Laurent Series and z-Transform
- Geometric Series
Time Shift A

20180914 Fri

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Causal Signal $a(n) \Rightarrow f(z), X(z)$

$(n \geq 0)$ $(|z| < p)$ $(|z| > p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \rightarrow \frac{2}{2 - z} \quad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \rightarrow \frac{z}{z - 0.5} \quad (|z| > 0.5)$$

Anti-Causal Signal

$a(n) \Rightarrow -f(z), -X(z)$

$(n<0) \quad (|z|>p) \quad (|z|<p^{-1})$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f_2(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)}$$

$$X_2(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1-(2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f_2(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \rightarrow \frac{2}{z-2} = -f(z) \quad (|z|>2)$$

$$X_2(z) = \frac{(2z)}{1-(2z)} \rightarrow \frac{z}{0.5-z} = -X(z) \quad (|z|<0.5)$$

$$a'_n = -\left(\frac{1}{2}\right)^n \quad (n<0)$$

$$f(z) = \frac{2}{2-z} \rightarrow \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (|z|>2)$$

$$X(z) = \frac{z}{z-0.5} \rightarrow \frac{-(2z)}{1-(2z)} \quad (|z|<0.5)$$

Inverse z

$z \leftarrow z^{-1}$, $\text{RoC}(z) \leftarrow \text{RoC}(z^{-1})$

Causal	z^{-1}	anti-Causal
$f(z) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$	⟷	$f(z^{-1}) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$
$X(z) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	⟷	$X(z^{-1}) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$

$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (\lvert z^{-1}\rvert < 2) \rightarrow f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$

$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (\lvert z^{-1}\rvert > 0.5) \rightarrow X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal) with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal) with the same formula as causal f(z)

Causal		anti-Causal
$f(z) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$	⟷	$X(z^{-1}) = \frac{2}{2-z} \quad (\lvert z\rvert < 2)$
$X(z) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$	⟷	$f(z^{-1}) = \frac{z}{z-0.5} \quad (\lvert z\rvert > 0.5)$

Inverse z

$f(z^{-1})$, $ROC(z^{-1})$ $\Rightarrow$ a_{-n}

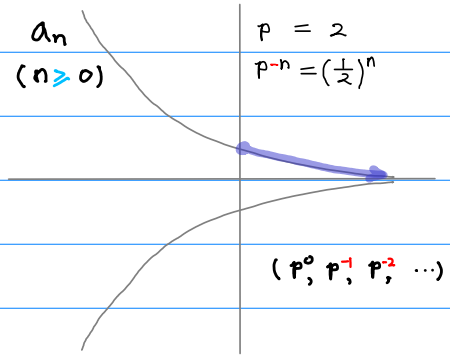
Causal	anti-Causal
$f(z) = \frac{2}{2-z}$ $(\lvert z\rvert < 2)$	$f(z^{-1}) = \frac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$
$X(z) = \frac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$	$X(z^{-1}) = \frac{2}{2-z}$ $(\lvert z\rvert < 2)$

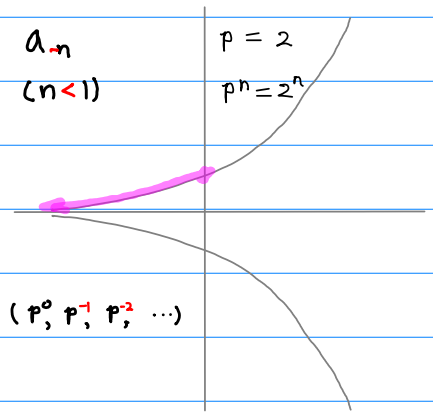
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \quad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \quad n = 0, 1, 2, \cdots$$

$$f(z^{-1}) = \frac{2}{2-z^{-1}} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \quad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^1 z^{-1} + p^2 z^{-2} + \cdots \quad n = 0, -1, -2, \cdots$$





Inverse z $X(z^{-1})$, $ROC(z^{-1}) \Rightarrow a_{-n}$

Causal	anti-Causal
$f(z) = \frac{2}{2-z}$ $(\lvert z\rvert < 2)$	$f(z^{-1}) = \frac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$
$X(z) = \frac{z}{z-0.5}$ $(\lvert z\rvert > 0.5)$	$X(z^{-1}) = \frac{2}{2-z}$ $(\lvert z\rvert < 2)$

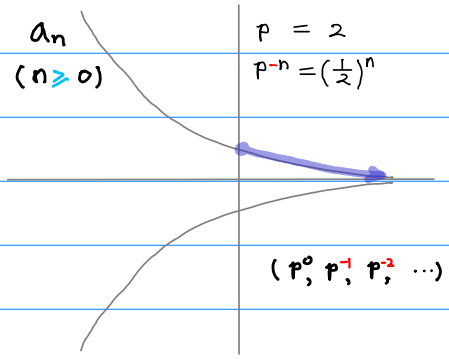
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

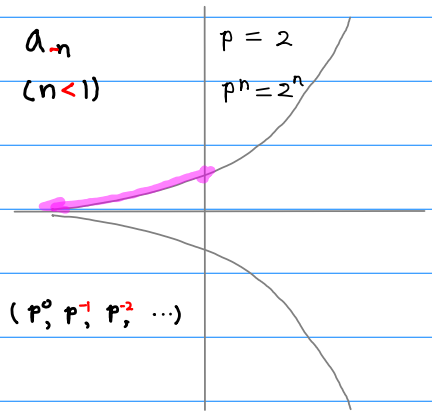
$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \qquad a_{-n} = \left(\tfrac{1}{2}\right)^{-n}$$

$$= p^0 z^0 + p^1 z^1 + p^2 z^2 + \cdots \qquad n = 0, -1, -2, \cdots$$

z





Inverse ROC

$\text{ROC}(z) \leftarrow \text{ROC}(z^{-1})$

Causal → **anti-Causal**

$$f(z) = \frac{2}{2-z} \quad (|z| < 2) \quad \Rightarrow \quad -f(z) = -\frac{2}{2-z} \quad (|z| > 2)$$

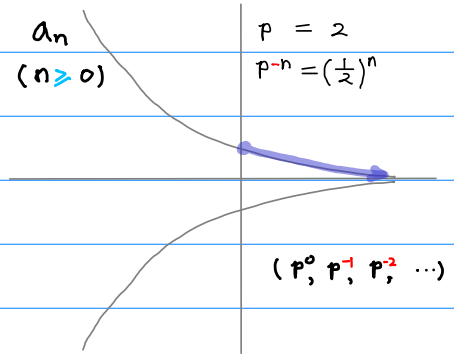
$$X(z) = \frac{z}{z-0.5} \quad (|z| > 0.5) \quad \Rightarrow \quad -X(z) = -\frac{z}{z-0.5} \quad (|z| < 0.5)$$

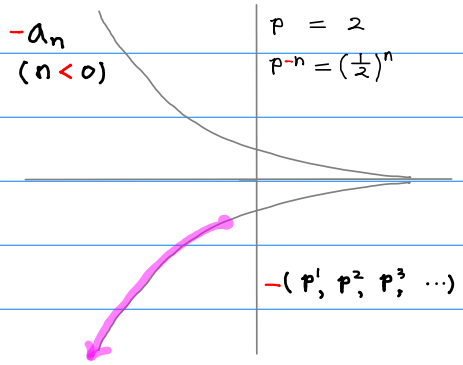
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots \qquad a_n = \left(\frac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \qquad n = 0, 1, 2, \cdots$$

$$-f(z) = \frac{2}{z-2} = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots \qquad a_n = -\left(\frac{1}{2}\right)^n$$

$$= p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots \qquad n = -1, -2, -3, \cdots$$





Inverse ROC $f(z)$, $ROC(z^{-1}) \Rightarrow -a_n$

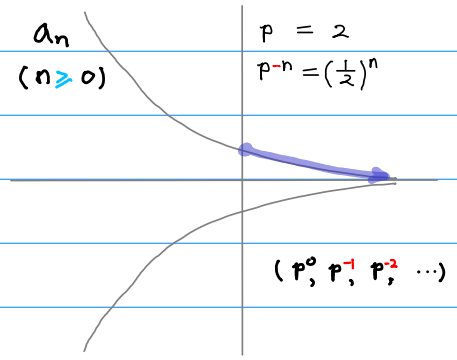
Causal	anti-Causal
$f(z) = \frac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \frac{2}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

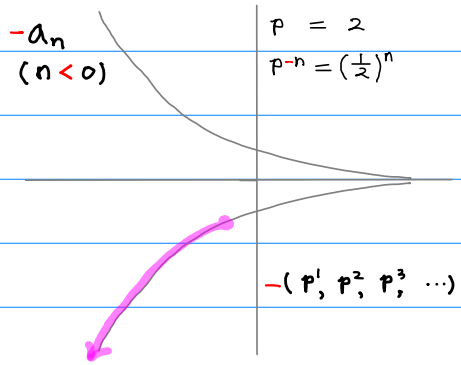
$$f(z) = \frac{2}{2-z} = \frac{1}{1-\left(\frac{z}{2}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^1 + \left(\tfrac{1}{2}\right)^2 z^2 + \cdots \quad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots \quad n = 0, 1, 2, \cdots$$

$$f(z) = \frac{2}{2-z} = \frac{-\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} = -\left[(2)^1 z^{-1} + (2)^2 z^{-2} + \left(\tfrac{1}{2}\right)^3 z^{-3} + \cdots\right] \quad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots\right] \quad n = -1, -2, -3, \cdots$$





Inverse ROC $X(z)$, $ROC(z^{-1}) \Rightarrow -a_n$

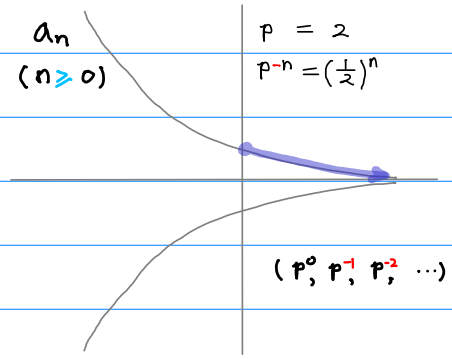
Causal	anti-Causal
$f(z) = \frac{z}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \frac{z}{2-z} \quad (\lvert z \rvert > 0.5)$
$X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert < 2)$

