Prof. Dr. H. Brenner **Osnabrück SS 2009**

Einführung in die Algebra

Vorlesung 5

Gruppenhomomorphismen

Definition 1. Seien $(G, \circ, e_G)$ und $(H, \circ, e_H)$ Gruppen. Eine Abbildung

$$\psi : G \longrightarrow H$$

heißt *Gruppenhomomorphismus*, wenn folgende Eigenschaften gelten.

(1) $\psi(e_G) = e_H$.
(2) $\psi(g \circ g') = \psi(g) \circ \psi(g')$ für alle $g, g' \in G$.

Die Menge der Gruppenhomomorphismen von G nach H wird mit

$$\mathrm{Hom}\,(G, H)$$

bezeichnet. Aus der linearen Algebra sind vermutlich die linearen Abbildungen zwischen Vektorräume bekannt, welche insbesondere Gruppenhomomorphismen sind, darüber hinaus aber auch noch mit der skalaren Multiplikation verträglich sind. Die folgenden beiden Lemmata folgen direkt aus der Definition.

Lemma 1. *Es seien G und H Gruppen und $\varphi : G \to H$ sei ein Gruppenhomomorphismus. Dann ist $(\varphi(g))^{-1} = \varphi(g^{-1})$ für jedes $g \in G$.*

Beweis. Es ist

$$\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(e_G) = e_H \, .$$

Das heißt, dass $\varphi(g^{-1})$ die Eigenschaft besitzt, die für das Inverse von $\varphi(g)$ charakteristisch ist. Da das Inverse in einer Gruppe eindeutig bestimmt ist, muss $\varphi(g^{-1}) = (\varphi(g))^{-1}$ gelten. □

Lemma 2. *Es seien F, G, H Gruppen. Dann gelten folgende Eigenschaften.*

(1) *Die Identität* $\mathrm{id} : G \to G$ *ist ein Gruppenhomomorphismus.*
(2) *Sind $\varphi : F \to G$ und $\psi : G \to H$ Gruppenhomomorphismen, so ist auch die Hintereinanderschaltung $\psi \circ \varphi : F \to H$ ein Gruppenhomomorphismus.*
(3) *Ist $F \subseteq G$ eine Untergruppe, so ist die Inklusion $F \hookrightarrow G$ ein Gruppenhomomorphismus.*
(4) *Sei $\{e\}$ die triviale Gruppe. Dann ist die Abbildung $\{e\} \to G$, die e auf e_G schickt, ein Gruppenhomomorphismus. Ebenso ist die (konstante) Abbildung $G \to \{e\}$ ein Gruppenhomomorphismus.*

Beweis. Das ist trivial. □

Beispiel 3. Betrachte die additve Gruppe der reellen Zahlen, also $(\mathbb{R}, 0, +)$, und die multiplikative Gruppe der positiven reellen Zahlen, also $(\mathbb{R}_+, 1, \cdot)$. Dann ist die Exponentialabbildung

$$\exp : \mathbb{R} \longrightarrow \mathbb{R}_+, \, x \longmapsto \exp(x) \, ,$$

ein Gruppenisomorphismus . Dies beruht auf grundlegenden analytischen Eigenschaften der Exponentialfunktion. Die Homomorphieeigenschaft ist lediglich eine Umformulierung des Exponentialgesetzes

$$\exp(x+y) = e^{x+y} = e^x e^y = \exp(x) \exp(y) \, .$$

Die Injektivität der Abbildung folgt aus der strengen Monotonie, die Surjektivität folgt aus dem Zwischenwertsatz. Die Umkehrabbildung ist der natürliche Logarithmus, der somit ebenfalls ein Gruppenisomorphismus ist.

Lemma 4. *Sei G eine Gruppe. Dann entsprechen sich eindeutig Gruppenelemente $g \in G$ und Gruppenhomomorphismen φ von $\mathbb{Z}$ nach G über die Korrespondenz*

$$g \longmapsto (n \mapsto g^n) \text{ und } \varphi \longmapsto \varphi(1) \, .$$

Beweis. Sei $g \in G$ fixiert. Dass die Abbildung

$$\varphi_g : \mathbb{Z} \longrightarrow G, \, n \longmapsto g^n \, ,$$

ein Gruppenhomomorphismus ist, ist eine Umformulierung der Potenzgesetze (Lemma 2.2). Wegen $\varphi_g(1) = g^1 = g$ erhält man aus der Potenzabbildung das Gruppenelement zurück. Umgekehrt ist ein Gruppenhomomorphismus $\varphi : \mathbb{Z} \to G$ durch $\varphi(1)$ eindeutig festgelegt, da $\varphi(n) = (\varphi(1))^n$ für n positiv und $\varphi(n) = ((\varphi(1))^{-1})^{-n}$ für n negativ gelten muss. □

