First Order ODE's (1A)

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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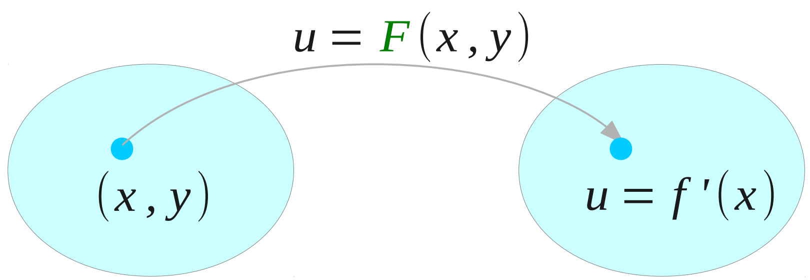
Direction Fields

Direction Field, First Order ODE



$y = f(x)$

$u = F(x, y)$ **F maps (x,y) to u**

$f'(x) = F(x, y)$ **F maps (x,y) to f'(x)**

the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

First Order ODE

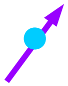
Find solution y=f(x)

$$\frac{dy}{dx} = g(x, y)$$

where the first derivative y' is given by some formula g(x,y) containing variable x, y

Direction Field Slope Field

A 2-d plot of y'=f'(x) at (x, y)



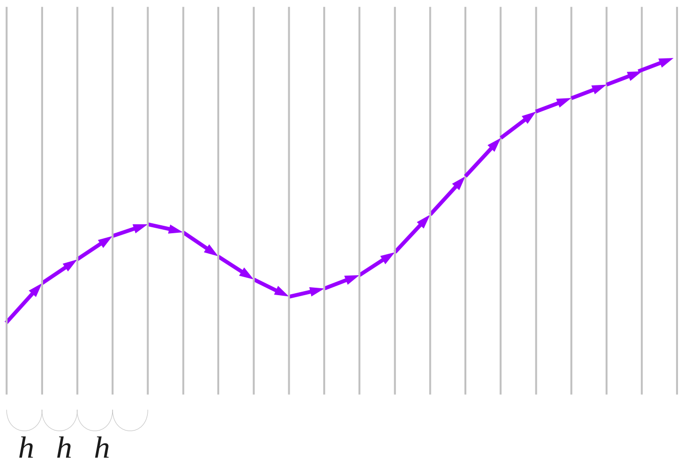
$$g(x, y) = \frac{dy}{dx}$$

Now, it also can be viewed as a function

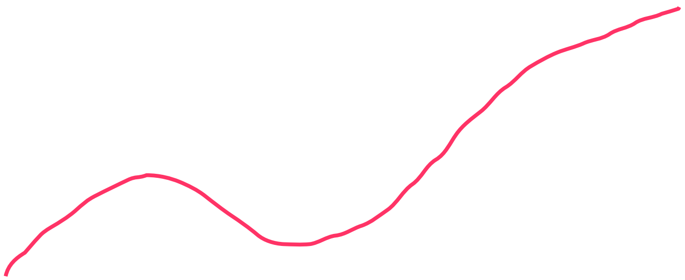
g maps (x,y) to f'(x)

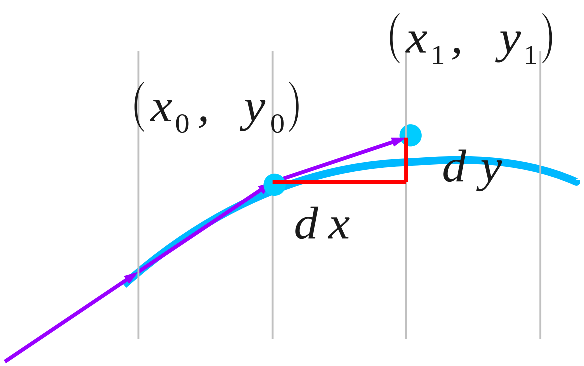
Euler's Method

$$F(x,y) = f'(x)$$



$$y = f(x)$$





$$dy = f'(x)dx$$

$$x_1 - x_0 = dx = h$$

$$y_1 - y_0 = dy = f'(x)dx = F(x_0,\ y_0)dx$$

$$y_1 = y_0 + F(x_0, y_0)h$$

$$y_{i+1} = y_i + F(x_i, y_i)h$$

Types of First Order ODEs

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x) y = g(x)$$

$$a_1(x) y' + a_0(x) y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y) dx + N(x, y) dy = 0$$