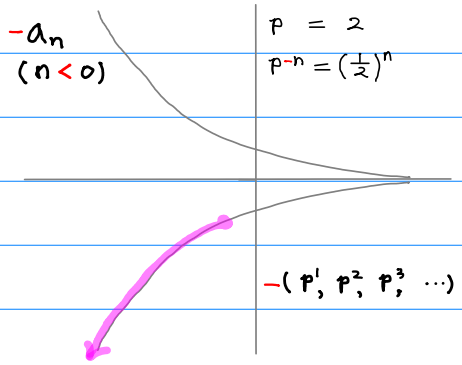
$$X(z) = \frac{z}{z-0.5} = \frac{1}{1-\left(\frac{1}{2z}\right)} = \left(\tfrac{1}{2}\right)^0 z^0 + \left(\tfrac{1}{2}\right)^1 z^{-1} + \left(\tfrac{1}{2}\right)^2 z^{-2} + \cdots \qquad a_n = \left(\tfrac{1}{2}\right)^n$$

$$= p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots \qquad n = 0, 1, 2, \cdots$$

$$X(z) = \frac{z}{z-0.5} = \frac{-(2z)}{1-(2z)} = -\left[(2)^1 z^1 + (2)^2 z^2 + \left(\tfrac{1}{2}\right)^3 z^3 + \cdots\right] \qquad -a_n = -\left(\tfrac{1}{2}\right)^n$$

$$= -\left[p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots\right] \qquad n = -1, -2, -3, \cdots$$





p=2

causal	anti-causal
$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$
$f(z) = \frac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z \rvert < 2)$	$f(z) = \frac{\left(\frac{2}{z}\right)}{1-\left(\frac{2}{z}\right)} \quad (\lvert z \rvert > 2)$
$X(z) = \frac{1}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z \rvert > 0.5)$	$X(z) = \frac{(2z)}{1-(2z)} \quad (\lvert z \rvert < 0.5)$
$f(z) = \frac{2}{2-z} \quad (\lvert z \rvert < 2)$	$f(z) = \frac{2}{z-2} \quad (\lvert z \rvert > 2)$
$X(z) = \frac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$X(z) = \frac{z}{0.5-z} \quad (\lvert z \rvert < 0.5)$

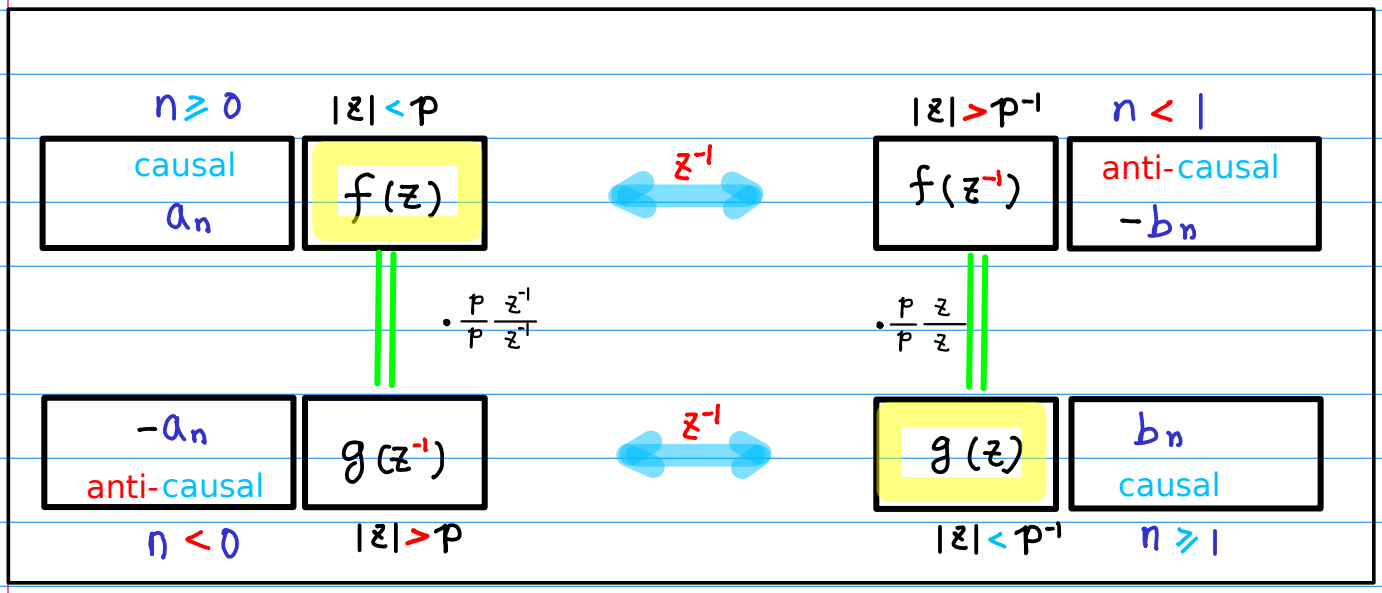
p=1/2

causal	anti-causal
$a_n = (2)^n \quad (n \geq 0)$	$a_n = (2)^n \quad (n < 0)$
$f(z) = \frac{1}{1-(2z)} \quad (\lvert z \rvert < 0.5)$	$f(z) = \frac{\left(\frac{1}{2z}\right)}{1-\left(\frac{1}{2z}\right)} \quad (\lvert z \rvert > 0.5)$
$X(z) = \frac{1}{1-\left(\frac{z}{2}\right)} \quad (\lvert z \rvert > 2)$	$X(z) = \frac{\left(\frac{z}{2}\right)}{1-\left(\frac{z}{2}\right)} \quad (\lvert z \rvert < 2)$
$f(z) = \frac{0.5}{0.5-z} \quad (\lvert z \rvert < 0.5)$	$f(z) = \frac{0.5}{z-0.5} \quad (\lvert z \rvert > 0.5)$
$X(z) = \frac{z}{z-2} \quad (\lvert z \rvert > 2)$	$X(z) = \frac{z}{2-z} \quad (\lvert z \rvert < 2)$

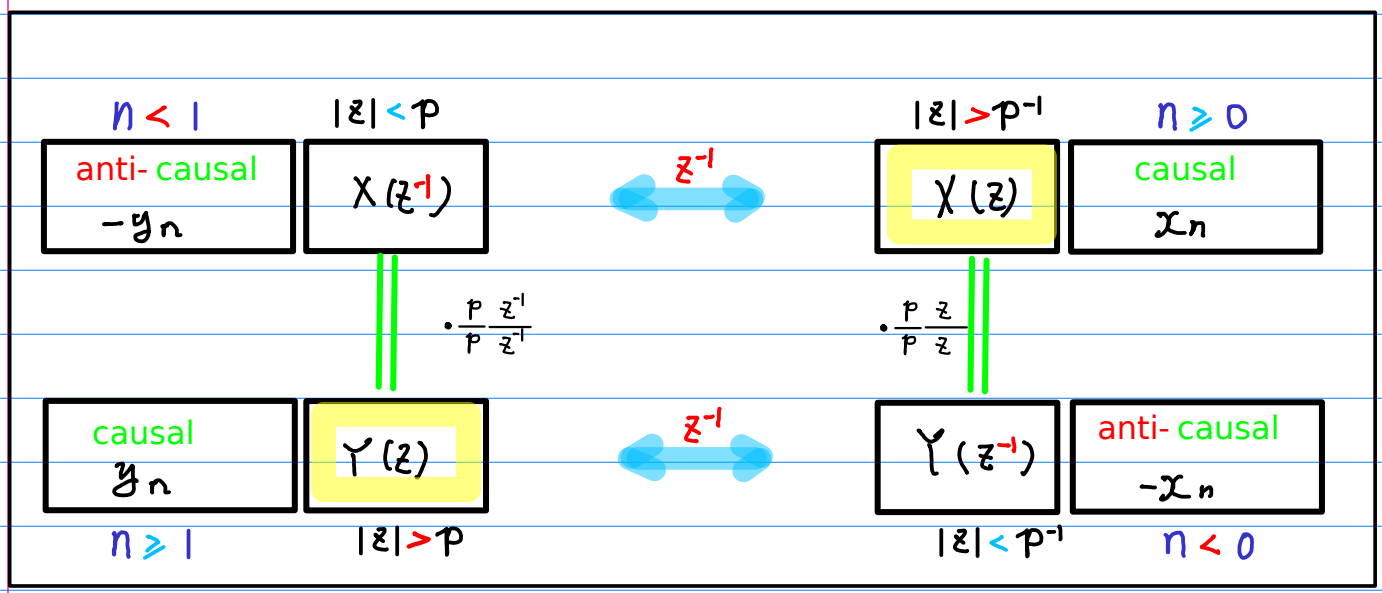
$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $b_n \leftrightarrow g(z)$



Z-Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n$



$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

Laurent Series $a_n \leftrightarrow f(z)$ $-a_{-n}=b_n \leftrightarrow g(z)$

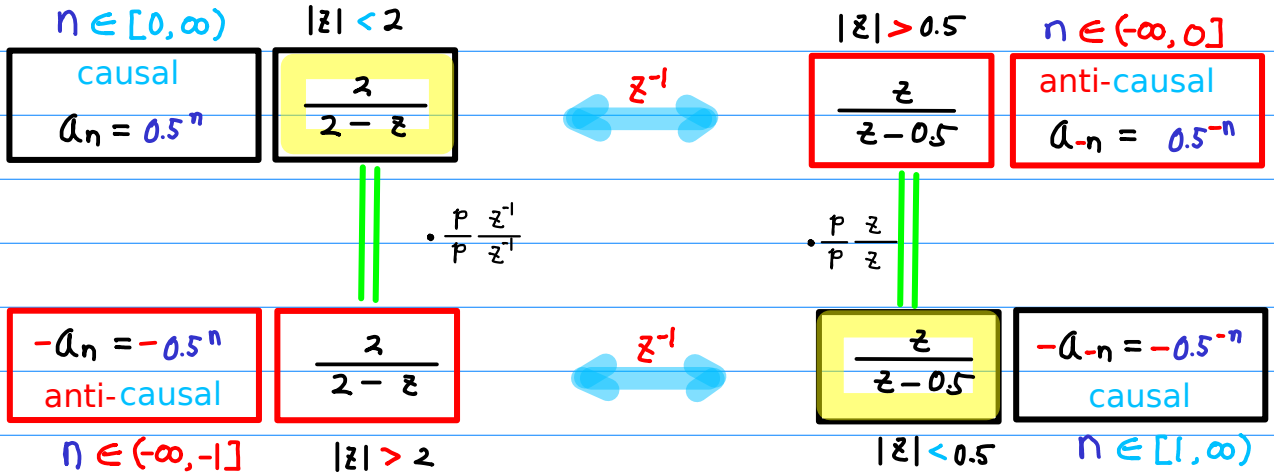
causal a_n	$\xleftrightarrow{z^{-1}}$	anti-causal a_{-n}
$-a_n$ anti-causal	$\xleftrightarrow{z^{-1}}$	$-a_{-n}$ causal

