XVI.—*On the Attraction of Ellipsoids, with a new Demonstration of* CLAIRAUT'S *Theorem, being an Account of the late* Professor MAC CULLAGH'S *Lectures on those Subjects. Compiled by* GEORGE JOHNSTON ALLMAN, LL. D., *of Trinity College, Dublin.*

Read June 13, 1853.

[THE following Memoir contains the substance of a Series of Lectures delivered by the late Professor MAC CULLAGH to the Candidates for Fellowship in Trinity College, Dublin, in Hilary and Michaelmas Terms, 1846.

It is now published by the Academy, with the view of securing to Professor MAC CULLAGH the merit of whatever is original in the investigation or its results.

The present Paper may be regarded as a Sequel to the Account of Professor MAC CULLAGH'S Lectures on Rotation, given by the Rev. SAMUEL HAUGHTON in a former part of the present volume of the Transactions of the Academy.]

PROPOSITION I.

If P *be any point on the surface of an ellipsoid, and* PC_1 *be drawn perpendicular to an axis* OC, *and an ellipsoid be described through* C_1 *concentric, similar and similarly placed to the given ellipsoid; then the component of the attraction of the given ellipsoid on* P *in a direction parallel to* OC *is equal to the attraction of the inner ellipsoid on the point* C_1.

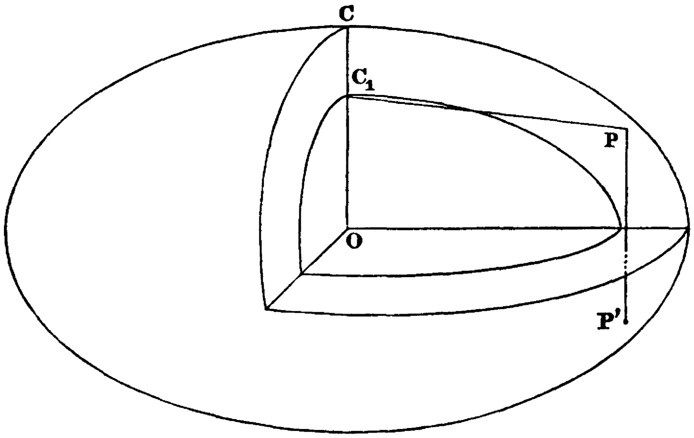

FIG. 1.

This theorem is an extension of that given by MAC LAURIN* relating to the attraction of a spheroid on a point placed on its surface. It may, moreover, be established by means of the same geometrical proposition from which MAC LAURIN deduced his theorem.

Through the point P let a chord PP′ of the given ellipsoid be drawn parallel to the axis OC; now suppose both ellipsoids to be divided into wedges by planes parallel to each other, and passing respectively through this chord and the parallel axis of the inner; and suppose the wedges to be divided into pyramids, the common vertex of one set being at P, and of the other at C_1. Observing that any two of these parallel planes cut the two surfaces in similar ellipses, such that the semi-axis of one is equal to the parallel ordinate of the other, it is easy to see that the reasoning employed by MAC LAURIN may be used to establish the truth of the theorem stated above.

* De Caus. Phys. Flux. et Refl. Maris, sect. 3. Or see Airy's Tract on the Figure of the Earth, Prop. 8.

PROPOSITION II.

*To calculate the attraction of an ellipsoid on a point placed at the extremity of an axis.**

Let the semi-axes of the ellipsoid be a, b, c, where $a > b > c$, and let the point on which it is required to find the attraction be C, the extremity of the least axis.

Suppose the ellipsoid to be divided by a series of cones of revolution which have a common vertex C and a common axis CC′, C′ being the vertex of the ellipsoid opposite to C; it will be sufficient to find an expression for the attraction of the part of the ellipsoid contained between two consecutive conical surfaces, whose semi-angles are θ and $\theta + d\theta$ respectively. Suppose now the part of the ellipsoid between two consecutive cones to be divided into elementary pyramids with a common vertex C. Let CP be one of these elementary pyramids, whose solid angle is ω; let PQ be drawn perpendicular to CC′; from the centre O draw a radius vector OR parallel to CP, and from the extremity R let fall a perpendicular RS on the axis CC′.

