II. *A Method for determining the Number of impossible Roots in adfected Æquations. By Mr.* George Campbell.

Lemma I.

IN every adfected quadratick Æquation $ax^2 - Bx + A = 0$, whose Roots are real, a fourth Part of the Square of the Coefficient of the second Term is greater than the Rectangle under the Coefficient of the first Term and the absolute Number or $\frac{1}{4}B^2 > a \times A$; and *vice versa* if $\frac{1}{4}B^2 > a \times A$, the Roots of the Æquation $ax^2 - Bx + A = 0$, will be real. But if $\frac{1}{4}B^2 < a \times A$, the Roots will be impossible. This is evident from the Roots of the Æquation being $\frac{\frac{1}{2}B + \sqrt{\frac{1}{4}B^2 - a \times A}}{a}$, $\frac{\frac{1}{2}B - \sqrt{\frac{1}{4}B^2 - a \times A}}{a}$.

Lemma II.

Whatever be the Number of impossible Roots in the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} +$ &c. $\pm dx^3 \mp cx^2 \pm bx \mp A = 0$, there are just as many in the Æquation $Ax^n - bx^{n-1} + cx^{n-2} - dx^{n-3} +$ &c. $\pm Dx^3 \mp Cx^2 \pm Bx \mp 1 = 0$. For the Roots of the last Æquation are the Reciprocals of those of the first, as is evident from common Algebra. Let the Roots of the biquadratick Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$ be a, b, c, d, whereof let c, d be impossible, then the Roots of the Æquation

$A x^4 - D x^3 + C x^2 - B x + 1 = 0$ will be $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, and therefore two of them to wit $\frac{1}{c}, \frac{1}{d}$ impoſſible.

Lemma III.

In every Æquation $x^n - B x^{n-1} + C x^{n-2} - D x^{n-3} + E x^{n-4} -$ &c. $\pm e x^4 \mp d x^3 \pm c x^2 \mp b x \pm A = 0$, all whoſe Roots are real, if each Term be multiply'd by the Index of x in that Term, and each Product be divided by x, the reſulting Æquation $n x^{n-1} - \overline{n-1} B x^{n-2} + \overline{n-2} C x^{n-3} - \overline{n-3} D x^{n-4} + \overline{n-4} E x^{n-5} -$ &c. $\pm 4 e x^3 \mp 3 d x^2 \pm 2 c x \mp b = 0$ ſhall have all its Roots real. Thus if all the Roots of the Æquation $x^4 - B x^3 + C x^2 - D x + A = 0$ be real, then all the Roots of the Æquation $4 x^3 - 3 B x^2 + 2 C x - D = 0$ will alſo be real. This Lemma doth not hold converſly, for there are an Infinity of Caſes where all the Roots of the Æquation $n x^{n-1} - \overline{n-1} B x^{n-2} + \overline{n-2} C x^{n-3} - \overline{n-3} D x^{n-4} +$ &c. $\pm 3 d x^2 \mp 2 c x \pm b = 0$ are real, at the ſame Time ſome or perhaps all the Roots of the Æquation $x^n - B x^{n-1} + C x^{n-2} - D x^{n-3} +$ &c. $\pm d x^3 \mp c x^2 \pm b x \mp A = 0$ are impoſſible: But whatever be the Number of impoſſible Roots in the Æquation $n x^{n-1} - \overline{n-1} B x^{n-2} + \overline{n-2} C x^{n-3} -$ &c. $\pm 2 c x \mp b = 0$, there are at leaſt as many in the Æquation $x^n - B x^{n-1} + C x^{n-2}$ &c. $\pm c x^2 \mp b x \pm A = 0$. Thus all the Roots of the Æquation $4 x^3 - 3 B x^2 + 2 C x - D = 0$ may be real, and yet two or perhaps all the four

four Roots of the Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$ may be impossible, but if two of the Roots of the Æquation $4x^3 - 3Bx^2 + 2Cx - D = 0$ be impossible, there must be at least two impossible Roots in the Æquation $x^4 - Bx^3 + Cx^2 - Dx + A = 0$. All this hath been demonstrated by Algebraical Writers, particularly by Mr. *Reyneau* in his *Analyse Demontré*, and is easily made evident by the Method of the *Maxima* and *Minima*.

