

Complex Trigonometric and Hyperbolic Functions (7A)

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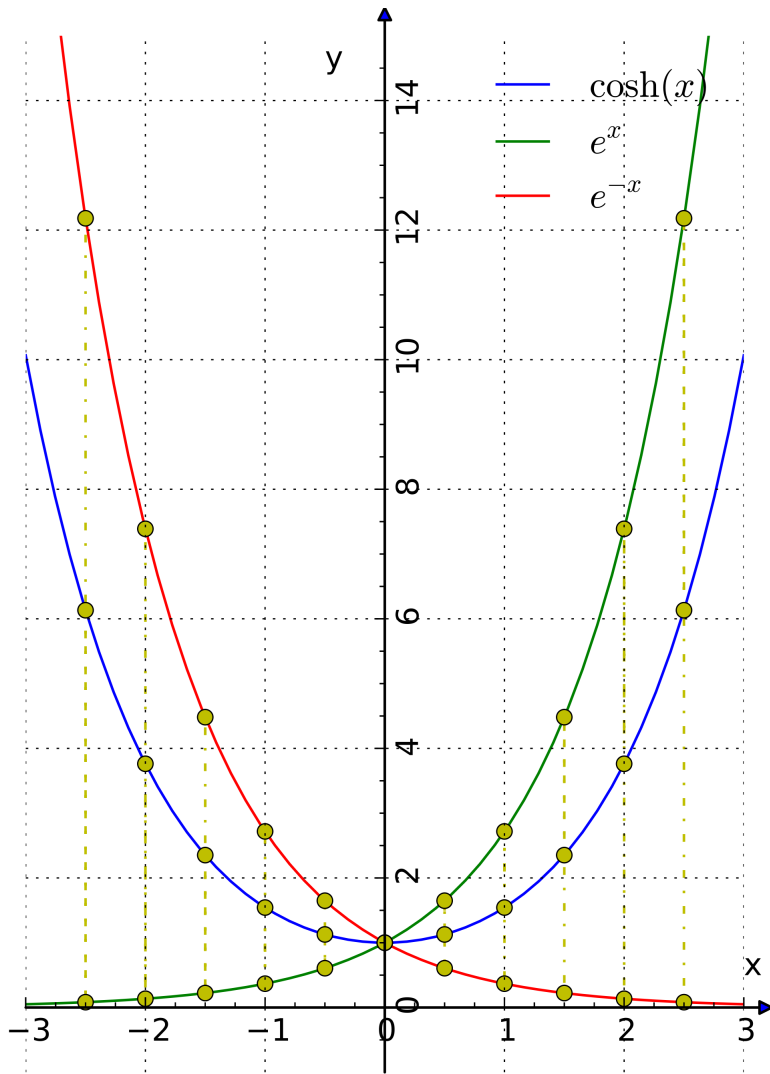
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cosh(x)

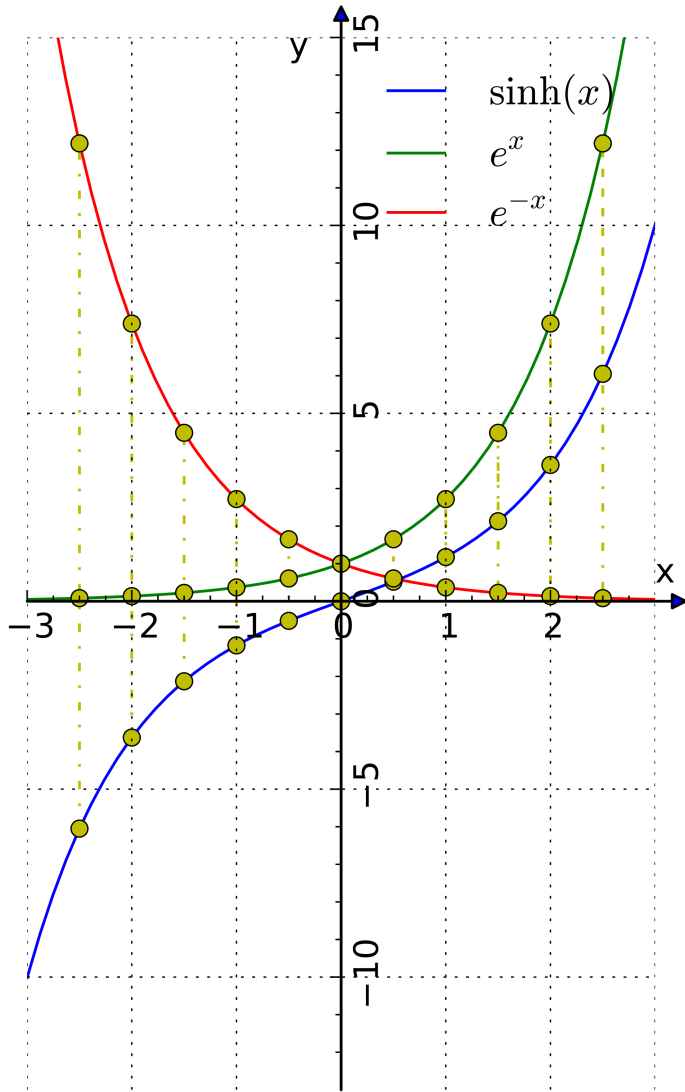


$$\cosh \alpha = \frac{1}{2}(e^{+\alpha} + e^{-\alpha})$$

$$\cos \theta = \frac{1}{2}(e^{+j\theta} + e^{-j\theta})$$

<http://en.wikipedia.org/>

sinh(x)

