2}b}{I} = M_1 \div b\left(\frac{A_1 + A_3}{6} + \frac{A_2}{2}\right)$ (3.).

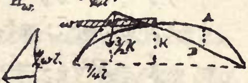
§ 374. Plain Arched Iron Ribs.—I. By (45.) p. 309, the quantity $\left(1 + \frac{16k^2}{3l^2} + \frac{aEA_1}{l}\right)$ is B. The horizontal movement, or change of span, from temperature change, $= etl$. To introduce this quantity in the parenthesis, multiply and divide by $\frac{lH}{EA_1}$; and, as $H = p_0 A_1$, the new term becomes $\frac{etE}{p_0}$. (1.)



Case IV. $H_{w_0} : \frac{1}{2}w_0 l = \frac{1}{4}l : k$, $\therefore H_{w_0} = \frac{w_0 l^2}{8k}$.

$H_w : \frac{3}{4}\cdot\frac{1}{2}wl = \frac{1}{4}l : \frac{3}{2}k$, or $H_w = \frac{wl^2}{16k}$.

$\therefore H = \frac{l^2}{8k}(w_0 + \frac{1}{2}w)$ (3.)



H_w for a load over half the span ought to be $\frac{1}{2}$ of H for load on both halves.

The dead load, on the whole arch, causes no bending moment. The live load causes a moment at $AB = \frac{wl^2}{16k}\cdot\frac{k}{4} = \frac{wl^2}{64}$ (4.) The equilibrium curve on the loaded side passes an equal distance, $\frac{1}{4}k$, above the rib at the quarter span, causing an equal moment with opposite sign. These quantities can also be obtained by method used in § 345.

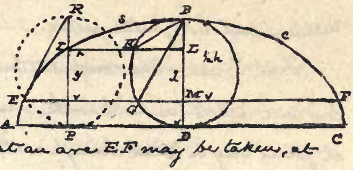
Section IV.

§ 375. Platforms.— Omit the Barlow rail.

II. Corrugated Iron.— The curves of the corrugated sheets are approximate arcs of circles, and no material error will be committed by assuming for their figure any curve which does not greatly differ from a circular arc. A cycloidal arc is such a curve, and its value of I is not difficult to be obtained. By Weisbach, p. 656,

$IB = s = 2BK$; $BF = c = 2BG$; $BK^2 = BL^2 + KL^2$; $KL^2 = BL\cdot LD$; $\therefore$

$BK^2 = BL^2 + BL\cdot LD = BL(BL + LD) = BL\cdot d$. Also $BG^2 = MB\cdot d$.



$BE . BF = c^2 = 4BG^2;$

$EI \cdot IF = (c-s)(c+s) = c^2 - s^2 = 4BG^2 - 4BK^2.$

$\dfrac{c^2 - s^2}{c^2} = \dfrac{4(BG^2 - BK^2)}{4\,BG^2} = \dfrac{MB\cdot d - BL\cdot d}{MB\cdot d}$

$= \dfrac{IH}{MB} = \dfrac{y}{\frac{1}{2}h}$. Thus it appears that an arc EF may be taken, at any height, ½h, without regard to d, the diameter of generating circle.

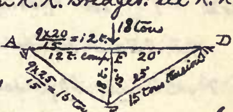
$I = t\int y^2 ds = t\dfrac{h^2}{4}\int_0^c \left(1 - \dfrac{s^2}{c^2}\right)^2 ds = t\left(\dfrac{h^2 s}{4} - \dfrac{h^2 s^3}{6c^2} + \dfrac{h^2 s^5}{20c^4}\right)_0^c = \dfrac{2}{15}h^2 ct. \quad r^2 = \dfrac{2}{15}h^2.$

If bt = cross-section, $M = \dfrac{fI}{y_1} = \dfrac{fr^2 S}{y_1} = f\,\dfrac{2}{15}\,\dfrac{h^2 bt}{\frac{1}{2}h} = \frac{4}{15} f h b t.$ (2.)

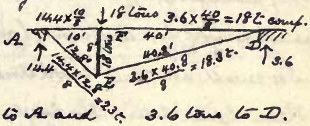
(10.) comes from a similar analysis.

§ 376. Iron Roofs.— See Graphics, Part I., "Roof Trusses," for stress diagrams for load and wind. See Eng. + Bldg. Record, Aug. 31, 1889. Abstract of Paper by C. M. Jarvis, 5th Meeting, Conn. Assocⁿ Eng. + Sur.

§ 377. Iron Braced Girders.— Loads on R. R. Bridges: see R. R. Gazette, Sept. 7 and Oct. 1, 1886.



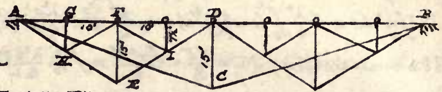
Note the way in which the several stresses are found in this triangular truss. 18 tons passes down to E, and thence a vertical component of 9 tons goes each way to the abutment.



Note also the computations in the second sketch, where, of the 18 tons at the joint E, 14.4 tons are carried to A and 3.6 tons to D. The long tie ED has a tension of 18.3 tons while carrying only 3.6 tons of vertical force; and hence is not an economical member. Invert the triangles and the values are still true, but the stresses are reversed.

A convenient rule for computation, where members occupy the sides of a right triangle, and a certain vertical force may be considered as passing from one to another; is, to multiply the force or stress acting along a member by the length of the piece to which it is transferred, and divide the product by the length of the piece from which it is transferred. See the operations indicated in the last sketch, as applied to the inclined and horizontal pieces.

Fink Truss.— The computation of stresses in a Fink truss, as given below, will tend to make clear the way in which the action of one or more loads at joints may be traced through a truss. Example:— Span 80 ft., depth 15 ft.; 8 panels; total load 72 tons, or each joint 9 tons. Maximum stresses occur with full load.

