# Residue Integrals and Laurent Series with non-annular region

20170213

Based on

T.J. Cavicchi, Digital Signal Processing

Complex Analysis for Mathematics and Engineering
J. Mathews

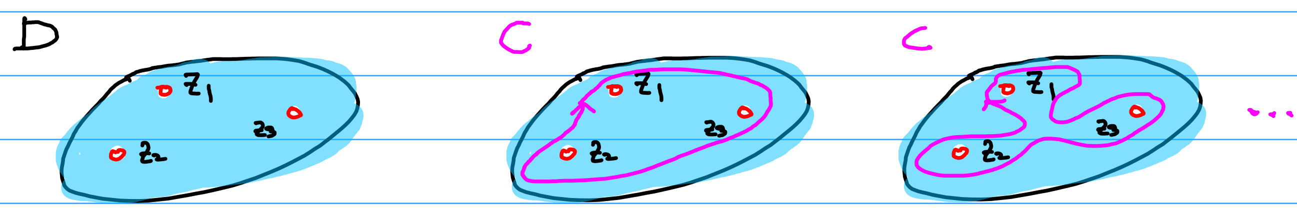
# Residue Theorem

D: simply connected domain
C: simple closed contour (ccw) in D

if $f(z)$ is analytic inside C and on C
  except at the points $\boxed{z_1, z_2, \cdots, z_k}$ in C

then
$$\boxed{\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \operatorname{Res}(f(z), z_j)}$$

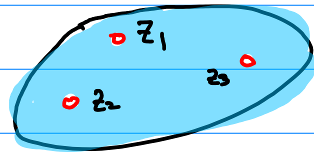
singular points of $f(z)$ : $z_1, z_2, \cdots, z_k$



# Integration of a function of a complex var.

$$\boxed{\oint_C f(z)\,dz} = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

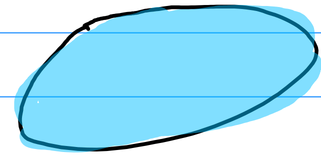
finite number $k$ of
singular points $z_k$
residue theorem




$$\boxed{\oint_C f(z)\,dz} = 0$$

if $f(z)$ is analytic within and on $C$

no singularity



$$\boxed{\oint_C f(z)\,dz} = 0$$

if $\boxed{f(z) = F'(z)}$ on $C$

: $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

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Digital Signal Processing, Wiley, 2000

$$\boxed{\oint_C f(z)\,dz} = 0$$

if $f(z)$ is continuous in $D$ and

$\boxed{f(z) = F'(z)}$ : $F(z)$ is an antiderivative of $f(z)$

fundamental theorem of calculus

# Series Expansion

can expand $f(z)$ about **any point $z_m$**
over powers of $(z - z_m)$

whether or not $f(z)$ is singular at $z_m$
or at other points between $z$ and $z_m$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

① **Laurent Series** Expansion of $f(z)$ at $z_m$
general $n_1$ — depend on $f(z)$ and $z_m$

② **z-transform** of $a_n^{\{m\}}$
general $n_1$ — depend on $f(z)$
$\boxed{z_m = 0}$

③ **Taylor Series** Expansion of $f(z)$ at $z_m$
positive $n_1$ — depend on $f(z)$ and $z_m$ $(n_1 \geq 0)$

④ **MacLaurin Series** Expansion of $f(z)$ at $z_m$
positive $n_1$ — depend on $f(z)$ $(n_1 \geq 0)$
$\boxed{z_m = 0}$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$n_1 > 0$ pos powers

| | |
|---|---|
| ① Laurent Series | ③ Taylor Series |
| ② z-transform | ④ MacLaurin Series |

$\boxed{z_m = 0}$

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* Expansion of $f(z)$ about any point $z_m$ over powers of $(z - z_m)$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}} \, dz$$ for general $f(z)$

$$a_n^{\{m\}} = \sum_k \mathrm{Res}\left[ \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right]$$ for general $f(z)$

$$a_n^{\{m\}} = \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$ for analytic $f(z)$ within $C$

analytic $f(z)$ $\longrightarrow$ $\frac{f(z)}{(z - z_m)^{n+1}}$ has a pole at $z_m$ order of $n+1$

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$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$\boxed{z_m}$ : possible poles of $f(z)$
not necessarily poles

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}}\, dz'$$

$$= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z-z_m)^{n+1}}, z_k \right)$$

$\boxed{z_k}$ : poles of $\frac{f(z)}{(z-z_m)^{n+1}}$

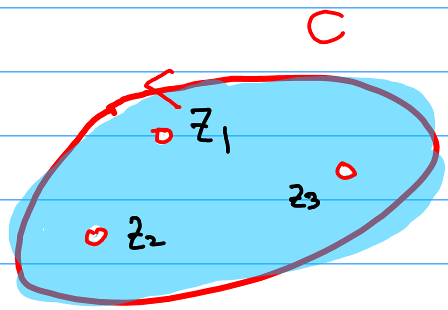
within $C$

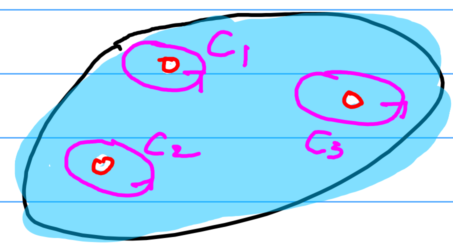
$$= \frac{1}{n!} f^{(n)}(z_m) \qquad n_1 \geq 0$$

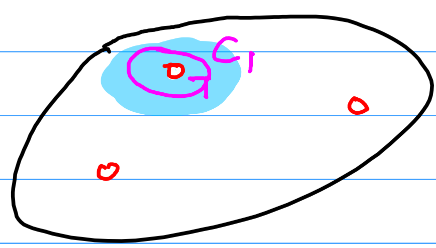
# Residue Theorem and Laurent Series

assumed there are (m) singularities (poles) of $f(z)$ in a region

let $\boxed{C_m}$ is taken to enclose only one pole $z_m$



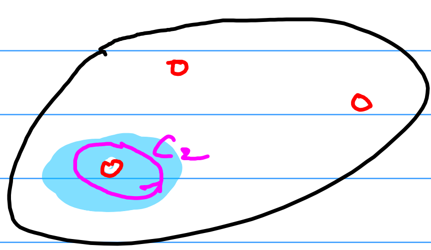




$a_n^{\{1\}}$ expanded at $z_1$

$C_1$ encloses $z_1$ only

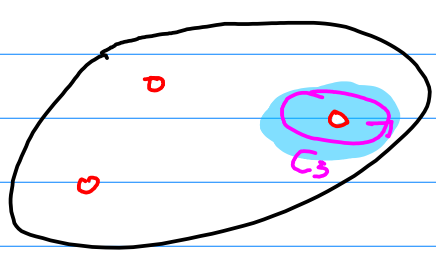
$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$




$a_n^{\{2\}}$ expanded at $z_2$

$C_2$ encloses $z_2$ only

$\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$



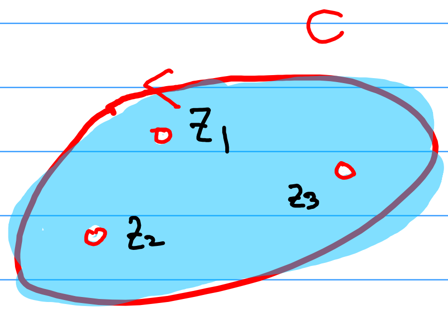

$a_n^{\{3\}}$ expanded at $z_3$

$C_3$ encloses $z_3$ only

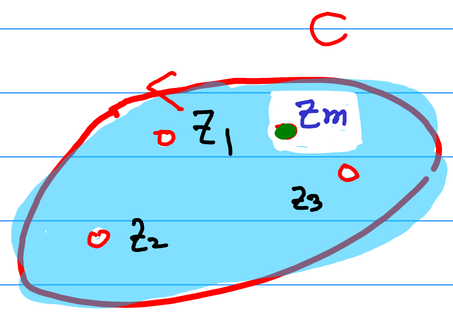
$\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$