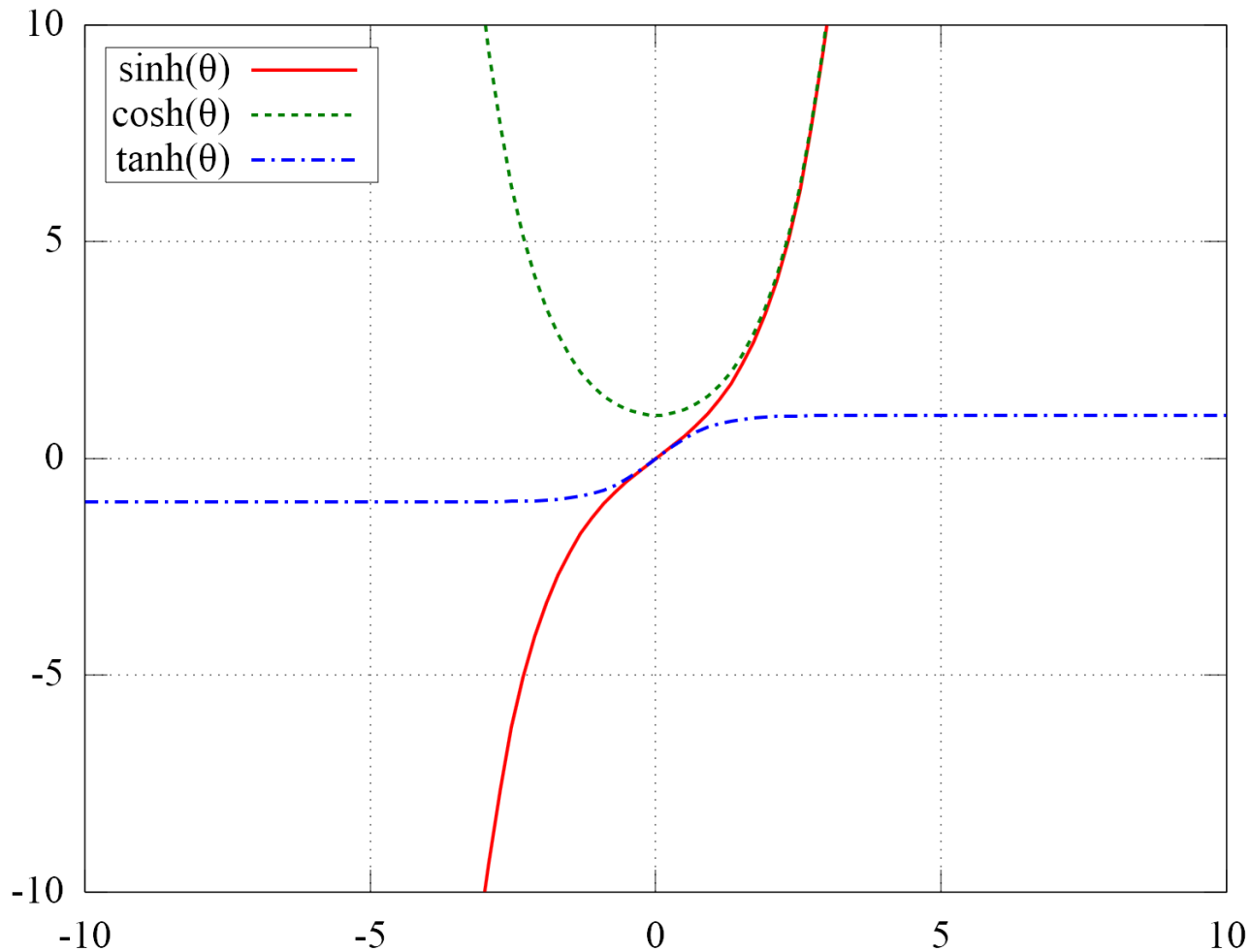


$$\sinh \alpha = \frac{1}{2}(e^{+\alpha} - e^{-\alpha})$$

$$\sin \theta = \frac{1}{2j}(e^{+j\theta} - e^{-j\theta})$$

<http://en.wikipedia.org/>

Definitions of Hyperbolic Functions



$$\cosh \alpha = \frac{1}{2}(e^{\alpha} + e^{-\alpha})$$

$$\sinh \alpha = \frac{1}{2}(e^{\alpha} - e^{-\alpha})$$

$$\tanh \alpha = \frac{(e^{\alpha} - e^{-\alpha})}{(e^{\alpha} + e^{-\alpha})}$$

<http://en.wikipedia.org/>

Hyperbolic vs. Trigonometric Functions

Trigonometric Function

$$ix$$

$$e^{+ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

$$\cos x = \frac{1}{2}(e^{+ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{+ix} - e^{-ix})$$