COROLARY. Let all the Roots of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + Ex^{n-4} - Fx^{n-5} + \&c. \pm fx^5 \mp ex^4 \pm dx^3 \mp cx^2 \pm bx \mp A = 0$ be real, and by this Lemma all the Roots of the Æquation $nx^{n-1} - \overline{n-1}Bx^{n-2} + \overline{n-2}Cx^{n-3} - \overline{n-3}Dx^{n-4} + \overline{n-4}Ex^{n-5} - \overline{n-5}Fx^{n-6} + \&c. \pm 5fx^4 \mp 4ex^3 \pm 3dx^2 \mp 2cx \pm b = 0$ will be real, and therefore (by the same Lemma) all the Roots of the Æquation $n \times \overline{n-1}\,x^{n-2} - \overline{n-1} \times \overline{n-2}\,Bx^{n-3} + \overline{n-2} \times \overline{n-3}\,Cx^{n-4} - \overline{n-3} \times \overline{n-4}\,Dx^{n-5} + \overline{n-4} \times \overline{n-5}\,Ex^{n-6} - \overline{n-5} \times \overline{n-6}\,Fx^{n-7} + \&c. \pm 20fx^3 \mp 12ex^2 \pm 6dx \mp 2c = 0$

or (dividing all by 2) of $n \times \frac{\overline{n-1}}{2}x^{n-2} - \overline{n-1} \times \frac{\overline{n-2}}{2}Bx^{n-3} + \overline{n-2} \times \frac{\overline{n-3}}{2}Cx^{n-4} - \&c. \pm 10fx^3 \mp 6ex^2 \pm 3dx \mp c = 0$ will be real. After the same Manner all the Roots of the Æquation

$$n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3}x^{n-3} - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3}Bx^{n-4} +$$

$Bx^{n-4} + \overline{n-2} \times \frac{\overline{n-3}}{2} \times \frac{\overline{n-4}}{3} C x^{n-5} - \&c. \pm 10 f x^2 \mp 4ex \pm d = 0$ will be real; and thus we may descend until we arrive at the quadratick Æquation $n \times \frac{\overline{n-1}}{2} x^2 - \overline{n-1} Bx + C = 0$. The same Æquations do ascend thus $n \times \frac{\overline{n-1}}{2} x^2 - \overline{n-1} Bx + C = 0$, $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} x^3 - \overline{n-1} \times \frac{\overline{n-2}}{2} Bx^2 + \overline{n-2} Cx - D = 0$, $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \frac{\overline{n-3}}{4} x^4 - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3} Bx^3 + \overline{n-2} \times \frac{\overline{n-3}}{2} \times Cx^2 - \overline{n-3} Dx + E = 0$, $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \frac{\overline{n-3}}{4} \times \frac{\overline{n-4}}{5} x^5 - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3} \times \frac{\overline{n-4}}{4} Bx^4 + \overline{n-2} \times \frac{\overline{n-3}}{2} \times \frac{\overline{n-4}}{3} Cx^3 - \overline{n-3} \times \frac{\overline{n-4}}{2} Dx^2 + \overline{n-4} Ex - F = 0$, and so on. Let M represent any of the Coefficients of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - Dx^{n-3} + Ex^{n-4} - \&c. \pm A = 0$, and let L N be the adjacent Coefficients, let M be the Exponent of the Coefficient M: By the Exponent of a Coefficient I mean the Number which expresseth