# Series Expansion at $z_m$

no annular region




$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$




$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz$$

$$= \sum_k \operatorname{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}},\ z_k\right)$$

Let $z_1, z_2, z_3$ poles of $f(z)$

Then the poles of $\dfrac{f(z)}{(z - z_m)^{n+1}}$

| | | |
|---|---|---|
| $n \geq 0$ | $z_1,\ z_2,\ z_3,$ | $z_m$ |
| $n < 0$ | $z_1,\ z_2,\ z_3$ | |

# Computing $a_n^{\{m\}}$

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n \qquad \boxed{n \leftarrow k}$$

$$f(z) = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^k$$

for a given $n$

$$\frac{f(z)}{(z-z_m)^{n+1}} = \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1}$$

$k$ : index variable

$n$ : fixed value

$$\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \oint_C \sum_{k=n_1}^{\infty} a_k^{\{m\}} (z - z_m)^{k-n-1}\,dz$$

$$= \sum_{k=n_1}^{\infty} \oint_C a_k^{\{m\}} (z - z_m)^{k-n-1}\,dz \qquad \boxed{k = n}$$

$$\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \oint_C a_n^{\{m\}} \frac{1}{(z-z_m)}\,dz = 2\pi i \cdot a_n^{\{m\}}$$

$$\boxed{a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz}$$

$$\oint_C \overbrace{\boxed{\cdots (z-z_m)^{-3} + (z-z_m)^{-2}}}^{\to 0} + \boxed{\frac{1}{(z-z_m)}} + \overbrace{\boxed{1 + (z-z_m) + (z-z_m)^2 + \cdots}}^{\to 0}\,dz$$

$$= \oint_C \boxed{\frac{1}{(z-z_m)}}\,dz = 2\pi i$$

# Computing $a_n^{\{m\}}$ using Residues

expansion at $z_m$

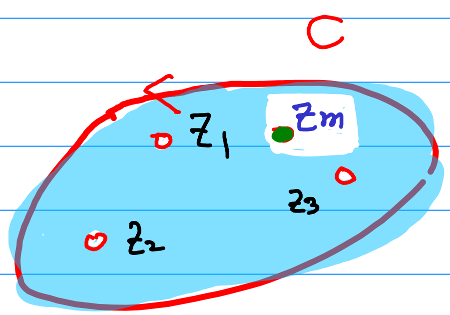
$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

$\boxed{n=-1}$ $n+1=0$ $(z-z_m)^{n+1}=1$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}\left(f(z), z_k\right)$$

$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_k \mathrm{Res}(f(z), z_k)$$

$$a_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m) = \sum_k \mathrm{Res}(f(z), z_k)$$

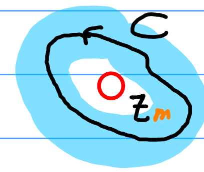



$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}}(z-z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\,dz = \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

Residue → Laurent Series → annular region if $C$ encloses only one pole $z_o$,

and the expansion at that pole $z_o$ is assumed,




then

$$a_{-1}^{\{o\}} = \frac{1}{2\pi i} \oint_{C_o} f(z)\, dz = \mathrm{Res}(f(z), z_o)$$

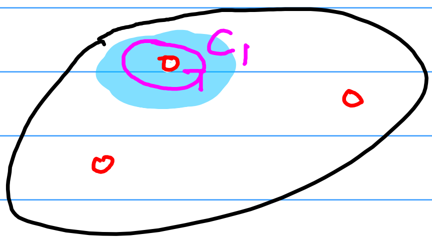
Let $\boxed{\tilde{a}_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_m)}$ notation $\sim$

the residue of $f(z)$ at $z_m$

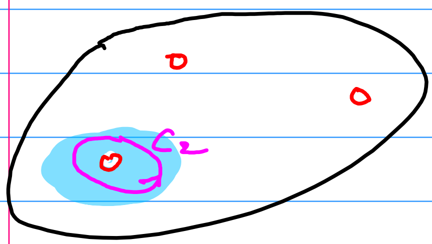
using $C_m$ which is in the anulus ROC

$$\boxed{f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{m\}} (z - z_m)^n}$$

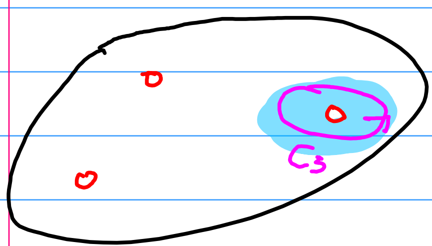
# Residues at the poles of f(z)



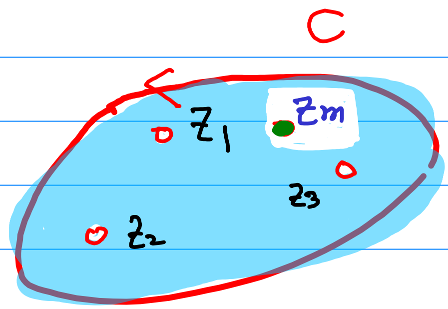

$$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1) = \frac{1}{2\pi i}\oint_{C_1} f(z)\,dz \qquad f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{1\}} (z - z_1)^n$$




$$\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2) = \frac{1}{2\pi i}\oint_{C_2} f(z)\,dz \qquad f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{2\}} (z - z_2)^n$$

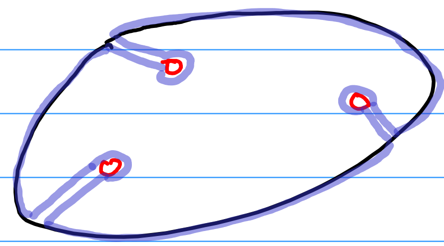



$$\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3) = \frac{1}{2\pi i}\oint_{C_3} f(z)\,dz \qquad f(z) = \sum_{n=-\infty}^{+\infty} a_n^{\{3\}} (z - z_3)^n$$



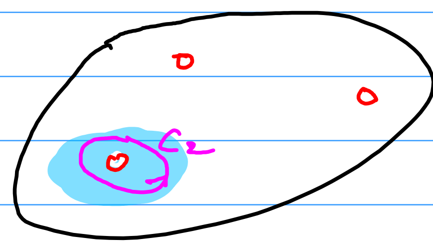

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$a_n^{\{m\}} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz$$
$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

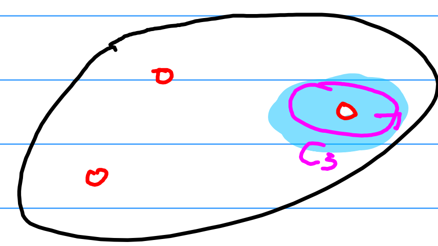


$$a_{-1}^{\{m\}} = \frac{1}{2\pi i}\oint_C f(z)\, dz$$
$$= \sum_k \mathrm{Res}\left(f(z), z_k\right)$$


$\tilde{a}_{-1}^{\{1\}} = \mathrm{Res}(f(z), z_1)$




$\tilde{a}_{-1}^{\{2\}} = \mathrm{Res}(f(z), z_2)$




$\tilde{a}_{-1}^{\{3\}} = \mathrm{Res}(f(z), z_3)$

$$\boxed{a_{-1}^{\{m\}} = \tilde{a}_{-1}^{\{1\}} + \tilde{a}_{-1}^{\{2\}} + \tilde{a}_{-1}^{\{3\}}}$$

$$a_{-1}^{\{m\}} = \mathrm{Res}(f(z), z_1) + \mathrm{Res}(f(z), z_2) + \mathrm{Res}(f(z), z_3)$$