$$\tan x = \frac{1}{i} \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$

Hyperbolic Function

$$x$$

$$e^{+x} = \cosh x + \sinh x$$

$$e^{-x} = \cosh x - \sinh x$$

$$\cosh x = \frac{1}{2}(e^{+x} + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^{+x} - e^{-x})$$

$$\tanh x = \frac{(e^{+x} - e^{-x})}{(e^{+x} + e^{-x})}$$



Trigonometric functions with imaginary arguments

$$iX \rightarrow X$$

$$\cos x = \frac{1}{2}(e^{+ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{+ix} - e^{-ix})$$

$$\tan x = \frac{1}{i} \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$



$$\cosh x = \frac{1}{2}(e^{+x} + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^{+x} - e^{-x})$$

$$\tanh x = \frac{(e^{+x} - e^{-x})}{(e^{+x} + e^{-x})}$$



$$\cos ix = \cosh x$$

$$\sin ix = i \sinh x$$

$$\tan ix = i \tanh x$$

$$\cos ix = \frac{1}{2}(e^{-x} + e^{+x})$$

$$\sin ix = \frac{1}{2i}(e^{-x} - e^{+x})$$

$$\tan ix = \frac{1}{i} \frac{(e^{-x} - e^{+x})}{(e^{-x} + e^{+x})}$$

$$\cosh x = \frac{1}{2}(e^{+x} + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^{+x} - e^{-x})$$

$$\tanh x = \frac{(e^{+x} - e^{-x})}{(e^{+x} + e^{-x})}$$

Hyperbolic functions with imaginary arguments

$$\cos x = \frac{1}{2}(e^{+ix} + e^{-ix})$$



$$\cosh x = \frac{1}{2}(e^{+x} + e^{-x})$$

$$\sin x = \frac{1}{2i}(e^{+ix} - e^{-ix})$$



$$\sinh x = \frac{1}{2}(e^{+x} - e^{-x})$$

$$\tan x = \frac{1}{i} \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$



$$\tanh x = \frac{(e^{+x} - e^{-x})}{(e^{+x} + e^{-x})}$$

$$x \leftarrow ix$$

$$\cos x = \frac{1}{2}(e^{+ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i}(e^{+ix} - e^{-ix})$$

$$\tan x = \frac{1}{i} \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$

$$\cosh ix = \frac{1}{2}(e^{+ix} + e^{-ix})$$

$$\sinh ix = \frac{1}{2}(e^{+ix} - e^{-ix})$$

$$\tanh ix = \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$

$$\cosh ix = \cos x$$

$$\sinh ix = i \sin x$$

$$\tanh ix = i \tan x$$

With imaginary arguments

$$\cos x = \frac{1}{2}(e^{+ix} + e^{-ix})$$



$$\cosh x = \frac{1}{2}(e^{+x} + e^{-x})$$

$$\sin x = \frac{1}{2i}(e^{+ix} - e^{-ix})$$



$$\sinh x = \frac{1}{2}(e^{+x} - e^{-x})$$

$$\tan x = \frac{1}{i} \frac{(e^{+ix} - e^{-ix})}{(e^{+ix} + e^{-ix})}$$



$$\tanh x = \frac{(e^{+x} - e^{-x})}{(e^{+x} + e^{-x})}$$

x

$$\cos ix = \cosh x$$

$$\sin ix = i \sinh x$$

$$\tan ix = i \tanh x$$

$$\cosh ix = \cos x$$

$$\sinh ix = i \sin x$$

$$\tanh ix = i \tan x$$

ix

Euler Formula

Euler Formula

$$e^{+ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

Euler Formula

$$e^{+ix} = \cosh ix + \sinh ix$$

$$e^{-ix} = \cosh ix - \sinh ix$$

$$\cos ix = \cosh x$$

$$\sin ix = i \sinh x$$

$$\tan ix = i \tanh x$$

$$\cosh ix = \cos x$$

$$\sinh ix = i \sin x$$

$$\tanh ix = i \tan x$$

Modulus of $\sin(z)$ – (1)

$$\begin{aligned}\sin(z) &= \sin(x+iy) \\ &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) + i\cos(x)\sinh(y)\end{aligned}$$

$$\begin{aligned}|\sin(z)|^2 &= \sin(z)\overline{\sin(z)} \\ &= \frac{1}{2i}(e^{+i(x+iy)} - e^{-i(x+iy)})\frac{-1}{2i}(e^{-i(x-iy)} - e^{+i(x-iy)}) \\ &= \frac{1}{4}(e^{-y+ix} - e^{+y-ix})(e^{-y-ix} - e^{+y+ix}) \\ &= \frac{1}{4}(e^{-2y} - e^{+2ix} - e^{-2ix} + e^{+2y}) \\ &= \frac{1}{4}(e^{+2y} - 2 + e^{-2y} - e^{+2ix} + 2 - e^{-2ix}) \\ &= +\frac{1}{4}(e^{+2y} - 2 + e^{-2y}) - \frac{1}{4}(e^{+2ix} - 2 + e^{-2ix}) \\ &= \left[\frac{1}{2}(e^{+y} - e^{-y})\right]^2 + \left[\frac{1}{2i}(e^{+ix} - e^{-ix})\right]^2 \\ &= \sin^2(x) + \sinh^2(y)\end{aligned}$$

