Temporal Characteristics of Random Processes

Young W Lim

Oct 15, 2022

Copyright (c) 2021 - 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.

Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Random Variable Definition

A random variable

a **function** over a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$

$$s \rightarrow X(s)$$

$$x = X(s)$$

a **function** of the possible **outcomes** s of an **experiment**

Random Variable Definition

A random variable

- a **random variable** : a capital letter X
- a <u>particular</u> <u>value</u> : a lowercase letter x
- a **sample space** $S = \{s_1, s_2, s_3, ..., s_n\}$
- an **outcome** (an <u>element</u> of S) : s

$s \rightarrow X(s)$

$x = X(s)$

$s \rightarrow x$

Random Variable Example

Example

$$X(s_1) = x_1 \qquad s_1 \longrightarrow x_1$$
$$X(s_2) = x_2 \qquad s_2 \longrightarrow x_2$$
$$\cdots \qquad \cdots$$
$$X(s_n) = x_n \qquad s_n \longrightarrow x_n$$

$S = \{s_1, s_2, s_3, ..., s_n\}$ a sample space

$X = \{x_1, x_2, x_3, ..., x_n\}$ a random variable

Random Process (1)

A random process

a function of both **outcome** s and **time** t

$$X(t, s)$$

assigning a time function to every **outcome** s_i

$$s_i \rightarrow x(t, s_i)$$

Random Process (2)

A random process

the family of such time functions is called a random process

$$x(t, s_i) = X(t, s_i)$$

$$x(t, s) = X(t, s)$$

Random Process (3)

We have seen that a random variable X is
a rule which assigns a number
to every outcome e of an experiment.
The random variable is a function X(e)
that maps the set of experiment outcomes
to the set of numbers.
A random process is a rule
that maps every outcome e of an experiment
to a function X(t, e).
A random process is usually conceived of
as a function of time,
but there is no reason to not consider random processes
that are functions of other independent variables,
such as spatial coordinates.
The function X(u, v, e) would be a function
whose value depended on the location (u, v) and the outcome e,
and could be used in representing random variations in an image.

Ensemble of time functions

Time functions

A random process $X(t, s)$ represents
a family or ensemble of time functions

$X(t, s)$ represents

- a single time function $x(t, s)$
- when t is a variable and s is fixed at an outcome

$x(t, s)$ represents

- a sample function,
- an ensemble member,
- a realization of the process

Short-form notation for time functions

The short-form notation $x(t)$

to represent a specific waveform of a random process $X(t)$
for a given **outcome** s_i

$$x(t) = x(t, s)$$

$$X(t) = X(t, s)$$

Random Process Example

Example

$$X(t, s_1) = x_1(t) \qquad s_1 \longrightarrow x_1(t)$$
$$X(t, s_2) = x_2(t) \qquad s_2 \longrightarrow x_2(t)$$
$$\ldots \qquad \ldots$$
$$X(t, s_n) = x_n(t) \qquad s_n \longrightarrow x_n(t)$$

$$S = \{s_1, s_2, s_3, \ldots, s_n\}$$ a sample space

$$X(t) = \{x_1(t), x_2(t), x_3(t), \ldots, x_n(t)\}$$ a random process

Random variables with time

a **random process** $X(t,s)$ represents a single time function
when t is a variable and s is fixed at an outcome

a random process $X(t,s)$ represents a single **random variable**
when both t and s are fixed at a time and an outcome, respectively

$$X_i = X(t_i, s) = X(t_i) \qquad \textbf{\textit{random variable}}$$

$$X(t,s) = X(t) \qquad \textbf{\textit{random process}}$$

An alphabet

the alphabet of $X(t)$

the set of its possible values

- the values of **time** t for which a random process is defined
- the **alphabet** of the random variable $X = X(t)$ at time t

Classification of Random Processes

(1) Types of time and alphabet

- the values of **time** t for which a random process is defined
  - continuous time
  - discrete time
- the **alphabet** of the random variable $X = X(t)$ at time t
  - continuous alphabet
  - discrete alphabet

Classification of Random Processes

(2) types of the random variable $X(t)$ and the time t

- a continuous **alphabet** continuous **time** random process
  - $X(t)$ has continuous values and t has continuous values
- a discrete **alphabet** continuous **time** random process
  - $X(t)$ has discrete values and t has continuous values
- a continuous **alphabet** discrete **time** random process
  - $X(t)$ has continuous values and t has discrete values
- a discrete **alphabet** discrete **time** random process
  - $X(t)$ has discrete values and t has discrete values

Deterministic and Non-deterministic Random Processes

- A process is **non-deterministic**
  if future values of any sample function
  cannot be predicted exactly from observed past values
- A process is **deterministic**
  if future values of any sample function
  can be predicted from observed past values

Deterministic Random Process Example (1)

$$X(t) = A\cos(\omega_0 t + \Theta)$$

A, Θ, or ω_0(or all) can be random variables.

a sample function corresponds to the above equation with particular values of these random variables.

$$x_i(t) = A_i\cos(\omega_{0,i} t + \Theta_i)$$

Deterministic Random Process Example (2)

$$x_i(t) = A_i \cos(\omega_{0,i} t + \Theta_i)$$

the knowledge of the sample function
prior to any time instance fully allows
the prediction of the sample function's future values
because all the necessary information is known

$$x_i(t) \quad t \leq 0 \qquad \Longrightarrow \qquad x_i(t) \quad t > 0$$

Functions and variables of a random process $X(t, \theta)$ (1)

$X(t, \theta)$	a family of functions, an ensemble
$X(t, \theta_k)$	a single time function selected by the outcome θ_k
$X(t_1, \theta)$	a random variable at the time $t = t_1$
$X(t_1, \theta_k)$	a number at the time $t = t_1$, of the outcome θ_k

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf

Functions and variables of a random process $X(t, \theta)$ (2)

- $X(t, \theta)$ is a family of functions. Imagine a giant strip chart recording in which each pen is identified with a different θ. This family of functions is traditionally called an ensemble.
- A single function $X(t, \theta_k)$ is selected by the outcome θ_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.
- If t is fixed, say $t = t_1$, then $X(t_1, \theta)$ is a random variable. Its value depends on the outcome θ.
- If both t_1 and θ_k are given then $X(t_1, \theta_k)$ is just a number.

https://www.cis.rit.edu/class/simg713/notes/chap7-random-process.pdf