$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n}$$

$$\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_m)^{n+1}}\, dz \\ &= \sum_k \operatorname{Res}\left( \frac{f(z)}{(z-z_m)^{n+1}}, z_k \right) \end{aligned}$$

$$\begin{aligned} a_{-1}^{\{m\}} &= \frac{1}{2\pi i} \oint_C f(z)\, dz \\ &= \sum_k \operatorname{Res}\left( f(z), z_k \right) \end{aligned}$$

$\vdots$

$$a_{-3}^{\{m\}} = \sum_k \operatorname{Res}\left( f(z)(z-z_m)^2, z_k \right)$$

$$a_{-2}^{\{m\}} = \sum_k \operatorname{Res}\left( f(z)(z-z_m)^1, z_k \right)$$

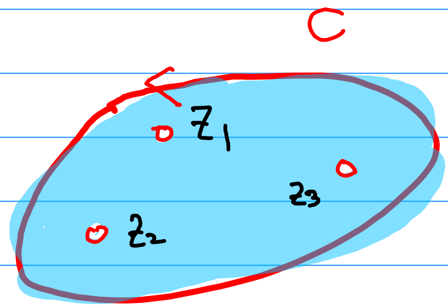
$$a_{-1}^{\{m\}} = \sum_k \operatorname{Res}\left( f(z) \quad , z_k \right)$$

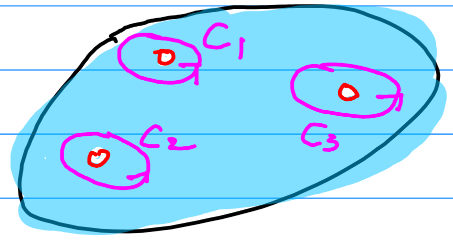
$$a_{0}^{\{m\}} = \sum_k \operatorname{Res}\left( \frac{f(z)}{(z-z_m)^1}, z_k \right)$$

$$a_{1}^{\{m\}} = \sum_k \operatorname{Res}\left( \frac{f(z)}{(z-z_m)^2}, z_k \right)$$

$$a_{2}^{\{m\}} = \sum_k \operatorname{Res}\left( \frac{f(z)}{(z-z_m)^3}, z_k \right)$$

$\vdots$




$$\boxed{\oint_C f(z)\,dz = 2\pi j \sum_{k=1}^{M} \tilde{a}_{-1}^{\{k\}} = 2\pi j \sum_{k=1}^{M} \mathrm{Res}\left(f(z), z_k\right)}$$

residue theorem

$$\boxed{a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)}$$

Laurent coefficient

$C$ encloses $k$ poles

$C_k$ encloses only the $k$-th pole

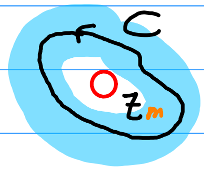
$\tilde{a}_{-1}^{\{k\}}$ the residue of the $k$-th pole enclosed by $C$, $z_k$

# Non-anular region

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z-z_m)^{n+1}} , z_k \right)$$




$\boxed{C}$ is in the same region of analyticity of $f(z)$

typically a circle centered on $z_m$ non-annular ok

$\boxed{z_k}$ within C : singularities of $\boxed{\frac{f(z)}{(z-z_m)^{n+1}}}$

$\boxed{n_1 = n_{f,m}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$

$\boxed{a_n^{\{m\}}}$ depends on $\boxed{f(z)}$, $\boxed{z_m}$, $\boxed{\text{region of analyticity}}$

Whether $f(z)$ is singular at $z = z_m$ or not

or at other points between $z$ and $z_m$

We can expand $f(z)$ about any point $z_m$

over powers of $(z - z_m)$.

# Poles for Residue Computation

$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z'-z_m)^{n+1}} dz'$$

$$= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right)$$

$z_k$ within $C$ : singularities of $\boxed{\dfrac{f(z)}{(z-z_m)^{n+1}}}$

$n \geq 0$ $\quad \{\text{poles of } f(z)\} \cup \{z = z_m\}$ $\quad n = 0, 1, 2 \cdots$

$n < 0$ $\quad \{\text{poles of } f(z)\}$ $\quad n = -1, -2, \cdots$

# Laurent's Theorem

$f$ : analytic within the $\boxed{\text{annular}}$ domain $D$

$$\boxed{r < |z - z_0| < R}$$

then

$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z - z_0)^k ,$$

valid for $\boxed{r < |z - z_0| < R}$

The coefficients $a_k$ are given by

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s - z_0)^{k+1}}\, ds , \qquad k = 0, \pm 1, \pm 2, \cdots$$

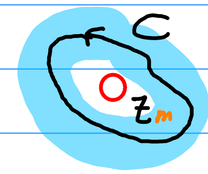
$C$ : a simple closed curve
that lies entirely within $D$
that encloses $z_0$

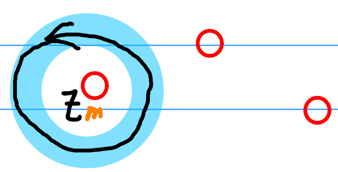
# Curve C & Domain D of the Laurent Series

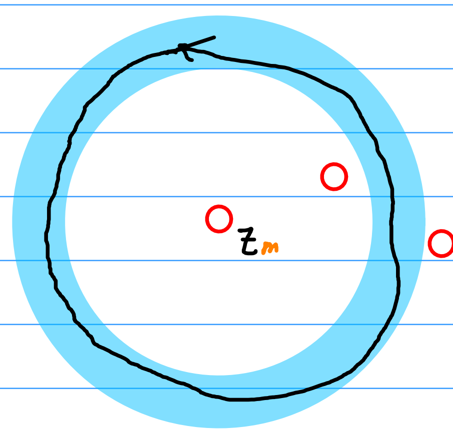
$$f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} (z - z_m)^n$$

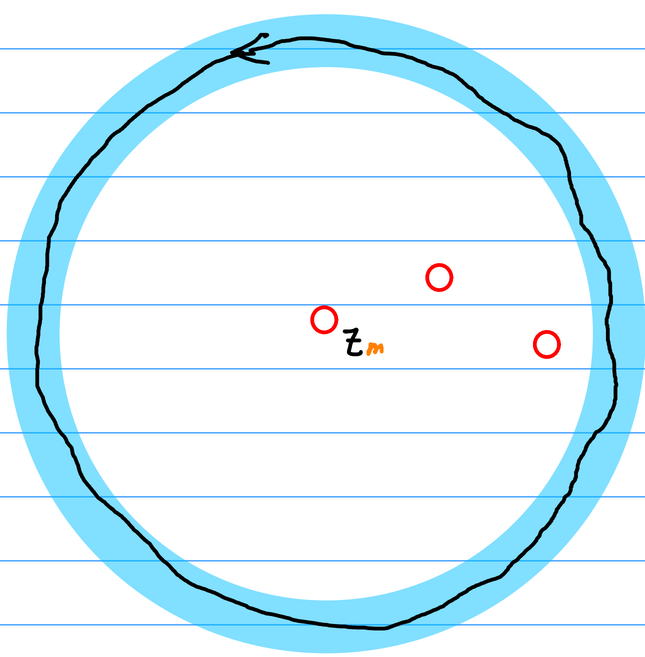
$$a_n^{\{m\}} = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z' - z_m)^{n+1}} \, dz'$$

$$= \sum_k \mathrm{Res}\left( \frac{f(z)}{(z - z_m)^{n+1}}, z_k \right)$$

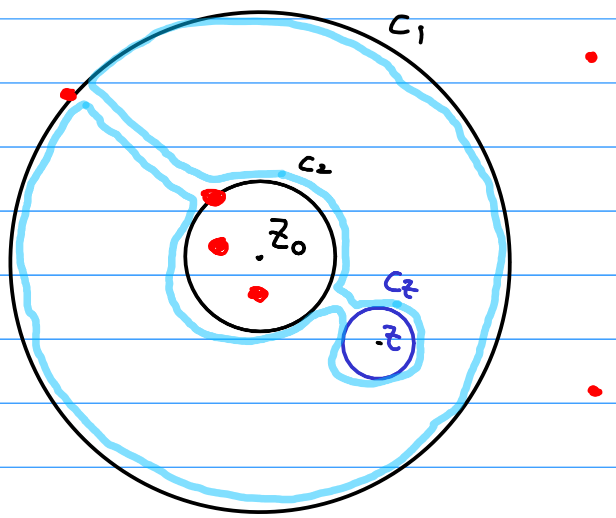


# Expansion Points and Evaluation Points




$z_0$ : expansion point

$z$ : evaluation point

which poles of $f(z)$ lie between the point of evaluation $z$ and the point $z_0$ about which the expansion is formed

$$\frac{f(z')}{(z'-z_0)}$$ is analytic between $C_1$ & $C_2$

deformation theorem $C_1 - C_2$ coincide

common contour $C$

# Residues

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds \Rightarrow \oint_C f(s)\, ds = 2\pi i \cdot a_{-1}$$