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0$$

$$z = f(x, y)$$

Separable First Order ODEs

Separable ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

$$\frac{1}{g_2(y)} \frac{dy}{dx} = g_1(x)$$

$$p(y) \frac{dy}{dx} = q(x)$$

$$\frac{1}{g_2(y)} y' = g_1(x)$$

$$p(y) y' = q(x)$$

Solving Separable ODEs (1)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx}dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$p(y)y' = q(x)$$

$$p(y)y' = q(x)$$

$$p(y)y'dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$y = f(x)$$

not a ratio

$$dy = \frac{df}{dx}dx$$

$$\int p(y)dy$$

includes a constant

$$\int q(x)dx$$

includes another constant

implicit function y

Solving Separable ODEs (2)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\,y' = q(x)$$

given a composite function
p(y(x)) find y=f(x)

$$P(y) = \int p(y)\,dy + c_1$$

$$\frac{d}{dy}[P(y)] = p(y)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right]\cdot\frac{dy}{dx} = q(x)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right]\cdot y' = q(x)$$

$$\frac{d}{dy}[P(y)]\cdot\frac{dy}{dx} = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}[P(y)]\cdot = q(x)$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$p(y) = \frac{d}{dy}\left[\int p(y)\,dy + c_1\right]$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$P(y) = Q(x) + C$$

Linear First Order ODEs

Homogeneous and Particular Solutions

Standard Form of First Order ODEs

$$1\frac{dy}{dx} + P(x)y = Q(x)$$

$$1y' + P(x)y = Q(x)$$

total solution

$$y = y_h + y_p$$

The Homogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

the common part of the solutions of many different differential equations whose homogeneous DE's are the same

The Nonhomogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

particular solution

$$y_p = f_p(x)$$

the particular solution of a specific differential equation, excluding common part of the solution

Three Different Linear ODEs

EQ 1

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \longleftarrow \quad y_1$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \longleftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \longleftarrow \quad y_1 + y_h$$

EQ 2

$$\frac{dy}{dx} + P(x)y = R(x) \quad \longleftarrow \quad y_2$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \longleftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = R(x) \quad \longleftarrow \quad y_2 + y_h$$

EQ 3

$$\frac{dy}{dx} + P(x)y = S(x) \quad \longleftarrow \quad y_3$$

$$\frac{dy}{dx} + P(x)y = 0 \quad \longleftarrow \quad y_h$$

$$\frac{dy}{dx} + P(x)y = S(x) \quad \longleftarrow \quad y_3 + y_h$$

Integrating Factor

$$\frac{dy}{dx} + P(x)\,y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)\,y_1$$

$$\frac{dy_p}{dx} + P(x)\,y_p = Q(x)$$

$$y' + P(x)\,y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)\,y_1$$

$$y_p' + P(x)\,y_p = Q(x)$$

homogeneous solution

$$y_h = f_h(x)$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

particular solution

$$y_p = f_p(x)$$

Total Solution

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y_1 = e^{-\int P(x)\,dx}$$

total solution

$c\,y_1 + u(x)\,y_1$

$u(x)\,y_1$

homogeneous solution

particular solution

$$c\,e^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x)\,dx}\right\} dx\right]$$