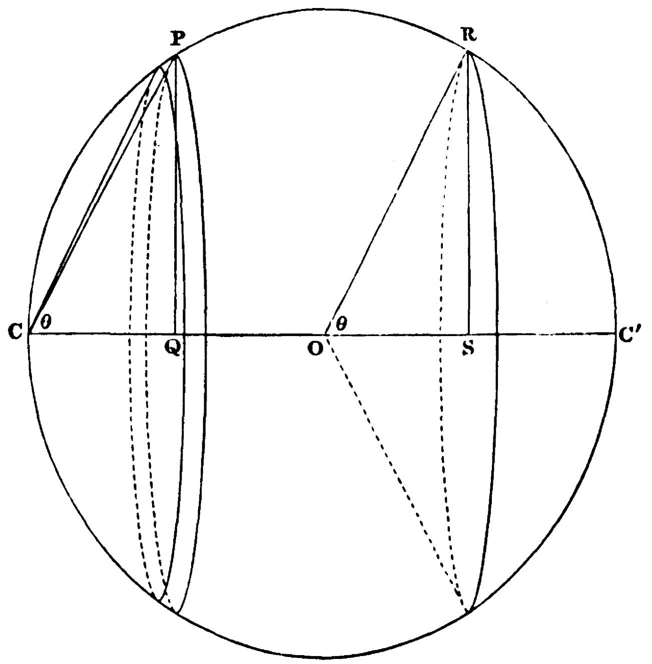

FIG. 2.

Now the attraction of the elementary pyramid CP on the material point μ, placed at its vertex $= \mu f\rho\omega \,.\, \text{CP}$; and the component of this attraction in the direction of the axis is

* Proceedings of the Royal Irish Academy, vol. iii. p. 367.

$$\mu f\rho\varpi \,.\, CQ = 2\,\mu f\rho\varpi \,.\, \frac{\overline{OR}^2 \cos^2\theta}{c}.$$

Now suppose the radius vector OR to revolve around the axis OC′, then the attraction on the point C of the portion of the ellipsoid bounded by the two cones of revolution, whose semi-angles are θ and $\theta + d\theta$ respectively, since it is made up of the components in the direction CC′ of the attractions of all the elementary pyramids CP, is

$$\frac{2\mu f\rho}{c}\cos^2\theta\,\Sigma\,(\overline{OR}^2\,\varpi) = \frac{2\mu f\rho}{c}\cos^2\theta d\theta\Sigma\,(\overline{OR}^2\,d\phi),$$

$d\phi$ being the angle between two consecutive sides of the cone generated by the revolution of OR.

But $\Sigma\,(\overline{OR}^2 d\phi)$ is equal to twice the superficial area of the part of this cone which is enclosed within the ellipsoid; moreover, the projection on the plane ab of this portion of the surface of the cone is an ellipse whose semi-axes are $r_1 \sin\theta$, $r_2 \sin\theta$, and whose area is $\pi r_1 r_2 \sin^2\theta$, r_1 and r_2 being the maximum and minimum values of OR: the superficial area of the portion of the cone within the ellipsoid is therefore $\pi r_1 r_2 \sin\theta$.

Hence it follows that

$$\Sigma\,(\overline{OR}^2 d\phi) = 2\pi r_1 r_2 \sin\theta.$$

The attraction on the point C of the part of the ellipsoid contained between the two cones of revolution, whose common vertex is at C, and whose semi-angles are θ and $\theta + d\theta$ respectively, is therefore

$$\frac{4\,\pi\mu f\rho}{c}\cos^2\theta\, d\theta\, r_1\, r_2 \sin\theta,$$

where

$$\frac{1}{r_1} = \sqrt{\left(\frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{a^2}\right)}, \text{ and } \frac{1}{r_2} = \sqrt{\left(\frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{b^2}\right)}.$$

On substituting these values, the expression given above becomes

$$4\pi\mu f\rho\,\frac{abc\cos^2\theta\sin\theta d\theta}{\sqrt{(a^2\cos^2\theta + c^2\sin^2\theta)}\;\sqrt{(b^2\cos^2\theta + c^2\sin^2\theta)}}.$$

Hence the attraction of the solid ellipsoid on the point C at the extremity of the least axis is

$$4\pi\mu f\rho\int_0^{\frac{\pi}{2}}\frac{abc\cos^2\theta\sin\theta d\theta}{\sqrt{(a^2\cos^2\theta+c^2\sin^2\theta)}\ \sqrt{(b^2\cos^2\theta+c^2\sin^2\theta)}}.$$

Let $\cos\theta=u$, and this expression becomes

$$4\pi\mu f\rho\int_0^1\frac{abc\,u^2\,du}{\sqrt{\{c^2+u^2(a^2-c^2)\}}\ \sqrt{\{c^2+u^2(b^2-c^2)\}}}. \qquad (1)$$

In the same way it may be shown that the attraction of the ellipsoid on a point μ placed at the extremity of the mean axis, is

$$4\pi\mu f\rho\int_0^1\frac{abc\,u^2\,du}{\sqrt{\{b^2+u^2(c^2-b^2)\}}\ \sqrt{\{b^2+u^2(a^2-b^2)\}}};$$

and on a point at the extremity of the greater axis,

$$4\pi\mu f\rho\int_0^1\frac{abc\,u^2\,du}{\sqrt{\{a^2+u^2(b^2-a^2)\}}\ \sqrt{\{a^2+u^2(c^2-a^2)\}}}.$$

It will be seen in a subsequent proposition, that these three expressions are not independent of each other, the values of the three attractions in question being connected by an equation.

