Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 9

Warm-up-exercises

EXERCISE 9.1. Let K be a field and let V and W be two K-vector spaces. Let

$$\varphi : V \longrightarrow W$$

be a linear map. Prove that for all vectors $v_1, \ldots, v_n \in V$ and coefficients $s_1, \ldots, s_n \in K$ the relationship

$$\varphi(\sum_{i=1}^{n} s_i v_i) = \sum_{i=1}^{n} s_i \varphi(v_i)$$

holds.

EXERCISE 9.2. Let K be a field and let V be a K-vector space. Prove that for $a \in K$ the map

$$V \longrightarrow V,\, v \longmapsto av,$$

is linear.[1]

EXERCISE 9.3. Interpret the following physical laws as linear functions from $\mathbb{R}$ to $\mathbb{R}$. Establish in each situation what is the measurable variable and what is the proportionality factor.

(1) Mass is volume times density.
(2) Energy is mass times the calorific value.
(3) The distance is speed multiplied by time.
(4) Force is mass times acceleration.
(5) Energy is force times distance.
(6) Energy is power times time.
(7) Voltage is resistance times electric current.
(8) Charge is current multiplied by time.

EXERCISE 9.4. Around the Earth along the equator is placed a ribbon. However, the ribbon is one meter longer than the equator, so that it is lifted up uniformly all around to be tense. Which of the following creatures can run/fly/swim/dance under it?

(1) An amoeba.
(2) An ant.
(3) A tit.
(4) A flounder.
(5) A boa constrictor.

[1]Such a map is called a homothety with stretching or extension factor a.

(6) A guinea pig.
(7) A boa constrictor that has swallowed a guinea pig.
(8) A very good limbo dancer.

Exercise 9.5. Consider the linear map

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

such that

$$\varphi \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 5 \text{ and } \varphi \begin{pmatrix} 2 \\ -3 \end{pmatrix} = 4 \, .$$

Compute

$$\varphi \begin{pmatrix} 7 \\ 6 \end{pmatrix} \, .$$

Exercise 9.6. Complete the proof of Theorem 9.3 to the compatibility with the scalar multiplication.

Exercise 9.7. Let K be a field and let U, V, W be vector spaces over K. Let

$$\varphi : U \to V \text{ and } \psi : V \to W$$

be linear maps. Prove that also the composite function

$$\psi \circ \varphi : U \longrightarrow W$$

is a linear map.

Exercise 9.8. Let K be a field and let V be a K-vector space. Let $v_1, \ldots, v_n$ be a family of vectors in V. Consider the map

$$\varphi : K^n \longrightarrow V, \, (s_1, \ldots, s_n) \longmapsto \sum_{i=1}^{n} s_i v_i,$$

and prove the following statements.

(1) φ is injective if and only if $v_1, \ldots, v_n$ are linearly independent.
(2) φ is surjective if and only if $v_1, \ldots, v_n$ is a system of generators for V.
(3) φ is bijective if and only if $v_1, \ldots, v_n$ form a basis.

Exercise 9.9. Prove that the functions

$$\mathbb{C} \longrightarrow \mathbb{R}, \, z \longmapsto \operatorname{Re}(z),$$

and

$$\mathbb{C} \longrightarrow \mathbb{R}, \, z \longmapsto \operatorname{Im}(z),$$

are $\mathbb{R}$-linear maps. Prove that also the complex conjugation is $\mathbb{R}$-linear, but not $\mathbb{C}$-linear. Is the absolute value

$$\mathbb{C} \longrightarrow \mathbb{R}, \, z \longmapsto |z|,$$

$\mathbb{R}$-linear?

Exercise 9.10. Let K be a field and let V and W be two K-vector spaces. Let

$$\varphi : V \longrightarrow W$$

be a linear map. Prove the following facts.

(1) Consider the subspace $S \subseteq V$ then also the image $\varphi(S)$ is a subspace of W.
(2) In particular the image bild $\varphi = \varphi(V)$ of the map is a subspace of W.
(3) Consider the subspace $T \subseteq W$ then the preimage $\varphi^{-1}(T)$ is a subspace of V.
(4) In particular $\varphi^{-1}(0)$ is a subspace of V.

Exercise 9.11. Determine the kernel of the linear map

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^3, \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \longmapsto \begin{pmatrix} 2 & 1 & 5 & 2 \\ 3 & -2 & 7 & -1 \\ 2 & -1 & -4 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Exercise 9.12. Determine the kernel of the linear map

$$\varphi : \mathbb{R}^4 \longrightarrow \mathbb{R}^2,$$

given by the matrix

$$M = \begin{pmatrix} 2 & 3 & 0 & -1 \\ 4 & 2 & 2 & 5 \end{pmatrix}.$$

Exercise 9.13. Find by elementary geometric considerations a matrix describing a rotation by 45 degrees counter-clockwise in the plane.

Exercise 9.14. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},$$

which sends a rational number $q \in \mathbb{Q}$ into q and all the irrational numbers into 0. Is this a linear map? Is it compatible with multiplication with a scalar?

Hand-in-exercises

Exercise 9.15. (3 points)

Consider the linear map

$$\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

such that

$$\varphi \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}, \varphi \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \varphi \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

Compute

$$\varphi \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}.$$

EXERCISE 9.16. (3 points)

Find by elementary geometric considerations a matrix describing a rotation by 30 degrees counter-clockwise in the plane.

EXERCISE 9.17. (3 points)

Determine the image and the kernel of the linear map

$$f : \mathbb{R}^4 \longrightarrow \mathbb{R}^4, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & 3 & 4 & -1 \\ 2 & 5 & 7 & -1 \\ -1 & 2 & 3 & -2 \\ -2 & 0 & 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

EXERCISE 9.18. (3 points)

Let $E \subset \mathbb{R}^3$ be the plane identified by the linear equation $5x + 7y - 4z = 0$. Determine a linear map

$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

such that the image of φ is equal to E.

EXERCISE 9.19. (3 points)

On the real vector space $G = \mathbb{R}^4$ of mulled wines we consider the two linear maps

$$\pi : G \longrightarrow \mathbb{R}, \begin{pmatrix} z \\ n \\ r \\ s \end{pmatrix} \longmapsto 8z + 9n + 5r + s,$$

and

$$\kappa : G \longrightarrow \mathbb{R}, \begin{pmatrix} z \\ n \\ r \\ s \end{pmatrix} \longmapsto 2z + n + 4r + 8s.$$

We put π as the price function and κ as the caloric function. Determine a basis for kern π, one for kern κ and one for kern $(\pi \times \kappa)$.[2]

[2]Do not mind that there may exist negative numbers. In a mulled wine of course the ingredients do not come in with a negative coefficient. But if you would like to consider for example, in how many ways you can change a particular recipe, without changing the total price or the total amount of energy, then the negative entries make sense.