expresseth the Place which it hath among the Coefficients, thus if M represent the Coefficient E (and therefore $L = D$ and $N = F$) then $m = 4$. It will be easy to see, that, amongst the foregoing ascending Æquations, that which hath its absolute Number N will be $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times$ &c. $\frac{\overline{n-m}}{m+1} x^{m+1} - \overline{n-1} \times \frac{\overline{n-2}}{2} \times$ &c. $\frac{\overline{n-m}}{m} B x^{m} + \overline{n-2} \times$ &c. $\frac{\overline{n-m}}{m-1} C x^{m-1} -$ &c. $\pm \overline{n-m+1} \times \frac{\overline{n-m}}{2} L x^2 \mp \overline{n-m} M x \pm N = 0$, all whose Roots are real when all the Roots of the Æquation $x^n - B x^{n-1} + C x^{n-2} -$ &c. $\pm A = 0$ are real. Let $N = F$ and therefore $M = E$, $L = D$ and $m = 4$, then that of the ascending Æquations whose absolute Number is F, will be $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \frac{\overline{n-3}}{4} \times \frac{\overline{n-4}}{5} x^5 - \overline{n-1} \times \frac{\overline{n-2}}{2} \times \frac{\overline{n-3}}{3} \times \frac{\overline{n-4}}{4} B x^4 + \overline{n-2} \times \frac{\overline{n-3}}{2} \times \frac{\overline{n-4}}{3} C x^3 - \overline{n-3} \times \frac{\overline{n-4}}{2} D x^2 + \overline{n-4} E x - F = 0$.

PROPOSITION I.

Let $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} + Ex^{n-4} -$ &c. $\pm ex^4 \mp dx^3 \pm cx^2 \mp bx \pm A = 0$ be an Æquation of any Dimenſions all whoſe Roots are real, let M be any Coefficient of this Æquation, L, N the adjacent Coefficients, and m the Exponent of M. Then the Square of any Coefficient M multiply'd by the Fraction $\frac{m \times \overline{n-m}}{\overline{m+1} \times \overline{n-m+1}}$ will always exceed the Rectangle under the adjacent Coefficients $L \times N$. Thus in the Æquation $x^4 - Bx^3 + Cx^2 - \mathcal{D}x + A = 0$, where $n = 4$, making $M = C$ and therefore $L = B$, $N = \mathcal{D}$, and $m = 2$, then $\frac{2 \times \overline{4-2}}{\overline{2+1} \times \overline{4-2+1}} \times C^2$ or $\frac{4}{9}C^2$ will exceed $B \times \mathcal{D}$ providing all the Roots of the Æquation be real.

Becauſe (by Lem. 3.) the Roots of the quadratick Æquation $n \times \frac{\overline{n-1}}{2} x^2 - \overline{n-1}\,Bx + C = 0$, are real, therefore (by Lem. 1.) $\frac{1}{4}\overline{n-1}|^2 \times B^2$ muſt be greater than $n \times \frac{\overline{n-1}}{2} \times C$ and (dividing both by $n \times \frac{\overline{n-1}}{2}$) $\frac{\overline{n-1}}{2n} \times B^2$ greater than $1 \times C$. Therefore in the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} +$ &c. $\pm A = 0$ of the n Degree, all whoſe Roots are real, the Square of B the Coefficient of the ſecond Term,

