Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 13

Warm-up-exercises

EXERCISE 13.1. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two convergent real sequences with $x_n \geq y_n$ for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n$ holds.

EXERCISE 13.2. Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be three real sequences. Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ be convergent to the same limit a. Prove that also $(y_n)_{n\in\mathbb{N}}$ converges to the same limit a.

EXERCISE 13.3. Let $k \in \mathbb{N}_+$. Prove that the sequence $\left(\frac{1}{n^k}\right)_{n\in\mathbb{N}}$ converges to 0.

EXERCISE 13.4. Give an example of a Cauchy sequence in $\mathbb{Q}$, such that (in $\mathbb{Q}$) it does not converge.

EXERCISE 13.5. Give an example of a real sequence, that does not converge, but it contains a convergent subsequence.

EXERCISE 13.6. Let $a \in \mathbb{R}$ be a non-negative real number and $c \in \mathbb{R}_+$. Prove that the sequence defined recursively as $x_0 = c$ and

$$x_{n+1} := \frac{x_n + a/x_n}{2},$$

converges to $\sqrt{a}$.

EXERCISE 13.7. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of the Fibonacci numbers and

$$x_n := \frac{f_n}{f_{n-1}}.$$

Prove that this sequence converges in $\mathbb{R}$ and that its limit x satisfies the relation

$$x = 1 + x^{-1}.$$

Calculate this x.

Hint: First prove, with the help of Simpson formula, that you can produce a sequence of nested intervals with these fractions.

Before the next task, we recall the following two definitions.

For two real numbers x and y we call

$$\frac{x+y}{2}$$

the *arithmetic mean.*

For two non-negative real numbers x and y we call

$$\sqrt{x \cdot y}$$

the *geometric mean.*

EXERCISE 13.8. Let x and y be two non-negative real numbers. Prove that the arithmetic mean of these numbers is greater than or equal to their geometric mean.

EXERCISE 13.9. Let I_n, $n \in \mathbb{N}$, be a a sequence of nested intervals in $\mathbb{R}$. Prove that the intersection

$$\bigcap_{n \in \mathbb{N}} I_n$$

consists of exactly one point $x \in \mathbb{R}$.

EXERCISE 13.10. Let $x > 1$ be a real number. Prove that the sequence x^n, $n \in \mathbb{N}$, diverges to $+\infty$.

EXERCISE 13.11. Give an example of a real sequence $(x_n)_{n \in \mathbb{N}}$, such that it contains a subsequence that diverges to $+\infty$ and also a subsequence that diverges to $-\infty$.

Hand-in-exercises

EXERCISE 13.12. (3 points)

Give examples of convergent sequences of real numbers $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ with $x_n \neq 0$, $n \in \mathbb{N}$, and with $\lim_{n \to \infty} x_n = 0$ such that the sequence

$$\left(\frac{y_n}{x_n} \right)_{n \in \mathbb{N}}$$

(1) converges to 0,
(2) converges to 1,
(3) diverges.

EXERCISE 13.13. (5 points)

Let $P = \sum_{i=0}^{d} a_i x^i$ and $Q = \sum_{i=0}^{e} b_i x^i$ be polynomials with $a_d, b_e \neq 0$. Determine, depending on d and e, whether

$$z_n = \frac{P(n)}{Q(n)}$$

(which is defined for n sufficiently large) is a convergent sequence or not, and determine the limit in the convergent case.

EXERCISE 13.14. (4 points)

Let I_n, $n \in \mathbb{N}$, be a sequence of nested intervals in $\mathbb{R}$ and let $(x_n)_{n\in\mathbb{N}}$ be a real sequence with $x_n \in I_n$ for all $n \in \mathbb{N}$. Prove that this sequence converges to the unique number belonging to the intersection of the family of nested intervals (see Exercise 9).

EXERCISE 13.15. (6 points)

Let $b > a > 0$ be positive real numbers. We define recursively two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ such that $x_0 = a$, $y_0 = b$ and that

$$x_{n+1} = \text{ geometric mean of } x_n \text{ and } y_n\,,$$

$$y_{n+1} = \text{ arithmetic mean of } x_n \text{ and } y_n\,.$$

Prove that $[x_n, y_n]$ is a sequence of nested intervals.

EXERCISE 13.16. (2 points)

Prove that the sequence $(\sqrt{n})_{n\in\mathbb{N}}$ diverges to ∞.

EXERCISE 13.17. (3 points)

Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence with $x_n > 0$ for all $n \in \mathbb{N}$. Prove that the sequence diverges to $+\infty$ if and only if the sequence $\left(\frac{1}{x_n}\right)_{n\in\mathbb{N}}$ converges to 0.

Christmas exercise

The following exercise shold be handed in on 4.1.2012 (separated from the other exercises). The acquired marks enter directly your point account.

EXERCISE 13.18. (8 points)

There are $n \geq 4$ persons in a room and they would like to play secret Santa. This means that for each person A, another person $B \neq A$ has to be determined to whom A has to give a gift. Every person is only allowed to know who to give the gift, no one is allowed to know more. The people stay the whole time in the room, they do not look away. They only have paper and pens. They are allowed to shuffle and to look in secret at choosen cards.

Describe a procedure to determine a gift relation which satisfies all conditions.