Integrating factor

$c\,y_1$

$$\frac{dy}{dx} + P(x)\,y = 0$$

Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

$$y = f(x)$$

$$\left\{e^{+\int P(x)dx}\right\} \cdot \left[\frac{dy}{dx} + P(x)y\right] = \left\{e^{+\int P(x)dx}\right\} Q(x)$$

$$y_1 = e^{-\int P(x)dx} \qquad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left\{e^{+\int P(x)dx}\right\} \frac{dy}{dx} + \left[\left\{e^{+\int P(x)dx}\right\} P(x)\right] y = \left\{e^{+\int P(x)dx}\right\} Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\} \cdot P(x) = \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\} \qquad f'(g(x))g'(x)$$

$$\left\{e^{+\int P(x)dx}\right\} \frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\} y = \left\{e^{+\int P(x)dx}\right\} Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\} \frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\} y = \frac{d}{dx}\left[e^{+\int P(x)dx} \cdot y\right]$$

$$f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\} \cdot y\right] = \left\{e^{+\int P(x)dx}\right\} Q(x)$$

$$\int \frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\} \cdot y\right] dx = \int \left\{e^{+\int P(x)dx}\right\} Q(x)\, dx + c$$

$$\left[\left\{e^{+\int P(x)dx}\right\} \cdot y\right] = \int \left\{e^{+\int P(x)dx}\right\} Q(x)\, dx + c$$

$$y(x) = c e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \{Q(x)\} \cdot \left\{e^{+\int P(x)dx}\right\} dx\right]$$

Exact First Order ODEs

To be exact

$$P(x,y)\,dx + Q(x,y)\,dy = df$$

to be exact →

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$P(x,y) = \frac{\partial f}{\partial x}$$

$$Q(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ all defined and continuous ⟺ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$P(x,y)\,dx + Q(x,y)\,dy$ is an exact (total) differential ⟺ $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

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Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$z = f(x,y)$$

$$\frac{\partial f}{\partial x} = M(x,y) \qquad \frac{\partial f}{\partial y} = N(x,y)$$

$$\frac{df}{dx} = M(x) \Rightarrow f(x) = \int M(x)dx + c$$

$$\int \frac{\partial f}{\partial x} dx = \int M(x,y)dx + c \qquad \int \frac{\partial f}{\partial y} dy = \int N(x,y)\,dy + c$$

$$f(x,y) = \int M(x,y)dx + g(y) \qquad f(x,y) = \int N(x,y)dy + h(x)$$

$$\frac{\partial f}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y)$$

$$\frac{\partial f}{\partial x} = M(x,y) = \frac{\partial}{\partial x}\int N(x,y)dy + h'(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \qquad h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy$$

Exact Equations (2)

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$$f(x,y) = \int N(x,y)\,dy + h(x)$$

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

$$h'(x) = M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy$$

$$g(y) = \int g'(y)\,dy$$

$$= \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

$$h(x) = \int h'(x)\,dx$$

$$= \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

$$f(x,y) = \int M(x,y)\,dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx \right\} dy$$

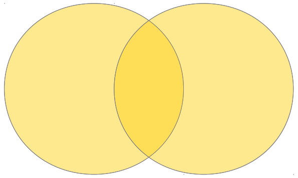
$$f(x,y) = \int N(x,y)\,dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)\,dy \right\} dx$$

Exact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

$$f(x,y) = \int M(x,y)dx + \int \left\{ N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx \right\} dy$$

$$f(x,y) = \int N(x,y)dy + \int \left\{ M(x,y) - \frac{\partial}{\partial x}\int N(x,y)dy \right\} dx$$



$$f(x,y) = \int M(x,y)dx + \int N(x,y)\,dy - \int \frac{\partial}{\partial y}\int M(x,y)dx\,dy \qquad \int \frac{\partial}{\partial y}\int \frac{\partial f}{\partial x}dx\,dy$$

$$f(x,y) = \int N(x,y)dy + \int M(x,y)\,dx - \int \frac{\partial}{\partial x}\int N(x,y)dy\,dx \qquad \int \frac{\partial}{\partial x}\int \frac{\partial f}{\partial y}dy\,dx$$