Modulus of $\sin(z)$ – (2)

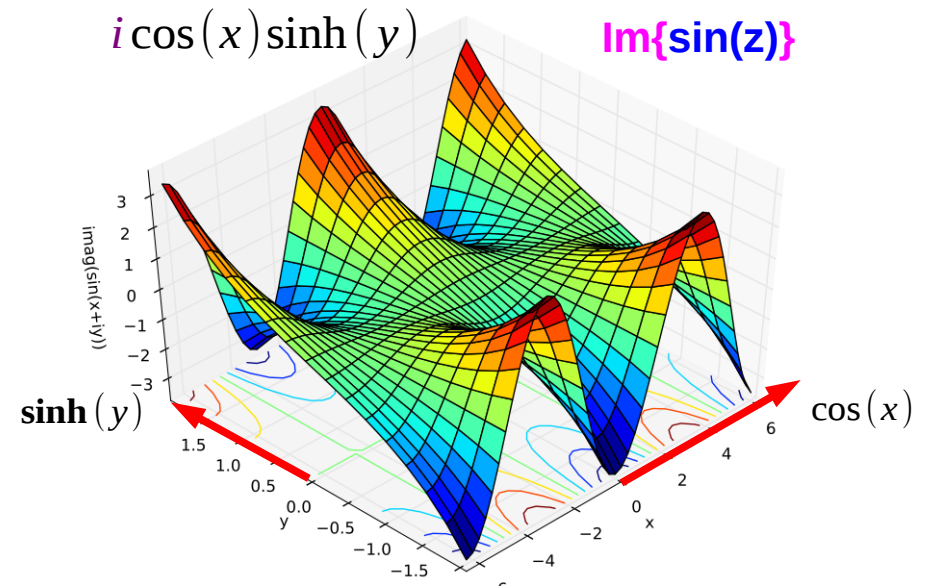
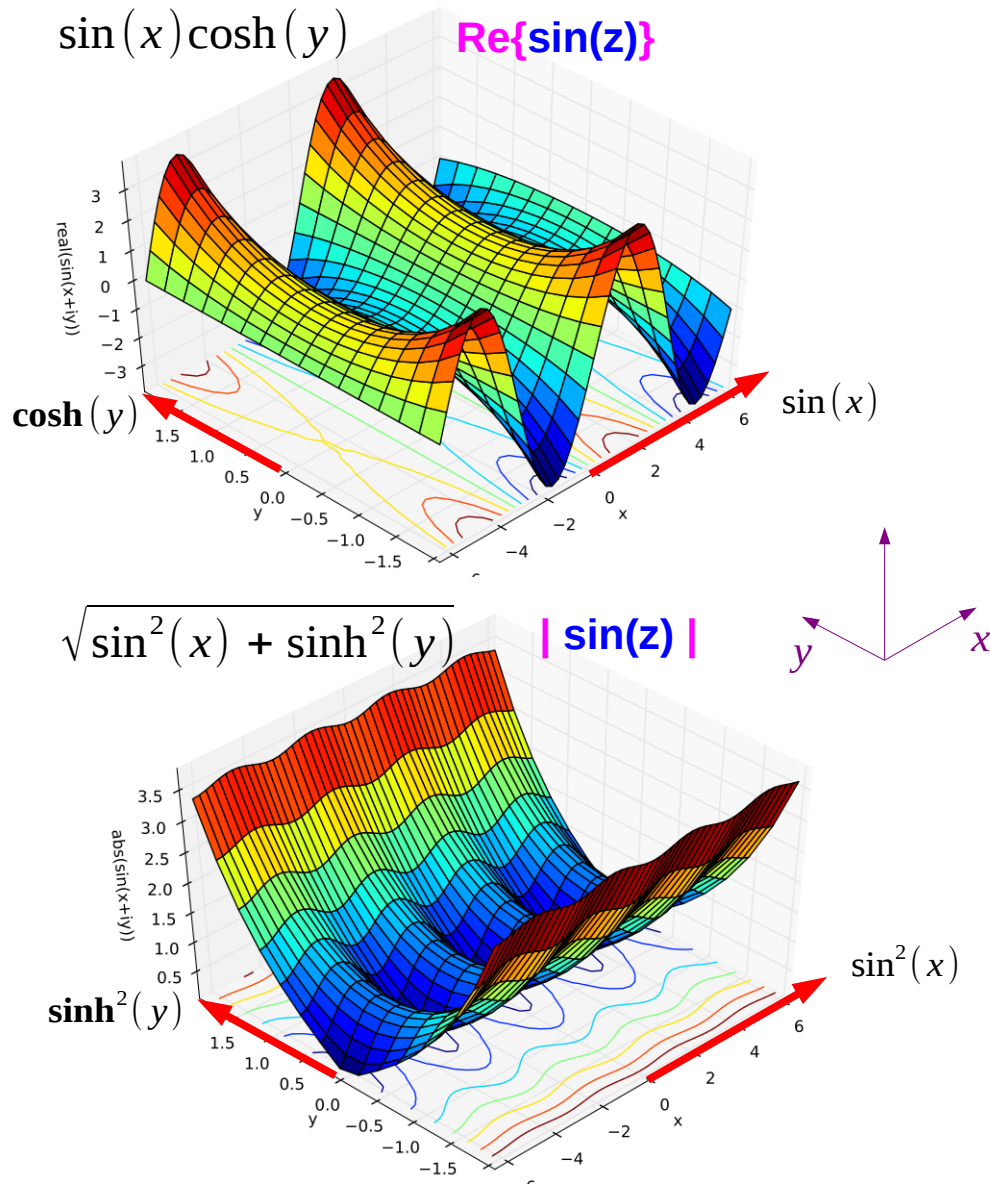
$$\begin{aligned}\sin(z) &= \sin(x+iy) \\ &= \sin(x)\cos(iy) + \cos(x)\sin(iy) \\ &= \sin(x)\cosh(y) + i\cos(x)\sinh(y)\end{aligned}$$