Term, multiply'd by the Fraction $\frac{n-1}{2n}$ is greater than $1 \times C$ the Rectangle under the adjacent Coefficients. But (by Lem. 2.) all the Roots of the Æquation $Ax^{n} - bx^{n-1} + cx^{n-2} - \&c. \pm Cx^{2} \mp Bx \pm 1 = 0$ or (dividing by A) of $x^{n} - \frac{b}{A}x^{n-1} + \frac{c}{A}x^{n-2} - \&c. \pm \frac{C}{A}x^{2} \mp \frac{B}{A}x \pm \frac{1}{A} = 0$ are real, therefore (from what hath been just now said) $\frac{\overline{n-1}}{2n} \times \frac{b^{2}}{A^{2}}$ must be greater than $1 \times \frac{c}{A}$ and consequently $\frac{\overline{n-1}}{2n} \times b^{2}$ greater than $c \times A$. Therefore in an Æquation $x^{n} - Bx^{n-1} + Cx^{n-2} - \&c. \pm cx^{2} \mp bx \pm A = 0$, of the n Degree, all whose Roots are real, the Square of the Coefficient of x multiply'd by the Fraction $\frac{n-1}{2n}$ is greater than the Rectangle under the Coefficient of x^{2} and the absolute Number. But by Cor. Lem. 3. all the Roots of the Æquation $n \times \frac{\overline{n-1}}{2} \times \frac{\overline{n-2}}{3} \times \&c. \times \frac{\overline{n-m}}{m+1} x^{m+1} - \overline{n-1} \times \frac{n-2}{2} \times \&c. \times \frac{\overline{n-m}}{m} Bx^{m} + \overline{n-2} \times \&c. \frac{\overline{n-m}}{m-1} Cx^{m-1} \&c. \pm \overline{n-m+1} \times \frac{\overline{n-m}}{2} \times Lx^{2} \mp$

$\overline{n-m}\, M x \pm N = o$ are real, therefore (seeing this Æquation is of the $m+1$ Degree) the Square of $\overline{n-m} \times M$ multiply'd by the Fraction $\frac{\overline{m+1}-1}{2 \times \overline{m+1}}$ will be greater than the Rectangle under $\overline{n-m+1} \times \frac{\overline{n-m}}{2} \times L$ and N, that is $\frac{m}{2 \times \overline{m+1}} \times \overline{n-m}|^2 \times M^2$ will be greater than $\overline{n-m+1} \times \frac{\overline{n-m}}{2} \times L \times N$ and therefore (dividing both by $\overline{n-m+1} \times \frac{\overline{n-m}}{2}$)

$$\frac{m \times \overline{n-m}}{\overline{m+1} \times \overline{n-m+1}} \times M^2 \text{ greater than } L \times N.$$

COROLARY. Make a Series of Fractions $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}$, &c. unto $\frac{1}{n}$ whose Denominators are Numbers going on in the Progression 1, 2, 3, 4, &c. unto the Number n which is the Dimensions of the Æquation $x^n - B x^{n-1} + C x^{n-2} -$ &c. $\pm A = o$, and whose Numerators are the same Progression inverted. Divide the second of these Fractions by the first, the third by the second, the fourth by the third, and so on, and place the Fractions which result from this Division above the middle Terms of the Æquation, thus $x^n - \overset{\frac{n-1}{2n}}{B} x^{n-1} + \overset{\frac{2 \times \overline{n-2}}{3 \times \overline{n-1}}}{C} x^{n-2} - D x^{n-3} +$

$\overset{\frac{3\times\overline{n-3}}{4\times\overline{n-2}}}{\mathcal{D}}x^{n-3}+\overset{\frac{4\times\overline{n-4}}{5\times\overline{n-3}}}{E}x^{n-4}-$ &c. $\pm A=0$. Then if all the Roots of the Æquation are real, the Square of any Coefficient multiply'd by the Fraction which ſtands above, will be greater than the Rectangle under the adjacent Coefficients. This Corolary doth not hold converſly, for there are an Infinity of Æquations in which the Square of each Coefficient multiply'd by the Fraction above it, may be greater than the Rectangle under the adjacent Coefficients, and notwithſtanding ſome or perhaps all of the Roots may be impoſſible. Therefore when the Square of a Coefficient multiply'd by the Fraction above, is greater than the Rectangle under the adjacent Coefficients, from this Circumſtance nothing can be determined as to the Poſſibility or Impoſſibility of the Roots of the Æquation: But when the Square of a Coefficient multiply'd by the Fraction above it, is leſs than the Rectangle under the adjacent Coefficients, it is a certain Indication of two impoſſible Roots. From what hath been ſaid, is immediately deduced the Demonſtration of that Rule which the moſt illuſtrious *Newton* gives for determining the Number of impoſſible Roots in any given Æquation.