Proposition III.

*To give geometrical representations of the attraction of an ellipsoid on points placed at the extremities of its least and mean axes.**

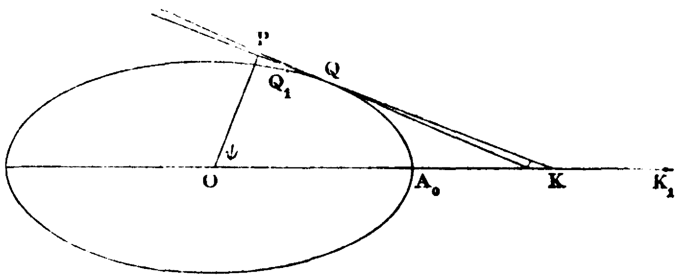

Fig. 3.

On the greater axis OA_0 of the focal ellipse assume a point K_1 such that

$$OK_1=\frac{b}{c}OA_0;$$

from the point K_1 draw a tangent K_1Q_1 to the focal ellipse, and let $T=$ tangent K_1Q_1 $-$ arc A_0Q_1, then the attraction of the ellipsoid on the particle μ placed at the extremity C of the least axis is

$$\frac{4\pi\mu f\rho abc^2}{(a^2-c^2)\,(b^2-c^2)}\,T. \qquad (2)$$

* Proceedings of the Royal Irish Academy, vol. iii. p. 367.

For let a point K be assumed on the greater axis OA_0 of the focal ellipse, such that

$$OK = \frac{OA_0}{c} \sqrt{\{c^2 + u^2(b^2 - c^2)\}};$$

from K let a tangent KQ be drawn to the focal ellipse, and let OP be the perpendicular let fall from O on KQ, then ψ denoting the angle A_0OP,

$$\overline{OK}^2 . \cos^2\psi = \frac{a^2 - c^2}{c^2} \{c^2 + u^2(b^2 - c^2)\} . \cos^2\psi.$$

Moreover,

$$\overline{OK}^2 . \cos^2\psi = \overline{OP}^2 = (a^2 - c^2)\cos^2\psi + (b^2 - c^2)\sin^2\psi.$$

Equating these values, and solving for $\sin^2\psi$, we get

$$\sin^2\psi = \frac{(a^2 - c^2)u^2}{c^2 + u^2(a^2 - c^2)}.$$

Now

$$d . (\tan KQ - \text{arc } A_0Q) = \sin\psi d . OK^*$$

$$= \frac{(a^2 - c^2)(b^2 - c^2)}{c} \frac{u^2 du}{\sqrt{\{c^2 + u^2(a^2 - c^2)\}} \sqrt{\{c^2 + u^2(b^2 - c^2)\}}}.$$

By comparing this expression with (1) given in the last proposition, it appears that the attraction on the point C of the portion of the ellipsoid contained between the two conical surfaces whose semi-angles are θ and $\theta + d\theta$ respectively, is

$$\frac{4\pi\mu f\rho abc^2}{(a^2 - c^2)(b^2 - c^2)} d . (\tan KQ - \text{arc } A_0Q).$$

Now in order to obtain the attraction of the whole ellipsoid on the point C, we have to integrate the expression given above between the limits $u = 0$ and $u = 1$, or $OK = OA_0$ and $OK = OK_1$; from which it appears that its value is

$$\frac{4\pi\mu f\rho abc^2}{(a^2 - c^2)(b^2 - c^2)} T.$$

It is easy to see that the attraction of the part of the ellipsoid contained within the conical surface, whose semi-angle θ is equal to $\cos^{-1} u$, is

* Transactions of the Royal Irish Academy, vol. xvi. p. 79. Proceedings of the Royal Irish Academy, vol. ii. p. 507.

$$4\pi\mu f\rho \frac{abc^2}{(a^2-c^2)(b^2-c^2)}(T-t), \tag{3}$$

where $t = \tan KQ - \text{arc } A_0 Q$.

To represent the attraction on a point μ placed at the extremity of the mean axis, assume on the transverse axis OA_0 of the focal hyperbola a point K_1 such that $OK_1 = OA_0 \frac{c}{b}$, and from K_1 draw a tangent $K_1 Q_1$ to the hyperbola, and let $T = \tan K_1 Q_1 - \text{arc } A_0 Q_1$, then the attraction of the ellipsoid on the point μ is

$$-4\pi\mu f\rho \frac{ab^2c}{(a^2-b^2)(c^2-b^2)} T. \tag{4}$$

To prove this, assume a point K such that $OK = \frac{OA_0}{b}\sqrt{\{b^2 + u^2(c^2-b^2)\}}$; from K draw a tangent KQ to the hyperbola, and from O let fall a perpendicular OP on this tangent, then if ψ = angle A_0OP,

$$\sin^2\psi = \frac{(a^2-b^2)u^2}{b^2+u^2(a^2-b^2)}.$$

Hence by following a method similar to that used in finding the representation of the attraction on a point at the extremity of the least axis, the expression given above may be easily obtained.