Z-Transform $X(z) \leftrightarrow x_n$ $Y(z) \leftrightarrow y_n = -x_{-n}$

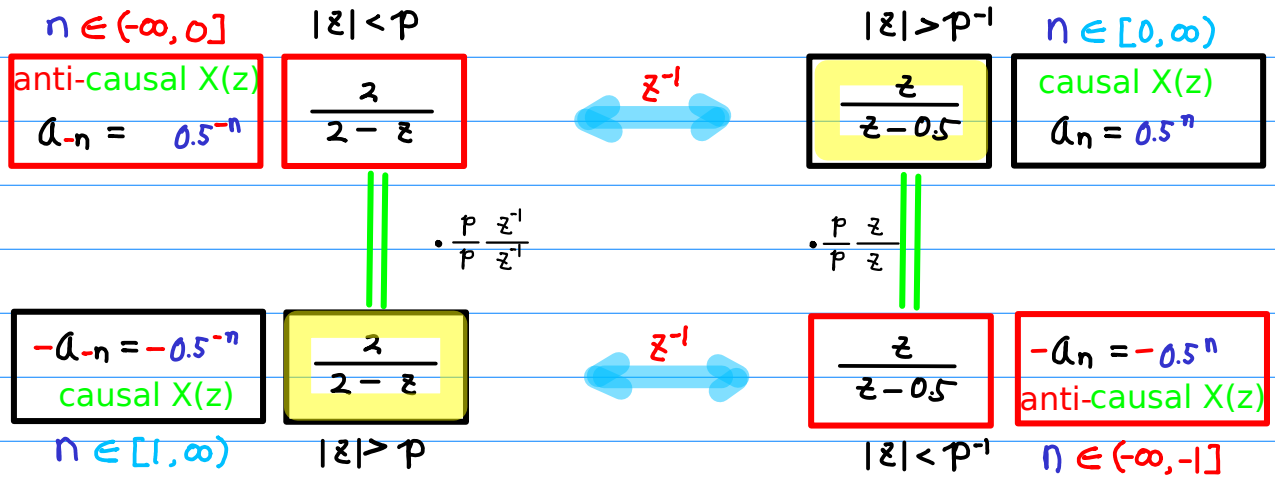
anti-causal x_{-n}	$\xleftrightarrow{z^{-1}}$	causal x_n
causal $-x_{-n}$	$\xleftrightarrow{z^{-1}}$	anti-causal $-x_n$

$$f(z),\ X(z) = \frac{2}{2-z},\ \frac{z}{z-0.5}$$

f(z) Laurent Series



X(z) z-Transform

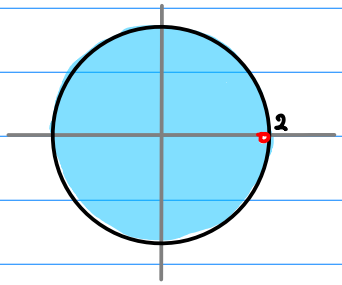


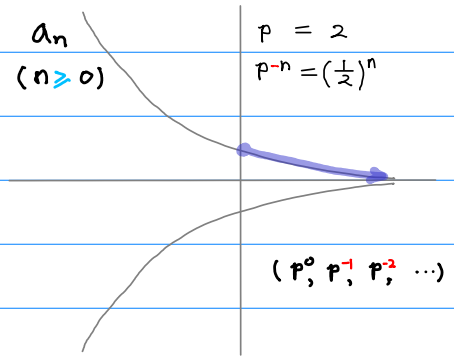
$$f(z) = \frac{2}{2-z}, \quad \frac{z}{z-0.5}$$