Man kann den Inhalt dieses Lemmas auch kurz durch $G \cong \mathrm{Hom}(\mathbb{Z}, G)$ ausdrücken. Die Gruppenhomomorphismen von einer Gruppe G nach $\mathbb{Z}$ sind schwieriger zu charakterisieren. Die Gruppenhomomorphismen von $\mathbb{Z}$ nach $\mathbb{Z}$ sind die Multiplikationen mit einer festen ganzen Zahl a, also

$$\mathbb{Z} \longrightarrow \mathbb{Z}, \, x \longmapsto ax \, .$$

Gruppenisomorphismen

Definition 2. Seien G und H Gruppen. Einen bijektiven Gruppenhomomorphismus

$$\varphi : G \longrightarrow H$$

nennt man einen *Isomorphismus* (oder eine *Isomorphie*). Die beiden Gruppen heißen *isomorph*, wenn es einen Isomorphismus zwischen ihnen gibt.

Lemma 5. *Seien G und H Gruppen und sei*

$$\varphi : G \longrightarrow H$$

ein Gruppenisomorphismus (also ein bijektiver Gruppenhomomorphismus). Dann ist auch die Umkehrabbildung

$$\varphi^{-1} : H \longrightarrow G,\ h \longmapsto \varphi^{-1}(h)\,,$$

ein Gruppenisomorphismus.

Beweis. Dies folgt aus $\varphi^{-1}(e_H) = e_G$ und aus

$$\begin{aligned}\varphi^{-1}(h_1h_2) = \varphi^{-1}\left(\varphi(\varphi^{-1}(h_1))\varphi(\varphi^{-1}(h_2))\right) &= \varphi^{-1}\left(\varphi(\varphi^{-1}(h_1)\varphi^{-1}(h_2))\right)\\ &= \varphi^{-1}(h_1)\varphi^{-1}(h_2).\end{aligned}$$

□

Isomorphe Gruppen sind bezüglich ihrer gruppentheoretischen Eigenschaften als gleich anzusehen. Isomorphismen einer Gruppe auf sich selbst nennt man auch *Automorphismen*. Wichtige Beispiele für Automorphismen sind die sogenannten inneren Automorphismen.

Definition 3. Sei G eine Gruppe und $g \in G$. Die durch g definierte Abbildung

$$\kappa_g : G \longrightarrow G,\ x \longmapsto gxg^{-1}\,,$$

heißt *innerer Automorphismus*.

Lemma 6. *Ein innerer Automorphismus ist in der Tat ein Automorphismus. Die Zuordnung*

$$G \longrightarrow \operatorname{Aut} G,\ g \longmapsto \kappa_g\,,$$

ist ein Gruppenhomomorphismus.

Beweis. Es ist

$$\kappa_g(e_G) = ge_Gg^{-1} = gg^{-1} = e_G$$

und

$$\kappa_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = \kappa_g(x)\kappa_g(x)\,,$$

so dass ein Gruppenhomomorphismus vorliegt. Wegen

$$\kappa_g(\kappa_h(x)) = \kappa_g(hxh^{-1}) = ghxh^{-1}g^{-1} = ghx(gh)^{-1} = \kappa_{gh}$$

ist einerseits

$$\kappa_{g^{-1}} \circ \kappa_g = \kappa_{g^{-1}g} = \mathrm{id}_G\,,$$

so dass κ_g bijektiv, also ein Automorphismus, ist. Andererseits ist deshalb (und wegen $\kappa_e = \mathrm{id}$) die Gesamtabbildung κ ein Gruppenhomomorphismus.

□

Wenn G eine kommutative Gruppe ist, so ist wegen $gxg^{-1} = xgg^{-1} = x$ die Identität der einzige innere Automorphismus. Der Begriff ist also nur bei nicht kommutativen Gruppen von Interesse.