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(s)\, ds = Res(f(z), z_0)$$

$$= \begin{cases} \lim_{z \to z_0} (z - z_0) f(z) & \text{(simple)} \\ \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) & \text{(order } n\text{)} \end{cases}$$

# Cauchy's Residue Theorem

$f(z)$ : analytic on and within $C$

except a finite number of singular points

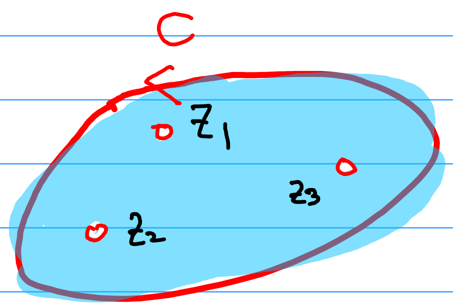
$z_1, z_2, \cdots, z_n$ within $C$

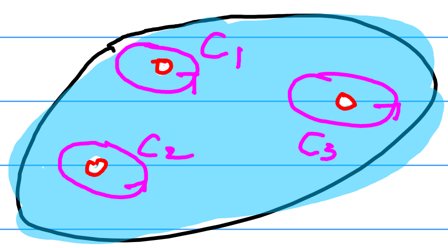
then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$

D : a simply connected domain

C : a simple closed contour in D




$\boxed{z_1}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k \qquad a_{-1}^{\{1\}} = \frac{1}{2\pi i} \oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$

$\boxed{z_2}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k \qquad a_{-1}^{\{2\}} = \frac{1}{2\pi i} \oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$

$\boxed{z_3}$ $\quad f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k \qquad a_{-1}^{\{3\}} = \frac{1}{2\pi i} \oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$

$\boxed{z_1}$

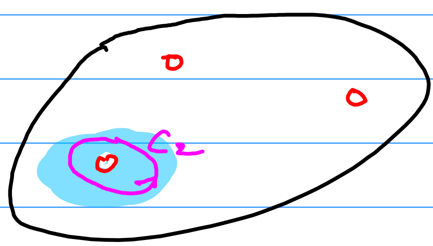
Laurent series expansion at $z_1$


$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_1)^k$$

$$a_{-1}^{\{1\}} = \frac{1}{2\pi i}\oint_{C_1} f(s)\,ds = \mathrm{Res}(f(z), z_1)$$

$\boxed{z_2}$

Laurent series expansion at $z_2$

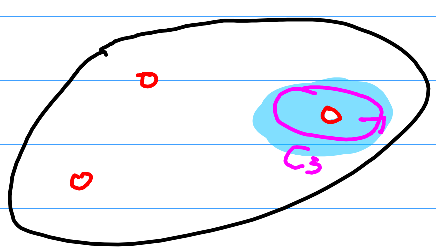



$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_2)^k$$

$$a_{-1}^{\{2\}} = \frac{1}{2\pi i}\oint_{C_2} f(s)\,ds = \mathrm{Res}(f(z), z_2)$$

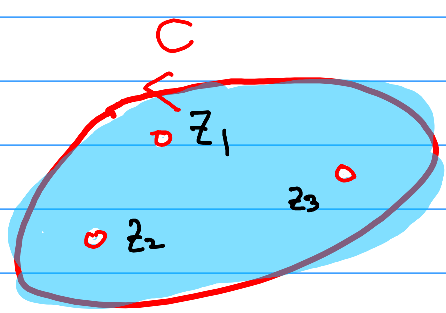
$\boxed{z_3}$

Laurent series expansion at $z_3$

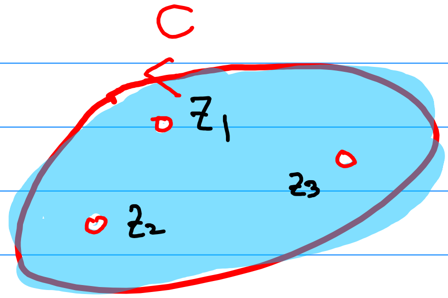



$$f(z) = \sum_{k=-\infty}^{+\infty} a_k (z-z_3)^k$$

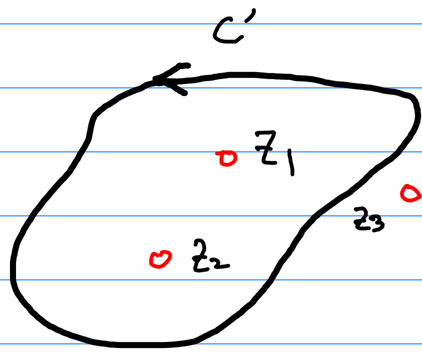
$$a_{-1}^{\{3\}} = \frac{1}{2\pi i}\oint_{C_3} f(s)\,ds = \mathrm{Res}(f(z), z_3)$$



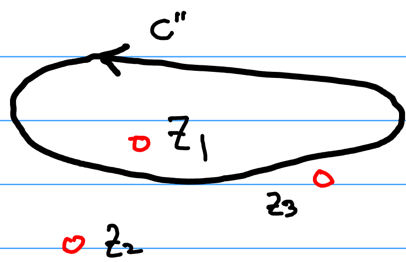

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f(z), z_k)$$



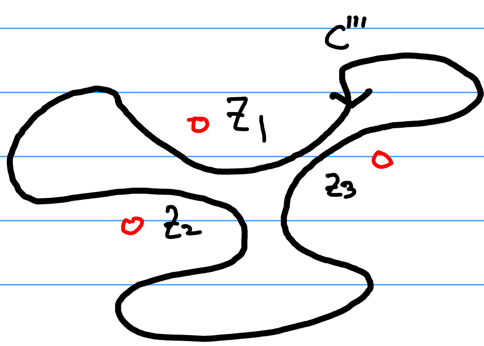

$$\int_C f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1) + 2\pi i \,\mathrm{Res}(f(z), z_2) + 2\pi i \,\mathrm{Res}(f(z), z_3)$$




$$\int_{C'} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1) + 2\pi i \,\mathrm{Res}(f(z), z_2)$$