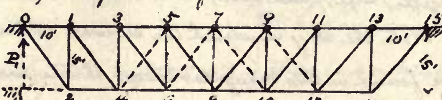


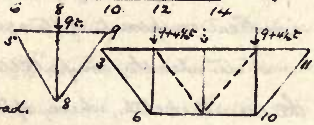
Compare first triangle on preceding page. $HF = AH = FI$.

Piece.	Load.	Piece.	Length.	Stress Ratio.	Vertical Component.	Stress.	Piece.	Stress Ratio.	
DC	36	AC	$\sqrt{(40^2+15^2)}$	$\frac{42.7}{15}$ ×	18 =	57.2 t.	AD	$\frac{40}{15}$	48 t.
FE	18	AE	$\sqrt{(20^2+15^2)}$	$\frac{25}{15}$ ×	9	15 t.	AF	$\frac{20}{15}$	12 t.
GH	9	AH	$\sqrt{(10^2+7\frac{1}{2}^2)}$	$\frac{12.5}{7.5}$ ×	4½	7.5 t.	AG	$\frac{10}{7.5}$	6 t.

AD, AF and AG being one, Compression from A to B = 66 t.

Pratt Truss: 80 ft. span; 15 ft. depth; panel length 10 ft.; panel weight, dead load, 4 tons; panel weight, live load 5 tons. Load on top.





See small sketches. Chord stress—full load.

Piece.	Load. Ratio.	Hor. Comp.	Constant difference of 6 t.	Adding stresses in same panel	Stress in Piece	
5-9	$\frac{9}{2} \times (\frac{10}{15} = \frac{2}{3}) =$	3 t.			48 t.	5-7
3-11	$(\frac{9}{2}+9) \times \frac{2}{3}$	9 t.			45 t.	3-5 + 6-8
1-13	$(\frac{9}{2}+9+9) \times \frac{2}{3}$	15 t.			36 t.	1-3 + 4-6
0-15	$(\frac{9}{2}+9+9+9) \times \frac{2}{3}$	21 t.			21 t.	0-1 + 2-4

Check: by formula $H = \frac{Wl}{8h}$, or $\frac{80 \times 72}{8 \times 15} = 48$.

Next take shear for dead steady load and then shear at head of live load; see Notes, II., p. 42. Web Stresses. F_d = dead ld shear; F_l = live ld. do.

Piece.	F_d.	F_l from live load. (The numbers in these ratios are panel lengths)	F_l.	F_d+F_l. (Compression in verticals)	Piece	Stress Ratio.	Tension. (Tension in diagonals)
1-2	14	7×5×4/8	17.5	31.5	0-2	18/15=1.2	37.8
3-4	10	6×5×3½/8	13.1	23.1	1-4	"	27.7
5-6	6	5×5×3/8	9.4	15.4	3-6	"	18.5
7-8	2	4×5×2½/8	6.2	8.2	5-8	"	9.8
9-10	−2	3×5×2/8	3.7	1.7	7-10	"	2.0
11-12	−6	2×5×1½/8	1.9	−4.1	9-12	Not needed.	

When the truss is fully loaded, 7-8 carries 9 tons.

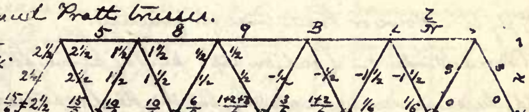
If the load were on the bottom chord, the stresses in chords and diagonals would be the same as above; but the verticals would belong with the diagonals nearer the middle; and 7-8 would carry 1.7 tons only.

The Howe truss with load on bottom chord is like the Pratt truss with load on top chord, and vice versa. The diagonals are struts, and verticals ties. In both classes of truss a diagonal and vertical, which together connect two contiguous loaded joints, will have the same shear.

Inclined or vertical ends alter no stresses beyond their own panels. Vertical ends may be regarded as superfluous as trussing.

The above method of analysis is simple, but not very convenient for the treatment of a truss of many panels. The method now to be given for the Warren girder can be readily applied to the Howe and Pratt trusses.

Chord Stress – Mult. by $(w+w')\frac{l}{2Nh}$. Web Stress – Mult. by $w\, s/h$ and $w' s/h$.

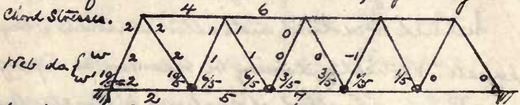


Warren Girder; Even number of panels; load on bottom.
Panel weight, dead load $= w$; do. do. live load $= w'$. $N =$ number panels; $h =$ vertical height; $s =$ slant height. $\frac{l}{2N} =$ hor. projection of braces.

Comparing Pratt truss on p. 110, we see that the fractions express the parts of w and w' which make up the shears in the several panels. The top row is for chord stresses: thus shear in 2d. panel, each brace, is $1\frac{1}{2}(w+w')$, which multiplied by $\frac{l}{2N}$ and divided by h gives increment of stress in chord. Begin to write this row at middle, and then sum up from end to middle, as before. The middle row is dead load shears when multiplied by w; write both ways from middle. The bottom row is live load shears for load advancing from right; When a load is added at C, 2/6 w' goes through web to left abutment; when at A, 3/6 do. write from the right, adding as one goes. Then add the shears for dead and live load, respecting the signs, and multiply by s/k for web stresses. Strut-braces slope up to the right for positive shear; tie-braces down to the right. AB will be a strut for load from right to C, and BC will be a tie; but AB will be a tie and BC a strut for load from left to A. [These matters are explained in detail in Graphics, Part II., "Bridge Trusses."] The denominator of the bottom row is the number of panels in truss, the numerators are the successive sums of the natural numbers, beginning at 0.

Warren Girder:

Odd no. of panels:



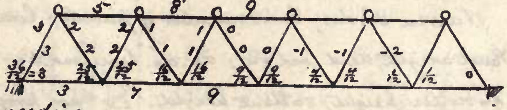
Load on bottom chord. Multipliers as before.

Computations are easily made by calculating the first term of each series, including the multiplier, and then adding differences.

