Density Functions

Young W Lim

Mar 17 2022

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Function of Random Variable (1)

- A new **random variable** Y can be defined by applying a real Borel measurable function $g : \mathbb{R} \to \mathbb{R}$ to the outcomes of a real-valued **random variable** X.

$$Y = g(X)$$

- The **cumulative distribution function** of Y is then

$$F_Y(y) = \mathrm{P}\{g(X) \le y\}$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Function of Random Variable (2)

- If function g is **invertible**
  (i.e., $h = g^{-1}$ exists, where h is g's inverse function)
- and if the function g is either *increasing* or *decreasing*,
  then the previous relation can be extended to obtain

$$F_Y(y) = P\{g(X) \le y\}$$
$$= \begin{cases} P\{X \le h(y)\} = F_X(h(y)), & \text{if } h = g^{-1} \text{ increasing}, \\ P\{X \ge h(y)\} = 1 - F_X(h(y)), & \text{if } h = g^{-1} \text{ decreasing}. \end{cases}$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Function of Random Variable (3)

- assuming **invertibility** and **differentiability** of g
- the relation between the **probability density functions** can be found by differentiating both sides of the above expression with respect to y, in order to obtain

$$f_Y(y) = f_X\left(h(y)\right)\left|\frac{dh(y)}{dy}\right|.$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Function of Random Variable (4)

- If there is no **invertibility** of g, but each y admits at most a countable number of roots (i.e., a *finite*, or *countably infinite*, number of x_i such that $y = g(x_i)$)
- then the generalized relation between the **probability density functions**

$$f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Function of Random Variable (5)

$$f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|$$

- where $x_i = g_i^{-1}(y)$, according to the **inverse function theorem**. The formulas for densities do not require g to be increasing.

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example I

Let X be a real-valued, continuous random variable and let $Y = X^2$

$$F_Y(y) = P\left\{X^2 \leq y\right\}$$

If $y < 0$, then $P\left\{X^2 \leq y\right\} = 0$, so

$$F_Y(y) = 0 \qquad \text{if} \quad y < 0.$$

If $y \geq 0$, then

$$P\left\{X^2 \leq y\right\} = P\left\{|X| \leq \sqrt{y}\right\} = P\left\{-\sqrt{y} \leq |X| \leq \sqrt{y}\right\}$$

so

$$F_Y(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \qquad \text{if} \quad y \geq 0.$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example II (1)

Suppose X is a random variable with a cumulative distribution

$$F_X(x) = P(X \leq x) = \frac{1}{(1+e^{-x})^{\theta}}$$

where $\theta > 0$ is a fixed parameter.
Consider the random variable $Y = \log(1+e^{-X})$. Then,

$$\begin{aligned} F_Y(y) &= P\{Y \leq y\} \\ &= P\left\{\log(1+e^{-X}) \leq y\right\} \\ &= P\left\{(1+e^{-X}) \leq e^{y}\right\} = P\left\{e^{-X} \leq e^{y} - 1)\right\} \\ &= P\{X \geq -\log(e^{y} - 1)\} \end{aligned}$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example II (2)

The last expression can be calculated
in terms of the cumulative distribution of X, so

$$\begin{aligned} F_Y(y) &= 1 - F_X(-\log(e^y - 1)) \\ &= 1 - \frac{1}{(1 + e^{\log(e^y - 1)})^\theta} \\ &= 1 - \frac{1}{(1 + e^y - 1)^\theta} \\ &= 1 - e^{-y\theta}. \end{aligned}$$

which is the cumulative distribution function (CDF)
of an exponential distribution.

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example III (1)

Suppose X is a random variable
with a standard normal distribution, whose density is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Consider the random variable $Y = X^2$
We can find the density using the above formula
for a change of variables:

$$f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|.$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example III (2)

In this case the change is not **monotonic**,
because every value of Y has two corresponding values of X
(one positive and negative).

$$x_1 = g_1^{-1}(y) = +\sqrt{y}$$
$$x_2 = g_2^{-1}(y) = -\sqrt{y}$$

$$f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right| .$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example III (3)

However, because of symmetry,
both halves will transform identically, i.e.,

$$f_Y(y) = 2 f_X(g^{-1}(y)) \left| \frac{dg^{-1}(y)}{dy} \right|$$

The inverse transformation is

$$x = g^{-1}(y) = \sqrt{y}$$

and its derivative is

$$\frac{dg^{-1}(y)}{dy} = \frac{1}{2\sqrt{y}}$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example III (4)

$$f_Y(y) = f_X(+\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right| + f_X(-\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right|$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

because of the symmetry of $f_X(x)$

$$f_Y(y) = 2 f_X(\sqrt{y}) \left| \frac{d\sqrt{y}}{dy} \right| = 2 f_X(\sqrt{y}) \frac{1}{2\sqrt{y}}$$

$$= f_X(\sqrt{y}) \frac{1}{\sqrt{y}}$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example III (4)

Then,

$$f_Y(y) = 2\frac{1}{\sqrt{2\pi}}e^{-y/2}\frac{1}{2\sqrt{y}} = \frac{1}{\sqrt{2\pi y}}e^{-y/2}$$

This is a chi-squared distribution with one degree of freedom.

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example IV (1)

- Suppose X is a random variable with a normal distribution, whose density is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}$$

- Consider the random variable $Y = X^2$.
  We can find the density using the above formula for a change of variables:

$$f_Y(y) = \sum_i f_X(g_i^{-1}(y)) \left| \frac{dg_i^{-1}(y)}{dy} \right|$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example IV (2)

- In this case the change is not monotonic, because every value of Y has two corresponding values of X (one positive and negative).

$$x_1 = g_1^{-1}(y) = +\sqrt{y}$$
$$x_2 = g_2^{-1}(y) = -\sqrt{y}$$

- Differently from the previous example, in this case however, there is no **symmetry** and we have to compute the two distinct terms:

$$f_Y(y) = f_X(g_1^{-1}(y)) \left| \frac{dg_1^{-1}(y)}{dy} \right| + f_X(g_2^{-1}(y)) \left| \frac{dg_2^{-1}(y)}{dy} \right|$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example IV (3)

- The inverse transformation is

$$x = g_{1,2}^{-1}(y) = \pm\sqrt{y}$$

- and its derivative is

$$\frac{dg_{1,2}^{-1}(y)}{dy} = \pm\frac{1}{2\sqrt{y}}.$$

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables

Example IV (4)

- Then,

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{1}{2\sqrt{y}} (e^{-(\sqrt{y}-\mu)^2/(2\sigma^2)} + e^{-(-\sqrt{y}-\mu)^2/(2\sigma^2)}).$$

- This is a noncentral chi-squared distribution with one degree of freedom.

https://en.wikipedia.org/wiki/Random_variable#Functions_of_random_variables