$$\begin{aligned}|\sin(z)|^2 &= |\sin(x)\cosh(y) + i\cos(x)\sinh(y)|^2 \\ &= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y) \\ &= \sin^2(x)(1 + \sinh^2(y)) + (1 - \sin^2(x))\sinh^2(y) \\ &= \sin^2(x) + \sin^2(x)\sinh^2(y) + \sinh^2(y) - \sin^2(x)\sinh^2(y) \\ &= \sin^2(x) + \sinh^2(y)\end{aligned}$$

$$\cosh^2 \alpha - \sinh^2 \alpha = 1$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

Graphs of $\sin(z)$



$$\begin{aligned} \sin(z) &= \sin(x+iy) \\ &= \sin(x) \cosh(y) + i \cos(x) \sinh(y) \end{aligned}$$

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

<http://en.wikipedia.org/>

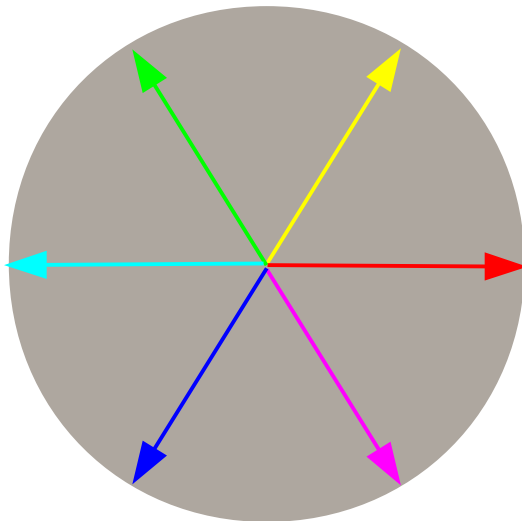
Domain Coloring - Argument

Domain

Coloring

z

$f(z)$



Argument of $f(z)$

hue to phase/angle/argument legend:

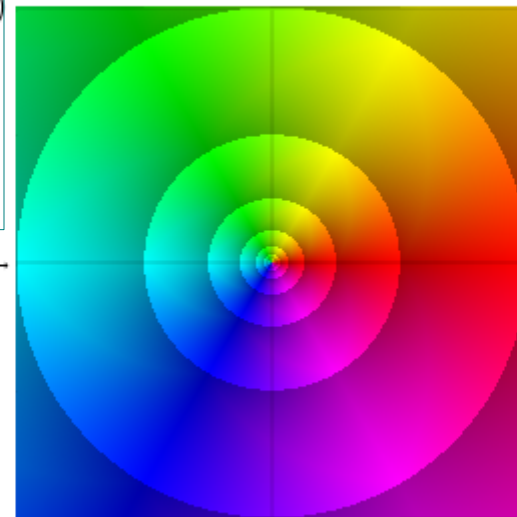
| hue | phase (radians) |
|---------|---------------------|
| red | $0 \bmod 2\pi$ |
| yellow | $\pi/3 \bmod 2\pi$ |
| green | $2\pi/3 \bmod 2\pi$ |
| cyan | $\pi \bmod 2\pi$ |
| blue | $4\pi/3 \bmod 2\pi$ |
| magenta | $5\pi/3 \bmod 2\pi$ |

$\theta = \pi \bmod 2\pi$

Each discontinuity in intensity occurs when $|z|=2^n$, for integer n (0,-1,-2,..)

