Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 17

Warm-up-exercises

Exercise 17.1. Compute the first five terms of the Cauchy product of the two convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^3}.$$

Exercise 17.2. Keep in mind that the partial sums of the Cauchy product of two series are not the product of the partial sums of the two series.

Exercise 17.3. Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series absolutely convergent in $x \in \mathbb{R}$. Prove that the Cauchy product of these series is exactly

$$\sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}.$$

Exercise 17.4. Let $x \in \mathbb{R}$, $|x| < 1$. Determine (in dependence of x) the sum of the two series

$$\sum_{k=0}^{\infty} x^{2k} \text{ and } \sum_{k=0}^{\infty} x^{2k+1}.$$

Exercise 17.5. Let

$$\sum_{n=0}^{\infty} a_n x^n$$

be an absolutely convergent power series. Compute the coefficients of the powers x^0, x^1, x^2, x^3, x^4 in the third power

$$\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} a_n x^n)^3.$$

Exercise 17.6. Prove that the real function defined by the exponential

$$\exp : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \exp x,$$

has no upper limit and that 0 is the infimum (but not the minimum) of the image set.[1]

Exercise 17.7. Prove that for the exponential function

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto a^x,$$

the following calculation rules hold (where $a, b \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$).

(1) $a^{x+y} = a^x \cdot a^y$.
(2) $a^{-x} = \frac{1}{a^x}$.
(3) $(a^x)^y = a^{xy}$.
(4) $(ab)^x = a^x b^x$.

Exercise 17.8. Prove that for the logarithm to base b the following calculation rules hold.

(1) We have $\log_b(b^x) = x$ and $b^{\log_b(y)} = y$, ie, the logarithm to base b is the inverse to the exponential function to the base b.
(2) We have $\log_b(y \cdot z) = \log_b y + \log_b z$
(3) We have $\log_b y^u = u \cdot \log_b y$ for $u \in \mathbb{R}$.
(4) We have

$$\log_a y \; = \; \log_a(b^{\log_b y}) \; = \; \log_b y \cdot \log_a b.$$

Exercise 17.9. A monetary community has an annual inflation of 2%. After what period of time (in years and days), the prices have doubled?

Hand-in-exercises

Exercise 17.10. (3 points)

Compute the coefficients $c_0, c_1, \ldots, c_5$ of the power series $\sum_{n=0}^\infty c_n x^n$, which is the Cauchy product of the geometric series with the exponential series.

Exercise 17.11. (4 points)

Let

$$\sum_{n=0}^\infty a_n x^n$$

[1] From the continuity it follows that $\mathbb{R}_+$ is the image set of the real exponential function.

be an absolutely convergent power series. Determine the coefficients of the powers $x^0, x^1, x^2, x^3, x^4, x^5$ in the fourth power

$$\sum_{n=0}^{\infty} c_n x^n = (\sum_{n=0}^{\infty} a_n x^n)^4 .$$

EXERCISE 17.12. (5 points)

For $N \in \mathbb{N}$ and $x \in \mathbb{R}$ let

$$R_{N+1}(x) = \exp x - \sum_{n=0}^{N} \frac{x^n}{n!} = \sum_{n=N+1}^{\infty} \frac{x^n}{n!}$$

be the remainder of the exponential series. Prove that for $|x| \leq 1 + \frac{1}{2}N$ the remainder term estimate

$$|R_{N+1}(x)| \leq \frac{2}{(N+1)!}|x|^{N+1}$$

holds.

EXERCISE 17.13. (3 points)

Compute by hand the first 4 digits in the decimal system of

$$\exp 1 .$$

EXERCISE 17.14. (4 points)

Prove that the real exponential function defined by the exponential series has the property that for each $d \in \mathbb{N}$ the sequence

$$\left(\frac{\exp n}{n^d}\right)_{n \in \mathbb{N}}$$

diverges to $+\infty$.[2]

EXERCISE 17.15. (6 points)

Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a continuous function $\neq 0$, with the property that

$$f(x+y) = f(x) \cdot f(y)$$

for all $x, y \in \mathbb{R}$. Prove that f is an exponential function, i.e. there exists a $b > 0$ such that $f(x) = b^x$.

[2]Therefore we say that the exponential function *grows faster* than any polynomial function.