Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 18

Warm-up-exercises

Exercise 18.1. Prove the following properties of the hyperbolic sine and the hyperbolic cosine

(1)
$$\cosh x + \sinh x = e^x .$$
(2)
$$\cosh x - \sinh x = e^{-x} .$$
(3)
$$(\cosh x)^2 - (\sinh x)^2 = 1 .$$

Exercise 18.2. Show that the hyperbolic sine on $\mathbb{R}$ is strictly increasing.

Exercise 18.3. Prove that the hyperbolic tangent satisfies the following estimate
$$-1 \leq \tanh x \leq 1 \text{ for all } x \in \mathbb{R} .$$

Exercise 18.4. Prove by elementary geometric considerations the Sine theorem, i.e. the statement that in a triangle the equalities
$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$$
hold, where a, b, c are the side lengths of the edges and α, β, γ are respectively the opposite angles.

Exercise 18.5. Compute the determinants of plane and spatial rotations.

Exercise 18.6. Prove the addition theorems for sine and cosine using the rotation matrices.

Exercise 18.7. We look at a clock with minute and second hands, both moving continuously. Determine a formula which calculates the angular position of the second hand from the angular position of the minute hand (each starting from the 12-clock-position measured in the clockwise direction).

Exercise 18.8. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$$

converges.

Exercise 18.9. Determine the coefficients up to z^6 in the series product $\sum_{n=0}^{\infty} c_n z^n$ of the sine series and the cosine series.

The next tasks require the definition of a *periodic function*.

A function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

is called *periodic* with *period* $L > 0$, if for all $x \in \mathbb{R}$ the equality

$$f(x) = f(x + L)$$

holds.

Exercise 18.10. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a periodic function and

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

any function.

a) Prove that the composite function $g \circ f$ is also periodic.

b) Prove that the composite function $f \circ g$ does not need to be periodic.

Exercise 18.11. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a continuous periodic function. Prove that f is bounded.

Hand-in-exercises

Exercise 18.12. (3 points)

Prove that in the power series $\sum_{n=0}^{\infty} c_n x^n$ of the hyperbolic cosine the coefficients c_n are 0 if n is odd.

Exercise 18.13. (6 points)

Prove that the hyperbolic cosine is strictly decreasing on $\mathbb{R}_{\leq 0}$ and strictly increasing on $\mathbb{R}_{\geq 0}$.

EXERCISE 18.14. (4 points)

Let

$$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

be the space rotation by 45 degree around the z-axis counterclockwise. How does the matrix describing φ with respect to the basis

$$\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 7 \end{pmatrix}$$

look like?

EXERCISE 18.15. (5 points)

Prove the addition theorem

$$\sin(x + y) = \sin x \cdot \cos y + \cos x \cdot \sin y$$

for the sine using the defining power series.

EXERCISE 18.16. (4 points)

Let

$$f_1, f_2 : \mathbb{R} \longrightarrow \mathbb{R}$$

be periodic functions with periods respectively L_1 and L_2. The quotient L_1/L_2 is a rational number. Prove that $f_1 + f_2$ is also a periodic function.

EXERCISE 18.17. (5 points)

Consider n complex numbers $z_1, z_2, \ldots, z_n$ lying in the disc B with center $(0,0)$ and radius 1, that is in $B = \{z \in \mathbb{C} \mid |z| \leq 1\}$. Prove that there exists a point $w \in B$ such that

$$\sum_{i=1}^{n} |z_i - w| \geq n \,.$$