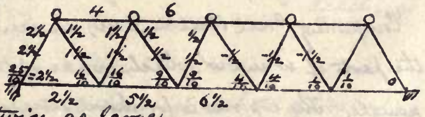
Apply to a Howe or Pratt truss for practice.

Warren Girder:– Load on top. 1st. If above sketches are inverted, the same numbers apply, but the kind of stress will be reversed.

2d. If above trusses are loaded on top chord the numbers will be as here shown.



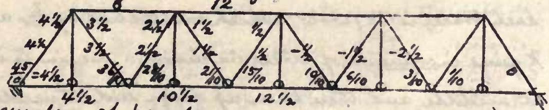
Compare these trusses with the preceding two. As the weights come at half-panel distances, the



live load denominators are twice as large.

Warren Girder:

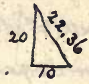
Every joint loaded.



No truss with odd number of panels.

If this truss is inverted, the same numbers will apply. The verticals, in the latter case, may be changed to the other joints, or not, without change of the above numbers.

Example. — Warren Girder, second truss, (top of page 112) 5 panels, span 100 ft., height 20 ft. Panel weights, dead load 16000 lbs., live load 30000 lbs.



$(w+w')\frac{l}{2\,\Sigma\, k} = 46000 \cdot \frac{100}{10 \times 20} = 23000.$ $\frac{22.36}{20} = 1.118$

	46000	46000	23000	23000	0
Top Chord		+92000		+138000	
Bottom Ch.d	−46000		−115000		−138000

$16000 \times 1.118 = 17888$; $\frac{30000}{5} \times 1.118 = 6708.$

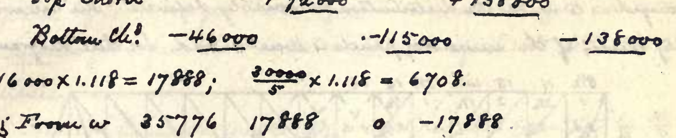
Web	From w	35776	17888	0	−17888	
	" w'	67080	40248	20124	6708	0
Web Stresses		±102856	±58136	±20124	—	

Check for chord stresses $\frac{46000 \times 5 \times 100}{8 \times 20} = 143750$; $\frac{143750}{5^2} = 5750$;

$143750 - 5750 = 138000.$ [See Notes, VIII., p. 41.]

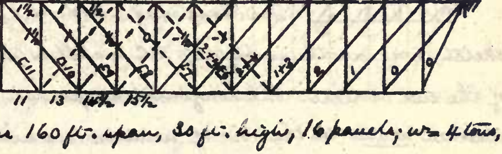
Double Quadrangular or Double Intersection Truss. Even no. panels.

Symbol Σ13 means sum of series 1, 3, 5 ---- 13. Denominator 16 omitted.

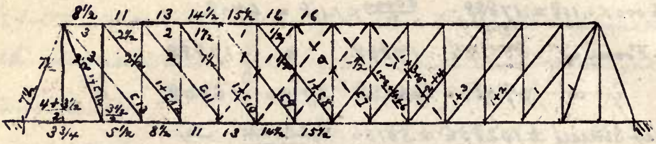


Example: — Truss 160 ft. span, 30 ft. high, 16 panels; w = 4 tons, w' = 5 tons, s = 36 ft. For chord stresses, $(4+5)\frac{20}{30} = 6$; $s/k = 1.2$; $4 \times 1.2 = 4.8$; $\frac{5}{16} \times 1.2 = 3/8$.

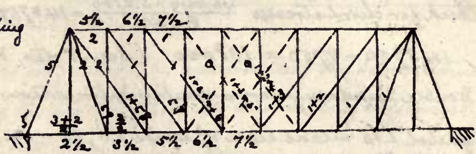
Note the lesser inclination of the end diagonal, and the consequent division of its chord increment by 2.

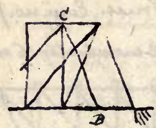


If the double quadrangular truss has inclined end-posts, as is commonly the case, the chord stresses will be unaffected, except in the end panels; but the distribution of the shear in the web members is rendered uncertain to a small amount, 1/n th. of a panel weight of live load, as that portion of the load at the foot of the end vertical may pass to the further abutment by either of the two diagonals which meet at its upper end. To be on the safe side, it is well to add this amount to each system in the web, and thus provide for its passage by either route. Some computers assume that one half goes over each system, and some trusses have the hip joint so arranged as to make the distribution reasonably definite. The uncertainty exists, if the number of panels is even or odd. See these diagrams.



Compare with preceding example.

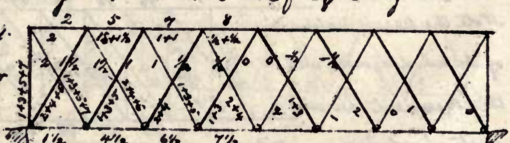




The truss may be modified, as by the annexed sketch, to remove the ambiguity. BC is still a tie, taking the place of the end vertical. The horizontal component of BC must be marked minus in the row of numbers to be summed for chord stresses. The change affects no pieces outside of the end panels.

The double quadrangular truss is less frequently built now than formerly.

Double Triangular or Warren Girder.



Reproduce these trusses on a larger scale and work out the stress in full. Check the chord stress at middle and the web stress at ends. Note that the web members in the double Warren Girder near the centre act as counters also, having a reversal of stress.

§ 377. III. Warren Girder — The method of the text is equally applicable to the Howe and Pratt trusses, and will be seen to be much like some of the work which has just been done.

The shear from a full load, as in § 343, is $F_n = (w+w')(\frac{N-1}{2} - (n-1))$. The maximum value will be $(w+w')\frac{N-1}{2}$, and subsequent values can be found by the repeated subtraction of $(w+w')$. If F is multiplied by $\frac{l}{N} \div h$, we get the horizontal components of stresses in the diagonals for a full load, or the increments of chord stress in the successive panels.

Hence:— Compute $(w+w')\frac{l}{Nh}$; multiply this quantity by $\frac{N-1}{2}$; subtract $(w+w')\frac{l}{Nh}$ repeatedly; and add the series thus obtained.

