Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 19

Warm-up-exercises

The following task should be solved both directly and through the derivation rules.

EXERCISE 19.1. Determine the derivative of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = x^n,$$

for all $n \in \mathbb{N}$.

EXERCISE 19.2. Determine the derivative of the function

$$f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = x^n$$

for all $n \in \mathbb{Z}$.

EXERCISE 19.3. Determine the derivative of the function

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R},\ x \longmapsto f(x) = x^{\frac{1}{n}},$$

for all $n \in \mathbb{N}_+$.

EXERCISE 19.4. Determine directly (without the use of derivation rules) the derivative of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = x^3 + 2x^2 - 5x + 3,$$

at any point $a \in \mathbb{R}$.

EXERCISE 19.5. Prove that the real absolute value

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto |x|,$$

is not differentiable at the point zero.

EXERCISE 19.6. Determine the derivative of the function

$$f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \frac{x^2 + 1}{x^3}.$$

EXERCISE 19.7. Prove that the derivative of a rational function is also a rational function.

EXERCISE 19.8. Consider $f(x) = x^3+4x^2-1$ and $g(y) = y^2-y+2$. Determine the derivative of the composite function $h(x) = g(f(x))$ directly and by the chain rule (Theorem 19.8).

EXERCISE 19.9. Prove that a polynomial $P \in \mathbb{R}[X]$ has degree $\leq d$ (or it is $P = 0$), if and only if the $(d+1)$-th derivative of P is the zero poynomial.

EXERCISE 19.10. Let

$$f, g : \mathbb{R} \longrightarrow \mathbb{R}$$

be two differentiable functions and consider

$$h(x) = (g(f(x)))^2 f(g(x)) .$$

a) Determine the derivative h' from the derivatives of f and g.

b) Let now

$$f(x) = x^2 - 1 \text{ and } g(x) = x + 2 .$$

Compute $h'(x)$ in two ways, one directly from $h(x)$ and the other by the formula of part a).

For the "linear approximation" of differentiable maps we need the definition of affine-linear maps.

Let K be a field and let V and W be vector spaces over K. A map

$$\alpha : V \longrightarrow W,\ v \longmapsto \alpha(v) = \varphi(v) + w,$$

where φ is a linear map and $w \in W$ is a vector, is called *affine-linear*.

EXERCISE 19.11. Let K be a field and let V be a K-vector space. Prove that given two vectors $u, v \in W$ there exists exactly one affine-linear map

$$\alpha : K \longrightarrow W$$

sucht that $\alpha(0) = u$ and $\alpha(1) = v$.

EXERCISE 19.12. Determine the affine-linear map

$$\alpha : \mathbb{R} \longrightarrow \mathbb{R}^3$$

such that $\alpha(0) = (2, 3, 4)$ and $\alpha(1) = (5, -2, -1)$.

Hand-in-exercises

Exercise 19.13. (3 points)

Determine the derivative of the function

$$f: D \longrightarrow \mathbb{R},\, x \longmapsto f(x) = \frac{x^2+x-1}{x^3-x+2},$$

where D is the set where the denominator does not vanish.

Exercise 19.14. (4 points)

Determine the tangents to the graph of the function $f(x) = x^3 - x^2 - x + 1$, which are parallel to $y = x$.

Exercise 19.15. (4 points)

Let $f(x) = \frac{x^2+5x-2}{x+1}$ and $g(y) = \frac{y-2}{y^2+3}$. Determine the derivative of the composite $h(x) = g(f(x))$ directly and by the chain rule (Theorem 19.8).

Exercise 19.16. (2 points)

Determine the affine-linear map

$$\alpha : \mathbb{R} \longrightarrow \mathbb{R},$$

whose graph passes through the two points $(-2, 3)$ and $(5, -7)$.

Exercise 19.17. (3 points)

Let $D \subseteq \mathbb{R}$ be a subset and let

$$f_i : D \longrightarrow \mathbb{R},\, i = 1, \ldots, n,$$

be differentiable functions. Prove the formula

$$(f_1 \cdots f_n)' = \sum_{i=1}^{n} f_1 \cdots f_{i-1} f_i' f_{i+1} \cdots f_n \,.$$