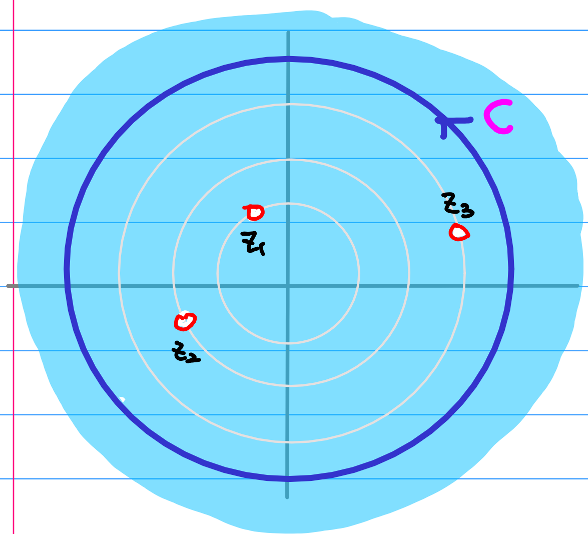



$$\int_{C''} f(z)\,dz = 2\pi i \,\mathrm{Res}(f(z), z_1)$$




$$\int_{C''} f(z)\,dz = 0$$

# Series Expansion at $z=0$

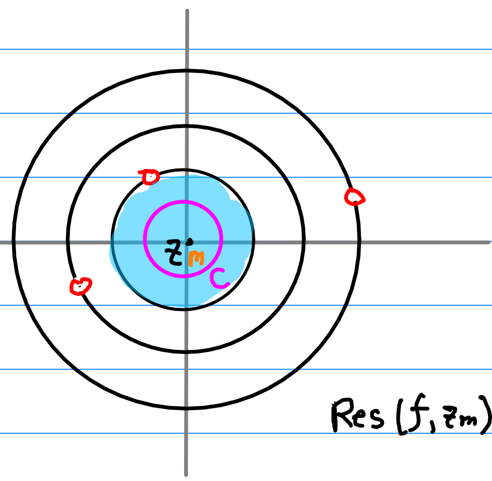



$$\boxed{f(z) = \sum_{n=n_1}^{\infty} a_n^{\{m\}} z^n}$$

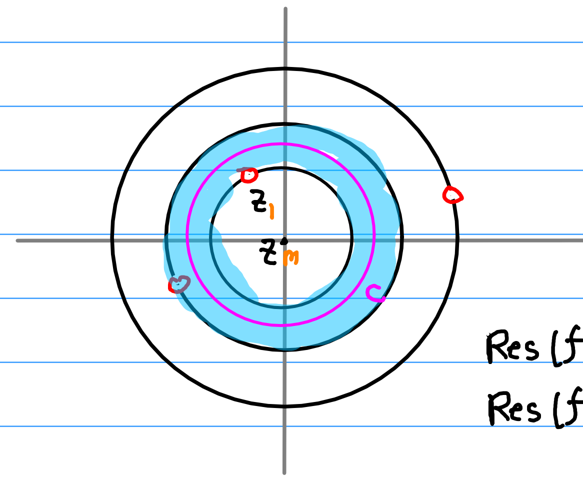
$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \operatorname{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

Poles $z_k$

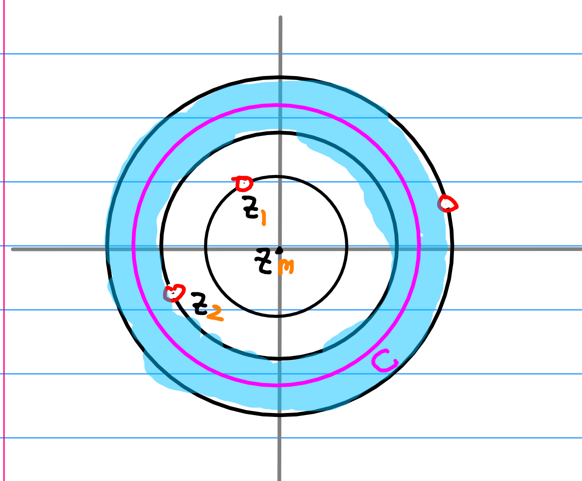
| | |
|---|---|
| $n \geq 0$ | $z_1, z_2, z_3, 0$ |
| $n < 0$ | $z_1, z_2, z_3$ |



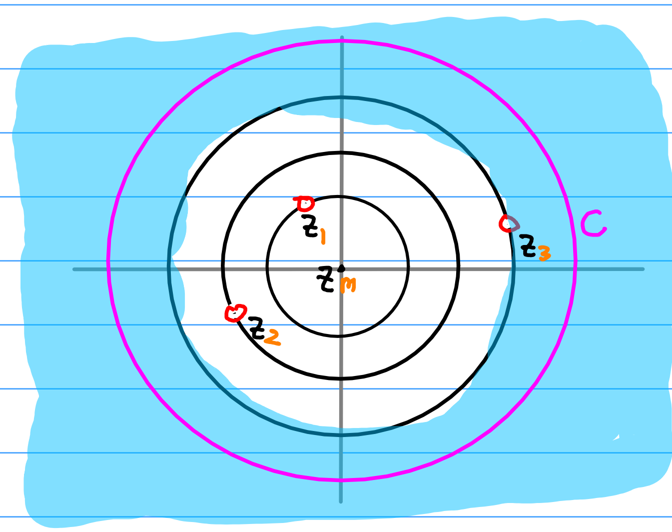

$\mathrm{Res}(f, z_m)$



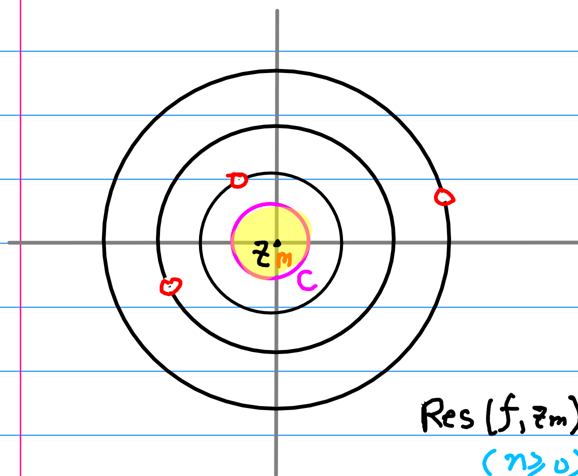

$\mathrm{Res}(f, z_1)$
$\mathrm{Res}(f, z_m)$




$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

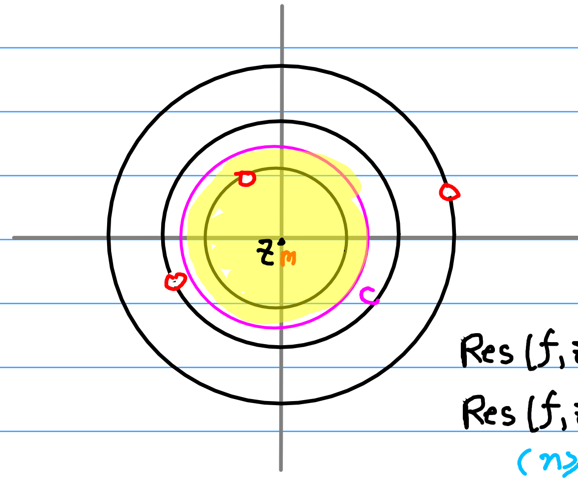



$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$
$+ \mathrm{Res}(f, z_m)$




$\mathrm{Res}(f, z_m)$

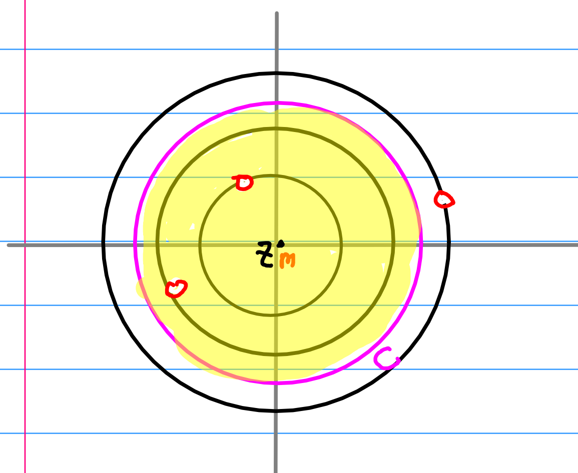
$(n \geq 0)$




$\mathrm{Res}(f, z_1)$

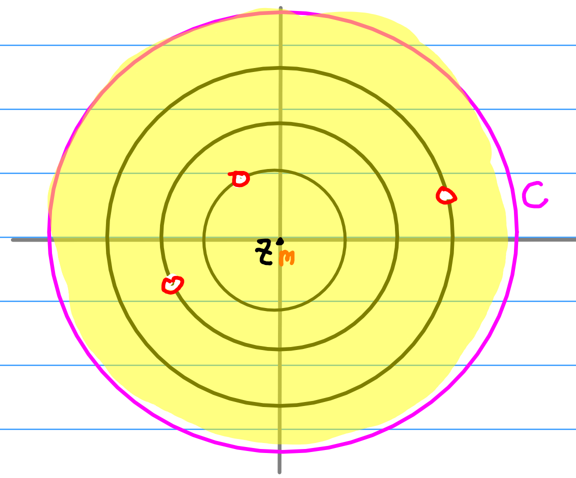
$\mathrm{Res}(f, z_m)$

$(n \geq 0)$




$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_m)$

$(n \geq 0)$




$\mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_1) + \mathrm{Res}(f, z_3)$

