Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 23

Warm-up-exercises

Exercise 23.1. Determine the Riemann sum (Treppenintegral) over $[-3, +4]$ of the staircase function

$$f(t) = \begin{cases} 5, \text{ if } -3 \leq t \leq -2\,, \\ -3, \text{ if } -2 < t \leq -1\,, \\ \frac{3}{7}, \text{ if } -1 < t < -\frac{1}{2}\,, \\ 13, \text{ if } t = -\frac{1}{2}\,, \\ \pi, \text{ if } -\frac{1}{2} < t < e\,, \\ 0, \text{ if } e \leq t \leq 3\,, \\ 1, \text{ if } 3 < t \leq 4\,, \end{cases} .$$

Exercise 23.2. a) Subdivide the interval $[-4, 5]$ in six subintervals of equal length.

b) Determine the Riemann sum of the staircase function on $[-4, 5]$, which takes alternately the values 2 and -1 on the subdivision constructed in a).

Exercise 23.3. Give an example of a function $f : [a, b] \to \mathbb{R}$ which assumes only finitely many values, but is not a staircase function.

Exercise 23.4. Let

$$f : [a, b] \longrightarrow [c, d]$$

be a staircase function and let

$$g : [c, d] \longrightarrow \mathbb{R}$$

be a function. Prove that the composite $g \circ f$ is also a staircase function.

Exercise 23.5. Give an example of a continuous function

$$f : [a, b] \longrightarrow [c, d]$$

and a staircase function

$$g : [c, d] \longrightarrow \mathbb{R}$$

such that the composite $g \circ f$ is not a staircase function.

Exercise 23.6. Determine the definite integral

$$\int_0^1 t\,dt$$

explicitly with upper and lower staircase functions.

Exercise 23.7. Determine the definite integral

$$\int_1^2 t^3\,dt$$

explicitly with upper and lower staircase functions.

Exercise 23.8. Let $I = [a, b]$ be a compact interval and let

$$f : I \longrightarrow \mathbb{R}$$

be a function. Consider a sequence of staircase functions $(s_n)_{n\in\mathbb{N}}$such that $s_n \leq f$ and a sequence of staircase functions $(t_n)_{n\in\mathbb{N}}$such that $t_n \geq f$. Assume that the two Riemann sums corresponding to the sequences converge and that their limits coincide. Prove that f is Riemann-integrable and that

$$\lim_{n\to\infty} \int_a^b s_n(x)\,dx \;=\; \int_a^b f(x)\,dx \;=\; \lim_{n\to\infty} \int_a^b t_n(x)\,dx.$$

Exercise 23.9. Let I be a compact interval and let

$$f : I \longrightarrow \mathbb{R}$$

be a function Prove that f is Riemann-integrable if and only if there is a subvision $a = a_0 < a_1 < \cdots < a_n = b$ such that the individual restrictions $f_i = f|_{[a_{i-1},a_i]}$ are Riemann-integrable.

Exercise 23.10. Let $I = [a, b] \subseteq \mathbb{R}$ be a compact interval and let $f, g : I \to \mathbb{R}$ be two Riemann-integrable functions. Prove the following statements.

(1) If $m \leq f(x) \leq M$ for all $x \in I$, then $m(b-a) \leq \int_a^b f(t)\,dt \leq M(b-a)$.
(2) If $f(x) \leq g(x)$ for all $x \in I$, then $\int_a^b f(t)\,dt \leq \int_a^b g(t)\,dt$.
(3) We have $\int_a^b f(t) + g(t)\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$.
(4) For $c \in \mathbb{R}$ we have $\int_a^b (cf)(t)\,dt = c\int_a^b f(t)\,dt$.

Exercise 23.11. Let $I = [a, b]$ be a compact interval and let $f : I \to \mathbb{R}$ be a Riemann-integrable function. Prove that

$$|\int_a^b f(t)\,dt| \leq \int_a^b |f(t)|\,dt\,.$$

Exercise 23.12. Let $I = [a, b]$ be a compact interval and let $f, g : I \to \mathbb{R}$ be two Riemann-integrable functions. Prove that fg is also Riemann-integrable.

Hand-in-exercises

Exercise 23.13. (2 points)

Let

$$f, g : [a, b] \longrightarrow \mathbb{R}$$

be two staircase functions. Prove that $f + g$ is also a staircase function.

Exercise 23.14. (3 points)

Determine the definite integral

$$\int_a^b t^2 \, dt$$

as a function of a and b explicitly with lower and upper staircase functions

Exercise 23.15. (4 points)

Determine the definite integral

$$\int_{-2}^{7} -t^3 + 3t^2 - 2t + 5 \, dt$$

explicitly with upper and lower staircase functions.

Exercise 23.16. (3 points)

Prove that for the function

$$]0, 1] \longrightarrow \mathbb{R}, \, x \longmapsto \frac{1}{x},$$

neither the lower nor the upper integral exist.

Exercise 23.17. (6 points)

Prove that for the function

$$]0, 1] \longrightarrow \mathbb{R}, \, x \longmapsto \frac{1}{\sqrt{x}},$$

the lower integral exists, but the upper integral does not exist.

Exercise 23.18. (5 points)

Let I be a compact interval and let

$$f : I \longrightarrow \mathbb{R}$$

be a monotone function. Prove that f is Riemann-integrable.