Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 6

Warm-up-exercises

Exercise 6.1. Compute the following product of matrices

$$\begin{pmatrix} Z & E & I & L & E \\ R & E & I & H & E \\ H & O & R & I & Z \\ O & N & T & A & L \end{pmatrix} \cdot \begin{pmatrix} S & E & I \\ P & V & K \\ A & E & A \\ L & R & A \\ T & T & L \end{pmatrix} .$$

Exercise 6.2. Compute over the complex numbers the following product of matrices

$$\begin{pmatrix} 2-i & -1-3i & -1 \\ i & 0 & 4-2i \end{pmatrix} \begin{pmatrix} 1+i \\ 1-i \\ 2+5i \end{pmatrix} .$$

Exercise 6.3. Determine the product of matrices

$$e_i \circ e_j ,$$

where the i-th standard vector (of length n) is considered as a row vector and the j-th standard vector (also of length n) is considered as a column vector.

Exercise 6.4. Let M be a $m \times n$- matrix. Show that the product of matrices Me_j, with the j-th standard vector (regarded as column vector) is the j-th column of M. What is e_iM, where e_i is the i-th standard vector (regarded as a row vector)?

Exercise 6.5. Compute the product of matrices

$$\begin{pmatrix} 2+i & 1-\frac{1}{2}i & 4i \\ -5+7i & \sqrt{2}+i & 0 \end{pmatrix} \begin{pmatrix} -5+4i & 3-2i \\ \sqrt{2}-i & e+\pi i \\ 1 & -i \end{pmatrix} \begin{pmatrix} 1+i \\ 2-3i \end{pmatrix}$$

according to the two possible parantheses.

For the following statements we will soon give a simple proof using the relationship between matrices and linear maps.

EXERCISE 6.6. Show that the multiplication of matrices is associative. More precisely: Let K be a field and let A be an $m \times n$-matrix, B an $n \times p$-matrix and C a $p \times r$-matrix over K. Show that $(AB)C = A(BC)$.

For a matrix M we denote by M^n the n-th product of M with iself. This is also called the n-th *power* of the matrix.

EXERCISE 6.7. Compute for the matrix

$$M = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

the powers

$$M^i,\ i = 1, \dots, 4.$$

EXERCISE 6.8. Let K be a field and let V and W be two vector spaces over K. Show that the product

$$V \times W$$

is also a K-vector space.

EXERCISE 6.9. Let K be a field and I an index set. Show that

$$K^I := \operatorname{Maps}(I, K)$$

with pointwise addition and scalar multiplication is a K-vector space.

EXERCISE 6.10. Let K be a field and let

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & 0 \end{array}$$

be a system of linear equations over K. Show that the set of all solutions of the system is a subspace of K^n. How is this solution space related to the solution spaces of the individual equations?

EXERCISE 6.11. Show that the addition and the scalar multiplication of a vector space V can be restricted to a subspace and that this subspace with the inherited structures of V is a vector space itself.

EXERCISE 6.12. Let K be a field and let V be a K-vector space. Let $U, W \subseteq V$ be two subspaces of V. Prove that the union $U \cup W$ is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.

Hand-in-exercises

Exercise 6.13. (3 points)

Compute over the complex numbers the following product of matrices

$$\begin{pmatrix} 3-2i & 1+5i & 0 \\ 7i & 2+i & 4-i \end{pmatrix} \begin{pmatrix} 1-2i & -i \\ 3-4i & 2+3i \\ 5-7i & 2-i \end{pmatrix} .$$

Exercise 6.14. (4 points)

We consider the matrix

$$M = \begin{pmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

over a field K. Show that the fourth power of M is 0, that is

$$M^4 = MMMM = 0 .$$

Exercise 6.15. (3 points)

Let K be a field and let V be a K-vector space. Show that the following properties hold (where $\lambda \in K$ and $v \in V$).

(1) We have $0v = 0$.
(2) We have $\lambda 0 = 0$.
(3) We have$(-1)v = -v$.
(4) If $\lambda \neq 0$ and $v \neq 0$ then $\lambda v \neq 0$.

Exercise 6.16. (3 points)

Give an example of a vector space V and of three subsets of V which satisfy two of the subspace axioms, but not the third.