NonExact First Order ODEs

Multiplying NonExact Equations by $\mu(x)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(x)M(x,y)dx + \mu(x)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(x)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \longleftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$\mu\frac{\partial M}{\partial y} = \frac{d\mu}{dx}N + \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu\frac{\partial M}{\partial y} - \mu\frac{\partial N}{\partial x}$$

$$\frac{d\mu}{dx}N = \mu M_y - \mu N_x$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$$

generally $P(x,y)$

sometimes $P(x)$ $\longrightarrow$ $\frac{d\mu}{dx} - P(x)\mu = 0$

Multiplying NonExact Equations by $\mu(y)$

NonExact Equations

$$M(x,y)dx + N(x,y)dy = 0 \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$z = f(x,y)$$

Exact Equations

$$\mu(y)M(x,y)dx + \mu(y)N(x,y)dy = 0 \qquad \frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}$$

If we can find $\mu(y)$

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x} \quad \Longleftarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{\partial \mu}{\partial y}M + \mu\frac{\partial M}{\partial y} = \frac{\partial \mu}{\partial x}N + \mu\frac{\partial N}{\partial x}$$

$$M\frac{\partial \mu}{\partial y} = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$

$$M\frac{d\mu}{dy} = \mu\frac{\partial N}{\partial x} - \mu\frac{\partial M}{\partial y}$$

$$M\frac{d\mu}{dy} = \mu N_x - \mu M_y$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$$

generally $P(x,y)$

sometimes $P(y)$ $\rightarrow$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

Solving NonExact Equations

$$\frac{\partial}{\partial y}[\mu(x) M(x,y)] = \frac{\partial}{\partial x}[\mu(x) N(x,y)]$$

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dx} - P(x)\mu = 0$$

$$\mu(x) = c e^{\int P(x)dx}$$

$$\mu(x) = c e^{\int \left(\frac{M_y - N_x}{N}\right)dx}$$

$$\mu(x) M(x,y) dx + \mu(x) N(x,y) dy = 0$$

$$\frac{\partial}{\partial y}[\mu(x) M(x,y)] = \frac{\partial}{\partial x}[\mu(x) N(x,y)]$$

$$\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu = P(x)\mu$$

$$\frac{d\mu}{dy} - P(y)\mu = 0$$

$$\mu(y) = c e^{\int P(y)dy}$$

$$\mu(y) = c e^{\int \left(\frac{N_x - M_y}{M}\right)dy}$$

$$\mu(y) M(x,y) dx + \mu(y) N(x,y) dy = 0$$

Substitution Method

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$z = f(x(t), y(t))$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = h_x(x, u) + h_u(x, u)\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$\frac{dy}{dx} = g(x, y) = g(x, h(x, u))$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x, u)) = h_x(x, y) + h_u(x, u)\frac{du}{dx}$$

Substitution Method (2)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

$$y' = s\left(\frac{y}{x}\right)$$

$$y = h(x,u) \qquad u = \Phi(x)$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x}\frac{dx}{dx} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx} = g(x,y) = g(x,h(x,u))$$

$$g(x,h(x,u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y = h(x,u) \Leftarrow u = \Phi(x)$$

$$y = ux \qquad u = \frac{y}{x}$$

$$y' = u + xu' \qquad \frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y' = g(x,ux) = s(u) \qquad y \Leftarrow h(x,u)$$

$$s(u) = u + xu'$$

$$s(u) - u = xu' \qquad \frac{du}{s(u) - u} = \frac{dx}{x}$$

Substitution Method (3)

a new literal   a function of x

$$u = \Phi(x)$$

contains x and y literals
(y is also a function of x)

a new literal u is introduced
using old literals x and y :
a new function of x

$$u = \frac{y}{x}$$

a old literal   a function of x and u

$$y = h(x, u)$$

the old literal y can be replaced by
the new literal u and the old literal x:
a new function of u and x

$$y = u\,x$$

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Substitution Method (4)