$\frac{N-1}{2}(w+w')\frac{l}{Nh} = H_1$

" $-(w+w')\frac{l}{Nh} + H_1 = H_2$

" -2 " $+ H_2 = H_3$

" -3 " $+ H_3 = H_4$

etc. etc. to middle of truss.

Check middle value by $\frac{Nl}{8h}$ if the number of panels N is even; if N is odd, diminish this value by the $1/N^2$ part.

Many prefer to compute the series from the middle of the truss.

For stresses in web members, find the shears in successive panels, as under [A.] below, for dead load; and those due to live load will be found by the calculation of the series (see § 343) F_n from $w' = \frac{w'}{N}\frac{(N-n)(N-n+1)}{2}$, values for which for eight panels are given under [B.]. Begin with right hand column and add from bottom.

[A.]

$\frac{N-1}{2}w = F_1$

" $-w = F_2$

" $-2w = F_3$

etc. etc.

Checked by change of sign upon passing middle, and values repeated in inverse order.

[B.]

$\frac{N-1}{2}w' = F_1'$,

Col. 3.	Col. 2.	Col. 1.
$28\frac{w'}{N}$		
21 "	$7\frac{w'}{N}$	$1\frac{w'}{N}$
15 "	6 "	1 "
10 "	5 "	1 "
6 "	4 "	1 "
3 "	3 "	1 "
$1\frac{w'}{N}$	2 "	1 "
	$1\frac{w'}{N}$	1 "
0		$1\frac{w'}{N}$

Checked by final value, $\frac{N-1}{2}w'$.

Novice computers write Column 3 at sight, omitting Cols. 1 and 2.

Now add corresponding shears for successive panels, $F_1+F'_1$, $F_2+F'_2$, etc., and, finally, multiply by $\frac{s}{h}$ for stresses in diagonals. Rankine divides [B.] at middle of truss, and reverses the second half, — not so handy for calculation, but showing more clearly which pieces have alternation of stresses.

See the book, for loads on each joint or on alternate joints. By using a little judgment this method can be applied in other cases.

IV. Lattice Girder.— By comparing the double Warren girder with the single Warren girder it will be seen that chord stresses in opposite panels of the first girder are nearer equality than are those of similar panels in the second; also that web stresses at any section in tie and strut do not differ much. When the systems of bracing become numerous, we approach closely the plate girder and may compute the stresses on the same basis, i.e., divide the bending moment at any section by the depth between chords for the chord stresses; and distribute the maximum shear at any section equally among the web members cut by the section.

V. Continuous Lattice Girder:— Divide (1) p. 536, by (3) to get (23).

§ 379. Bowstring Girder.— As there is no stress in the diagonals, for a full load, if the bow is a parabola, the lower chord stress will be found by the formula $\frac{Wl}{8h}$ or $\frac{N(w+w')l}{8h}$, and the upper chord stresses by multiplying this value by the secants of inclination to the horizon. The verticals will have tension $w+w'$.

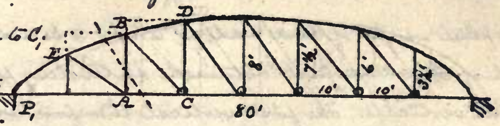
No stress in diagonals for a uniform load, § 125, Notes, p. 8.

Let height of truss be 8 ft., span 80 ft., and panel weight of live load, 5 tons: the vertical components of stresses (tensions) in the diagonals may be found as follows, beginning at the left. The computations are arithmetical. Next will follow the compressive stresses in the verticals. [Compare analysis of bowstring girder in Graphics, Part II., "Bridge Trusses."] The load in each case extends from right to foot of diagonal.

Thus, for BC, load to C,

$P_1 = \frac{5 \times 5t. \times 30'}{80'} = 9\frac{3}{8}t.$

= shear in panel BC.



Moment at $C = 9\frac{3}{8}t \times 30'$; Hor. force at $D = \frac{9\frac{3}{8} \times 30}{7\frac{1}{2}}$; Vert compt. in BD $= \frac{9\frac{3}{8} \times 30 \times 1\frac{1}{2}}{7\frac{1}{2} \times 10} = 5.625$. Shear in panel − Vert. compt. in $BD = 9\frac{3}{8} - 5\frac{5}{8} = 3\frac{3}{4}t.$ shear or vertical component in diagonal BC. Hence

P_1	Vert. Compt. in Bow.	Vert. Compt. in Diagonal.
$\frac{7 \times 5 \times 40}{80} = 17.5t.$	$\frac{17.5 \times 10 \times 3\frac{1}{2}}{3\frac{1}{2} \times 10} = 17.5$	$17.5 - 17.5 = 0$, 1st. panel
$\frac{6 \times 5 \times 35}{80} = 13.125.$	$\frac{13.125 \times 20 \times 2\frac{1}{2}}{6 \times 10} = 10.94$	$13.12 - 10.94 = 2.18$; 2d. = EA
$\frac{5 \times 5 \times 30}{80} = 9.375$	$\frac{9.375 \times 30 \times 1\frac{1}{2}}{7\frac{1}{2} \times 10} = 5.625$	$9.37 - 5.62 = 3.75$, 3d. = BC
$\frac{4 \times 5 \times 25}{80} = 6.25$	$\frac{6.25 \times 40 \times \frac{1}{2}}{8 \times 10} = 1.56$	$6.25 - 1.56 = 4.69$, 4th. "
$\frac{3 \times 5 \times 20}{80} = 3.75$	$\frac{3.75 \times 50 \times (-\frac{1}{2})}{7\frac{1}{2} \times 10} = -1.25$	$3.75 + 1.25 = 5.00$, 5th. "
$\frac{2 \times 5 \times 15}{80} = 1.875$	$\frac{1.875 \times 60 \times (-1\frac{1}{2})}{6 \times 10} = -2.81$	$1.87 + 2.81 = 4.68$, 6th. "
$\frac{1 \times 5 \times 10}{80} = 0.625$	$\frac{0.625 \times 70 \times (-2\frac{1}{2})}{3\frac{1}{2} \times 10} = -3.125$	$0.62 + 3.13 = 3.75$, 7th. "

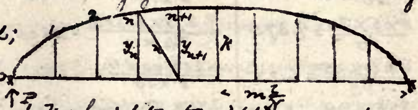
When the bow changes its inclination and thrusts up towards the left, its component is added to the shear. To find stresses in diagonals, multiply by length of each and divide by its vertical projection. Note that these values repeat from the middle, in inverse order; that diagonals of same length have same stress; and that both diagonals are needed in each panel.