$+ \mathrm{Res}(f, z_m)$

$(n \geq 0)$

# Inverse z-Transform $x[n] = \frac{1}{2\pi i} \int_C X(z) z^{n-1} dz$

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$z^{n-1} X(z) = \left( \sum_{k=0}^{\infty} x_k z^{-k} \right) z^{n-1}$$

$$\boxed{\int z^{n-1} \, \text{LHS} \, dz = \int \text{RHS} \, z^{n-1} dz}$$

$$= \sum_{k=0}^{\infty} x_k z^{-k+n-1}$$

$$\boxed{[0, \infty) = [0, n-1] \cup [n] \cup [n+1, \infty)}$$

$$= \sum_{k=0}^{n-1} x_k z^{-k+n-1} + \sum_{k=n}^{n} x_k z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k z^{-k+n-1} + \frac{x_n}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z) z^{n-1} dz = \int_c \sum_{k=0}^{n-1} x_k z^{-k+n-1} dz + \int_c \frac{x_n}{z^1} dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}} dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1} dz + x_n \int_c \frac{1}{z^1} dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}} dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_n \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x[n] = \frac{1}{2\pi i} \int_c X(z) z^{n-1} dz$$

Z-transform

$\boxed{z_m = 0}$

$$\boxed{\begin{aligned} x[n] &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right) \end{aligned}}$$

$n > 0$ $\quad z_k$ : poles of $f(z)$

$n = 0$ $\quad z_k$ : poles of $f(z)$ + $z = 0$

$z^{n-1} = z^{-1} = \frac{1}{z}$

$x[n]$ includes $u[n]$ $\rightarrow$ $X[z]$ contains $z$ on its numerator

Also, think about modified partial fraction $\dfrac{X[z]}{z}$

Laurent Expansion

expansion at $z_m$

$$\boxed{\begin{aligned} a_n^{\{m\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z - z_m)^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{(z - z_m)^{n+1}}, z_k\right) \end{aligned}}$$

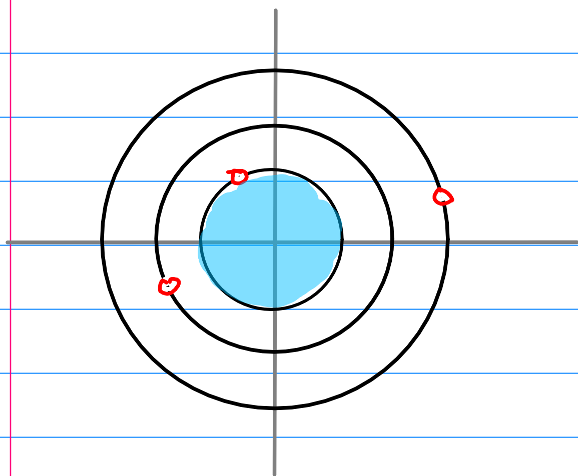
$\boxed{z_m = 0}$

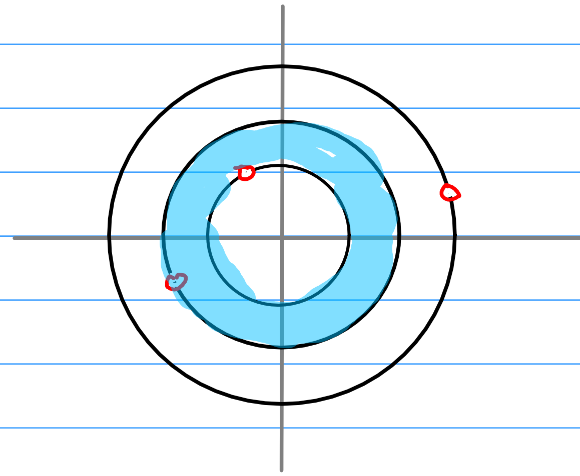
$$\boxed{\begin{aligned} a_n^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, z_k\right) \end{aligned}}$$

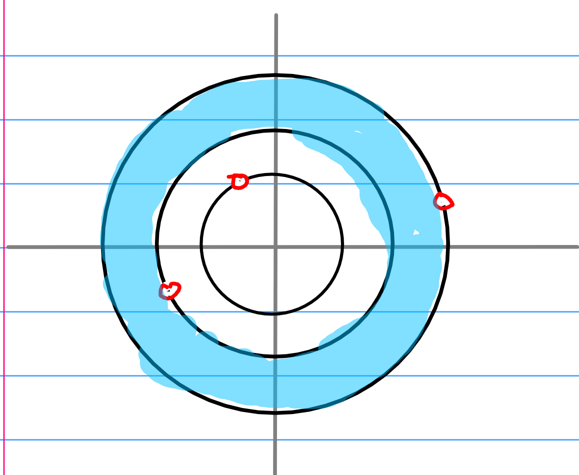
$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C f(z)\, z^{n-1}\, dz \\ &= \sum_k \mathrm{Res}\left(f(z)\, z^{n-1}, z_k\right) \end{aligned}$$

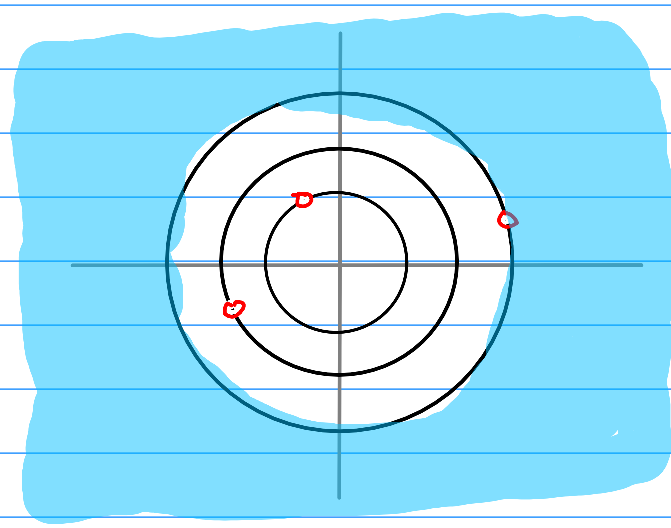
$$\begin{aligned} a_{-n}^{\{0\}} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{-n+1}}\, dz \\ &= \sum_k \mathrm{Res}\left(\frac{f(z)}{z^{-n+1}}, z_k\right) \end{aligned}$$