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$y' = u + xu' \Leftarrow y = ux \Leftarrow u = \frac{y}{x}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

$$y' = g(x, ux) = s(u) \Leftarrow \frac{dy}{dx} = s\left(\frac{y}{x}\right)$$

Find $y = f(x)$ in

$$\frac{dy}{dx} = g(x, y)$$

Find $u = \Phi(x)$ in

$$\frac{du}{dx} = F(x, u)$$

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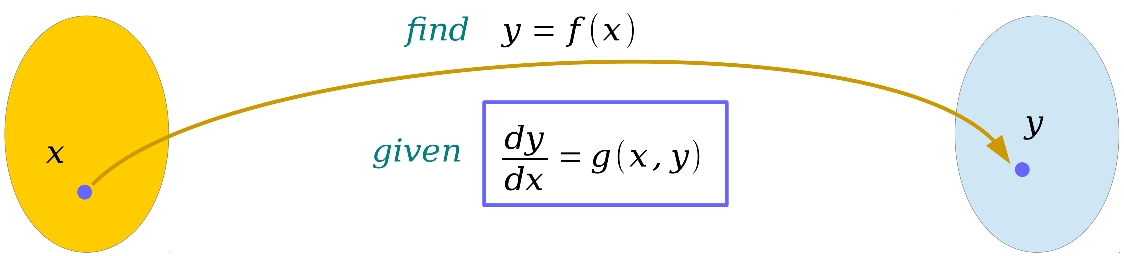

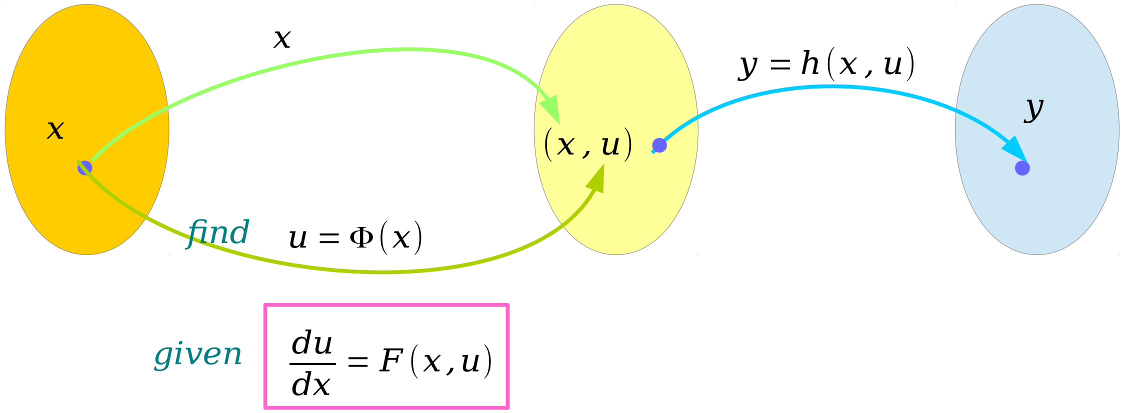

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(1) replace y'

$$\frac{dy}{dx} \Leftarrow \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

(2) replace y

$$g(x, y) \Leftarrow g(x, h(x, u))$$

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Substitution Method (6)

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

$$y = h(x, \Phi(x))$$

$$y = h(x, u) \qquad u = \Phi(x)$$

$$g(x, h(x, u)) = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$g(x, h(x, u)) = h_x(x, u) + h_u(x, u)\frac{du}{dx}$$

$$\frac{du}{dx} = \left\{ g(x, h(x, u)) - \frac{\partial h}{\partial x} \right\} / \frac{\partial h}{\partial u}$$

$$\frac{du}{dx} = \{ g(x, h(x, u)) - h_x(x, u) \} / h_u(x, u)$$

$$\frac{du}{dx} = F(x, u)$$

$$\frac{du}{dx} = F(x, u)$$

$$u = \Phi(x)$$

Homogeneous First Order ODEs

Homogeneous Functions

A homogeneous function of degree α

$$f(tx, ty) = t^{\alpha} f(x, y)$$

$$f(x, y) = x^2 + y^2$$

$$f(tx, ty) = (tx)^2 + (ty)^2$$

$$= t^2(x^2 + y^2)$$

$$= t^2 f(x, y)$$

A homogeneous Equations of degree α

$$M(x, y)dx + N(x, y)dy = 0$$

$$M(tx, ty) = t^{\alpha} M(x, y)$$

$$N(tx, ty) = t^{\alpha} N(x, y)$$

$$M(x, y) = M(x, x \cdot y/x) = x^{\alpha} M(1, y/x)$$

$$M(x, y) = M(y \cdot x/y, y) = y^{\alpha} M(x/y, 1)$$

$$N(x, y) = N(x, x \cdot y/x) = x^{\alpha} N(1, y/x)$$

$$N(x, y) = N(y \cdot x/y, y) = y^{\alpha} N(x/y, 1)$$

Homogeneous Equations (1)