0.5^n	0.5^{-n}
-0.5^n	-0.5^{-n}

$n \in [0, \infty)$ causal $a_n = 0.5^n$

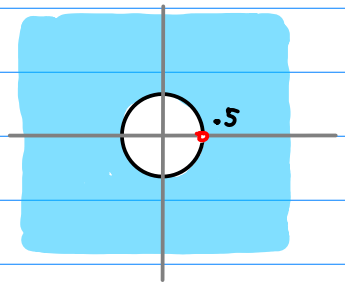
$|z| < 2$ $\quad \frac{2}{2-z}$

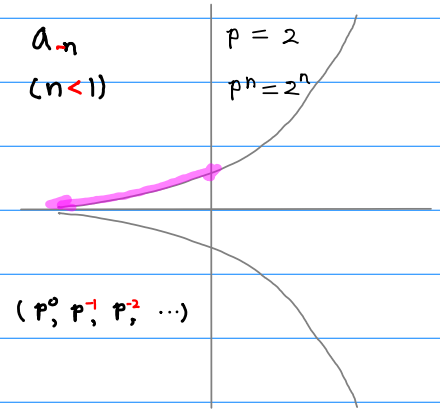




$|z| > 0.5$ $\quad \frac{z}{z-0.5}$

$n \in (-\infty, 0]$ anti-causal $a_{-n} = 0.5^{-n}$



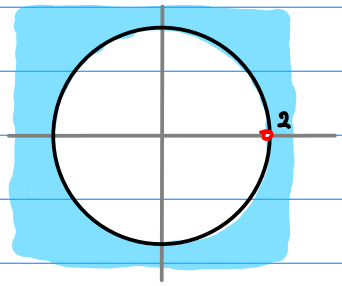


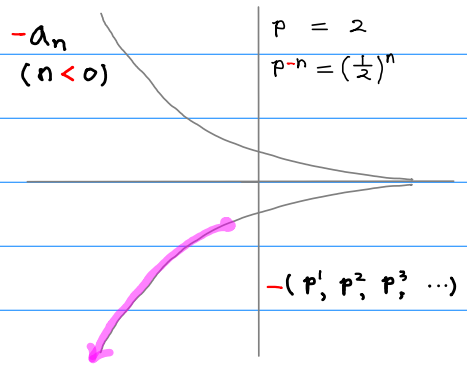
$-a_n = -0.5^n$ anti-causal

$n \in (-\infty, -1]$

$\frac{2}{2-z}$

$|z| > 2$

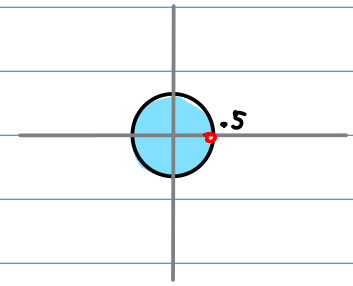


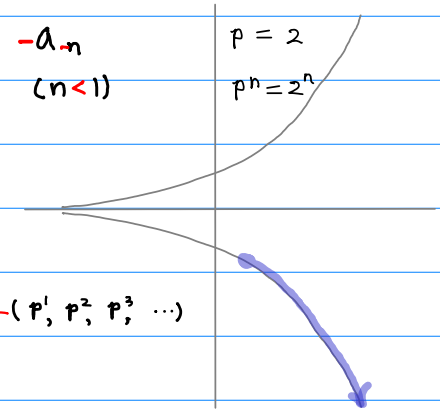


$\frac{z}{z-0.5}$

$-a_{-n} = -0.5^{-n}$ causal

$|z| < 0.5$ $\quad n \in [1, \infty)$





$$X(z) = \frac{2}{2-z}, \quad \frac{z}{z-0.5}$$

0.5^{-n}	0.5^{n}
-0.5^{-n}	-0.5^{n}

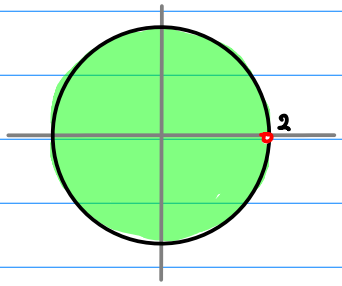
$n < 1$ | $|z| < 2$

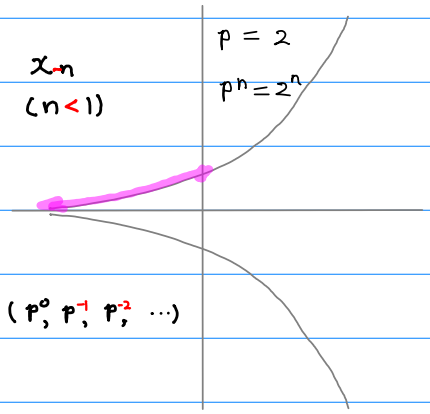
anti-causal

$x_{-n} = 0.5^{-n}$

$-\frac{0.5}{1-0.5z}$

x_{-n}





z^{-1}

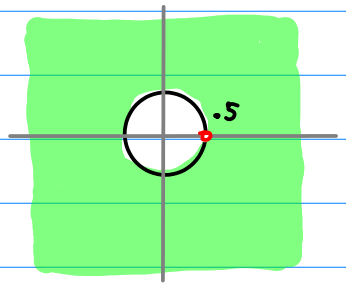
$|z| > 0.5$ | $n \geq 0$

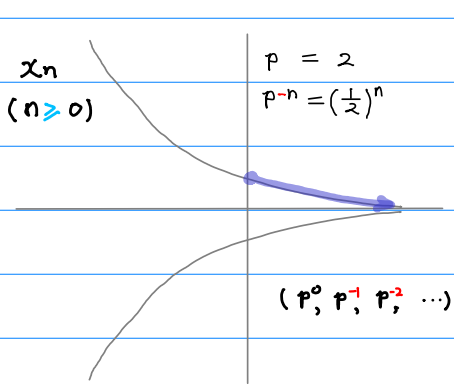
$-\frac{0.5}{1-0.5z^{-1}}$

causal

$x_n = 0.5^{n}$

x_n





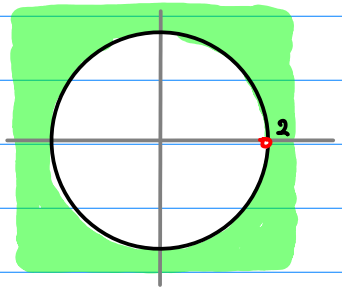
$n \geq 1$ | $|z| > 2$

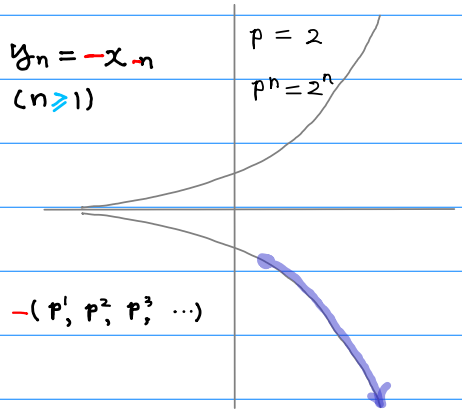
y_n

$-x_{-n} = -0.5^{-n}$

causal

$\frac{z^{-1}}{1-2z^{-1}}$





z^{-1}

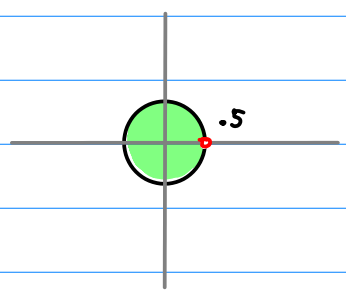
$|z| < 0.5$ | $n < 0$

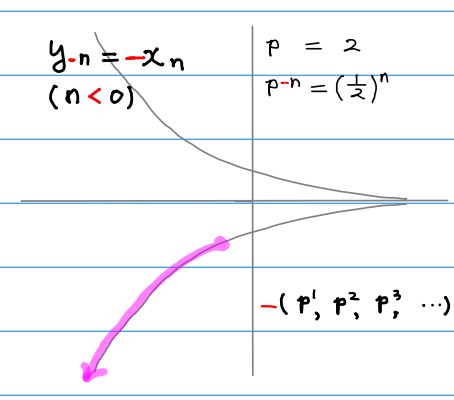
$\frac{z}{1-2z}$

$-x_n = -0.5^{n}$

anti-causal

y_n





$n \in [0, \infty)$ $n \in (-\infty, 0]$

0.5^{n} 0.5^{-n}

-0.5^{n} -0.5^{-n}

$n \in (-\infty, -1]$ $n \in [1, \infty)$

$$\frac{1}{1-0.5z} \qquad \frac{1}{1-0.5z^{-1}}$$

$$\frac{-2z^{-1}}{1-2z^{-1}} \qquad \frac{-2z}{1-2z}$$

$n \in [0, \infty)$

0.5^n $(0.5)^0, (0.5)^1, (0.5)^2$

$$\frac{1}{1 - 0.5z}$$

$$\frac{2}{2 - z}$$

$$\boxed{\frac{1}{1 - 0.5z}}$$

$n \in (-\infty, 0]$

0.5^{-n} $(0.5)^0, (0.5)^1, (0.5)^2, \cdots$

$$\frac{1}{1 - 0.5z^{-1}}$$

$$\frac{z}{z - 0.5}$$

$$\boxed{\frac{1}{1 - 0.5z^{-1}}}$$

-0.5^n $(0.5)^{-1}, (0.5)^{-2}, (0.5)^{-3}$

$n \in (-\infty, -1]$

$$\frac{-2z^{-1}}{1 - 2z^{-1}}$$

$$\frac{2}{2 - z}$$

$$\boxed{\frac{-2z^{-1}}{1 - 2z^{-1}}}$$

-0.5^{-n} $(0.5)^{-1}, (0.5)^{-2}, (0.5)^{-3}$

$n \in [1, \infty)$

$$\frac{-2z}{1 - 2z}$$

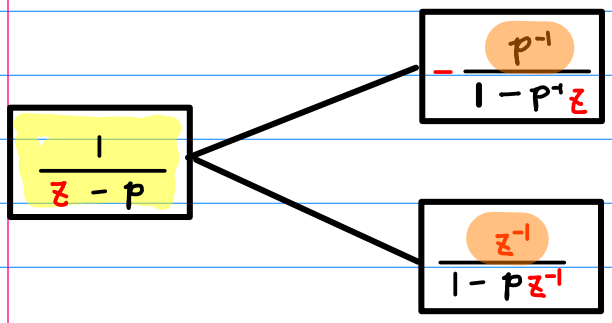
$$\frac{z}{z - 0.5}$$

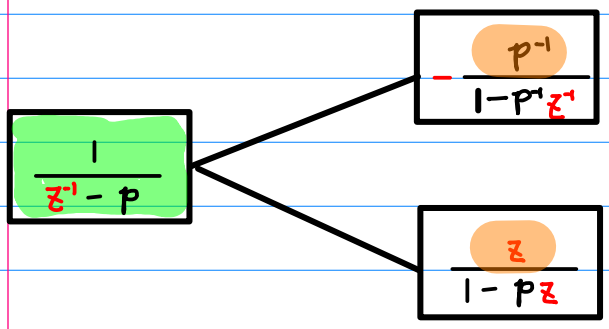
$$\boxed{\frac{-2z}{1 - 2z}}$$

2 formulas (A)

$\frac{1}{z - p}$ $\qquad$ $\frac{1}{z^{-1} - p}$

2 representations each

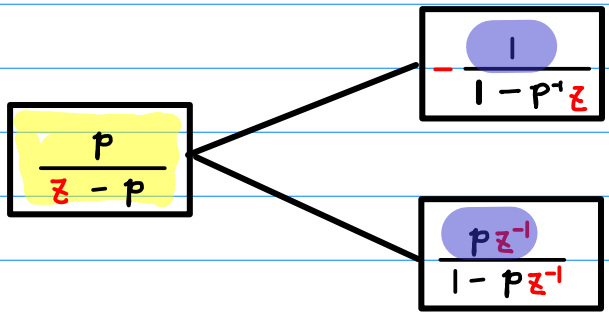


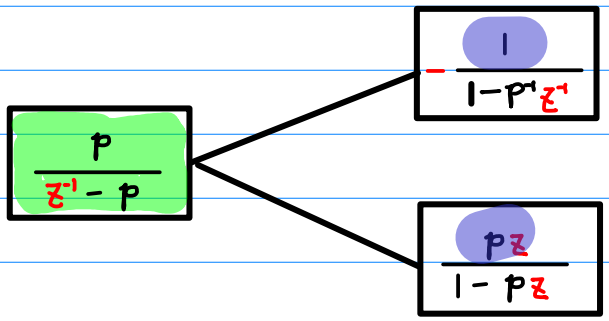


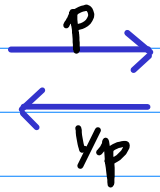
2 formulas (B)

$\frac{p}{z - p}$ $\qquad$ $\frac{p}{z^{-1} - p}$

2 representations each







Ⓐ $f(z) \longrightarrow a_n$

2 formulas Ⓐ

$\frac{1}{z - p}$ $\qquad$ $\frac{1}{z^{-1} - p}$

2 representations each

$\frac{1}{z - p}$

$-\frac{p^{-1}}{1 - p^{-1}z} \triangleq f(z)$ causal $\quad -(p^{-1})^{n+1} \quad (n \geq 0)$

$\frac{z^{-1}}{1 - pz^{-1}} \triangleq g(z^{-1})$ anti- causal $\quad (p^{-1})^{n+1} \quad (n < 0)$

$\frac{1}{z^{-1} - p}$

$-\frac{p^{-1}}{1 - p^{-1}z^{-1}} \triangleq f(z^{-1})$ anti- causal $\quad -(p^{-1})^{-n+1} \quad (n < 1)$

$\frac{z}{1 - pz} \triangleq g(z)$ causal $\quad (p^{-1})^{-n+1} \quad (n \geq 1)$

$-\frac{1}{2 - z}$

$\frac{-0.5}{1 - 0.5z}$ causal $\quad -0.5^{n+1} \quad (n \geq 0)$

$\frac{z^{-1}}{1 - 2z^{-1}}$ anti $\quad 0.5^{n+1} \quad (n < 0)$

$-\frac{0.5z}{z - 0.5}$

$\frac{-0.5}{1 - 0.5z^{-1}}$ anti $\quad -0.5^{-n+1} \quad (n < 1)$

$\frac{z}{1 - 2z}$ causal $\quad 0.5^{-n+1} \quad (n \geq 1)$

Ⓐ $f(z) \longrightarrow a_n$ method 1

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1})$ $\Rightarrow$ $-(p^{-1})^{n+1}$ $\Rightarrow$ $-(p^{-1})^{n+1}$

$n = 0, 1, 2, \cdots$

$-(p^{-1}z^0 + p^{-2}z^1 + p^{-3}z^2 + \cdots)$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ (p) $\Rightarrow$ $(p)^{-n-1}$ $\Rightarrow$ $(p^{-1})^{n+1}$

$n = -1, -2, -3, \cdots$

$(p^0 z^{-1} + p^1 z^{-2} + p^2 z^{-3} + \cdots)$

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ $\Rightarrow$ $-(p^{-1})^{-n+1}$ $\Rightarrow$ $-(p^{-1})^{-n+1}$

$n = 0, -1, -2, \cdots$

$-(p^{-1}z^0 + p^{-2}z^{-1} + p^{-3}z^{-2} + \cdots)$

p $\boxed{\dfrac{z}{1-pz}}$ (p) $\Rightarrow$ $(p)^{n-1}$ $\Rightarrow$ $(p^{-1})^{-n+1}$

$n = 1, 2, 3, \cdots$

$(p^0 z^1 + p^1 z^2 + p^2 z^3 + \cdots)$

Ⓐ $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z}}$ $-(p^{-1})$ z → $n = 0, 1, 2, \cdots$ → $-(p^{-1})^{n+1}$ $(p)^{-n-1}$

p $\boxed{\dfrac{z^{-1}}{1-pz^{-1}}}$ (p) z^{-1}/z^{-1} → $n = -1, -2, -3, \cdots$ → $(p^{-1})^{n+1}$ $(p)^{-n-1}$

p $\boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ z^{-1} → $n = 0, -1, -2, \cdots$ → $-(p^{-1})^{-n+1}$ $(p)^{n-1}$

p^{-1} $\boxed{\dfrac{z}{1-pz}}$ (p) z/z → $n = 1, 2, 3, \cdots$ → $(p^{-1})^{-n+1}$ $(p)^{n-1}$