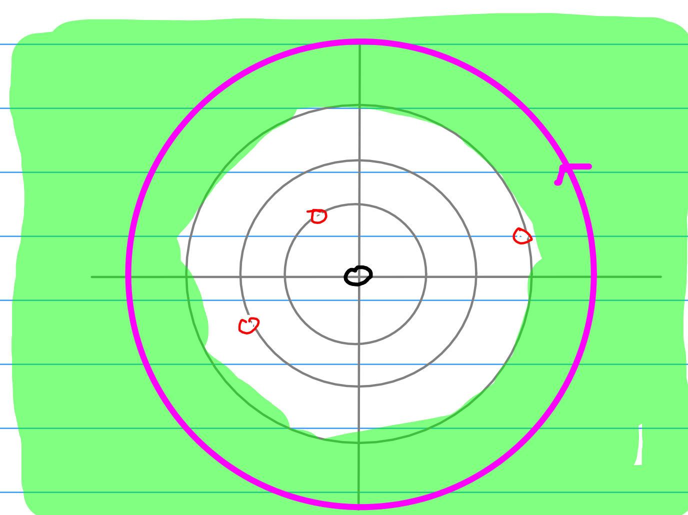
# Different D, Different Laurent Series











$$x[n] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$= \sum_k \mathrm{Res}\left(X(z) z^{n-1}, z_k\right)$$

z-transform

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

Complex Variables and Ap
Brown & Churchill

$$f(z) = \frac{-1}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

$D_1$ : $|z| < 1$

$D_2$ : $1 < |z| < 2$

$D_3$ : $2 < |z|$

① $D_1$ $\quad |z| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z| < 1$$

② $D_2$ $\quad 1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

③ $D_3$ $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

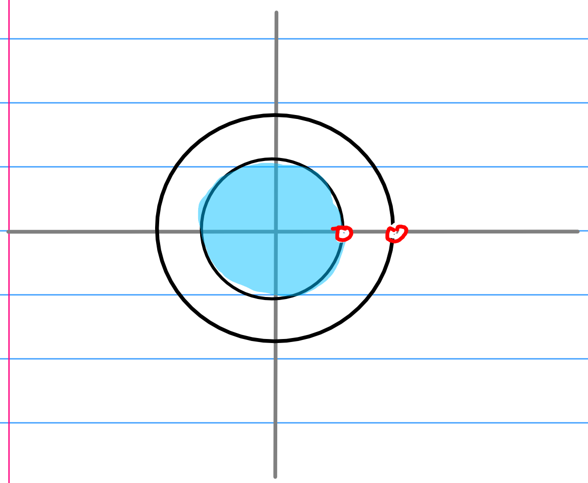
$$= \sum_{n=0}^{0} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

① $D_1$ $|z|<1$, $\left|\frac{z}{2}\right|<1$

$$\frac{f(z)}{z^{n+1}} = \frac{-1}{(z-1)(z-2)z^{n+1}}$$

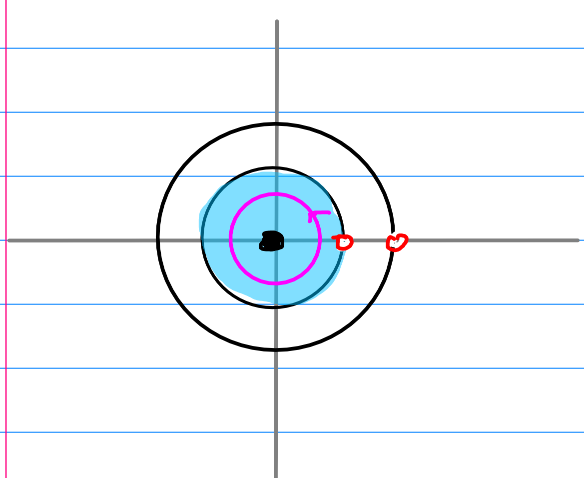


$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \quad |z|<1$$

$$a_n = \frac{f(z)}{z^{n+1}} = \frac{1}{(z-1)(z-2)z^{n+1}} \qquad \frac{1}{z-1} - \frac{1}{z-2}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_m)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$n \geq 0$ then the pole $z=0$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{d}{dz}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)\left((z-1)^{-2} - (z-2)^{-2}\right)$$

$$\frac{d^2}{dz^2}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)\left((z-1)^{-3} - (z-2)^{-3}\right)$$

$$\frac{d^3}{dz^3}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)(-2)(-3)\left((z-1)^{-4} - (z-2)^{-4}\right)$$

$$\frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n n! \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n}\left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

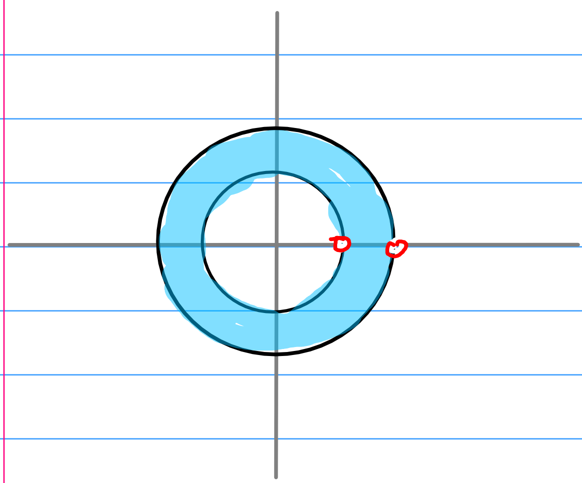
$$= -1 + 2^{-n-1}$$

$$a_n = -1 + 2^{-n-1} \qquad (n \geq 0)$$

$$f(z) = \sum_{n=n_1}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(2^{-n-1} - 1\right) z^n$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

② $D_2$ $\quad 1 < |z| < 2 \Rightarrow \quad \left|\frac{1}{z}\right| < 1, \quad \left|\frac{z}{2}\right| < 1$



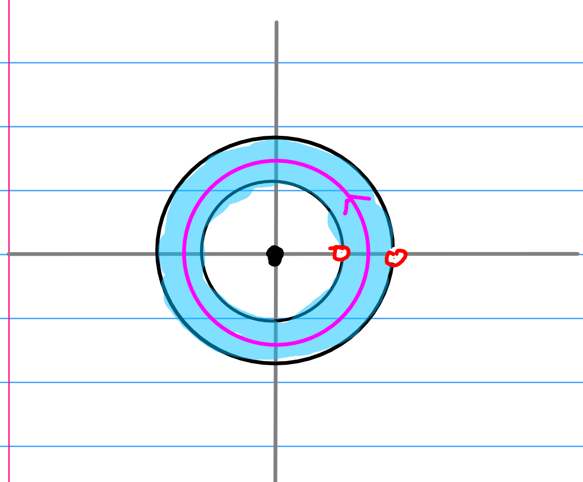
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z} \cdot \frac{1}{1-\left(\frac{1}{z}\right)} + \frac{1}{2} \frac{1}{1-\left(\frac{z}{2}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$



$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_k)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$\frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z) \quad (\text{order } n)$$

$$\frac{1}{n!} \lim_{z \to 0} \frac{d^n}{dz^n} \left((z-1)^{-1} - (z-2)^{-1}\right) = (-1)^n \lim_{z \to 0} \left((z-1)^{-n-1} - (z-2)^{-n-1}\right)$$

$$= (-1)^n \left((-1)^{-n-1} - (-2)^{-n-1}\right)$$

$$= -1 + 2^{-n-1}$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1) \frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