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = y/x \qquad y = u\,x$$

$$x^{\alpha} M(1, y/x)dx + x^{\alpha} N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$dy = \frac{\partial}{\partial x}(u\,x)dx + \frac{\partial}{\partial u}(u\,x)du$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = y^{\alpha} M(x/y, 1)$$

$$N(x,y) = y^{\alpha} N(x/y, 1)$$

$$v = x/y \qquad x = v\,y$$

$$y^{\alpha} M(x/y, 1)dx + y^{\alpha} N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$dx = \frac{\partial}{\partial y}(v\,y)dy + \frac{\partial}{\partial v}(vy)dv$$

$$dx = v\,dy + y\,dv$$

$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

Homogeneous Equations (2)

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$x^\alpha M(1, y/x)dx + x^\alpha N(1, y/x)dy = 0$$

$$M(1,u)dx + N(1,u)dy = 0$$

$$u = y/x \qquad y = u\,x$$

$$dy = u\,dx + x\,du$$

$$M(1,u)dx + N(1,u)(u\,dx + x\,du) = 0$$

$$[M(1,u) + u\,N(1,u)]dx + x\,N(1,u)du = 0$$

$$\frac{dx}{x} + \frac{N(1,u)du}{[M(1,u) + u\,N(1,u)]} = 0$$

A homogeneous Equations of degree α

$$M(x,y)dx + N(x,y)dy = 0$$

$$y^\alpha M(x/y, 1)dx + y^\alpha N(x/y, 1)dy = 0$$

$$M(v,1)dx + N(v,1)dy = 0$$

$$v = x/y \qquad x = v\,y$$

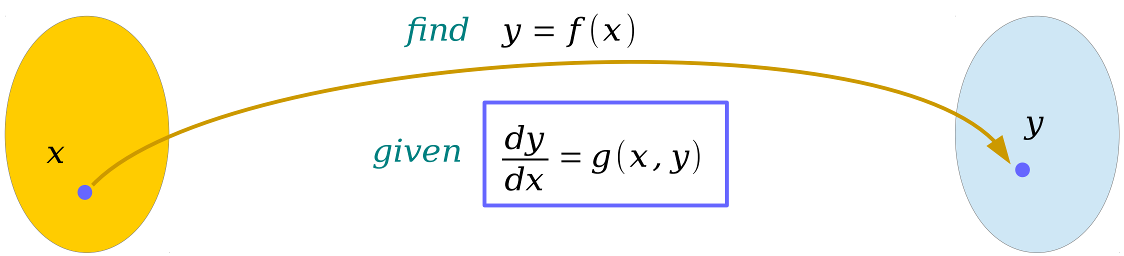
$$dx = v\,dy + y\,dv$$

$$M(v,1)(v\,dy + y\,dv) + N(v,1)dy = 0$$

$$[v\,M(v,1) + N(v,1)]dy + y\,M(v,1)dv = 0$$

$$\frac{dy}{y} + \frac{M(v,1)dv}{[v\,M(v,1) + N(v,1)]} = 0$$

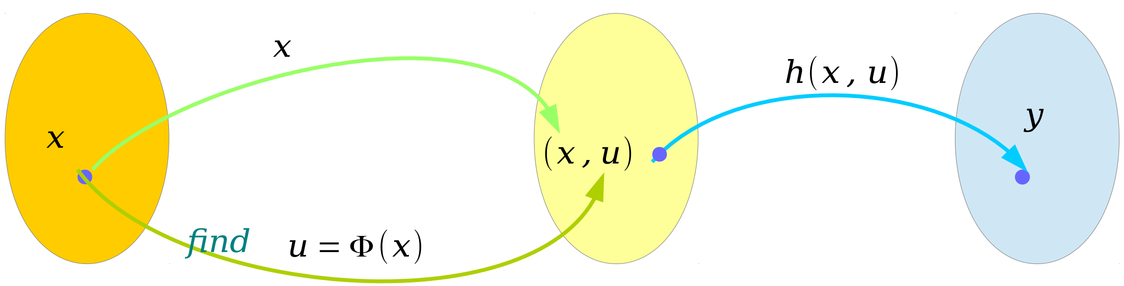
Homogeneous Equations (3)



$$M(x,y)dx + N(x,y)dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$



given $\frac{du}{dx} = F(x, u)$

$$u = \Phi(x) = y/x$$

$$y = h(x, u) = u\,x$$

$$\frac{dy}{dx} = \frac{\partial h}{\partial x} + \frac{\partial h}{\partial u}\frac{du}{dx}$$

$$\frac{dy}{dx}dx = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial u}\frac{du}{dx}dx$$

$$dy = u\,dx + x\,du$$

$$dy = u\,dx + x\,du$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{[M(1,u) + u\,N(1,u)]} = 0$$

Homogeneous Equations (4)