(B) $f(z) \longrightarrow a_n$

2 formulas

$\frac{p}{z - p}$ $\qquad$ $\frac{p}{z^{-1} - p}$

2 representations each

$\frac{p}{z - p}$

$-\frac{1}{1 - p^{-1}z} \triangleq f(z) \quad -(p^{-1})^n \quad (n \geq 0)$ — causal

$\|$

$\frac{pz^{-1}}{1 - pz^{-1}} \triangleq g'(z^{-1}) \quad (p^{-1})^n \quad (n < 0)$ — anti- causal

$\frac{p}{z^{-1} - p}$

$-\frac{1}{1 - p^{-1}z^{-1}} \triangleq f'(z^{-1}) \quad -(p^{-1})^{-n} \quad (n < 1)$ — anti- causal

$\|$

$\frac{pz}{1 - pz} \triangleq g'(z)$ — causal $\quad (p^{-1})^{-n} \quad (n \geq 1)$

$\frac{2}{2 - z}$

$\frac{1}{1 - 0.5z}$ causal $0.5^n \ (n \geq 0)$

$\frac{-2z^{-1}}{1 - 2z^{-1}}$ anti $-0.5^n \ (n < 0)$

$\frac{z}{z - 0.5}$

$\frac{1}{1 - 0.5z^{-1}}$ anti $0.5^{-n} \ (n < 1)$

$\frac{-2z}{1 - 2z}$ causal $-0.5^{-n} \ (n \geq 1)$

③ $f(z) \rightarrow a_n$ method 1

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1})$ → $-(p^{-1})^n$ → $-(p^{-1})^n$

$n = 0, 1, 2, \cdots$

$-(p^0 z^0 + p^{-1} z^1 + p^{-2} z^2 + \cdots)$

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ (p) → $(p)^{-n}$ → $(p^{-1})^n$

$n = -1, -2, -3, \cdots$

$(p^1 z^{-1} + p^2 z^{-2} + p^3 z^{-3} + \cdots)$

p $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ → $-(p^{-1})^{-n}$ → $-(p^{-1})^{-n}$

$n = 0, -1, -2, \cdots$

$-(p^0 z^0 + p^{-1} z^{-1} + p^{-2} z^{-2} + \cdots)$

p^{-1} $\boxed{\frac{pz}{1-pz}}$ (p) → $(p)^n$ → $(p^{-1})^{-n}$

$n = 1, 2, 3, \cdots$

$(p^1 z^1 + p^2 z^2 + p^3 z^3 + \cdots)$

(B) $f(z) \longrightarrow a_n$ method 2

p^{-1} $\boxed{-\frac{1}{1-p^{-1}z}}$ $-(p^{-1})$ z → $n = 0, 1, 2, \cdots$ → $-p^{-n}$ $(p)^{-n}$

p $\boxed{\frac{pz^{-1}}{1-pz^{-1}}}$ (p) z^{-1}/z^{-1} → $n = -1, -2, -3, \cdots$ → p^{-n} $(p)^{-n}$

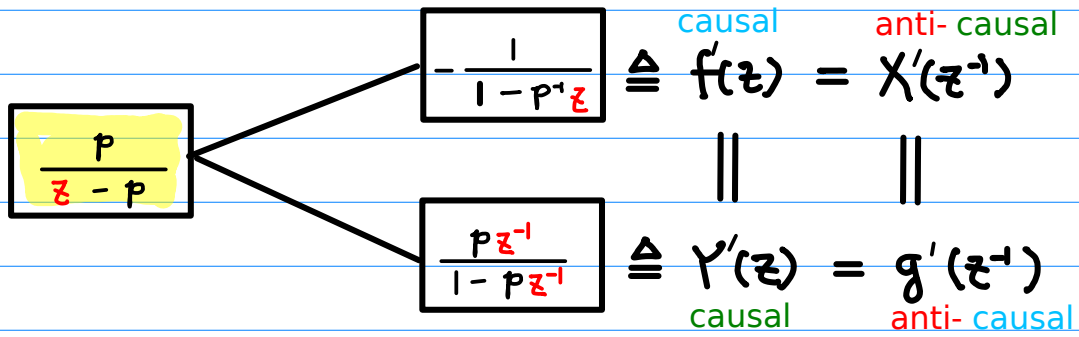
p $\boxed{-\frac{1}{1-p^{-1}z^{-1}}}$ $-(p^{-1})$ z^{-1} → $n = 0, -1, -2, \cdots$ → $-(p^{-1})^{-n}$ $(p)^{n}$

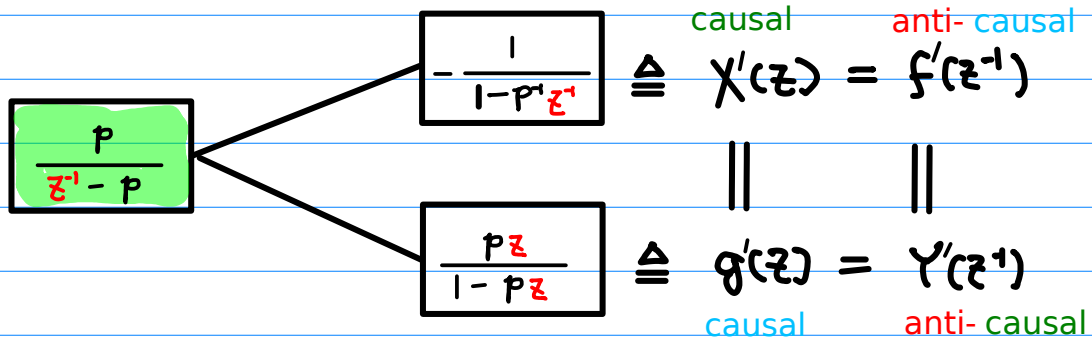
p^{-1} $\boxed{\frac{pz}{1-pz}}$ (p) z/z → $n = 1, 2, 3, \cdots$ → $(p^{-1})^{-n}$ $(p)^{n}$

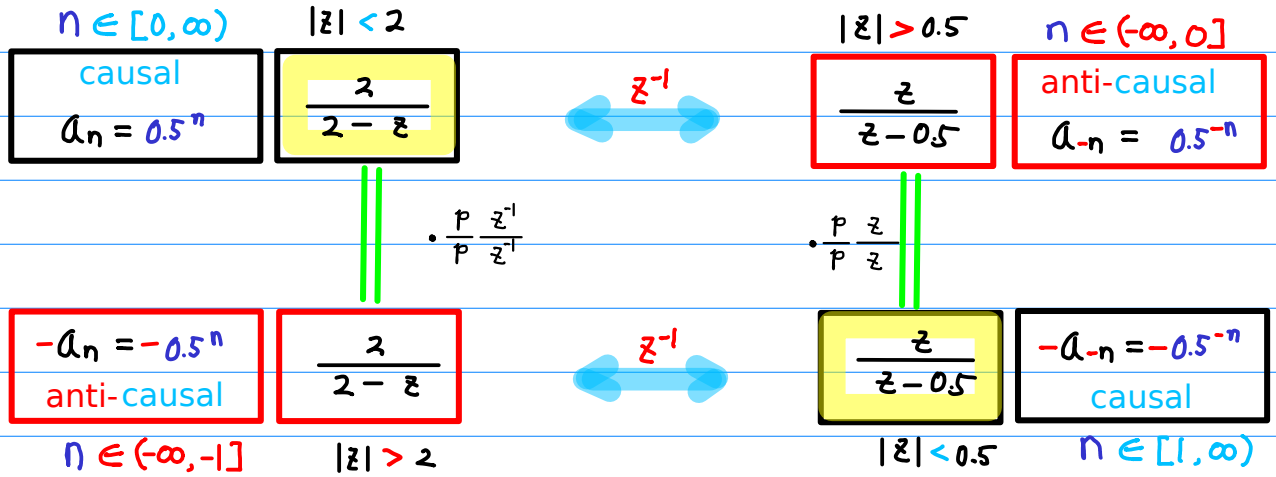
2 formulas

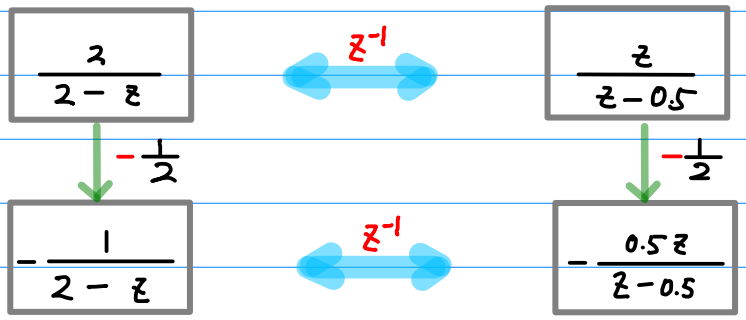
$$\frac{p}{z - p} \qquad \frac{p}{z^{-1} - p}$$