The Unit Circle

$\theta = \pi/2 \bmod 2\pi$



$\theta = 3\pi/2 \bmod 2\pi$

<http://en.wikipedia.org/>

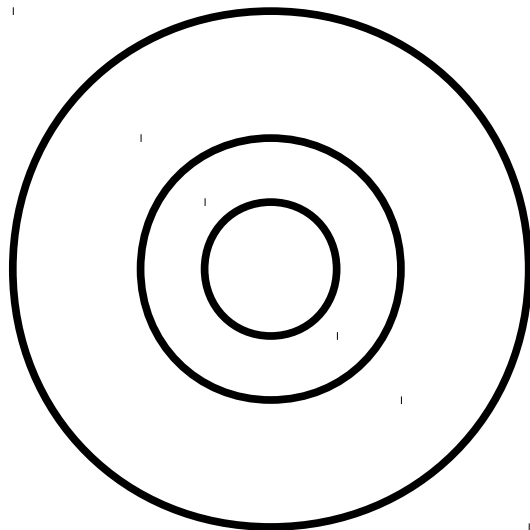
Domain Coloring - Modulus

Domain

Coloring

z

$f(z)$



Modulus of $f(z)$

hue to phase/angle/argument legend:

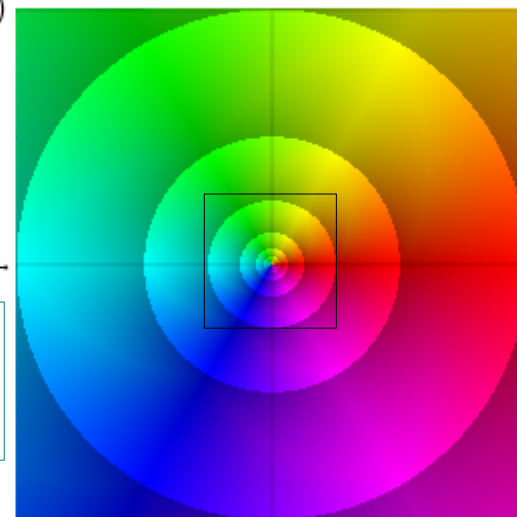
| hue | phase (radians) |
|---------|---------------------|
| red | $0 \bmod 2\pi$ |
| yellow | $\pi/3 \bmod 2\pi$ |
| green | $2\pi/3 \bmod 2\pi$ |
| cyan | $\pi \bmod 2\pi$ |
| blue | $4\pi/3 \bmod 2\pi$ |
| magenta | $5\pi/3 \bmod 2\pi$ |

$\theta = \pi \bmod 2\pi$

Each discontinuity in intensity occurs when $|z|=2^n$, for integer n (0,-1,-2,..)

The Unit Circle

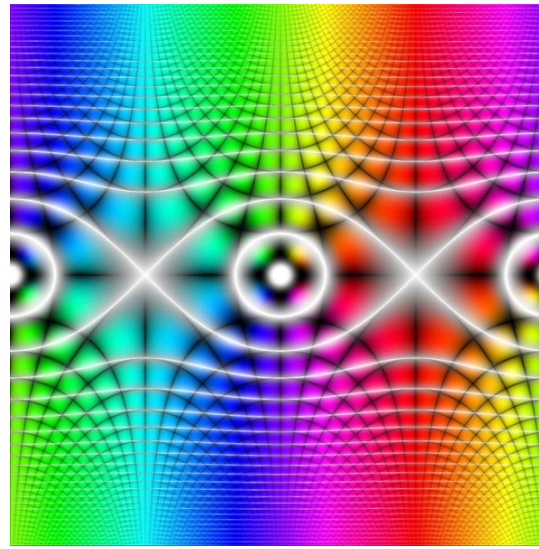
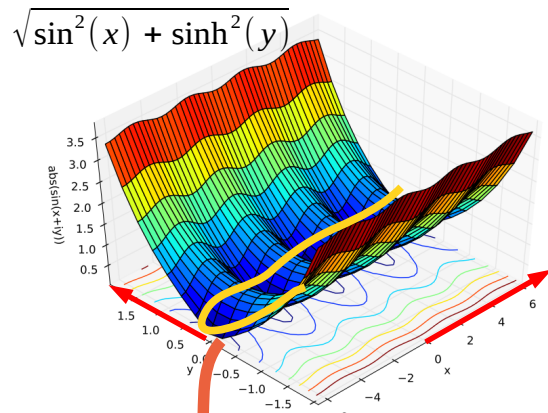
$\theta = \pi/2 \bmod 2\pi$



