Prof. Dr. H. Brenner

Mathematics for natural sciences I

Exercise sheet 12

Warm-up-exercises

EXERCISE 12.1. Prove that in \mathbb{Q} there is no element x such that $x^2 = 2$.

EXERCISE 12.2. Calculate by hand the approximations x_1, x_2, x_3, x_4 in the Heron process for the square root of 5 with initial value $x_0 = 2$.

EXERCISE 12.3. Let $(x_n)_{n \in \mathbb{N}}$ be a real sequence. Prove that the sequence converges to x if and only if for all $k \in \mathbb{N}_+$ a natural number $n_0 \in \mathbb{N}$ exists, such that for all $n \ge n_0$ the estimation $|x_n - x| \le \frac{1}{k}$ holds.

EXERCISE 12.4. Examine the convergence of the following sequence

$$x_n = \frac{1}{n^2}$$

where $n \geq 1$.

EXERCISE 12.5. Prove the statements (1), (3) and (5) of Lemma 12.10.

EXERCISE 12.6. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence of real numbers with limit equal to x. Prove that also the sequence

 $(|x_n|)_{n\in\mathbb{N}}$

converges, and specifically to |x|.

The next two exercises concern the Fibonacci numbers.

The sequence of the *Fibonacci numbers* f_n is defined recursively as

 $f_1 := 1, f_2 := 1$ and $f_{n+2} := f_{n+1} + f_n$.

EXERCISE 12.7. Prove by induction the Simpson formula or Simpson identity for the Fibonacci numbers f_n . It says $(n \ge 2)$

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

EXERCISE 12.8. Prove by induction the *Binet formula* for the Fibonacci numbers. This says that

$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

holds $(n \ge 1)$.

EXERCISE 12.9. Examine for each of the following subsets $M \subseteq \mathbb{R}$ the concepts upper bound, lower bound, supremum, infimum, maximum and minimum.

$$\begin{array}{ll} (1) & \{2, -3, -4, 5, 6, -1, 1\}, \\ (2) & \{\frac{1}{2}, \frac{-3}{7}, \frac{-4}{9}, \frac{5}{9}, \frac{6}{13}, \frac{-1}{3}, \frac{1}{4}\}, \\ (3) &] - 5, 2], \\ (4) & \{\frac{1}{n} | \, n \in \mathbb{N}_+\}, \\ (5) & \{\frac{1}{n} | \, n \in \mathbb{N}_+\} \cup \{0\}, \\ (6) & \mathbb{Q}_-, \\ (7) & \{x \in \mathbb{Q} | \, x^2 \leq 2\}, \\ (8) & \{x \in \mathbb{Q} | \, x^2 \leq 4\}, \\ (9) & \{x^2 | \, x \in \mathbb{Z}\}. \end{array}$$

Hand-in-exercises

EXERCISE 12.10. (3 points)

Examine the convergence of the following sequence

$$x_n = \frac{1}{\sqrt{n}}$$

where $n \geq 1$.

EXERCISE 12.11. (3 points)

Determine the limit of the real sequence given by

$$x_n = \frac{7n^3 - 3n^2 + 2n - 11}{13n^3 - 5n + 4}$$

EXERCISE 12.12. (3 points)

Prove that the real sequence

$$\left(\frac{n}{2^n}\right)_{n\in\mathbb{N}}$$

converges to 0.

EXERCISE 12.13. (6 points)

Examine the convergence of the following real sequence

$$x_n = \frac{\sqrt{n^n}}{n!} \,.$$

EXERCISE 12.14. (5 points)

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of real numbers and let the sequence $(z_n)_{n\in\mathbb{N}}$ be defined as $z_{2n-1} := x_n$ and $z_{2n} := y_n$. Prove that $(z_n)_{n\in\mathbb{N}}$ converges if and only if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to the same limit.