For vertical AB, load to C; P_1 as above $= 9\frac{3}{8}t.$ Moment at A $= 9\frac{3}{8} \times 20$; Hor. force at $B = \frac{9\frac{3}{8} \times 20}{6}$; Vert. compt. in $BE = \frac{9\frac{3}{8} \times 20 \times 2\frac{1}{2}}{6 \times 10} = 7.81$. Shear P_1 (in pieces cut by dotted section) − Vert. compt. in BE $= 9.37 - 7.81 = 1.56t.$ compression in AB. Hence — Compression in verticals, beginning at left.

	P_1 Vert. compt. in bow to left of jt.
$\frac{13.125 \times 10 \times 3\frac{1}{2}}{3\frac{1}{2} \times 10} = 13.125.$	$13.125 - 13.125 = 0$ 1st. vertical.
$\frac{9.375 \times 20 \times 2\frac{1}{2}}{6 \times 10} = 7.81.$	$9.37 - 7.81 = 1.56t.$ 2d. " AB.
$\frac{6.25 \times 30 \times 1\frac{1}{2}}{7\frac{1}{2} \times 10} = 3.75$	$6.25 - 3.75 = 2.50$ 3d. " CD.
$\frac{3.75 \times 40 \times \frac{1}{2}}{8 \times 10} = 0.94$	$3.75 - 0.94 = 2.81$ 4th. "
$\frac{1.875 \times 50 \times (-\frac{1}{2})}{7\frac{1}{2} \times 10} = -0.625$	$1.875 + 0.625 = 2.50$. Compare with 3d.

One panel weight of dead load, w, is now to be subtracted from each of the values just obtained, as tending to cause tension in the verticals. The first vertical therefore will always be a tie and carry $\underline{w}$, as does the end vertical of a straight-chord truss with inclined end posts; the second vertical will probably also always be in tension. All verticals should also be designed for $(w+w')$ tension under a full load.

§§ 347 and 374. To deduce formulae for stresses in web members of the parabolic or bowstring girder: The book is wrong.



Let w' = live load per panel; l = span; N = number of panels, to be even; $\frac{l}{N}$ = panel length; h = height at middle; number the joints on the bow. The nth. vertical and nth. diagonal start from the nth. joint. Let the load extend from the right $\underline{m}$ panels, a distance not exceeding that to foot of diagonal. Since

$h : y_n = (\tfrac{1}{2}N)^2 : n(N-n)$, $y_n = n(N-n)\frac{4h}{N^2}$ and $y_{n+1} = (n+1)(N-n-1)\frac{4h}{N^2}$.

$P_1 = F_{n+1} = mw'\frac{m+1}{2}\cdot\frac{l}{N} \div l = m(m+1)\frac{w'}{2N}$.

$M_{n+1} = m(m+1)\frac{w'}{2N}(n+1)\frac{l}{N}$; $H_{n+1} = \frac{M_{n+1}}{y_{n+1}} = \frac{m(m+1)w'l}{(N-n-1)\,8h}$.

$H_n = \frac{M_n}{y_n} = \frac{m(m+1)w'l}{(N-n)\,8h}$. [If $m = N-n-1$, $H_{n+1} = (N-n)\frac{w'l}{8h}$; $H_n = (N-n-1)\frac{w'l}{8h}$. $\therefore$ $H_{n+1} - H_n = \frac{w'l}{8h}$ = a constant = horizontal component of stress in n^{th} diagonal, and from it stress may be found.]

$H_{n+1} : V_{n+1} = \frac{l}{N} : y_{n+1} - y_n = \frac{l}{N} : [(n+1)(N-n-1) - n(N-n)]\frac{4h}{N^2} = l : (N-2n-1)\frac{4h}{N}$.

$\therefore V_{n+1} = \frac{m(m+1)(N-2n-1)w'}{(N-n-1)\,2N}$. The vert. compt. of stress in n^{th} diagonal, $F_{n+1} - V_{n+1} = m(m+1)\frac{w'}{2N}\left[1 - \frac{N-2n-1}{N-n-1}\right] = \frac{m(m+1)n}{N-n-1}\cdot\frac{w'}{2N}$.

The greater the value of $\underline{m}$, the greater will be $F-V$, until $m = N-n-1$. Beyond this point the value just obtained does not apply. But, if the load extends from the right beyond the joint $n+1$, F_{n+1} will be less; M_{n+1} and consequently H_{n+1} and V_{n+1} will be greater; $\therefore$ $F-V$ will be less. $\therefore$ The load from right to joint $n+1$ gives maximum stress in n^{th} diagonal.

m then being $N-n-1$, the vertical component of stress in n^{th} diagonal $= (N-n)\,n\,\frac{w'}{2N}$ always positive; and

Stress in n^{th} diagonal $= (N-n)\,n\,\frac{w'}{2N}\cdot\frac{s}{Y_n} = \frac{w' N s}{8K}$.

Horizontal component " $= \frac{w' l}{8K}$, as above. [See Bridge Trusses]

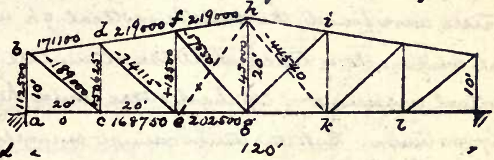
If the diagonals are to be struts, number on the bottom chord.

