Z Transform (H.1)

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$$X(z) = \sum_{k=-\infty}^{+\infty} x[k]\, z^{-k}$$

$$z = |r|\, e^{j2\pi F} = |r|\, e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k]\, z^{-k}$$

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi i}\int_C X(z)\, z^{n-1}\, dz$$

$$X(z) = \mathcal{Z}\left[\{x_n\}_0^\infty\right] = \sum_{k=0}^{\infty} x_k\, z^{-k}$$

$$z^{n-1} X(z) = \left(\sum_{k=0}^{\infty} x_k\, z^{-k}\right) z^{n-1}$$

$$= \sum_{k=0}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \sum_{k=n}^{n} x_k\, z^{-k+n-1} + \sum_{k=n+1}^{\infty} x_k\, z^{-k+n-1}$$

$$= \sum_{k=0}^{n-1} x_k\, z^{-k+n-1} + \frac{x_k}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\int_c X(z)\, z^{n-1}\, dz = \int_c \sum_{k=0}^{n-1} x_k\, z^{-k+n-1}\, dz + \int_c \frac{x_k}{z^1}\, dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n-1}\, dz + x_k \int_c \frac{1}{z^1}\, dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}}\, dz$$

$$= \sum_{k=0}^{n-1} x_k \cdot 0 + x_k \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0$$

$$x_k = \frac{1}{2\pi i} \int_c X(z)\, z^{n-1}\, dz$$

Admissible Form of z-transform

$$X(z) = \sum_{k=0}^{\infty} x[n] z^{-n}$$

admissible z-transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \cdots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \cdots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p
$Q(z)$: a polynomial of degree q

D: simply connected domain

C: simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C

except at the points $z_1, z_2, \cdots, z_k$ in C

then

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \sum_{j=1}^{k} \mathrm{Res}\left(f(z), z_j\right)$$