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Vector bundles, forcing algebras and local cohomology

Lecture 7

Affineness and superheight

We have seen in the last lecture that the complement of an affine open subset must have pure codimension 1. We have also seen in Example 6.2 that the non-affineness can be established by looking at the behavior of the codimension when the situation is restricted to closed subschemes. The following definition and theorem is an algebraic version of this observation.

DEFINITION 7.1. Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal. The (noetherian) *superheight* is the supremum

$$\sup \left(\operatorname{ht}(IS) : \, S \text{ is a notherian } R-\text{algebra} \right).$$

THEOREM 7.2. *Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal and $U = D(I) \subseteq X = \operatorname{Spec}(R)$. Then the following are equivalent.*

(1) *U is an affine scheme.*
(2) *I has superheight ≤ 1 and $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra.*

It is not true at all that the ring of global sections of an open subset U of the spectrum X of a noetherian ring is of finite type over this ring. This is not even true if X is an affine variety. This problem is directly related to Hilbert's fourteenth problem, which has a negative answer. We will later present examples where U has superheight one, yet is not affine, hence its ring of global sections is not finitely generated.

If R is a two-dimensional local ring with parameters f, g and if B is the forcing algebra for some $\mathfrak{m}$-primary ideal, then the ring of global sections of the torsor is just

$$\Gamma(D(\mathfrak{m}B), \mathcal{O}_B) \, = \, B_f \cap B_g.$$

Plus closure

For an ideal $I \subseteq R$ in a domain R define

$$I^+ = \{ f \in R : \text{ there exists a finite domain extension } R \subseteq T \text{ such that } f \in IT \}$$

Equivalent: let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered by Artin). Then

$$f \in I^+ \text{ if and only if } f \in IR^+ .$$

The plus closure commutes with localization.

We also have the inclusion $I^+ \subseteq I^*$. Here the question arises:

Question: Is $I^+ = I^*$?

This question is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure means that there exists a d-dimensional closed subscheme inside the torsor which meets the exceptional fiber (the fiber over the maximal ideal) in one point, and this means that the superheight of the extended ideal is d. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension. So also the plus closure depends only on the torsor.

In characteristic zero, the plus closure behaves very differently compared with positive characteristic. If R is a normal domain of characteristic 0, then the trace map shows that the plus closure is trivial, $I^+ = I$ for every ideal I.

Examples

In the following two examples we use results from tight closure theory to establish (non)-affineness properties of certain torsors.

Example 7.3. Let K be a field and consider the Fermat ring

$$R = K[X,Y,Z]/(X^d + Y^d + Z^d)$$

together with the ideal $I = (X,Y)$ and $f = Z^2$. For $d \geq 3$ we have $Z^2 \notin (X,Y)$. This element is however in the tight closure $(X,Y)^*$ of the ideal in positive characteristic (assume that the characteristic p does not divide d) and is therefore also in characteristic 0 inside the tight closure and inside the solid closure. Hence the open subset

$$D(X,Y) \subseteq \operatorname{Spec}\,(K[X,Y,Z,S,T]/(X^d + Y^d + Z^d, SX + TY - Z^2))$$

is not an affine scheme. In positive characteristic, Z^2 is also contained in the plus closure $(X,Y)^+$ and therefore this open subset contains punctured surfaces (the spectrum of the forcing algebra contains two-dimensional closed subschemes which meet the exceptional fiber $V(X,Y)$ in only one point; the ideal (X,Y) has superheight 2 in the forcing algebra). In characteristic zero however, the superheight is one because plus closure is trivial for normal domains in characteristic 0, and therefore by Theorem 7.2 the algebra $\Gamma(D(X,Y), \mathcal{O}_B)$ is not finitely generated. For $K = \mathbb{C}$ and $d = 3$ one can also show that $D(X,Y)_{\mathbb{C}}$ is, considered as a complex space, a Stein space.

EXAMPLE 7.4. Let K be a field of positive characteristic $p \geq 7$ and consider the ring

$$R = K[X, Y, Z]/(X^5 + Y^3 + Z^2)$$

together with the ideal $I = (X, Y)$ and $f = Z$. Since R has a rational singularity, it is F-regular, i.e. all ideals are tightly closed. Therefore $Z \notin (X, Y)^*$ and so the torsor

$$D(X, Y) \subseteq \operatorname{Spec}\left(K[X, Y, Z, S, T]/(X^5 + Y^3 + Z^2, SX + TY - Z)\right)$$

is an affine scheme. In characteristic zero this can be proved by either using that R is a quotient singularity or by using the natural grading ($\deg(X) = 6$, $\deg(Y) = 10$, $\deg(Z) = 15$) where the corresponding cohomology class $\frac{Z}{XY}$ gets degree -1 and then applying the geometric criteria on the corresponding projective curve (rather the corresponding curve of the standard-homogenization $U^{30} + V^{30} + W^{30} = 0$).