The stress in the n^{th} vertical is the same as the vertical component of the stress in the $n-1^{th}$ diagonal, for the same value of m.

$T_n - V_n = \frac{m(m+1)(n-1)}{N-n}\cdot\frac{w'}{2N} = (N-n-1)(n-1)\frac{w'}{2N} = [(N-n)n-(N-1)]\frac{w'}{2N}$;

from which subtract w.

Both diagonals are needed in each panel. Their stresses are proportioned to their lengths. Stresses in the verticals vary the same way, less a constant part.



Example.— Here the dead load must be included in the web calculations. Diagonals to be ties. End height 10 ft.; middle height 20 ft.; 6 panels of 20 ft. each; $w = 15000$ lbs. per panel, $w' = 30000$ lbs. per panel. Slope of top chord $\frac{10}{20}\cdot\frac{1}{3} = 1/6$; ∴ multiplier for top chord $= \frac{\sqrt{37}}{6} = 1.014$.

Using the formula $M_n = \frac{n(N-n)\,l}{2N}(w+w')$

$\frac{45000 \times 6 \times 120}{8} = 4{,}050{,}000$; $\frac{4050000}{3^2} = 450000$; $4050000 - 450000 = 3{,}600{,}000$;

$4{,}050{,}000 - 4 \times 450{,}000 = 2{,}250{,}000$.

$4{,}050{,}000 \div 20 = 202{,}500$ $\qquad 202{,}500 \times 1.014 = 205{,}300 [= fh]$

$3{,}600{,}000 \div 16\tfrac{2}{3} = 216{,}000 [= eg]$; $216{,}000 \times 1.014 = 219{,}000 = df$.

$2{,}250{,}000 \div 13\tfrac{1}{3} = 168{,}750 = ce$; $168{,}750 \times 1.014 = 171{,}100 = bd$.

The increase of stress in df over fh makes it evident that fg must be acting as a strut and not as a tie, which action is contrary to supposition. Panel fg will be revised below.

T_w	37500	22500	7500	-7500	-22500	-37500
$T_{w'}$	75000	50000	30000	15000	5000	0
	112500	72500	37500	7500	-17500	-37500

Stress in de will be found as follows:– Shear = 72500. Reaction = 50000 + 37500. M at e = 87500 × 40 − 15000 × 20 = 3,200,000. Vert. compt. in df $= \dfrac{3,200,000 \times 3\frac{1}{3}}{16\frac{2}{3} \times 20} = 32000$. 72500 − 32000 = 40500, which mult. by 1.83 = 74115.

∴ bc = 112500 − 28125 = 84375. 84375 × 2.24 = 189000 tension.

de = 72500 − 32000 = 40500. 40500 × 1.83 = 74115 "

fg = 37500 − 26250 = 11250. 11250 × 1.50 = 17550 "

hk = 7500 + 24000 = 31500. 31500 × 1.414 = 44540 "

il = −17500 + 15625 not needed.

ab = 112500 compression. cd = 72500 − 21875 = 50625 compn.

ef = 37500 − 24000 = 13500 compn. gh = 7500 − 16250 = −8750 tension

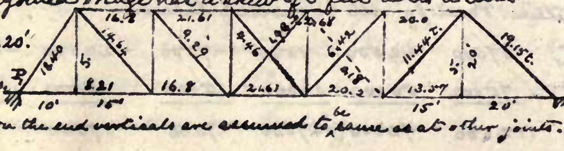
If fg and gi were in action for a full load, as when the chord stresses were found, the middle vertical gh must hold the vertical components of top chord stress at middle joint at that time. Vertical component of top chord stress would then be $(202,500)\frac{3\frac{1}{3}}{20} \times 2$ = 67500 tension. But this tension cannot be supplied by hg, which carries, at the most, one panel weight, 45000 lbs., and ties he and hk must therefore then be in action. Therefore, maximum tension in gh = 45000, and it is never in compression. From the above change of diagonals there result the chord stresses $df = fk$ = 219000 and $eg = gk$ = 202500.

Skew Bridges.— Trusses are sometimes built with panels of unequal length, to adapt the structure to some peculiarity of place. Truss-bridges on a skew-crossing not unfrequently have end panels unequal, in order to bring the floorbeams at right angles to axis of bridge. It will then be necessary to calculate the chord-stresses for all panels, and to pass the moving load across the truss from either end, to complete the strain sheet.

The subjoined bridge has a skew of 5 feet in its width.

Span 105'; ht. 20'.

w = 2 tons. w' = 3 tons.



The loads on the end verticals are assumed to be same as at other joints.

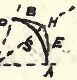
$P_7 = \frac{5\times6\times57\frac{1}{2}}{105} = 16\frac{3}{7}$.

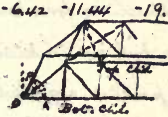
E from full CL. =	$16\frac{3}{7}$	$11\frac{3}{7}$	$6\frac{3}{7}$	$1\frac{3}{7}$	$-3\frac{4}{7}$	$-8\frac{4}{7}$	$-13\frac{4}{7}$
multiplied by	$\frac{10}{20}$	$\frac{15}{20}$				$\frac{19}{20}$	$\frac{24}{20}$
= increments of chord stress	$8\frac{6}{28}$	$8\frac{4}{28}$	$4\frac{23}{28}$	$1\frac{4}{28}$	$-2\frac{19}{28}$	$-6\frac{12}{28}$	$-13\frac{16}{28}$
Chord Stresses =	$8\frac{6}{28}$	$16\frac{23}{28}$	$21\frac{17}{28}$	$22\frac{19}{28}$		20	$13\frac{16}{28}$

Note where the shear changes sign, in applying these numbers to truss.