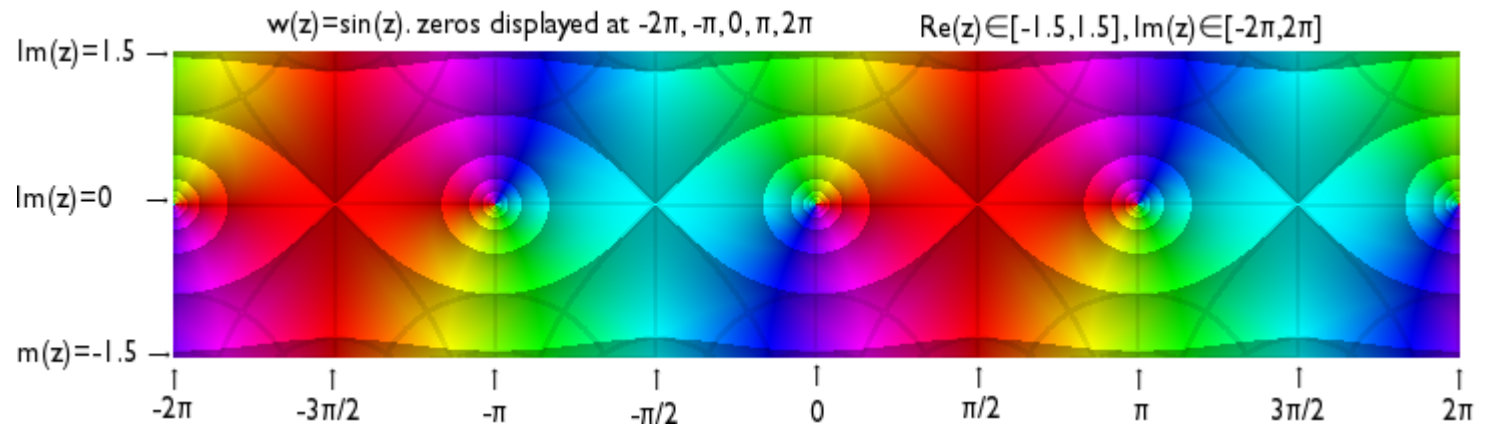
$\theta = 3\pi/2 \bmod 2\pi$

<http://en.wikipedia.org/>

Domain Coloring of $\sin(z)$

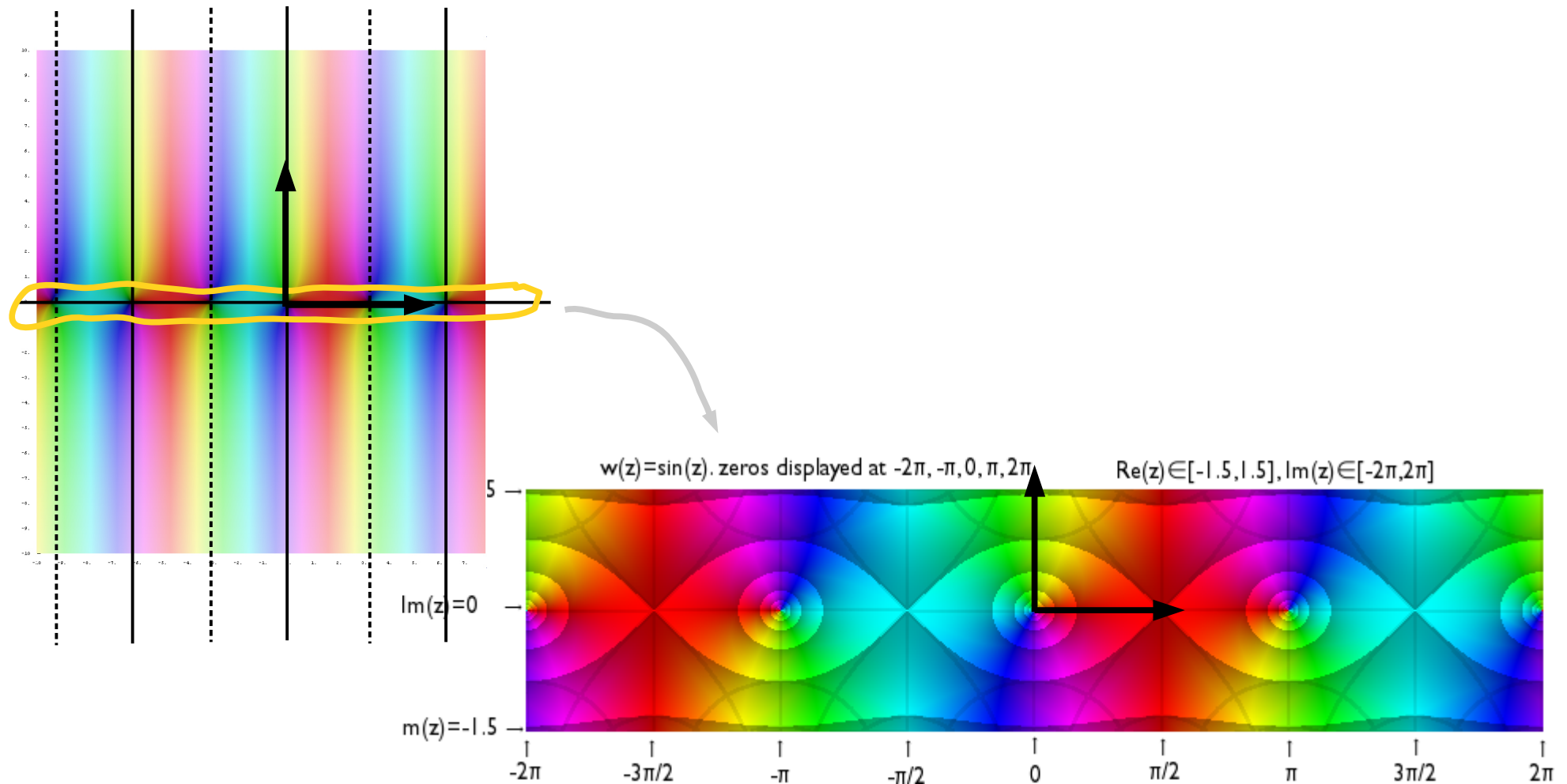


Domain coloring of $\sin(z)$ over $(-\pi, \pi)$ on x and y axes. Brightness indicates absolute magnitude, saturation represents imaginary and real magnitude.