2 representations each



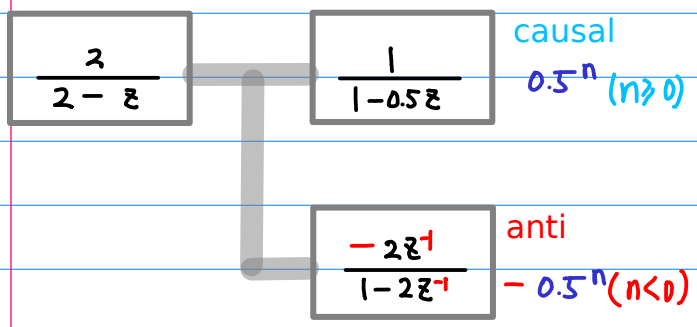


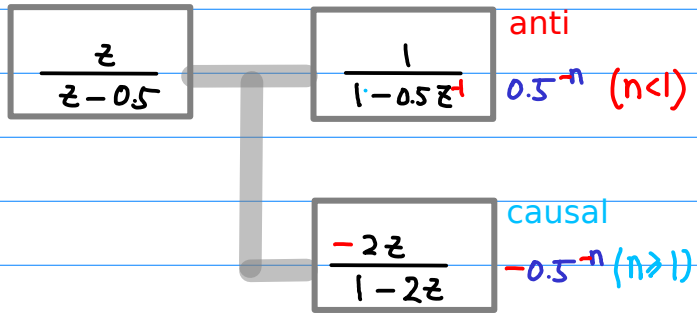


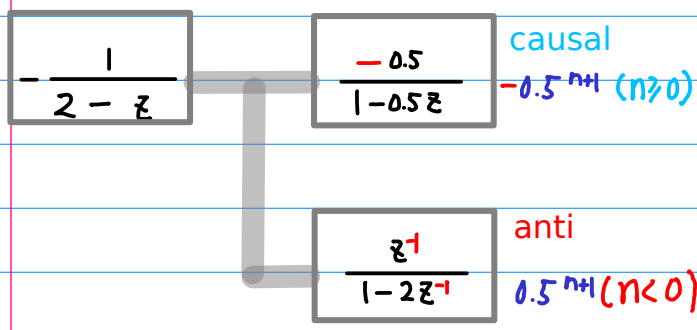


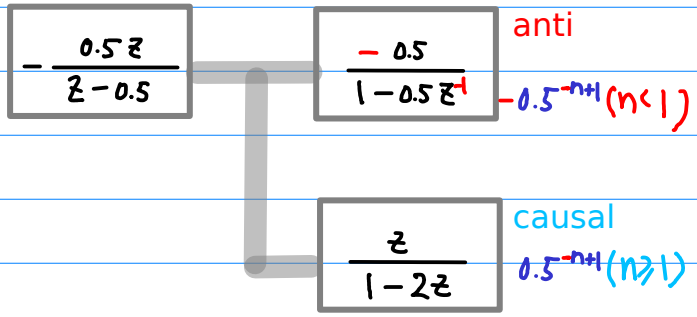
$$0.5^{-n+1} = \left(\frac{1}{2}\right)^{-n+1}$$

$$= 2^{+n-1}$$

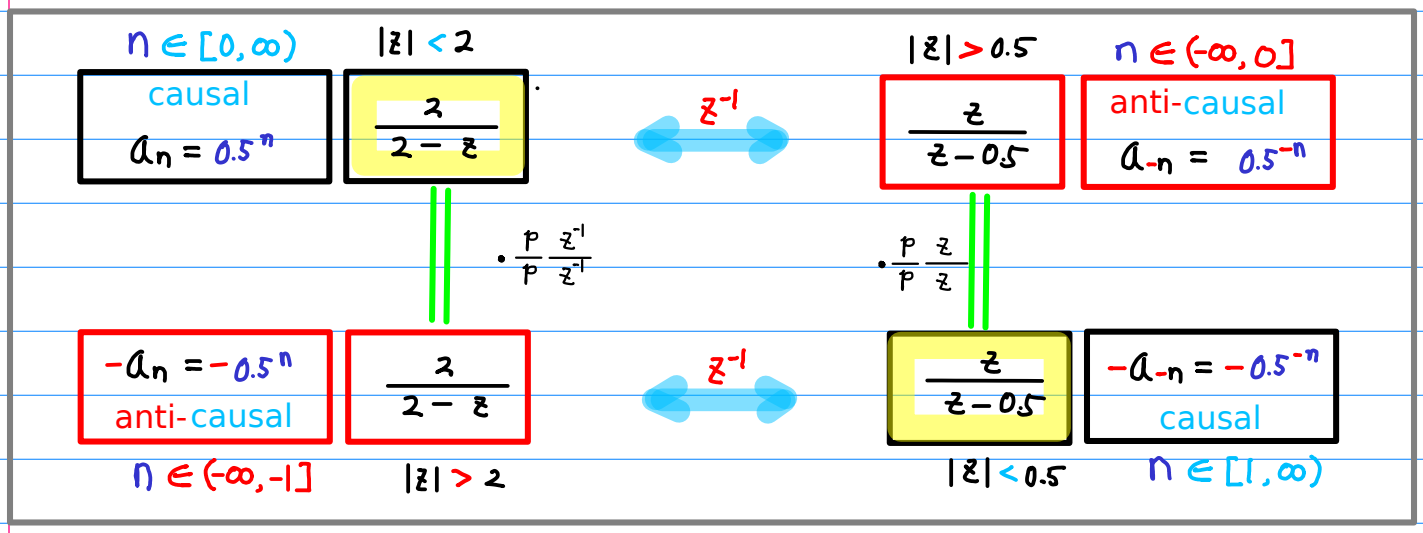

$\frac{2}{2-z}$
$\frac{1}{1-0.5z^{-1}}$
causal
$0.5^n\ (n\geq 0)$
$\frac{-2z^{-1}}{1-2z^{-1}}$
anti
$-0.5^n\ (n<0)$

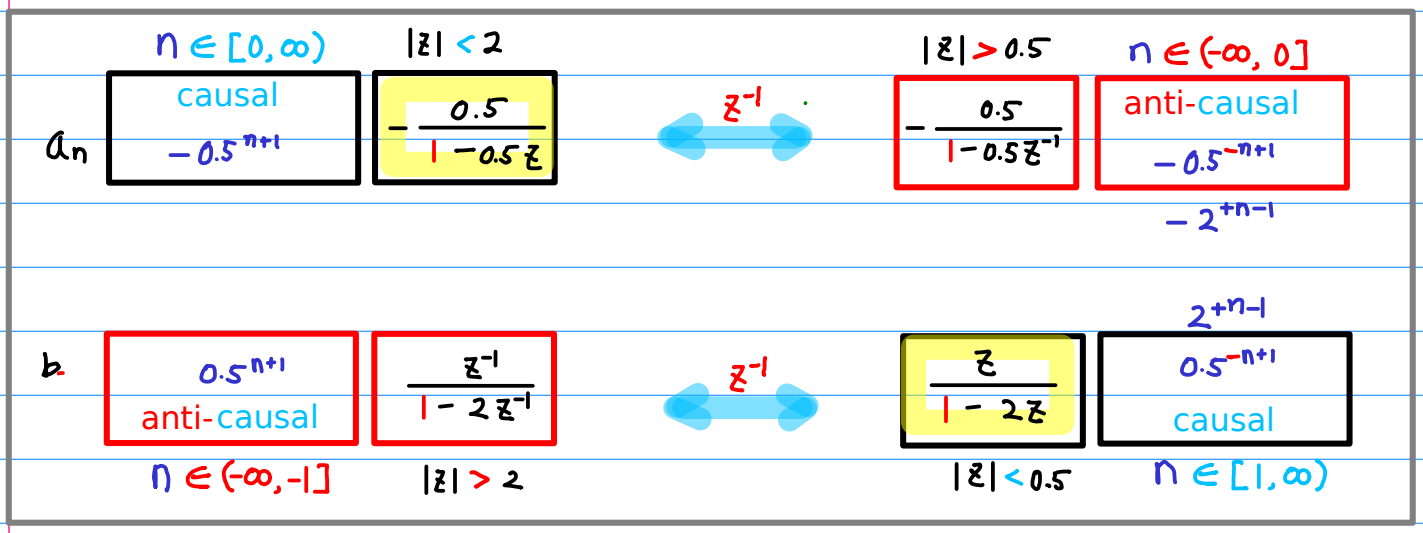

$\frac{z}{z-0.5}$
$\frac{1}{1-0.5z^{-1}}$
anti
$0.5^{-n}\ (n<1)$
$\frac{-2z}{1-2z}$
causal
$-0.5^{-n}\ (n\geq 1)$

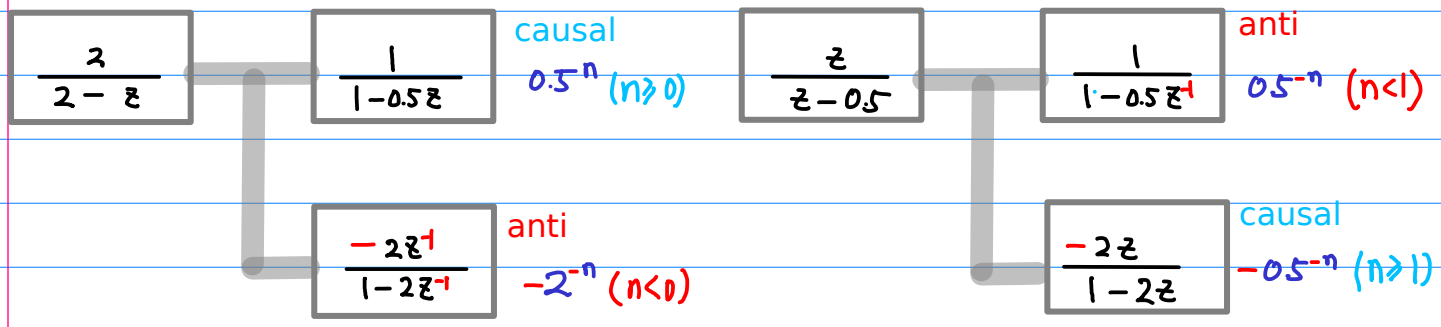

$-\frac{1}{2-z}$
$\frac{-0.5}{1-0.5z^{-1}}$
causal
$-0.5^{n+1}\ (n\geq 0)$
$\frac{z^{-1}}{1-2z^{-1}}$
anti
$0.5^{n+1}\ (n<0)$

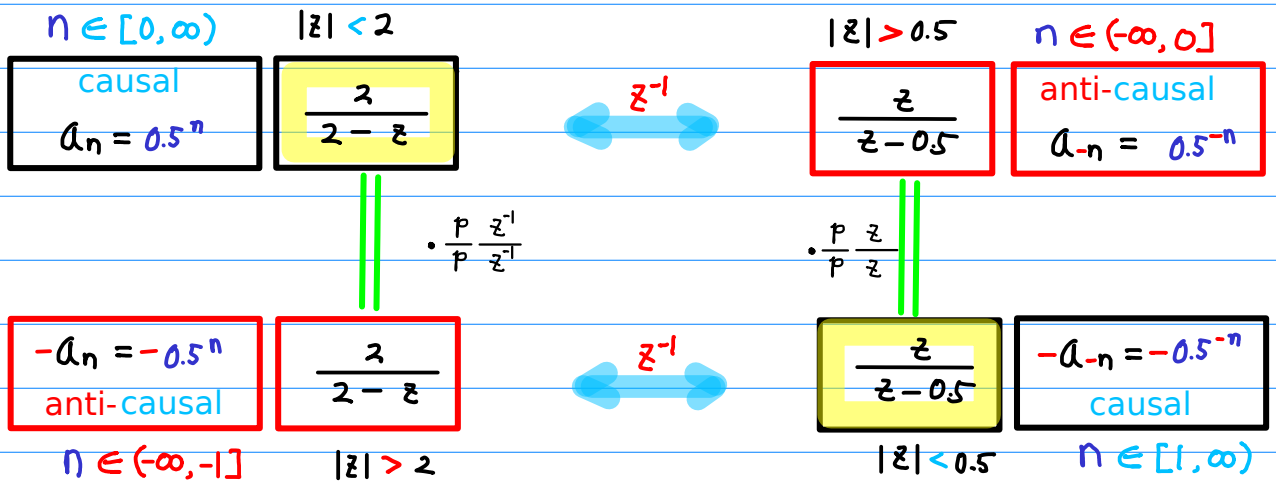

$-\frac{0.5z}{z-0.5}$
$\frac{-0.5}{1-0.5z^{-1}}$
anti
$-0.5^{-n+1}\ (n<1)$
$\frac{z}{1-2z}$
causal
$0.5^{-n+1}\ (n\geq 1)$

