Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 3

Warm-up-exercises

EXERCISE 3.1. Show that the binomial coefficients satisfy the following recursive relation

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

EXERCISE 3.2. Show that the binomial coefficients are natural numbers.

EXERCISE 3.3. Prove the formula

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

EXERCISE 3.4. Prove by induction for $n \geq 10$ the inequality

$$3^n \geq n^4.$$

In the following computing tasks regarding complex numbers, the result must always be in the form $a + bi$, with real numbers a, b, and these should be as simple as possible.

EXERCISE 3.5. Calculate the following expressions in the complex numbers.

(1) $(5 + 4i)(3 - 2i)$.
(2) $(2 + 3i)(2 - 4i) + 3(1 - i)$.
(3) $(2i + 3)^2$.
(4) i^{1011}.
(5) $(-2 + 5i)^{-1}$.
(6) $\frac{4-3i}{2+i}$.

EXERCISE 3.6. Show that the complex numbers constitute a field.

EXERCISE 3.7. Prove the following statements concerning the real and imaginary parts of a complex number.

(1) $z = \mathrm{Re}\,(z) + \mathrm{Im}\,(z)i$.
(2) $\mathrm{Re}\,(z+w) = \mathrm{Re}\,(z) + \mathrm{Re}\,(w)$.
(3) $\mathrm{Im}\,(z+w) = \mathrm{Im}\,(z) + \mathrm{Im}\,(w)$.
(4) For $r \in \mathbb{R}$ we have

$$\mathrm{Re}\,(rz) = r\,\mathrm{Re}\,(z) \text{ and } \mathrm{Im}\,(rz) = r\,\mathrm{Im}\,(z)\,.$$

(5) $z = \mathrm{Re}\,(z)$ if and only if $z \in \mathbb{R}$, and this is exactly the case when $\mathrm{Im}\,(z) = 0$.

EXERCISE 3.8. Prove the following calculating rules for the complex numbers.

(1) $|z| = \sqrt{z\,\overline{z}}$.
(2) $\mathrm{Re}\,(z) = \frac{z+\overline{z}}{2}$.
(3) $\mathrm{Im}\,(z) = \frac{z-\overline{z}}{2i}$.
(4) $\overline{z} = \mathrm{Re}\,(z) - i\,\mathrm{Im}\,(z)$.
(5) For $z \neq 0$ we have $z^{-1} = \frac{\overline{z}}{|z|^2}$.

EXERCISE 3.9. Prove the following properties of the absolute value of a complex number.

(1) For a real number z its real absolute value and its complex absolute value coincide.
(2) We have $|z| = 0$ if and only if $z = 0$.
(3) $|z| = |\overline{z}|$.
(4) $|zw| = |z||w|$.
(5) $\mathrm{Re}\,(z), \mathrm{Im}\,(z) \leq |z|$.
(6) For $z \neq 0$ we have $|1/z| = 1/|z|$.

EXERCISE 3.10. Check the formula we gave in Example 3.15 for the square root of a complex number $z = a + bi$, in the case $b < 0$.

EXERCISE 3.11. Determine the two complex solutions of the equation

$$z^2 + 5iz - 3 = 0\,.$$

Hand-in-exercises

EXERCISE 3.12. (3 points)

Prove the following formula

$$n2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}\,.$$

Exercise 3.13. (3 points)

Calculate the complex numbers

$$(1+i)^n$$

for $n = 1, 2, 3, 4, 5$.

Exercise 3.14. (3 points)

Prove the following properties of the complex conjugation.

(1) $\overline{z+w} = \overline{z} + \overline{w}$.
(2) $\overline{-z} = -\overline{z}$.
(3) $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$.
(4) For $z \neq 0$ we have $\overline{1/z} = 1/\overline{z}$.
(5) $\overline{\overline{z}} = z$.
(6) $\overline{z} = z$ if and only if $z \in \mathbb{R}$.

Exercise 3.15. (2 points)

Let $a, b, c \in \mathbb{C}$ with $a \neq 0$. Show that the equation

$$az^2 + bz + c = 0$$

has at least one complex solution z.

Exercise 3.16. (3 points)

Calculate the square root, the fourth root and the eighth root of i.

Exercise 3.17. (4 points)

Find the three complex numbers z such that

$$z^3 = 1\,.$$