Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 11

Warm-up-exercises

Exercise 11.1. Determine explicitly the column rank and the row rank of the matrix

$$\begin{pmatrix} 3 & 2 & 6 \\ 4 & 1 & 5 \\ 6 & -1 & 3 \end{pmatrix} .$$

Describe linear dependencies (if they exist) between the rows and between the columns of the matrix.

Exercise 11.2. Show that the elementary operations on the rows do not change the column rank.

Exercise 11.3. Compute the determinant of the matrix

$$\begin{pmatrix} 1+3i & 5-i \\ 3-2i & 4+i \end{pmatrix} .$$

Exercise 11.4. Compute the determinant of the matrix

$$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 3 \\ 8 & 7 & 4 \end{pmatrix} .$$

Exercise 11.5. Prove by induction that the determinant of an upper triangular matrix is equal to the product of the diagonal elements.

Exercise 11.6. Check the multi-linearity and the property to be alternating, directly for the determinant of a 3×3-matrix.

Exercise 11.7. Let M be the following square matrix

$$M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} ,$$

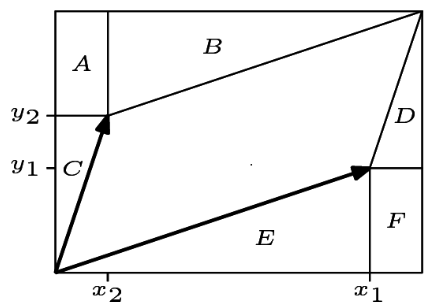
where A and D are square matrices. Prove that $\det M = \det A \cdot \det D$.

EXERCISE 11.8. Determine for which $x \in \mathbb{C}$ the matrix

$$\begin{pmatrix} x^2+x & -x \\ -x^3+2x^2+5x-1 & x^2-x \end{pmatrix}$$

is invertible.

EXERCISE 11.9. Use the image to convince yourself that, given two vectors (x_1, y_1) and (x_2, y_2), the determinant of the 2×2-matrix defined by these vectors is equal (up to sign) to the area of the plane *parallelogram* spanned by the vectors.



EXERCISE 11.10. Prove that you can develop the determinant according to each row and each column.

EXERCISE 11.11. Let K be a field and $m, n, p \in \mathbb{N}$. Prove that the transpose of a matrix satisfy the following properties (where $A, B \in \mathrm{Mat}_{m\times n}(K)$, $C \in \mathrm{Mat}_{n\times p}(K)$ and $s \in K$).

(1) $(A^{tr})^{tr} = A$.
(2) $(A+B)^{tr} = A^{tr} + B^{tr}$.
(3) $(sA)^{tr} = s \cdot A^{tr}$.
(4) $(A \circ C)^{tr} = C^{tr} \circ A^{tr}$.

EXERCISE 11.12. Compute the determinant of the matrix

$$\begin{pmatrix} 0 & 2 & 7 \\ 1 & 4 & 5 \\ 6 & 0 & 3 \end{pmatrix},$$

by developing the matrix along every column and along every row.

EXERCISE 11.13. Compute the determinant of all the 3×3-matrices, such that in each column and in each row there are exactly one 1 and two 0s.

EXERCISE 11.14. Let $z \in \mathbb{C}$ and let

$$\mathbb{C} \longrightarrow \mathbb{C},\ w \longmapsto zw,$$

be the associated multiplication. Compute the determinant of this map, considering it as a real-linear map $\mathbb{R}^2 \to \mathbb{R}^2$.

The next exercises require the following definition.

Let K be a field and let V be a K-vector space. For $a \in K$ the linear map

$$\varphi : V \longrightarrow V,\ v \longmapsto av,$$

is called the *stretching* (or *homothety*) with *extension factor* a.

EXERCISE 11.15. What is the determinant of a homothety?

EXERCISE 11.16. Check the multiplication theorem for determinants of two homotheties on a finite-dimensional vector space.

EXERCISE 11.17. Check the multiplication theorem for determinants of the following matrices

$$A = \begin{pmatrix} 5 & 7 \\ 2 & -4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -3 & 1 \\ 6 & 5 \end{pmatrix}.$$

Hand-in-exercises

EXERCISE 11.18. (4 points)

Let K be a field and let V and W be vector spaces over K of dimensions n and m. Let

$$\varphi : V \longrightarrow W$$

be a linear map, described by the matrix $M \in \mathrm{Mat}_{m \times n}(K)$ with respect to two bases. Prove that

$$\mathrm{rang}\ \varphi = \mathrm{rang}\ M.$$

EXERCISE 11.19. (3 points)

Compute the determinant of the matrix

$$\begin{pmatrix} 1+i & 3-2i & 5 \\ i & 1 & 3-i \\ 2i & -4-i & 2+i \end{pmatrix}.$$

EXERCISE 11.20. (4 points)

Compute the determinant of the matrix

$$A = \begin{pmatrix} 2 & 1 & 0 & -2 \\ 1 & 3 & 3 & -1 \\ 3 & 2 & 4 & -3 \\ 2 & -2 & 2 & 3 \end{pmatrix}.$$

EXERCISE 11.21. (2 points)

Compute the determinant of the elementary matrices.

EXERCISE 11.22. (5 points)

Check the multiplication theorem for determinants of the following matrices

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & 0 & -1 \\ 1 & 3 & 4 \end{pmatrix} \text{ and } B = \begin{pmatrix} -2 & 1 & 0 \\ 2 & 3 & 5 \\ 2 & 0 & -3 \end{pmatrix}.$$

Abbildungsverzeichnis