E from w =	6.57	4.57	2.57	0.57	−1.43	−3.43	−5.43
$E_{w'}$ advancing from right =	$\frac{18\times57\frac{1}{2}}{105}$	$\frac{15\times50}{105}$	$\frac{12\times42\frac{1}{2}}{105}$	$\frac{9\times35}{105}$	$\frac{6\times27\frac{1}{2}}{105}$	$\frac{3\times20}{105}$	0
or	9.86	7.14	4.86	3.0	1.57	0.57	0
Add to E_w	16.43	11.71	7.43	3.57	0.14	−	
multiply by	1.12	1.25	1.25	1.25	1.25		
Stresses in Diagonals =	18.40	14.64	9.29	4.46	0.18 = counter on right.		
$E_{w'}$ advancing from left =	0	−0.29	−1.00	−2.14	−3.71	−5.72	−8.15
Add to E_w			+	−1.57	−5.14	−9.15	−13.58
multiply by				1.25	1.25	1.25	1.41
				−1.96	−6.42	−11.44	−19.15

Skew Portal. — Eng. News, Mar. 29/90.





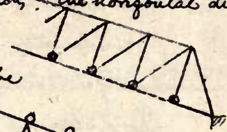
$S = 90° -$ skew angle; B = bracket angle; E = angle of end post with hor. plane; H = angle between planes of portal + truss; P = angle between plane of portal and horizontal plane; $A = 90°$ = angle between plane of truss and horizontal plane. Measure off equal distances OA, OH, OP, and connect A, H, and P by circular arcs having O for centre. Then in right spherical triangle AHP,

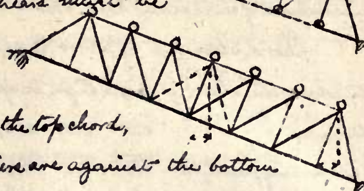
$\cot H = \cot S \sin E$; $\cos B = \cos S \cos B$; $\cot P = \cot E \sin S$.

Correct the angle S on account of ends of portal being away from the centre lines of the trusses, on each end by an amount equal to one-half the width of end posts. S' = true angle to be used; S = angle of centre lines; f = ½ width c. to c. of trusses; g = ½ width each end post. Then $\cot S' = \frac{f \cot S}{f-g}$. [F.A. Steiger.

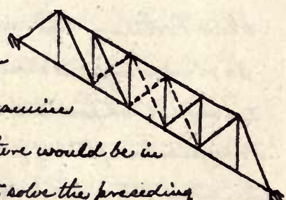
Trusses on a Grade. — Trusses may be built on a steep grade. If the chords remain horizontal, and the floor beams have increasing elevations, the stresses will be unchanged; except in the portions of the verticals above and below each floor beam; for a panel weight of live load will be above part of the vertical and below the remainder.

If the bridge is inclined, as in this sketch, the horizontal distances must be used in calculating moments and chord stresses, and the shears must be decomposed perpendicularly to the chords.





If the roadway is carried on the top chord, however, while the supports at piers are against the bottom chord, the verticals let fall from the loaded joints will be to the right (see Fig.) of the bottom chord joints. Hence the arrangement of load on the truss will be unsymmetrical, and will resemble that on the skew bridge truss just examined. In this case, also, it will be necessary to calculate chord stresses for all the pieces, and to cause the moving load to traverse in both directions for a complete solution of the web stresses. Such trusses are rare, but not unknown.

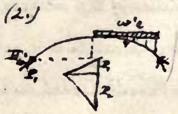


The posts in a grade truss might be kept vertical, as in the sketch. The student can examine this case, to discover what, if any, difference there would be in analysis. The graphical method enables one to solve the preceding cases without special difficulty.

Trusses with auxiliary systems, draw spans and the treatment for concentrated loads are left for the class-room.

§ 380. Braced Iron Arch.— Case I. Arch wholly loaded,

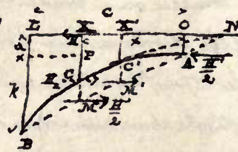
$M = \frac{1}{8} wl^2 = \frac{1}{8} w \cdot 2c \cdot 2c = \frac{wc^2}{2}$. $H_0 = \frac{wc^2}{2k}$; (1.) $H_1 = \frac{w'c^2}{2k}$. (2.)



If only one-half is loaded with w', $P_1 = \frac{w'c}{4}$;

$H_{w'} : P_1 = c : k$; $\therefore H_{w'} = \frac{w'c^2}{4k} = \frac{1}{2} H_1$.

$\frac{1}{2} H'$ acts anywhere on the straight line AB, this being the unloaded side. As ACB is curved, to find stress at C, make a section XM, and take moments about X.



$\frac{1}{2} H_1 \cdot MX = H_2 \cdot CX$. $\therefore H_2 = \frac{H}{2} \frac{MX}{CX}$. (5.)

Also by moments about C, $H \cdot CX = \frac{1}{2} H_2 MC$; $\therefore H = \frac{H}{2} \cdot \frac{MC}{CX}$. (4) and this is tension on the unloaded side because CX and CM are opposite in direction. On the loaded side, M will come above the rib.

H_2 is a maximum when $\frac{MX}{CX}$ is max. Consider the point C'. If C always fell on the straight line $C'N$, the ratio $\frac{MX}{CX}$ would be constant; but below either side of C', CX increases faster than does MX, and above C', MX diminishes faster than does CX; therefore C', the point of tangency of NC', is the point of maximum.

$CP : k = x^2 : c^2$; $CP = \frac{kx^2}{c^2}$. $CX = a + \frac{kx^2}{c^2} = \frac{ac^2 + kx^2}{c^2}$. $CM = MP - CP$.

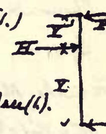
$MP : k = x : c$; $\therefore MP = \frac{kx}{c}$; $\therefore CM = \frac{kcx - kx^2}{c^2}$. $\therefore$ (4 A.)

$MX = MC + CX = \frac{kcx - kx^2 + ac^2 + kx^2}{c^2} = \frac{kcx + ac^2}{c^2}$; $\therefore$ (5 A.).

Differentiate (5 A.) for max. x. $\frac{kc(ac^2 + kx^2) - (ac^2 + kcx)2kx}{(ac^2 + kx^2)^2} = 0$.