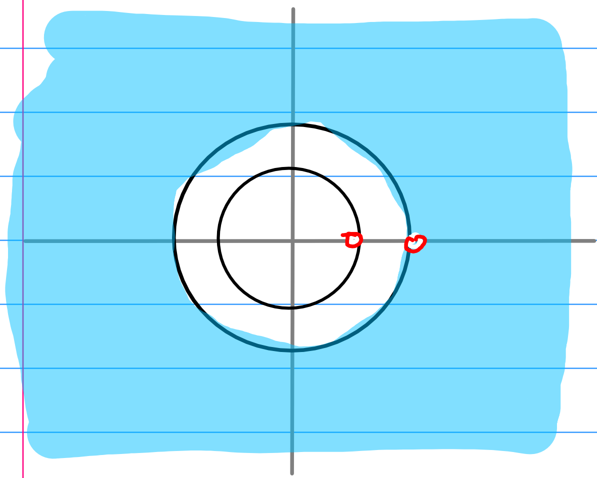
| $n=-3$ | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ | |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $-1+2^{-1}$ | $-1+2^{-2}$ | $-1+2^{-3}$ | $\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$ |
| $1$ | $1$ | $1$ | $2^{-1}$ | $2^{-2}$ | $2^{-3}$ | |

$$\begin{cases} a_n = 2^{-n-1} & n \geq 0 \\ a_n = 1 & n < 0 \end{cases} \qquad \begin{cases} 2^{-n-1} z^n \\ z^{-n} \end{cases}$$

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{z^n} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$

③ $D_3$ $\quad 2 < |z| \quad \left|\frac{2}{z}\right| < 1 \quad \left|\frac{1}{z}\right| < 1$



$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-(\frac{1}{z})} - \frac{1}{z}\frac{1}{1-(\frac{2}{z})}$$

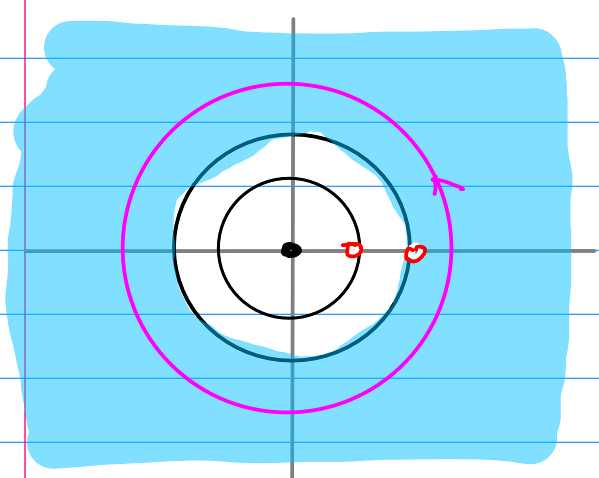
$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$a_n = \sum_{k=1}^{M} \mathrm{Res}\left(\frac{f(z)}{(z-z_n)^{n+1}}, z_k\right) = \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right)$$

$$+ \mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right)$$



$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 0\right) = -1 + 2^{-n-1} \quad (n \geq 0)$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 1\right) = \lim_{z \to 1} (z-1)\frac{-1}{(z-1)(z-2)z^{n+1}} = 1$$

$$\mathrm{Res}\left(\frac{-1}{(z-1)(z-2)z^{n+1}}, 2\right) = \lim_{z \to 2} (z-2)\frac{-1}{(z-1)(z-2)z^{n+1}} = -\frac{1}{2^{n+1}}$$

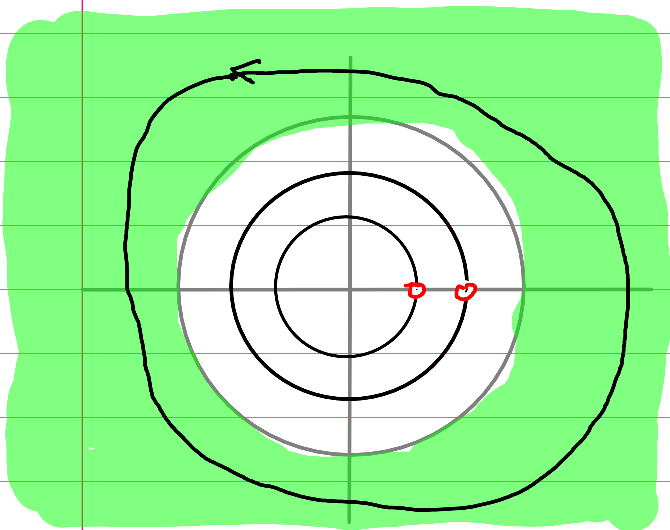
| $n=-3$ | $n=-2$ | $n=-1$ | $n=0$ | $n=1$ | $n=2$ | |
|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $-1+2^{-1}$ | $-1+2^{-2}$ | $-1+2^{-3}$ | $\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 0\right)$ |
| $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 1\right)$ |
| $-2^2$ | $-2$ | $-1$ | $-2^{-1}$ | $-2^{-2}$ | $-2^{-3}$ | $\mathrm{Res}\left(\frac{f(z)}{z^{n+1}}, 2\right)$ |
| $1-2^2$ | $1-2$ | $0$ | $0$ | $0$ | $0$ | |

$$a_n = 1 - 2^{-n+1} \quad n < 0$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \sum_{n=-1}^{-\infty} (1-2^{-n+1}) z^n = \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$f(z) = \frac{-1}{(z-1)(z-2)}$$



$$x[n] = \frac{1}{2\pi i}\int_C \boxed{X(z)\, z^{n-1}}\, dz = \sum_{j=1}^{k} \mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, z_j\right)$$

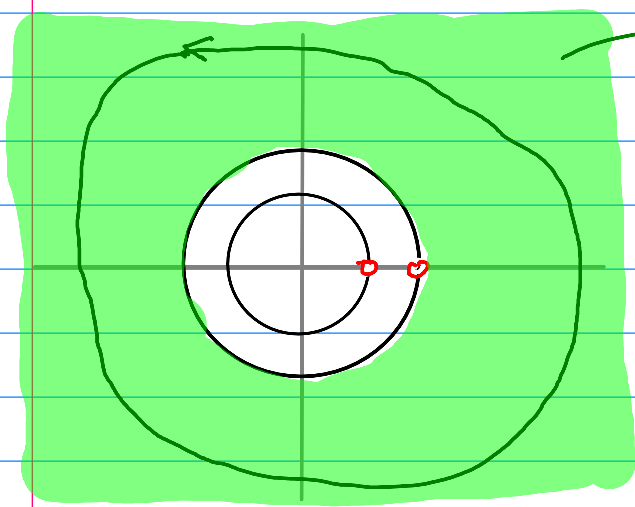
$$X(z) = \frac{-1}{(z-1)(z-2)}$$

$$X(z)\, z^{n-1} = \frac{-1}{(z-1)(z-2)}\, z^{n-1}$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 1\right) = (z-1)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=1} = 1$$

$$\mathrm{Res}\left(\boxed{X(z)\, z^{n-1}}, 2\right) = (z-2)\frac{-1}{(z-1)(z-2)}\, z^{n-1}\Big|_{z=2} = -2^{n-1}$$

$$x[n] = 1 - 2^{n-1}$$



ROC (Region of Convergence)

$$|z| > 2 \Rightarrow \frac{2}{|z|} < 1$$

$$\left(\frac{2}{z}\right)^0 + \left(\frac{2}{z}\right)^1 + \left(\frac{2}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{2}{z}}$$

$$|z| > 2 \Rightarrow \frac{1}{|z|} < 1$$

$$\left(\frac{1}{z}\right)^0 + \left(\frac{1}{z}\right)^1 + \left(\frac{1}{z}\right)^2 + \cdots \xrightarrow[\text{converge}]{} \frac{1}{1-\frac{1}{z}}$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\left(\frac{1}{z}\right)} - \frac{1}{z}\frac{1}{1-\left(\frac{2}{z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n}$$

$$\left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots$$
$$+ \frac{1}{2}\left\{ \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \cdots \right\} \xrightarrow[\text{converge}]{} \frac{1}{z-1} - \frac{1}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$(1-2^0)z^{-1} + (1-2^1)z^{-2} + (1-2^2)z^{-3} + \cdots \xrightarrow[\text{converge}]{} \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$

$$x[n] = 1 - 2^{n-1} \longleftrightarrow X(z) = \frac{-1}{(z-1)(z-2)} \quad (|z| > 2)$$