The attractions C, B of the ellipsoid on points placed at the extremity of the least and mean axes are thus represented by means of arcs of the focal ellipse and hyperbola respectively. In consequence of the third focal conic of the ellipsoid being imaginary, no direct geometrical representation can be given for the attraction A on a point placed at the extremity of its greater axis. It will, however, be found, as was intimated above, that a simple relation exists between the three attractions, which enables us to represent this last by means of arcs of both focal conics.

The relation alluded to is

$$\frac{A}{a} + \frac{B}{b} + \frac{C}{c} = 4\pi\mu f\rho.^* \tag{5}$$

* Proceedings of the Royal Irish Academy, vol. ii. p. 525.

This can be easily proved by the help of the following geometrical theorem :

If from the extremities A, B, C of the three axes of an ellipsoid, three parallel chords Ap, Bq, Cr, be drawn, and if these chords be projected each on the axis from whose extremity it is drawn, then the sum of these three projections, Aα, Bβ, Cγ, divided respectively by the lengths of the axes AA′, BB′, CC′, on which they are measured, will be equal to unity.

Now conceive three chords Ap, Ap', Ap'', to be drawn from A, making each with the other two very small angles, and so forming a pyramid with a very small vertical solid angle ω; and from B and C let two systems of chords Bq, Bq', Bq'', and Cr, Cr', Cr'', be drawn, each system forming a very small pyramid whose three edges are parallel to the three edges Ap, Ap', Ap'', of the pyramid which has its vertex at A.

The attractions of the three pyramids, reduced each to the direction of the axis passing through its vertex, will be equal to $\mu f \rho \omega . \mathrm{A}\alpha$, $\mu \rho f \omega . \mathrm{B}\beta$, $\mu f \rho \omega . \mathrm{C}\gamma$ respectively, and, therefore, the sum of those attractions divided respectively by the lengths of the axes will be

$$\mu f \rho \omega \left(\frac{\mathrm{A}\alpha}{\mathrm{AA}'} + \frac{\mathrm{B}\beta}{\mathrm{BB}'} + \frac{\mathrm{C}\gamma}{\mathrm{CC}'} \right) = \mu f \rho \omega .$$

Let pyramids thus related be indefinitely multiplied, and the ellipsoid will be simultaneously exhausted from the three points A, B, C.

Hence the sum of the whole attractions at A, B, C, divided respectively by the lengths of the corresponding axes, will be $2\pi \mu f \rho$, or,

$$\frac{A}{a} + \frac{B}{b} + \frac{C}{c} = 4\, \pi \mu f \rho .$$

PROPOSITION IV.

To find an expression for the potential V of a system of particles at a point M *whose distance from the centre of gravity of the system is very great compared with the mutual distances of the particles.*

It is proved by POISSON,* that if the origin of co-ordinates be at the centre of gravity,

* Mecanique, tome i. p. 178.

$$V = \frac{M}{r'} + \frac{3}{2r'^5}\Sigma\,(xx' + yy' + zz')^2\,dm - \frac{1}{2r'^3}\Sigma\,(x^2 + y^2 + z^2)\,dm,$$

$x'\,y'\,z'$ being the co-ordinates of the distant point, and r' its distance from the origin. Let now the principal axes at that centre be taken as axes of co-ordinates; then, since

$$\Sigma xydm = 0, \qquad \Sigma xzdm = 0, \qquad \Sigma yzdm = 0;$$

$$V = \frac{M}{r'} + \frac{3}{2r'^5}\Sigma\,(x^2x'^2 + y^2y'^2 + z^2z'^2)dm - \frac{1}{2r'^3}\Sigma\,(x^2 + y^2 + z^2)\,dm.$$

Hence, if A, B, C be the three principal moments of inertia, and I the moment of inertia round OM,

$$V = \frac{M}{r'} + \frac{1}{2r'^3}(A + B + C - 3I). \qquad (6)$$

PROPOSITION V.

A system of material particles attract a point M, *whose distance from the centre of gravity* O *of the attracting mass is very great compared with the mutual distances of the particles; then if a tangent plane be drawn to the "ellipsoid of gyration,"* perpendicular to* OM, *the whole attraction lies in the plane* OST, *where* S *is the point in which this tangent plane intersects* OM, *and* T *the point of contact.*

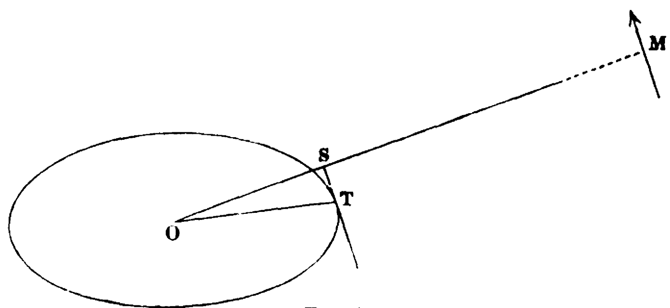

FIG. 4.