SCHOLIUM.

Let the Roots of the Æquation $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} + Ex^{n-4} - Fx^{n-5} +$ &c. $\pm A = 0$ (with their Signs) be repreſented by the Letters $a, b, c, d, e, f, g,$ *&c.* then (as is commonly known) B will be the Sum of all the Roots or $= a + b + c + d + e + f +$ &c. C the Sum of the Products of all the Pairs of Roots or $= ab + ac + ad + af + ag +$ *&c.* *D* the Sum of the Products of all the Ternaryes of Roots or $= abc + abd + abe + abf + abg +$ *&c.* $E = abcd + abce + abcf + abeg +$ *&c.* $F = abcde + abcdf + abcdg + bcdef +$ *&c.* and so on. Let (as in this Proposition) *M* represent any of these Coefficients, *L*, *N* the adjacent Coefficients, and *m* the Exponent of *M*; let *Z* represent the Sum of the Squares of all the possible Differences between the Terms of the Coefficient *M*, let α be the Sum of all those of the foresaid Squares whose Terms differ by one Letter, β the Sum of all those Squares whose Terms differ by two Letters, γ the Sum of those Squares whose Terms differ by three Letters, δ the Sum of those Squares whose Terms differ by four Letters and so on. Thus if $M = F = abcde + abcdf + abcdg +$ *&c.* then $Z = \overline{abcde - abcdf}|^2 + \overline{abcde - abcdg}|^2 + \overline{abcde - abcfg}|^2 + \overline{bcdef - abfgh}|^2 +$ *&c.* $\alpha = \overline{abcde - abcdf}|^2 + \overline{abcde - abcdg}|^2 + \overline{abcde - abcdh}|^2 + \overline{bcdef - bcdeg}|^2 +$ *&c.* $\beta = \overline{abcde - abcfg}|^2 + \overline{abcde - abcfh}|^2 + \overline{bcdef - acdfh}|^2 +$ *&c.* $\gamma = \overline{abcde - abfgh}|^2 + \overline{abcdf - abegh}|^2 +$ *&c.* $\delta = \overline{abcde - afghk}|^2 + \overline{acdfg - abehk}|^2 +$ *&c.* This being laid down I say that the Square of any Coefficient *M* multiply'd by the Fraction $\frac{m \times \overline{n - m}}{\overline{m + 1} \times \overline{n - m + 1}}$ exceeds the Rectangle under the adjacent Coefficients $L \times N$ by

$\overline{n + 1} \times Z$

$\frac{\overline{n+1}\times Z}{\overline{m+1}\times\overline{n-m+1}} - \frac{1}{2}\alpha - \frac{1}{3}\beta - \frac{1}{4}\gamma - \frac{1}{5}\delta -$ &c. The Series $-\frac{1}{2}\alpha - \frac{1}{3}\beta - \frac{1}{4}\gamma -$ &c. muſt conſiſt of m Number of Terms.

