Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 21

Warm-up-exercises

Exercise 21.1. Determine the derivatives of hyperbolic sine and hyperbolic cosine.

Exercise 21.2. Determine the derivative of the function
$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto x^2 \cdot \exp(x^3 - 4x).$$

Exercise 21.3. Determine the derivative of the function
$$\ln\, : \mathbb{R}_+ \longrightarrow \mathbb{R}.$$

Exercise 21.4. Determine the derivatives of the sine and the cosine function by using Theorem 21.1.

Exercise 21.5. Determine the 1034871-th derivative of the sine function.

Exercise 21.6. Determine the derivative of the function
$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto \sin(\cos x).$$

Exercise 21.7. Determine the derivative of the function
$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto (\sin x)(\cos x).$$

Exercise 21.8. Determine for $n \in \mathbb{N}$ the derivative of the function
$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto (\sin x)^n.$$

Exercise 21.9. Determine the derivative of the function
$$D \longrightarrow \mathbb{R},\, x \longmapsto \tan x = \frac{\sin x}{\cos x}.$$

EXERCISE 21.10. Prove that the real sine function induces a bijective, strictly increasing function

$$[-\pi/2, \pi/2] \longrightarrow [-1, 1],$$

and that the real cosine function induces a bijective, strictly decreasing function

$$[0, \pi] \longrightarrow [-1, 1].$$

EXERCISE 21.11. Determine the derivatives of arc-sine and arc-cosine functions.

EXERCISE 21.12. We consider the function

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R},\ x \longmapsto f(x) = 1 + \ln x - \frac{1}{x}.$$

a) Prove that f gives a continuous bijection between $\mathbb{R}_+$ and $\mathbb{R}$.

b) Determine the inverse image u of 0 under f, then compute $f'(u)$ and $(f^{-1})'(0)$. Draw a rough sketch for the inverse function f^{-1}.

EXERCISE 21.13. Let

$$f, g : \mathbb{R} \longrightarrow \mathbb{R}$$

be two differentiable functions. Let $a \in \mathbb{R}$. We have that

$$f(a) \geq g(a) \text{ and } f'(x) \geq g'(x) \text{ for all } x \geq a\,.$$

Prove that

$$f(x) \geq g(x) \text{ for all } x \geq a\,.$$

EXERCISE 21.14. We consider the function

$$f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = e^{-\frac{1}{x}}.$$

a) Investigate the monotony behavior of this function.

b) Prove that this function is injective.

c) Determine the image of f.

d) Determine the inverse function on the image for this function.

e) Sketch the graph of the function f.

EXERCISE 21.15. Consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = (2x + 3)e^{-x^2}.$$

Determine the zeros and the local (global) extrema of f. Sketch up roughly the graph of the function.

EXERCISE 21.16. Discuss the behavior of the function graph of

$$f:\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = e^{-2x} - 2e^{-x}.$$

Determine especially the monotony behavior, the extrema of f, $\lim_{x\to\infty} f(x)$ and also for the derivative f'.

EXERCISE 21.17. Prove that the function

$$f(x) = \begin{cases} x \sin \frac{1}{x} \text{ for } x \in]0,1], \\ 0 \text{ for } x = 0, \end{cases}$$

is continuous and that it has infinitely many zeros.

EXERCISE 21.18. Determine the limit of the sequence

$$\frac{\sin n}{n},\ n \in \mathbb{N}_+ .$$

EXERCISE 21.19. Determine for the following functions if the function limit exists and, in case, what value it takes.

(1) $\lim_{x\to 0} \frac{\sin x}{x}$,
(2) $\lim_{x\to 0} \frac{(\sin x)^2}{x}$,
(3) $\lim_{x\to 0} \frac{\sin x}{x^2}$,
(4) $\lim_{x\to 1} \frac{x-1}{\ln x}$.

EXERCISE 21.20. Determine for the following functions, if the limit function for $x \in \mathbb{R} \setminus \{0\}$, $x \to 0$, exists, and, in case, what value it takes.

(1) $\sin \frac{1}{x}$,
(2) $x \cdot \sin \frac{1}{x}$,
(3) $\frac{1}{x} \cdot \sin \frac{1}{x}$.

Hand-in-exercises

EXERCISE 21.21. (3 points)

Determine the linear functions that are tangent to the exponential function.

EXERCISE 21.22. (3 points)

Determine the derivative of the function

$$\mathbb{R}_+ \longrightarrow \mathbb{R},\ x \longmapsto x^x.$$

The following task should be solved without reference to the second derivative.

Exercise 21.23. (4 points)

Determine the extrema of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \sin x + \cos x\,.$$

Exercise 21.24. (6 points)

Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a polynomial function of degree $d \geq 1$. Let m be the number of local maxima of f and n the number of local minima of f. Prove that if d is odd then $m = n$ and that if d is even then $|m - n| = 1$.