* The centre of this ellipsoid is at the centre of gravity; its axes are in the directions of the principal axes, and their lengths are determined by the equations

$$Ma^2 = A, \quad Mb^2 = B, \quad Mc^2 = C.$$

This ellipsoid is used by Professor MAC CULLAGH in his Theory of Rotation; see Rev. S. HAUGHTON's Account of Professor MAC CULLAGH's Lectures on that subject, Transactions R. I. A. vol. xxii. p. 139.

Let α, β, γ be the direction angles of OT; α', β', γ', of OM; α_1, β_1, γ_1, of TS; and α_0, β_0, γ_0, of the normal to the plane OST; and let OS, OT, and the angle SOT, be denoted by p, r, and ϕ respectively. It will be sufficient to prove, that the component Q of the attraction in the direction of the normal to the plane OST is cypher.

We shall first find the components X, Y, Z, of the attraction in the directions of the axes, and thence deduce the value of Q.

Now,

$$X = -\frac{dV}{dx'} = \frac{M}{r'^2}\cos\alpha' + \frac{3}{2r'^4}(A+B+C-3I)\cos\alpha' + \frac{3}{2r'^3}\frac{dI}{dx'},$$

$$Y = -\frac{dV}{dy'} = \frac{M}{r'^2}\cos\beta' + \frac{3}{2r'^4}(A+B+C-3I)\cos\beta' + \frac{3}{2r'^3}\frac{dI}{dy'},$$

$$Z = -\frac{dV}{dz'} = \frac{M}{r'^2}\cos\gamma' + \frac{3}{2r'^4}(A+B+C-3I)\cos\gamma' + \frac{3}{2r'^3}\frac{dI}{dz'};$$

but,

$$\frac{dI}{dx'} = \frac{2(A-I)\cos\alpha'}{r'};\quad \frac{dI}{dy'} = \frac{2(B-I)\cos\beta'}{r'};\quad \frac{dI}{dz'} = \frac{2(C-I)\cos\gamma'}{r'}.$$

Hence we have

$$\begin{aligned} X &= \frac{M}{r'^2}\cos\alpha' + \frac{3}{2r'^4}(A+B+C-5I)\cos\alpha' + \frac{3A\cos\alpha'}{r'^4}, \\ Y &= \frac{M}{r'^2}\cos\beta' + \frac{3}{2r'^4}(A+B+C-5I)\cos\beta' + \frac{3B\cos\beta'}{r'^4}, \\ Z &= \frac{M}{r'^2}\cos\gamma' + \frac{3}{2r'^4}(A+B+C-5I)\cos\gamma' + \frac{3C\cos\gamma'}{r'^4}. \end{aligned} \qquad (7)$$

Now,

$$Q = X\cos\alpha_0 + Y\cos\beta_0 + Z\cos\gamma_0;$$

but,

$$\begin{aligned} \sin\phi\cos\alpha_0 &= \cos\beta\cos\gamma' - \cos\gamma\cos\beta', \\ \sin\phi\cos\beta_0 &= \cos\gamma\cos\alpha' - \cos\alpha\cos\gamma', \\ \sin\phi\cos\gamma_0 &= \cos\alpha\cos\beta' - \cos\beta\cos\alpha'; \end{aligned}$$

the following relations moreover exist,

$$a^2\cos\alpha' = rp\cos\alpha,\quad b^2\cos\beta' = rp\cos\beta,\quad c^2\cos\gamma' = rp\cos\gamma; \qquad (8)$$

hence, by substitution, we have

$$\cos\alpha_0 = \frac{b^2 - c^2}{pr\sin\phi}\cos\beta'\cos\gamma', \quad \cos\beta_0 = \frac{c^2 - a^2}{pr\sin\phi}\cos\gamma'\cos\alpha',$$

$$\cos\gamma_0 = \frac{a^2 - b^2}{pr\sin\phi}\cos\alpha'\cos\beta'.$$

Substituting these values for $\cos\alpha_0$, $\cos\beta_0$, $\cos\gamma_0$, in the expression for Q, and observing that

$$\cos\alpha'\cos\alpha_0 + \cos\beta'\cos\beta_0 + \cos\gamma'\cos\gamma_0 = 0,$$

we get

$$Q = \frac{3M}{r'^4}\frac{a^2(b^2 - c^2) + b^2(c^2 - a^2) + c^2(a^2 - b^2)}{pr\sin\phi}\cos\alpha'\cos\beta'\cos\gamma' = 0. \quad (9)$$

PROPOSITION VI.

