DTFT (4B)

- Discrete Time Fourier Transform

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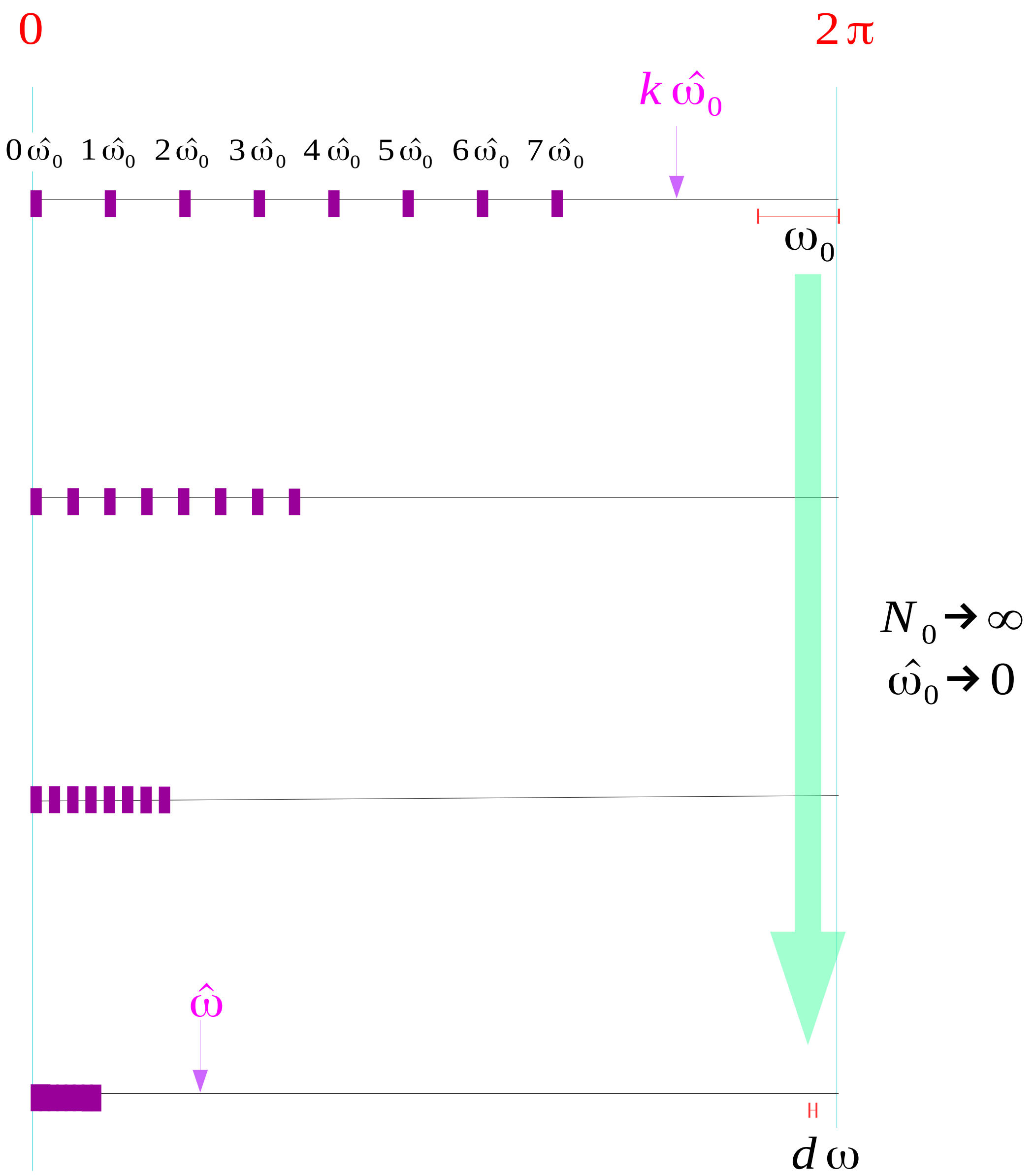
Aperiodic Signal Conversion $x[n]$