Let the Æquation be $x^5 - Bx^4 + Cx^3 - Dx^2 + Ex - A = 0$, whoſe Roots let be a, b, c, d, e, in which Caſe $n = 5$. Let $M = B = a + b + c + d + e$, then $L = 1$, $N = G$, $m = 1$, $Z = \overline{a-b}|^2 + \overline{a-c}|^2 + \overline{a-d}|^2 + \overline{a-e}|^2 + \overline{b-c}|^2 +$ &c. $= \alpha$; therefore $\frac{1\times\overline{5-1}}{\overline{1+1}\times\overline{5-1+1}}\times B^2$ or $\frac{2}{5}B^2$ exceeds $1\times C$ by $\frac{\overline{5+1}\times Z}{\overline{1+1}\times\overline{5-1+1}} - \frac{1}{2}\alpha = \frac{3}{5}Z - \frac{1}{2}\alpha =$ (becauſe $Z = \alpha$) $\frac{1}{10}Z = \frac{1}{10}\overline{a-b}|^2 + \frac{1}{10}\times\overline{a-c}|^2 + \frac{1}{10}\overline{a-d}|^2 +$ &c. which is always a poſitive Number when the Roots a, b, c, d, e are real, poſitive or negative Numbers. Let $M = C = ab + ac + ad + ae + bc +$ &c. then $L = B$, $N = D$, $m = 2$, $Z = \overline{ab-ac}|^2 + \overline{ab-ad}|^2 + \overline{ab-cd}|^2 + \overline{ab-de}|^2 +$ &c. $\alpha = \overline{ab-ac}|^2 + \overline{ab-ad}|^2 + \overline{ab-ae}|^2 +$ &c. $\beta = \overline{ab-cd}|^2 + \overline{ab-ce}|^2 + \overline{ab-de}|^2 +$ &c. therefore $\frac{2\times\overline{5-2}}{\overline{2+1}\times\overline{5-2+1}}\times C^2$

or

or $\frac{1}{2}C^2$ ſurpaſſeth $B\times\mathcal{D}$ by $\frac{\overline{5+1}\times Z}{\overline{2+1}\times\overline{5-2+1}}$ $-\frac{1}{2}\alpha-\frac{1}{3}\beta=$ (becauſe $Z=\alpha+\beta$) $=\frac{1}{6}\beta=\frac{1}{6}\times\overline{ab-cd}|^2+\frac{1}{6}\overline{ab-ce}|^2+\frac{1}{6}\times\overline{ab-de}|^2+$ &c. which is always a poſitive Number when the Roots a, b, c, d, e are real Numbers, poſitive or negative. Let $M=\mathcal{D}=abc+abd+abe+acd+ace+$ &c. then $L=C$, $N=E$, $m=3$, $Z=\overline{abc-abd}|^2+\overline{abc-abe}|^2+\overline{abc-ade}|^2+$ &c. $\alpha=\overline{abc-abd}|^2+\overline{abc-abe}|^2+\overline{abc-acd}|^2+$ &c. $\beta=\overline{abc-ade}|^2+\overline{abc-cde}|^2+\overline{abc-bde}|^2+$ &c. $\gamma=o$, therefore $\frac{3\times\overline{5-3}}{\overline{3+1}\times\overline{5-3+1}}\times\mathcal{D}^2$ or $\frac{1}{2}\mathcal{D}^2$ exceeds $C\times E$ by $\frac{\overline{5+1}}{\overline{3+1}\times\overline{5-3+1}}\times Z-\frac{1}{2}\alpha-\frac{1}{3}\beta=$ (becauſe $Z=\alpha+\beta$) $=\frac{1}{6}\times\beta=\frac{1}{6}\times\overline{abc-ade}|^2+\frac{1}{6}\times\overline{abc-cde}|^2+\frac{1}{6}\overline{abc-bde}|^2+$ &c. which is a poſitive Number when the Roots are real Numbers. Let $M=E=abcd+abce+abde+bcde+$ &c. then $L=\mathcal{D}$, $N=A$, $m=4$, $Z=\overline{abcd-abce}|^2+\overline{abcd-bcde}|^2+\overline{abcd-acde}|^2+$ &c. $=\alpha$, $\beta=$

$\beta = o = \gamma = \delta$, therefore $\frac{4\times\overline{5-4}}{\overline{4+1}\times\overline{5-4+1}}\times E^2$ or $\frac{2}{5}E^2$ exceeds $\mathcal{D}\times A$ by $\frac{\overline{5+1}}{\overline{4+1}\times\overline{5-4+1}}\times Z - \frac{1}{2}\alpha = \frac{3}{5}Z - \frac{1}{2}\alpha = \frac{1}{10}Z = \frac{1}{10}\times\overline{abcd - abce}|^2 + \frac{1}{10}\times\overline{abcd - bcde}|^2 +$ &c. which is a positive Number when the Roots are real Numbers.