The same things being supposed, to find the components of the attraction, namely, R in the direction of the centre of gravity MO, *and P in the transverse direction* TS.

To find R;

$$R = X\cos\alpha' + Y\cos\beta' + Z\cos\gamma',$$

$$\therefore\; R = \frac{M}{r'^2} + \frac{3}{2r'^4}(A + B + C - 5I) + \frac{3I}{r'^4},$$

$$R = \frac{M}{r'^2} + \frac{3}{2r'^4}(A + B + C - 3I). \quad (10)$$

To find P;

$$P = X\cos\alpha_1 + Y\cos\beta_1 + Z\cos\gamma_1;$$

but,

$$\sin\phi\cos\alpha_1 = \cos\alpha'\cos\phi - \cos\alpha,$$

$$\sin\phi\cos\beta_1 = \cos\beta'\cos\phi - \cos\beta,$$

$$\sin\phi\cos\gamma_1 = \cos\gamma'\cos\phi - \cos\gamma.$$

Substituting for $\cos\alpha$, $\cos\beta$, $\cos\gamma$, their values from (8), we get

$$\cos\alpha_1 = -\frac{a^2 - p^2}{pr\sin\phi}\cos\alpha', \quad \cos\beta_1 = -\frac{b^2 - p^2}{pr\sin\phi}\cos\beta',$$

$$\cos\gamma_1 = -\frac{c^2 - p^2}{pr\sin\phi}\cos\gamma'.$$

Substituting these values of $\cos\alpha_1$, $\cos\beta_1$, $\cos\gamma_1$, and observing that

$$\cos\alpha'\cos\alpha_1+\cos\beta'\cos\beta_1+\cos\gamma'\cos\gamma_1=0,$$

we have

$$P=-\frac{3M}{r'^4}\frac{p^2(r^2-p^2)}{pr\sin\phi}=-\frac{3M}{r'^4}(pr\sin\phi);$$

or,

$$P=-\frac{3M}{r'^4}(\mathrm{OS}\times\mathrm{ST}). \tag{11}$$

The negative sign indicates* that the force P acts in the direction TS, i. e. from the radius vector towards the perpendicular of the ellipsoid of gyration. If the force P be resolved into three others in the directions of the axes, it is evident from the values given in Proposition V. for X, Y, Z, that these components are

$$\frac{3(A-I)}{r'^4}\cos\alpha',\quad \frac{3(B-I)}{r'^4}\cos\beta',\quad \frac{3(C-I)}{r'^4}\cos\gamma'.\dagger \tag{12}$$

Proposition VII.

An ellipsoid is composed of ellipsoidal strata of different densities and of variable but small ellipticities; find the components, central and transverse, of its attraction on an external point.

The values found in the last Proposition for the components of the attraction of any mass on a very distant point, will be found to hold in the present

* The direction of the force P, which Professor Mac Cullagh determines by the interpretation of the negative sign, may be very clearly seen from the following considerations. This force exists in every case where the three principal moments of inertia are not all equal, that is, when the ellipsoid of gyration is not a sphere. The greatest axis of that ellipsoid is manifestly towards that part of the body in which there is a deficiency of attracting matter. If we consider now the position of a perpendicular on a tangent plane of an ellipsoid with relation to the corresponding radius vector, we shall find that it always lies away from the greatest axis. But the transverse force has been shown to be in the plane of radius vector and perpendicular. Therefore, the direction of the transverse force, being towards the preponderating matter, must be parallel to TS.

† The results given by Professor Mac Cullagh in Propositions V. and VI. may be otherwise

case, whatever be the position of the attracted point. In order to show this, we shall first prove it for a homogeneous ellipsoid of small ellipticities. Such an ellipsoid being given, another, confocal with it, can be constructed so small, that the distance to the attracted point may be regarded as very great, compared with the axes of this ellipsoid: the components of the attraction of this small ellipsoid on the distant point are given by the expressions (10) and

obtained, and, perhaps, with greater facility, by introducing the consideration of the *statical moment* of the attracting force.*

If the three principal moments of inertia were equal to each other, then the whole attraction would be in the direction of the centre of gravity, and its magnitude would be

$$\frac{M}{r'^2}.$$

In general, however, the attracting mass will be of an irregular shape; there will exist then, in addition to the principal part of the attraction which will be central, a transverse force which will cause a motion of rotation about the centre of gravity.

