

Z Transform (H.1)

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$$X(z) = \sum_{k=-\infty}^{+\infty} x[k] z^{-k}$$

$$z = |r| e^{j2\pi F} = |r| e^{j\Omega}$$

$$x[n] \longleftrightarrow X(z)$$

One Sided z-transform

$$X(z) = \sum_{k=0}^{+\infty} x[k] z^{-k}$$

$$X(z) = \mathcal{Z}\left[\{x_n\}_{n=0}^{\infty}\right] = \sum_{n=0}^{\infty} x_n z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$x_n = x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi i} \int_C X(z) z^{n+1} dz$$

$$X(z) = \mathcal{Z} [\{x_n\}_0^{\infty}] = \sum_{k=0}^{\infty} x_k z^{-k}$$

$$z^{n+1} X(z) = \left(\sum_{k=0}^{\infty} x_k z^{-k} \right) z^{n+1}$$

$$= \sum_{k=0}^{\infty} x_k z^{-k+n+1}$$

$$= \sum_{k=0}^{n-1} x_k z^{-k+n+1} + \sum_{k=n}^{\infty} x_k z^{-k+n+1}$$

$$= \sum_{k=0}^{n-1} x_k z^{-k+n+1} + \frac{x_n}{z^1} + \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}}$$

$$\begin{aligned} \int_c X(z) z^{n+1} dz &= \int_c \sum_{k=0}^{n-1} x_k z^{-k+n+1} dz + \int_c \frac{x_n}{z^1} dz + \int_c \sum_{k=n+1}^{\infty} \frac{x_k}{z^{k-n+1}} dz \\ &= \sum_{k=0}^{n-1} x_k \int_c z^{-k+n+1} dz + x_n \int_c \frac{1}{z^1} dz + \sum_{k=n+1}^{\infty} x_k \int_c \frac{1}{z^{k-n+1}} dz \\ &= \sum_{k=0}^{n-1} x_k \cdot 0 + x_n \cdot 2\pi i + \sum_{k=n+1}^{\infty} x_k \cdot 0 \end{aligned}$$

$$x_k = \frac{1}{2\pi i} \int_c X(z) z^{n+1} dz$$

Admissible Form of z -transform

$$X(z) = \sum_{k=0}^{\infty} x[n] z^{-n}$$

admissible z -transform

if $X(z)$ is a rational function

$$X(z) = \frac{P(z)}{Q(z)} = \frac{b_0 + b_1 z^1 + b_2 z^2 + \dots + b_{p-1} z^{p-1} + b_p z^p}{a_0 + a_1 z^1 + a_2 z^2 + \dots + a_{q-1} z^{q-1} + a_q z^q}$$

$P(z)$: a polynomial of degree p

$Q(z)$: a polynomial of degree q

D: Simply connected domain

C: Simple closed contour (CCW) in D

if $f(z)$ is analytic inside C and on C
except at the points z_1, z_2, \dots, z_k in C

then

$$\frac{1}{2\pi i} \int_C f(z) dz = \sum_{j=1}^k \text{Res}(f(z), z_j)$$