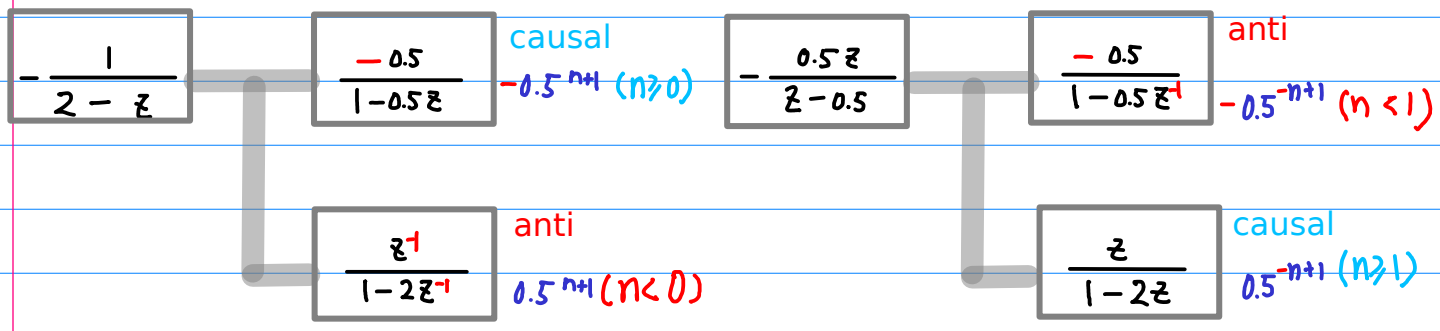
$\frac{2}{2-z}$, $\frac{z}{z-0.5}$ VS. $-\frac{0.5}{1-0.5z}$, $\frac{z}{1-2z}$

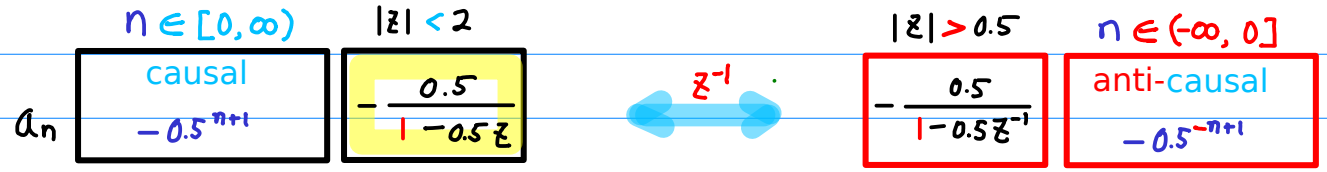


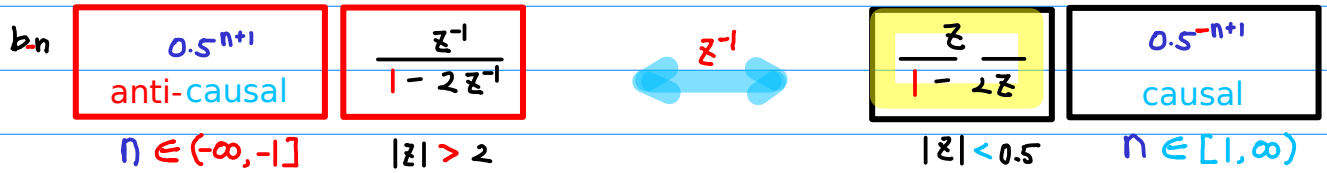


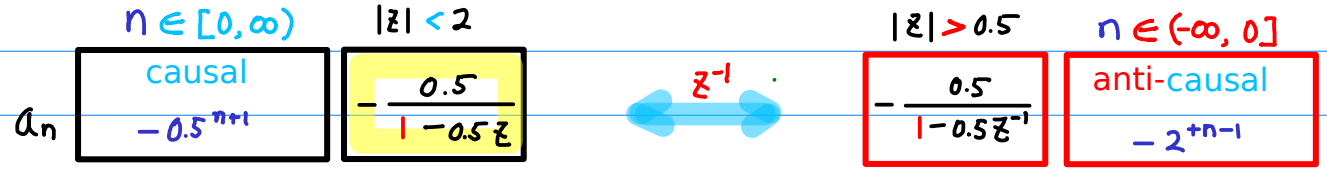

$\frac{2}{2-z}$
$\frac{1}{1-0.5z^{-1}}$
causal
0.5^n $(n \geq 0)$
$\frac{-2z^{-1}}{1-2z^{-1}}$
anti
-2^{-n} $(n<0)$
$\frac{z}{z-0.5}$
$\frac{1}{1-0.5z^{-1}}$
anti
0.5^{-n} $(n<1)$
$\frac{-2z}{1-2z}$
causal
-0.5^{-n} $(n \geq 1)$

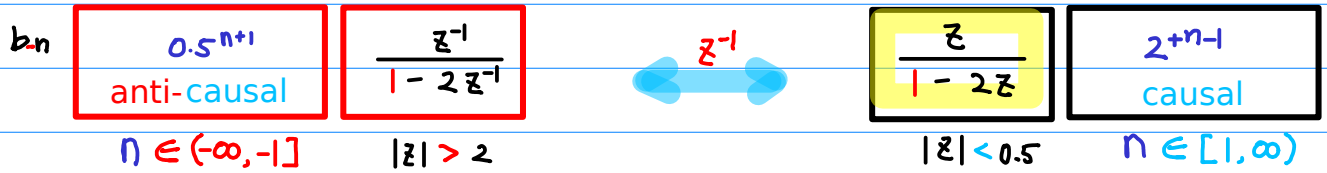

$n \in [0, \infty)$
$|z| < 2$
causal
$a_n = 0.5^n$
$\frac{2}{2-z}$
z^{-1}
$|z| > 0.5$
$n \in (-\infty, 0]$
$\frac{z}{z-0.5}$
anti-causal
$a_{-n} = 0.5^{-n}$
$\cdot \frac{z^{-1}}{z^{-1}}$
$\cdot \frac{z}{z}$
$-a_n = -0.5^n$
anti-causal
$\frac{2}{2-z}$
z^{-1}
$\frac{z}{z-0.5}$
$-a_{-n} = -0.5^{-n}$
causal
$n \in (-\infty, -1]$
$|z| > 2$
$|z| < 0.5$
$n \in [1, \infty)$


$-\frac{1}{2-z}$
$\frac{-0.5}{1-0.5z}$
causal
-0.5^{n+1} $(n\geq 0)$
$\frac{z^{-1}}{1-2z^{-1}}$
anti
0.5^{n+1} $(n<0)$
$-\frac{0.5z}{z-0.5}$
$\frac{-0.5}{1-0.5z^{-1}}$
anti
-0.5^{-n+1} $(n<1)$
$\frac{z}{1-2z}$
causal
0.5^{-n+1} $(n\geq 1)$


$n\in[0,\infty)$
$|z|<2$
a_n
causal
-0.5^{n+1}
$-\frac{0.5}{1-0.5z}$
z^{-1}
$|z|>0.5$
$n\in(-\infty,0]$
$-\frac{0.5}{1-0.5z^{-1}}$
anti-causal
-0.5^{-n+1}


b_n
0.5^{n+1}
anti-causal
$n\in(-\infty,-1]$
$\frac{z^{-1}}{1-2z^{-1}}$
$|z|>2$
z^{-1}
$\frac{z}{1-2z}$
$|z|<0.5$
0.5^{-n+1}
causal
$n\in[1,\infty)$


$n\in[0,\infty)$
$|z|<2$
a_n
causal
-0.5^{n+1}
$-\frac{0.5}{1-0.5z}$
z^{-1}
$|z|>0.5$
$n\in(-\infty,0]$
$-\frac{0.5}{1-0.5z^{-1}}$
anti-causal
-2^{+n-1}


b_n
0.5^{n+1}
anti-causal
$n\in(-\infty,-1]$
$\frac{z^{-1}}{1-2z^{-1}}$
$|z|>2$
z^{-1}
$\frac{z}{1-2z}$
$|z|<0.5$
2^{+n-1}
causal
$n\in[1,\infty)$

Time Shift

$P = 2$

1	$(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
3	$(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
5	$(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
7	$(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
9	$(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = \frac{2}{(2-z)z}$	$X(z) = \frac{z^2}{z-0.5}$
11	$(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = -\frac{2}{(2-z)z}$	$X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

2	$(n \geq 0)$ $a_n = (2)^n$	$f(z) = \frac{0.5}{0.5 - z}$	$X(z) = \frac{z}{z-2}$
4	$(n < 0)$ $a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5 - z}$	$X(z) = -\frac{z}{z-2}$
6	$(n \geq 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5 - z}$	$X(z) = \frac{1}{z-2}$
8	$(n < 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5 - z}$	$X(z) = -\frac{1}{z-2}$
10	$(n \geq -1)$ $a_{n+1} = (2)^{n+1}$	$f(z) = \frac{0.5}{(0.5 - z)z}$	$X(z) = \frac{z^2}{z-2}$
12	$(n < -1)$ $a_{n+1} = (2)^{n+1}$	$f(z) = -\frac{0.5}{(0.5 - z)z}$	$X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \leftrightarrow \frac{1}{2}$