The components of the moment of this transverse force in the three principal planes are

$$x'Y - y'X, \quad y'Z - z'Y, \quad z'X - x'Z;$$

but from (7),

$$x'Y - y'X = -\frac{3(A-B)}{r'^3}\cos\alpha'\cos\beta' = -\frac{3M}{r'^3}(a^2-b^2)\cos\alpha'\cos\beta',$$

$$y'Z - z'Y = -\frac{3(B-C)}{r'^3}\cos\beta'\cos\gamma' = -\frac{3M}{r'^3}(b^2-c^2)\cos\beta'\cos\gamma',$$

$$z'X - x'Z = -\frac{3(C-A)}{r'^3}\cos\gamma'\cos\alpha' = -\frac{3M}{r'^3}(c^2-a^2)\cos\gamma'\cos\alpha'.$$

Now it is well known, that $\frac{1}{2}(a^2-b^2)\cos\alpha'\cos\beta'$, $\frac{1}{2}(b^2-c^2)\cos\beta'\cos\gamma'$, $\frac{1}{2}(c^2-a^2)\cos\gamma'\cos\alpha'$, are the areas of the projections of the triangle OST on the principal planes. Hence it follows, that the resultant moment lies in the plane of the radius vector OT, and the perpendicular OS to a tangent plane of the ellipsoid of gyration; the tangent plane being perpendicular to OM. It appears also, that the magnitude of the resultant moment is

$$-\frac{3M}{r'^3}(\text{OS} \times \text{ST}),$$

and therefore that the transverse component of the attraction

$$P = -\frac{3M}{r'^4}(\text{OS} \times \text{ST}).$$

Or the values of the central force and the moment of the transverse force may be obtained directly from the expression (6) for the potential V. This function is of such a nature, that its differential coefficient with relation to any line (the sign being changed) is equal to the re-

* See Rev. R. Townsend, in the University Examination Papers, 1849, p. 51.

(11); now the attractions of two confocal ellipsoids on an external point are in the same direction, and proportional to their masses; the components of the attraction of the proposed ellipsoid will, therefore, be

$$R = \frac{M}{r'^4} + \frac{3}{2r'^4}\frac{M}{M_1}(A_1 + B_1 + C_1 - 3I_1),$$

$$P = -\frac{3M}{r'^4}(\mathrm{OS}_1 \times \mathrm{S}_1\mathrm{T}_1),$$

the letters with suffixes referring to the small ellipsoid.

The attracting ellipsoids being confocal, their ellipsoids of gyration are confocal also; hence it follows, that

$$\frac{M}{M_1}(A_1 + B_1 + C_1 - 3I_1) = A + B + C - 3I,$$

and

$$\mathrm{OS}_1 \times \mathrm{S}_1\mathrm{T}_1 = \mathrm{OS} \times \mathrm{ST}.$$

It appears from this, that the central and transverse components of the attraction of a soid ellipsoid of uniform density, and whose ellipticities are small,

solved part of the attraction in that direction; and the differential coefficient with relation to any angle (the sign being changed as before) gives the component in the plane of that angle of the moment of the attractive force.

Hence,

$$R = -\frac{dV}{dr'} = \frac{M}{r'^2} + \frac{3}{2r'^4}(A + B + C - 3I),$$

since

$$\frac{dI}{dr'} = 0.$$

Again, if N be the component of the moment of the attractive force round OZ,

$$N = -\left(x'\frac{d}{dy'} - y'\frac{d}{dx'}\right)V;$$

but

$$\left(x'\frac{d}{dy'} - y'\frac{d}{dx'}\right)F(x'^2 + y'^2) = 0, \text{ where } F \text{ is any function.}$$

$$\therefore \quad N = \frac{3}{2r'^3}\left(x'\frac{d}{dy'} - y'\frac{d}{dx'}\right) \quad = \frac{3}{2r'^3}\left(x'\frac{d}{dy'} - y'\frac{d}{dx'}\right)\left(\frac{Ax'^2 + By'^2 + Cz'^2}{r'^2}\right).$$

$$\therefore \quad N = -\frac{3(A - B)}{r'^3}\cos\alpha'\cos\beta'.$$

The two other components of the moment may be similarly obtained. The remainder of the proof is the same as in the former part of this note.

on any external point whatever, are given by the same formulæ as the corresponding components of the action of any mass on a distant point.

Now it is a property of moments of inertia, that they are *subtractive*, that is, the difference of the moments of inertia of two masses with relation to any axis is equal to the moment of inertia of the difference of those masses with relation to the same axis. And the values at which we have arrived for the central force, and for the three components of the transverse force, contain in each term either the mass or a moment of inertia in the first power, and, therefore, these values also are subtractive. Hence the two components of the attraction of a homogeneous mass contained between two concentric and coaxal ellipsoids of small ellipticities, are given by formulæ (10) and (11). Now suppose an ellipsoidal mass to be composed of strata bounded by ellipsoids of different but small ellipticities, each stratum being homogeneous throughout its extent, while the density varies from one stratum to another according to any law; then, since those formulæ hold for the action of each stratum separately, and since the terms of which they are made up are in their nature *additive*, they hold for the entire mass.*

PROPOSITION VIII.

An oblate spheroid is composed of spheroidal strata of different densities and of variable but small ellipticities; find the components of its attraction on any external point.