all are functions of x

$$y = f(x) \Rightarrow y(x)$$

$$u = \Phi(x) \Rightarrow u(x)$$

$$u = y/x \Rightarrow u(x) = y(x)/x$$

$$y = ux \Rightarrow y(x) = u(x)\,x$$

$$y = h(x, u) = h(x, \Phi(x))$$

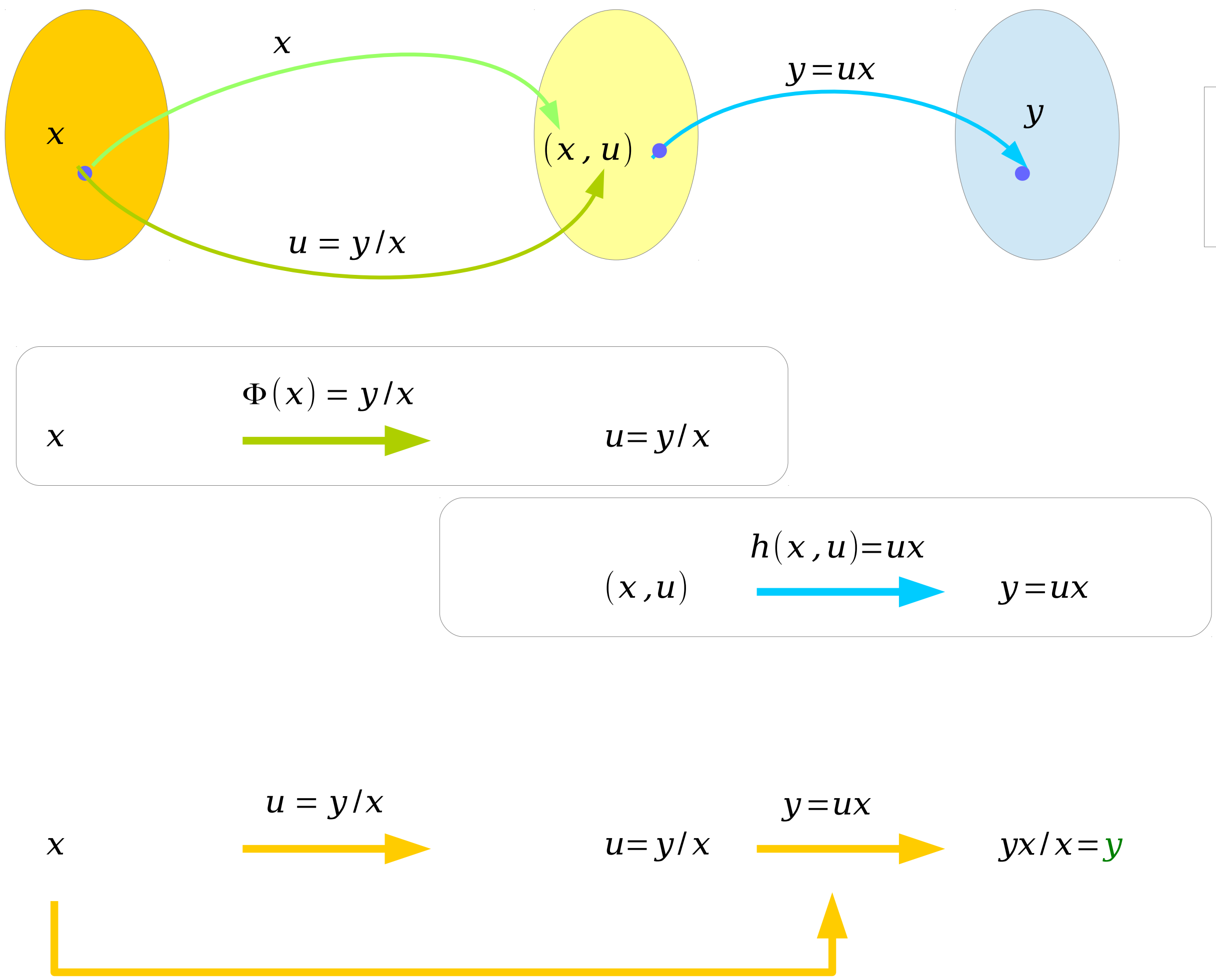
$$= u\,x \qquad = \Phi(x)\,x$$

$$= \frac{y}{x}\,x$$

$$y = ux$$

$$dy = u\,dx + x\,du$$

Homogeneous Equations (5)



$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$M(x,y) = x^{\alpha} M(1, y/x)$$

$$N(x,y) = x^{\alpha} N(1, y/x)$$

$$u = \Phi(x) = y/x$$

$$y = h(u,x) = u\,x$$

$$dy = u\,dx + x\,du$$

$$\frac{dx}{x} + \frac{N(1,u)\,du}{[M(1,u) + u\,N(1,u)]} = 0$$

Bernoulli's First Order ODEs

Bernoulli's Equations (1)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

$$\frac{dy}{dx} + P(x)y = Q(x)y^0$$

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad n = 0$$

***Linear* Equation**

$$\frac{dy}{dx} + P(x)y = Q(x)y^1$$

$$\frac{dy}{dx} + [P(x) - Q(x)]y = 0 \qquad n = 1$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)y = Q(x)y^n$$