1	$(n \geq 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
2	$(n \geq 0)$	$a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
3	$(n < 0)$	$a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
4	$(n < 0)$	$a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
5	$(n \geq 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
6	$(n \geq 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$
7	$(n < 1)$	$a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
8	$(n < 1)$	$a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$
9	$(n \geq -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = \frac{2}{(2-z)z}$	$X(z) = \frac{z^2}{z-0.5}$
10	$(n \geq -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = \frac{0.5}{(0.5-z)z}$	$X(z) = \frac{z^2}{z-2}$
11	$(n < -1)$	$a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = -\frac{2}{(2-z)z}$	$X(z) = -\frac{z^2}{z-0.5}$
12	$(n < -1)$	$a_{n+1} = (2)^{n+1}$	$f(z) = -\frac{0.5}{(0.5-z)z}$	$X(z) = -\frac{z^2}{z-2}$

Shift to the right

Shift to the right → delete a_0 | $\times z$ | $\times z^{-1}$

(1)	$(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
(5)	$(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
(2)	$(n \geq 0)$ $a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
(6)	$(n \geq 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$

Shift to the right → insert a_0 | $\times z$ | $\times z^{-1}$

(3)	$(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
(7)	$(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
(4)	$(n < 0)$ $a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
(8)	$(n < 1)$ $a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

(1)	$(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
(9)	$(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = \frac{2}{(2-z)z}$	$X(z) = \frac{z}{z-0.5}$
(2)	$(n \geq 0)$ $a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
(10)	$(n \geq -1)$ $a_{n+1} = (2)^{n+1}$	$f(z) = \frac{0.5}{(0.5-z)z}$	$X(z) = \frac{z}{z-2}$

Shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

(3)	$(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
(11)	$(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$	$f(z) = -\frac{2}{(2-z)z}$	$X(z) = -\frac{z}{z-0.5}$
(4)	$(n < 0)$ $a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
(12)	$(n < -1)$ $a_{n+1} = (2)^{n+1}$	$f(z) = -\frac{0.5}{(0.5-z)z}$	$X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$ $\qquad$ b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$ $\qquad$ b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$ $\qquad$ b^{n-1} $n=1,2,3,\cdots$

$1 \longleftrightarrow \frac{1}{1}$

1	$(n \geq 0)$ $a_n = (1)^n$	$f(z) = \frac{1}{1-z}$	$X(z) = \frac{z}{z-1}$
2	$(n \geq 0)$ $a_n = (1^{-1})^n$	$f(z) = \frac{1}{1-z}$	$X(z) = \frac{z}{z-1}$
3	$(n < 0)$ $a_n = (1)^n$	$f(z) = -\frac{1}{1-z}$	$X(z) = -\frac{z}{z-1}$
4	$(n < 0)$ $a_n = (1^{-1})^n$	$f(z) = -\frac{1}{1-z}$	$X(z) = -\frac{z}{z-1}$
5	$(n \geq 1)$ $a_{n-1} = (1)^{n-1}$	$f(z) = \frac{z}{1-z}$	$X(z) = \frac{1}{z-1}$
6	$(n \geq 1)$ $a_{n-1} = (1^{-1})^{n-1}$	$f(z) = \frac{z}{1-z}$	$X(z) = \frac{1}{z-1}$
7	$(n < 1)$ $a_{n-1} = (1)^{n-1}$	$f(z) = -\frac{z}{1-z}$	$X(z) = -\frac{1}{z-1}$
8	$(n < 1)$ $a_{n-1} = (1^{-1})^{n-1}$	$f(z) = -\frac{z}{1-z}$	$X(z) = -\frac{1}{z-1}$
9	$(n \geq -1)$ $a_{n+1} = (1)^{n+1}$	$f(z) = \frac{1}{(1-z)z}$	$X(z) = \frac{z}{z-1}$
10	$(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$	$f(z) = \frac{1}{(1-z)z}$	$X(z) = \frac{z}{z-1}$
11	$(n < -1)$ $a_{n+1} = (1)^{n+1}$	$f(z) = -\frac{1}{(1-z)z}$	$X(z) = -\frac{z}{z-1}$
12	$(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$	$f(z) = -\frac{1}{(1-z)z}$	$X(z) = -\frac{z}{z-1}$

(1) $(n \geq 0) \quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

(2) $(n \geq 0) \quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

Shift to the right → $\times z \qquad \times z^{-1}$

delete a_0

(5) $(n \geq 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

(6) $(n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

(3) $(n < 0) \quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

(4) $(n < 0) \quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

Shift to the right → $\times z \qquad \times z^{-1}$

insert a_0

(7) $(n < 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

(8) $(n < 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

(1) $(n \geq 0)$ $a_n = (1)^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

(2) $(n \geq 0)$ $a_n = (1^{-1})^n$ $f(z) = \frac{1}{1-z}$ $X(z) = \frac{z}{z-1}$

Shift to the left ← $\times z^{-1}$ $\times z$

delete a_0

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(9) $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

(10) $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = \frac{1}{(1-z)z}$ $X(z) = \frac{z}{z-1}$

(3) $(n < 0)$ $a_n = (1)^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

(4) $(n < 0)$ $a_n = (1^{-1})^n$ $f(z) = -\frac{1}{1-z}$ $X(z) = -\frac{z}{z-1}$

Shift to the left ← $\times z^{-1}$ $\times z$

insert a_0

---

(11) $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

(12) $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $f(z) = -\frac{1}{(1-z)z}$ $X(z) = -\frac{z}{z-1}$

Causality

$f(z)$ $(|z| < p)$ $\leftrightarrow$ a_n $(n \geq 0)$ $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z^{-1})$ $(|z| < p)$ $\leftrightarrow$ x_{-n} $(n < 1)$ $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z^{-1})$ $(|z| > p^{-1})$ $\leftrightarrow$ a_{-n} $(n < 1)$ $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$X(z)$ $(|z| > p^{-1})$ $\leftrightarrow$ x_n $(n \geq 0)$ $-(p^{-1}, p^{-2}, p^{-3}, \cdots)$

$f(z)$ $(|z| > p)$ $\leftrightarrow$ $-a_n$ $(n < 0)$ $(p^0, p^1, p^2, \cdots)$

$X(z^{-1})$ $(|z| > p)$ $\leftrightarrow$ $-x_{-n}$ $(n \geq 1)$ $(p^0, p^1, p^2, \cdots)$

$f(z^{-1})$ $(|z| < p^{-1})$ $\leftrightarrow$ $-a_{-n}$ $(n \geq 1)$ $(p^0, p^1, p^2, \cdots)$

$X(z)$ $(|z| < p^{-1})$ $\leftrightarrow$ $-x_n$ $(n < 0)$ $(p^0, p^1, p^2, \cdots)$

$f(z)$	$f(z^{-1})$
$g(z^{-1})$	$g(z)$

$X(z^{-1})$	$X(z)$
$Y(z)$	$Y(z^{-1})$

a_n	a_{-n}
b_{-n}	b_n

x_{-n}	x_n
y_n	y_{-n}

$f(z)$	$f(z^{-1})$
$f(z)$	$f(z^{-1})$

$X(z^{-1})$	$X(z)$
$X(z^{-1})$	$X(z)$

$-(p^1, p^2, p^3, \cdots)$	$-(p^1, p^2, p^3, \cdots)$
$(p^0, p^1, p^2, \cdots)$	$(p^0, p^1, p^2, \cdots)$

$-(p^1, p^2, p^3, \cdots)$	$-(p^1, p^2, p^3, \cdots)$
$(p^0, p^1, p^2, \cdots)$	$(p^0, p^1, p^2, \cdots)$

$-\frac{p^{-1}}{1-p^{-1}z}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$
$\frac{z^{-1}}{1-pz^{-1}}$	$\frac{z}{1-pz}$

$-\frac{p^{-1}}{1-p^{-1}z}$	$-\frac{p^{-1}}{1-p^{-1}z^{-1}}$
$\frac{z^{-1}}{1-pz^{-1}}$	$\frac{z}{1-pz}$

$f(z)$	$g(z)$
$f(z)$	$g(z)$

$Y(z)$	$X(z)$
$Y(z)$	$X(z)$

a_n	a_{-n}
$-a_n$	$-a_{-n}$

x_{-n}	x_n
$-x_{-n}$	$-x_n$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$\lvert z\rvert < p$	$\lvert z\rvert > p^{-1}$
$\lvert z\rvert > p$	$\lvert z\rvert < p^{-1}$

$[0, \infty)$	$(-\infty, 0]$
$(-\infty, -1]$	$[1, \infty)$

$(-\infty, 0]$	$[0, \infty)$
$[1, \infty)$	$(-\infty, -1]$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^0, p^1, p^2, \cdots)$

$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$
$(p^0, p^1, p^2, \cdots)$

a_n a_{-n}	2^{-n} 2^{n}
$-a_n$ $-a_{-n}$	-2^{-n} -2^{n}

$a_n = -2^{-n}$

x_{-n} x_n	2^{-n} 2^{n}
$-x_{-n}$ $-x_n$	-2^{-n} -2^{n}

$x_n = -2^{n}$

$-(p^1, p^2, p^3, \cdots)$ $-(p^1, p^2, p^3, \cdots)$	$-(2^1, 2^2, 2^3, \cdots)$ $-(2^1, 2^2, 2^3, \cdots)$
$(p^0, p^1, p^2, \cdots)$ $(p^0, p^1, p^2, \cdots)$	$(2^0, 2^1, 2^2, \cdots)$ $(2^0, 2^1, 2^2, \cdots)$

$-\frac{p^{-1}}{1-p^{-1}z}$ $-\frac{p^{-1}}{1-p^{-1}z^{-1}}$	$\frac{2^{-1}}{1-2^{-1}z}$ $\frac{2^{-1}}{1-2^{-1}z^{-1}}$
$\frac{z^{-1}}{1-pz^{-1}}$ $\frac{z}{1-pz}$	$-\frac{z^{-1}}{1-2z^{-1}}$ $-\frac{z}{1-2z}$

$\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)}$ $\frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)}$

$-\frac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)}$ $-\frac{z}{1-2z}$

$\lvert z\rvert < p$ $\lvert z\rvert > p^{-1}$	$\lvert z\rvert < 2$ $\lvert z\rvert > 2^{-1}$
$\lvert z\rvert > p$ $\lvert z\rvert < p^{-1}$	$\lvert z\rvert > 2$ $\lvert z\rvert < 2^{-1}$

$[0, \infty)$ $(-\infty, 0]$	$[0, \infty)$ $(-\infty, 0]$
$(-\infty, -1]$ $[1, \infty)$	$(-\infty, -1]$ $[1, \infty)$