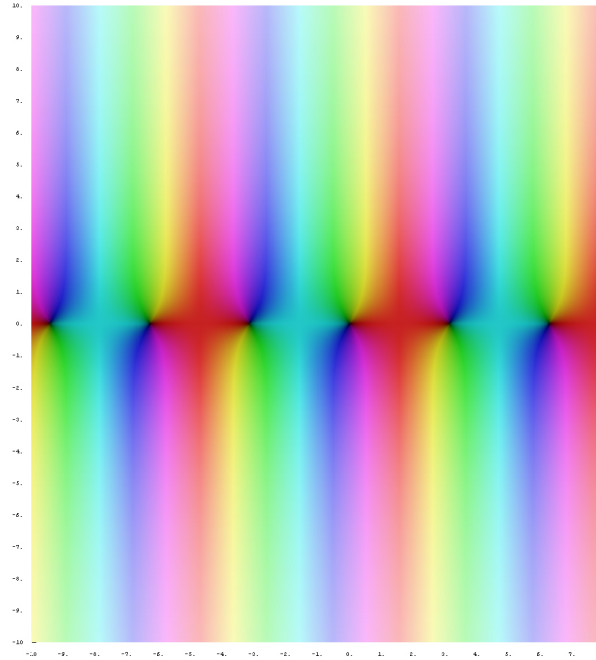
<http://en.wikipedia.org/>

Another domain coloring of $\sin(z)$

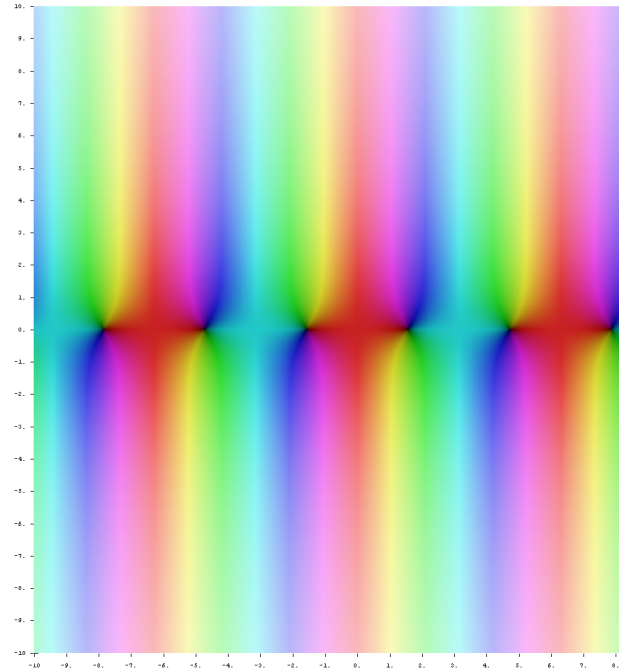


<http://en.wikipedia.org/>

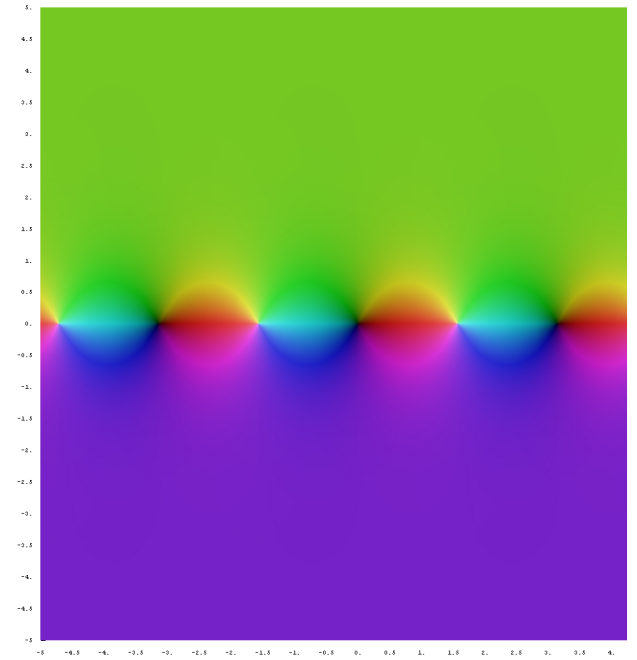
Domain Coloring of $\sin(x)$, $\cos(x)$, $\tan(x)$



$$\sin(z) = \sin(x+iy)$$



$$\cos(z) = \cos(x+iy)$$



$$\tan(z) = \tan(x+iy)$$

<http://en.wikipedia.org/>

References

- [1] <http://en.wikipedia.org/>
- [2] J. S. Walther, A Unified Algorithm for Elementary Functions
- [3] J. Calvert, <http://mysite.du.edu/~jcalvert/math/hyperb.htm>