PROPOSITION II.

Let $x^n - Bx^{n-1} + Cx^{n-2} - \mathcal{D}x^{n-3} + Ex^{n-4} -$ &c. $\pm A = o$ be an Æquation of any Degree, whose Roots with their Signs let be expressed by the Letters $a, b, c, d, e, f,$ &c. let M represent any Coefficient of this Æquation, L, N the Coefficients adjacent to M; K, O the Coefficients adjacent to L, N; $I, \mathcal{P}$ those adjacent to K, O; H, Q those adjacent to $I, \mathcal{P}$, and so on. Let m represent the Exponent of M and let Z (as in the preceeding Proposition) represent the Sum of the Squares of all the possible Differences between the Terms of the Coefficient M. Then the Product of the Square of any Coefficient M multiply'd by the Fraction $\frac{1}{2}\times$

$$\overline{1 - \frac{1}{n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times \&c.\times\frac{\overline{n-m+1}}{m}}}$$

doth always

always exceed $L \times N - K \times O + I \times P - H \times Q +$ &c. by $\frac{\frac{1}{2} Z}{n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \&c. \times \frac{n-m+1}{m}}$ which is always a positive Number, when the Roots *a*, *b*, *c*, *d*, *e* &c. are real Numbers positive or negative. Let the Æquation be of the seventh Degree or $x^7 - Bx^6 + Cx^5 - Dx^4 + Ex^3 - Fx^2 + Gx - A = 0$, whose Roots let be *a*, *b*, *c*, *d*, *e*, *f*, *g*, in which Case $n = 7$. Let $M = E = abcd + abce + abcf + abcg + bcde +$ &c. then $m = 4$, $L = -D$, $N = -F$, $K = C$, $O = G$, $I = -B$, $P = -A$, $Z = \overline{abcd - abce}|^2 + \overline{abcd - abcf}|^2 + \overline{abcd - abcg}|^2 +$ &c. Therefore $\frac{1}{2} \times \overline{1 - \frac{1}{7 \times \frac{6}{2} \times \frac{5}{3} \times \frac{4}{4}}} \times E^2$ or $\frac{17}{35} E^2$ exceeds $D \times F - C \times G + B \times A$ by $\frac{\frac{1}{2} Z}{7 \times \frac{6}{2} \times \frac{5}{3} \times \frac{4}{4}}$ or $\frac{Z}{70} = \frac{1}{70} \times \overline{abcd - abce}|^2 + \frac{1}{70} \times \overline{abcd - abcf}|^2 +$ &c.

From this Proposition, is deduced the following Rule for determining the Number of impossible Roots in any given Æquation. From each of the Unciæ of the middle Terms of that Power of a Binomial, whose

whose Index is the Dimensions of the proposed Æquation, subtract Unity, then divide each Remainder by twice the Correspondent *Uncia*, and set the Fractions which result from this Division, above the middle Terms of the given Æquation. And under any of the middle Terms if its Square multiplyed by the Fraction standing above it, be greater than the Rectangle under the immediately adjacent Terms, ***Minus*** the Rectangle under the next adjacent Terms, ***Plus*** the Rectangle under the Terms then next adjacent — *&c.* place the Sign $+$, but if it be less, place the Sign $-$. And under the first and last Term place $+$. And there will be at least as many impossible Roots, as there are Changes in the Series of the under-written Signs from $+$ to $-$, or from $-$ to $+$. Let it be required to determine the Number of impossible Roots in the Æquation $x^7 - 5x^6 + 15x^5 - 23x^4 + 18x^3 + 10x^2 - 28x + 24 = 0$. The *Unciæ* of the middle Terms of the 7th Power of a Binomial are 7, 21, 35, 35, 21, 7, from which subtracting Unity, and dividing each of the Remainders by twice the correspondent *Uncia*, the Quotients will be $\frac{6}{14}$, $\frac{20}{42}$, $\frac{34}{70}$, $\frac{34}{70}$, $\frac{20}{42}$, $\frac{6}{14}$ or $\frac{3}{7}$, $\frac{10}{21}$, $\frac{17}{35}$, $\frac{17}{35}$, $\frac{10}{21}$, $\frac{3}{7}$ which Fractions place above the middle Terms of the Æquation, has