$x^2 + \frac{2ac}{k}x = \frac{ac^2}{k}$; $x = c\left\{\sqrt{\left(\frac{a^2}{k^2} + \frac{a}{k}\right)} - \frac{a}{k}\right\}$. (6.)

§ 381. Iron Piers.— $M_0 = HY = \frac{fI}{y_1}$; $I = r^2 A = \frac{d^2}{8} A$. (p. 523, VI.)

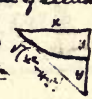


$\therefore f = \frac{HY \cdot \frac{1}{2} d}{\frac{1}{8} d^2 \cdot A} = \frac{4HY}{Ad}$. Then add or subtract $\frac{P}{A}$. (1.)

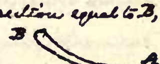
If tension $= 0$, $\frac{4HY}{d} - P = 0$; $\therefore$ (2.)

$P_1 : H = Y : Y + Y'$. $\therefore P_1 = \frac{HY}{Y + Y'}$; $M_0 = P_1 Y' =$ (3.). For (4) see (1).

§ 382. Suspension Bridges.— If the chain is of uniform section, $\frac{\text{Weight}}{C} = \frac{s}{x}$. By (13.) p. 140, s approx. $= x + \frac{2}{3}\frac{y^2}{x}$, $\therefore$ wt. of chain of section A, sufficient to carry $H = C\left(1 + \frac{2}{3}\frac{y^2}{x^2}\right)$. If sectional area varies as the pull, it will vary as secant of inclination, or $\sec i = \frac{\sqrt{x^2 + 4y^2}}{x}$



$= \sqrt{1 + \frac{4y^2}{x^2}} = 1 + 2\frac{y^2}{x^2}$ nearly. If chain is made throughout of section equal to B, the above formula for weight must be increased by $\sec i$, $\therefore$



$C' = C\left(1 + \frac{2}{3}\frac{y^2}{x^2}\right)\left(1 + \frac{2y^2}{x^2}\right) = C\left(1 + \frac{2}{3}\frac{y^2}{x^2} + \frac{2y^2}{x^2} + \text{[illegible]}\right) = C\left(1 + \frac{8y^2}{3x^2}\right)$ nearly. (1.)

To find C''. $C'' = \int_0^x \frac{C}{x} \sec^2 i \, dx$, since pull varies as $\sec i$, and $ds = dx \sec i$.

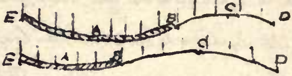
$\frac{C}{x}$ is a constant. $C'' = \frac{C}{x}\int_0^x \left(1 + \frac{4y^2}{x^2}\right) dx$. $x^2 = py$; $\therefore \frac{4y^2}{x^2} = \frac{4x^2}{p^2}$. $C'' = \frac{C}{x}\int_0^x \left(1 + \frac{4x^2}{p^2}\right) dx$

$= \frac{C}{x}\left(x + \frac{4x^3}{3p^2}\right) = \frac{C}{x}\left(x + \frac{4y^2}{3x}\right) = C\left(1 + \frac{4y^2}{3x^2}\right)$ (2.)

For (3.) and (4.) Rankine uses 90000 for wire and 60000 for links, with a unit stress of 1/6 the ultimate.

By (3.) p. 188, $H = \frac{2px^2}{4y} = \frac{w'x^2}{2y} = H'$. (5.)

p. 578. The inclination of the planes of the cables to each other also steadies the structure laterally. Horizontal parabolic stay cables are also employed.



p. 579. Point B is distant x from mid-span, so that c+x and c−x are the two segments, loaded and unloaded. For longer segment loaded, load = $w'(c+x)$. It can be proved (see Arches, p. 154), that the amount of load carried by the half chain will be equal to the magnitude of the reaction at B, the point of contraflexure, if EB were an unassisted girder. The point of contraflexure is at the end of the loaded portion, because EB carries a positive load and BD a negative one, the upward pull of the rods. Then ~~reaction at~~ load on the half chain is $\frac{w'(c+x)}{2}$, and intensity of load on chain is $\frac{w'(c+x)}{2c}$. Therefore $M_d.\ \frac{w'(c+x)}{2c}.\ \frac{(c-x)^2}{8}$

Intensity on EB = $w' - \frac{w'(c+x)}{2c} = \frac{w'(c-x)}{2c}$, and $M_a = \frac{w'(c-x)}{2c}\frac{(c+x)^2}{8}$

Shear at E = $\frac{w'(c-x)(c+x)}{4c}$ = shear at B and at D.

If $x = 0$, $M_{c\ or\ A} = \frac{wc^2}{16}$; $F = \frac{wc}{4}$. (7.) + (8.)

To make M max., diff^{te} relatively to x and put equal to zero.

$M = \frac{w'}{16c}(c^3 + c^2x - cx^2 - x^3)$; $\frac{dM}{dx} = c^2 - 2cx - 3x^2 = 0$; $x = \frac{1}{3}c$. or $-\frac{1}{3}c$

M_A max = $-M_c$ max = $\frac{2wc^2}{27} = \frac{wc^2}{14}$ approx. (9.)

§ 383. Proportion of Dead to Live Load. — See Notes on § 167. Maximum economy is obtained when length of spans and their heights are such that cost of piers and foundations = cost of superstructure: Cost of substructure = cost of superstructure.

§ 402. Bearing Piles. — Pile supported by friction, P = safe load. Major Sanders gives $P = \frac{H}{3h}W$ P and W in tons. H = height in feet through which the ram W fell at last blow; h = distance in feet pile is driven by that blow. Mr. McAlpine writes

$P = 80(W + .228\sqrt{H} - 1)$. P and W–tons, H = feet.

See Eng. News, 1878, and Dec. 29/88, or Hering's "Bearing Piles."

Eng. News gives $P_{in\ lbs.} = \frac{2WH}{h+1}$; W in lbs.; H in ft.; h in inches.