$$y' + P(x)y = Q(x)y^0$$

$$y' + P(x)y = Q(x) \qquad n = 0$$

***Linear* Equation**

$$y' + P(x)y = Q(x)y^1$$

$$y' + [P(x) - Q(x)]y = 0 \qquad n = 1$$

***Linear* Equation**

Bernoulli's Equations (2)

***Bernoulli's* Equation**

$$\frac{dy}{dx} + P(x)\,y \;=\; Q(x)\,y^n$$

$$\frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} \;=\; Q(x)$$

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} \;=\; Q(x)$$

$$u = y^{1-n} \qquad \frac{du}{dx} = (1-n)\,\boxed{y^{-n}\frac{dy}{dx}}$$

$$\frac{1}{(1-n)}\frac{du}{dx} + P(x)\,u \;=\; Q(x)$$

$$\frac{du}{dx} + (1-n)P(x)\,u \;=\; (1-n)Q(x)$$

***Linear* Equation**

***Bernoulli's* Equation**

$$y' + P(x)\,y \;=\; Q(x)\,y^n$$

$$\frac{1}{y^n}y' + P(x)\frac{y}{y^n} \;=\; Q(x)$$

$$y^{-n}y' + P(x)\,y^{1-n} \;=\; Q(x)$$

$$u = y^{1-n} \qquad u' = (1-n)\,\boxed{y^{-n}y'}$$

$$\frac{1}{(1-n)}u' + P(x)\,u \;=\; Q(x)$$

$$u' + (1-n)P(x)\,u \;=\; (1-n)Q(x)$$

***Linear* Equation**

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