Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 12

Warm-up-exercises

Exercise 12.1. Prove that in $\mathbb{Q}$ there is no element x such that $x^2 = 2$.

Exercise 12.2. Calculate by hand the approximations x_1, x_2, x_3, x_4 in the Heron process for the square root of 5 with initial value $x_0 = 2$.

Exercise 12.3. Let $(x_n)_{n \in \mathbb{N}}$ be a real sequence. Prove that the sequence converges to x if and only if for all $k \in \mathbb{N}_+$ a natural number $n_0 \in \mathbb{N}$ exists, such that for all $n \geq n_0$ the estimation $|x_n - x| \leq \frac{1}{k}$ holds.

Exercise 12.4. Examine the convergence of the following sequence

$$x_n = \frac{1}{n^2}$$

where $n \geq 1$.

Exercise 12.5. Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be two convergent real sequences with $x_n \geq y_n$ for all $n \in \mathbb{N}$. Prove that $\lim_{n \to \infty} x_n \geq \lim_{n \to \infty} y_n$ holds.

Exercise 12.6. Let $(x_n)_{n \in \mathbb{N}}$, $(y_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ be three real sequences. Let $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ and $(x_n)_{n \in \mathbb{N}}$ and $(z_n)_{n \in \mathbb{N}}$ be convergent to the same limit a. Prove that also $(y_n)_{n \in \mathbb{N}}$ converges to the same limit a.

Exercise 12.7. Let $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence of real numbers with limit equal to x. Prove that also the sequence

$$(|x_n|)_{n \in \mathbb{N}}$$

converges, and specifically to $|x|$.

The next two exercises concern the Fibonacci numbers.

The sequence of the *Fibonacci numbers* f_n is defined recursively as

$$f_1 := 1 \, , f_2 := 1 \text{ and } f_{n+2} := f_{n+1} + f_n \, .$$

Exercise 12.8. Prove by induction the *Simpson formula* or Simpson identity for the Fibonacci numbers f_n. It says $(n \geq 2)$

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n \, .$$

Exercise 12.9. Prove by induction the *Binet formula* for the Fibonacci numbers. This says that

$$f_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$$

holds ($n \geq 1$).

Exercise 12.10. Examine for each of the following subsets $M \subseteq \mathbb{R}$ the concepts upper bound, lower bound, supremum, infimum, maximum and minimum.

(1) $\{2, -3, -4, 5, 6, -1, 1\}$,
(2) $\left\{\frac{1}{2}, \frac{-3}{7}, \frac{-4}{9}, \frac{5}{9}, \frac{6}{13}, \frac{-1}{3}, \frac{1}{4}\right\}$,
(3) $]-5, 2]$,
(4) $\left\{\frac{1}{n} | n \in \mathbb{N}_+\right\}$,
(5) $\left\{\frac{1}{n} | n \in \mathbb{N}_+\right\} \cup \{0\}$,
(6) $\mathbb{Q}_-$,
(7) $\{x \in \mathbb{Q} | x^2 \leq 2\}$,
(8) $\{x \in \mathbb{Q} | x^2 \leq 4\}$,
(9) $\{x^2 | x \in \mathbb{Z}\}$.

Hand-in-exercises

Exercise 12.11. (3 points)

Examine the convergence of the following sequence

$$x_n = \frac{1}{\sqrt{n}},$$

where $n \geq 1$.

Exercise 12.12. (3 points)

Determine the limit of the real sequence given by

$$x_n = \frac{7n^3 - 3n^2 + 2n - 11}{13n^3 - 5n + 4}.$$

Exercise 12.13. (4 points)

Prove that the real sequence

$$\left(\frac{n}{2^n}\right)_{n \in \mathbb{N}}$$

converges to 0.

Exercise 12.14. (6 points)

Examine the convergence of the following real sequence

$$x_n = \frac{\sqrt{n}^n}{n!}.$$

EXERCISE 12.15. (5 points)

Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be sequences of real numbers and let the sequence $(z_n)_{n\in\mathbb{N}}$ be defined as $z_{2n-1} := x_n$ and $z_{2n} := y_n$. Prove that $(z_n)_{n\in\mathbb{N}}$ converges if and only if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to the same limit.

EXERCISE 12.16. (3 points)

Determine the limit of the real sequence given by

$$x_n = \frac{2n + 5\sqrt{n} + 7}{-5n + 3\sqrt{n} - 4}\,.$$