The expressions given in the last Proposition for R and P become simplified in this case. Let OZ be the axis of revolution, and let λ denote the angle which OM makes with the plane XY; then since A and B are equal, we have

$$I = A\cos^2\lambda + C\sin^2\lambda,$$

and therefore,

$$A + B + C - 3I = (C - A)(1 - 3\sin^2\lambda),$$

also,

$$M(\mathrm{OS}\times\mathrm{ST}) = (A - C)\sin\lambda\cos\lambda.$$

Substituting these values in the expressions for R and P, we have

* See Professor MAC CULLAGH, in the University Examination Papers, 1833, p. 268.

$$R = \frac{M}{r'^2} + \frac{3}{2}\frac{C-A}{r'^4}(1 - 3\sin^2\lambda), \tag{13}$$

$$P = 3\frac{C-A}{r'^4}\cos\lambda\sin\lambda. \tag{14}$$

The direction of the force P is towards the plane of the equator; this appears from the shape of the "ellipsoid of gyration," which in this case is a prolate surface of revolution.

Prop. IX. Clairaut's Theorem.

Whatever be the law of variation of the earth's density at different distances from the centre, if the ellipticity of the surface be added to the ratio which the excess of the polar above the equatorial gravity bears to the equatorial gravity, their sum will be $\frac{5}{2}q$, *where* q *is the ratio of the centrifugal force at equator to the equatorial gravity.*

For suppose the attracted point M to be on the surface of the earth, which is known to be an oblate spheroid of small ellipticity. Then, from the principles of Hydrostatics, since the tangential force is cypher, we have

$$R\cos\theta - P\sin\theta - \omega^2 r\cos\lambda\cos(\theta-\lambda) = 0, \tag{15}$$

where ω denotes the angular velocity, and θ the angle which the tangent to the meridian through the attracted point makes with the radius vector; developing $\cos(\theta-\lambda)$ and arranging, we obtain

$$(R - \omega^2 r\cos^2\lambda)\cos\theta = (P + \omega^2 r\cos\lambda\sin\lambda)\sin\theta. \tag{16}$$

But from the property of the elliptic section made by the plane of the meridian, we have

$$\cot\theta = \frac{e^2\sin\lambda\cos\lambda}{1 - e^2\cos^2\lambda} = 2\epsilon\sin\lambda\cos\lambda, \quad q. p.,$$

where e is the excentricity and ϵ the ellipticity of this ellipse.

Substituting in (16) this value of $\cot\theta$, and the values of R and P from (13) and (14), the equation of equilibrium becomes

$$\left\{\frac{M}{r^2}+\frac{3}{2}\frac{C-A}{r^4}(1-3\sin^2\lambda)-\omega^2 r\cos^2\lambda\right\}2\epsilon\sin\lambda\cos\lambda=(3\frac{C-A}{r^4}+\omega^2 r)\sin\lambda\cos\lambda,$$

or, approximately,

$$\left\{\frac{M}{a^2}+\frac{3}{2}\frac{C-A}{a^4}(1-3\sin^2\lambda)-\omega^2 a\cos^2\lambda\right\}2\epsilon=3\frac{C-A}{a^4}+\omega^2 a.$$

If we neglect quantities of the second order, this equation becomes

$$\frac{2\epsilon M}{a^2}=3\frac{C-A}{a^4}+\omega^2 a. \tag{17}$$

We have thus arrived at a relation which enables us to express the unknown quantity $C-A$, in terms of quantities which are all known, and, therefore, to eliminate the former from any other equation in which it may occur.

Now let R_e and R_p denote the equatorial and polar attractions respectively; we have from the general value of R (13),

$$R_e=\frac{M}{a^2}+\frac{3}{2}\frac{C-A}{a^4},$$

$$R_p=\frac{M}{c^2}-3\frac{C-A}{c^4};$$

but

$$c=a(1-\epsilon),\ \therefore\ \frac{1}{c^2}=\frac{1}{a^2}(1+2\epsilon)\text{ and }\frac{1}{c^4}=\frac{1}{a^4}(1+4\epsilon),$$

$$\therefore\ R_p=\frac{M}{a^2}+\frac{2M\epsilon}{a^2}-3\frac{C-A}{a^4}.$$

But,

$$G_p=R_p\text{ and }G_e=R_e-\omega^2 a;$$

$$\therefore\ G_p-G_e=\frac{2\epsilon M}{a^2}-\frac{9}{2}\frac{C-A}{a^4}+\omega^2 a.$$

Eliminating $\frac{C-A}{a^4}$ by means of equation (17), we get

$$\frac{G_p-G_e}{G_e}=-\epsilon+\frac{5}{2}\frac{\omega^2 a}{G_e};$$

or,

$$\frac{G_p-G_e}{G_e}+\epsilon=\frac{5}{2}q. \tag{18}$$