Der Kern eines Gruppenhomomorphismus

Definition 4. Seien G und H Gruppen und sei $\varphi : G \to H$ ein Gruppenhomomorphismus. Dann nennt man das Urbild des neutralen Elementes den *Kern* von φ, geschrieben

$$\ker \varphi = \varphi^{-1}(e_H) = \{g \in G : \varphi(g) = e_H\} .$$

Lemma 7. *Seien G und H Gruppen und sei $\varphi : G \to H$ ein Gruppenhomomorphismus. Dann ist der Kern von φ eine Untergruppe von G.*

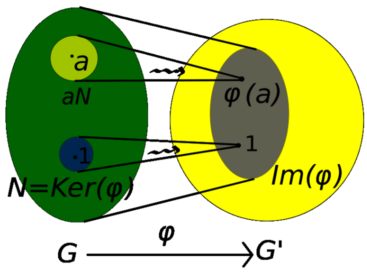
Beweis. Wegen $\varphi(e_G) = e_H$ ist $e_G \in \ker \varphi$. Seien $g, g' \in \ker \varphi$. Dann ist

$$\varphi(gg') = \varphi(g)\varphi(g') = e_H e_H = e_H$$

und daher ist auch $gg' \in \ker \varphi$. Der Kern ist also ein Untermonoid. Sei nun $g \in \ker \varphi$ und betrachte das inverse Element g^{-1}. Es ist

$$\varphi(g^{-1}) = (\varphi(g))^{-1} = e_H^{-1} = e_H ,$$

also auch $g^{-1} \in \ker \varphi$. □



Lemma 8. *Seien G und H Gruppen. Ein Gruppenhomomorphismus $\varphi : G \to H$ ist genau dann injektiv, wenn der Kern von φ trivial ist.*

Beweis. Wenn φ injektiv ist, so darf auf jedes Element $h \in H$ höchstens ein Element aus G gehen. Da e_G auf e_H geschickt wird, darf kein weiteres Element auf e_H gehen, d.h. $\ker \varphi = \{e_G\}$. Sei umgekehrt dies der Fall und sei angenommen, dass $g, \tilde{g} \in G$ beide auf $h \in H$ geschickt werden. Dann ist

$$\varphi(g\tilde{g}^{-1}) = \varphi(g)\varphi(\tilde{g})^{-1} = hh^{-1} = e_H$$

und damit ist $g\tilde{g}^{-1} \in \ker \varphi$, also $g\tilde{g}^{-1} = e_G$ nach Voraussetzung und damit $g = \tilde{g}$. □

Das Bild eines Gruppenhomomorphismus

Lemma 9. *Seien G und H Gruppen und sei $\varphi : G \to H$ ein Gruppenhomomorphismus. Dann ist das Bild von φ eine Untergruppe von H.*

Beweis. Sei $B := \text{bild}\,\varphi$. Dann ist $e_H = \varphi(e_G) \in B$. Seien $h_1, h_2 \in B$. Dann gibt es $g_1, g_2 \in G$ mit $\varphi(g_1) = h_1$ und $\varphi(g_2) = h_2$. Damit ist $h_1 + h_2 = \varphi(g_1) + \varphi(g_2) = \varphi(g_1 + g_2) \in B$. Ebenso gibt es für $h \in B$ ein $g \in G$ mit $\varphi(g) = h$. Somit ist $h^{-1} = (\varphi(g))^{-1} = \varphi(g^{-1}) \in B$. □

Beispiel 10. Betrachte die analytische Abbildung

$$\mathbb{R} \longrightarrow \mathbb{C},\ t \longmapsto e^{it} = \cos t + i \sin t\,.$$

Aufgrund des Exponentialgesetzes ist $e^{i(t+s)} = e^{it}e^{is}$ und $e^{i0} = e^0 = 1$. Daher liegt ein Gruppenhomomorphismus von der additiven Gruppe $(\mathbb{R}, +, 0)$ in die multiplikative Gruppe $(\mathbb{C}, \cdot, 1)$ vor. Wir bestimmen den Kern und das Bild dieser Abbildung. Für den Kern muss man diejenigen reellen Zahlen t bestimmen, für die

$$\cos t = 1 \text{ und } \sin t = 0$$

ist. Aufgrund der Periodizität der trigonometrischen Funktionen ist dies genau dann der Fall, wenn t ein Vielfaches von 2π ist. Der Kern ist also die Untergruppe $2\pi\mathbb{Z}$. Für einen Bildpunkt gilt $|\, e^{it} \,| = \sin^2 t + \cos^2 t = 1$, so dass der Bildpunkt auf dem komplexen Einheitskreis liegt. Andererseits durchlaufen die trigonometrischen Funktionen den gesamten Einheitskreis, so dass die Bildgruppe der Einheitskreis mit der komplexen Multiplikation ist.

Abbildungsverzeichnis

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