$$\begin{array}{ccccc} & & \frac{3}{7} & & \frac{10}{21} \\ x^7 & - & 5x^6 & + & 15x^5 & - \\ + & & - & & + & \end{array}$$

$$\overset{\frac{17}{35}}{\underset{-}{23x^4}} + \overset{\frac{17}{35}}{\underset{+}{18x^3}} + \overset{\frac{10}{21}}{\underset{-}{10x^2}} - \overset{\frac{3}{7}}{\underset{+}{28x}} + \underset{+}{24} = 0.$$ Then because the Square of $-5x^6$ multiply'd into the Fraction over its Head $\frac{3}{7}$, to wit $\frac{75}{7}x^{12}$ is less than $x^7 \times 15x^5$ or $15x^{12}$ I place the Sign $-$ under the Term $5x^6$. Because the Square of $15x^5$ multiply'd by the Fraction over its Head $\frac{10}{21}$ to wit $\frac{705}{7}x^{10}$ is greater than $\overline{-5x^6 \times -23x^4} - \overline{x^7 \times 18x^3} = 97x^{10}$ I place the Sign $+$ under the Term $15x^5$. Seeing $\frac{8993}{35}x^8$ (the Square of the Term $-23x^4$ multiply'd by the Fraction over its Head $\frac{17}{35}$) is less than $15x^5 \times 18x^3 - \overline{-5x^6 \times 10x^2} + \overline{x^7 \times -28x} = 292x^8$, I place the Sign $-$ under the Term $23x^4$. Because $\overline{18x^3}|^2 \times \frac{17}{35}$ or $\frac{5508}{35}x^6$ exceeds $\overline{-23x^4 \times 10x^2} - \overline{15x^5 \times -28x} + \overline{-5x^6 \times 24} = 70x^6$ I place the Sign $+$ under the Term $18x^3$. Since $\overline{10x^2}|^2 \times \frac{10}{21}$ or $\frac{1000}{21}x^4$ is less than $\overline{+18x^3 \times -28x} - \overline{-23x^4 \times 24} = 48x^4$ I place the Sign $-$ under the

the Term $10x^2$. Because $\overline{28x}|^2 \times \frac{3}{7}$ or $336x^2$ is greater than $10x^2 \times 24 = 240x^2$ under $28x$ I place $+$, then under the first and last Terms I place $+$; and the six Changes of under-written Signs shews that there are six impossible Roots.

If the impossible Roots were to be found by the *Newtonian* Rule, the Operation would stand thus:

$$\underset{+}{x^7} - \underset{-}{\overset{\frac{3}{7}}{5x^6}} + \underset{+}{\overset{\frac{5}{9}}{15x^5}} - \underset{+}{\overset{\frac{3}{4}}{23x^4}} + \underset{+}{\overset{\frac{3}{3}}{18x^3}} + \underset{+}{\overset{\frac{5}{9}}{10x^2}} - \underset{+}{\overset{\frac{3}{7}}{28x}} + \underset{+}{24} = 0,$$

by which Rule there are found only two impossible Roots, whereas there are six to wit $1 + \sqrt{-3}$, $1 - \sqrt{-3}$, $1 + \sqrt{-2}$, $1 - \sqrt{-2}$, $1 + \sqrt{-1}$, $1 + \sqrt{-1}$, the seventh Root being -1.

 III. A