Limit Values $\hat{\omega}$, $d\hat{\omega}$



γ_k at $k\hat{\omega}_0$

$N_0 \rightarrow \infty$

$$\hat{\omega}_0 = \left(\frac{2\pi}{N_0}\right) \rightarrow 0$$

$$\hat{\omega}_0 \rightarrow d\hat{\omega}$$

$$k\hat{\omega}_0 \rightarrow \hat{\omega}$$

$X(e^{j\hat{\omega}})$ at $\hat{\omega}$

The Product $X(e^{j\hat{\omega}})\, d\hat{\omega}$

$$\gamma_k N_0 \rightarrow X(e^{j\hat{\omega}})$$

$$N_0 \hat{\omega}_0 = N_0 \left(\frac{2\pi}{N_0}\right) = 2\pi$$

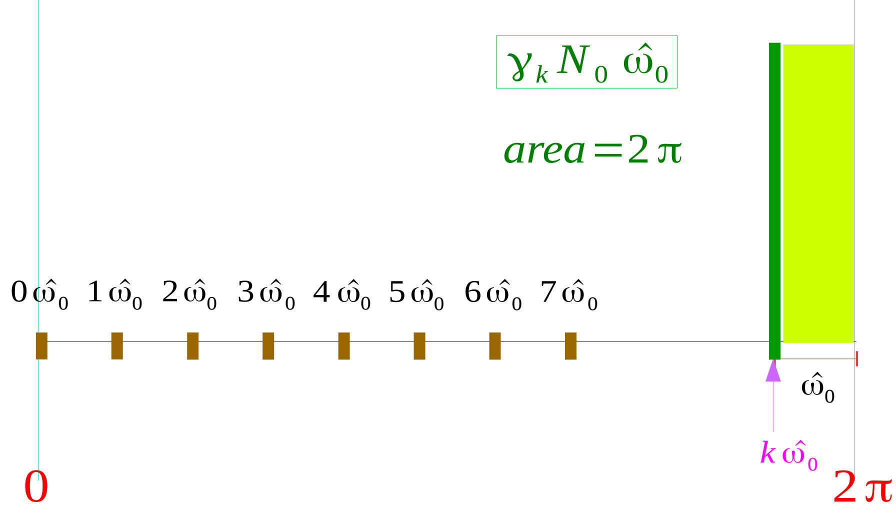
$\gamma_k N_0$

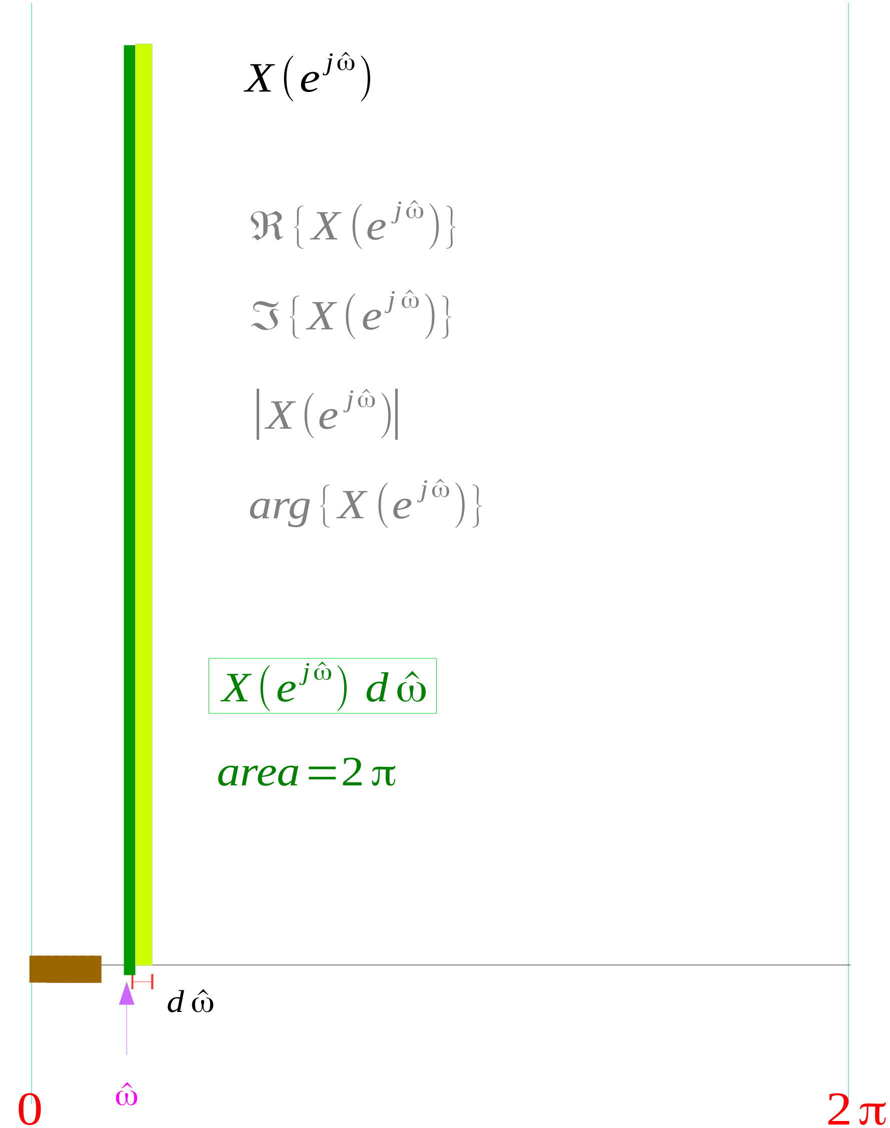
$\Re\{\gamma_k N_0\}$

$\Im\{\gamma_k N_0\}$

$|\gamma_k N_0|$

$arg\{\gamma_k N_0\}$





From DTFS to DTFT

$$x_{N_0}[n] = \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)k\,n} \cdot 1$$

$$= \sum_{k=0}^{N_0} \gamma_k \, e^{+j\left(\frac{2\pi}{N_0}\right)k\,n} \cdot \left(\frac{N_0}{2\pi}\right) \cdot \left(\frac{2\pi}{N_0}\right)$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)k\,n} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N_0 \, e^{+j\left(\frac{2\pi}{N_0}\right)k\,n} \cdot \left(\frac{2\pi}{N_0}\right)$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}}) \, e^{+j\hat{\omega}n} \, d\hat{\omega}$$

$N_0 \rightarrow \infty$ $\qquad \hat{\omega}_0 = \left(\frac{2\pi}{N_0}\right) \rightarrow 0$

$\hat{\omega}_0 \rightarrow d\hat{\omega}, \quad k\hat{\omega}_0 \rightarrow \hat{\omega}$

$x_{N_0}[n] \rightarrow x[n], \quad \gamma_k N_0 \rightarrow X(e^{j\hat{\omega}})$

From DTFS to DTFT

Discrete Time Fourier Series

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n]\, e^{-j\left(\frac{2\pi}{N}\right)kn} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N} \gamma_k\, e^{+j\left(\frac{2\pi}{N}\right)kn}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt$$

$$x_{N_0}[n] = \sum_{k=0}^{N_0} \gamma_k\, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{2\pi} \cdot \frac{N_0}{N_0}$$

$$x_{N_0}[n] = \frac{1}{2\pi} \sum_{k=0}^{N_0} \gamma_k N\, e^{+j\left(\frac{2\pi}{N_0}\right)kn} \cdot \frac{2\pi}{N_0}$$

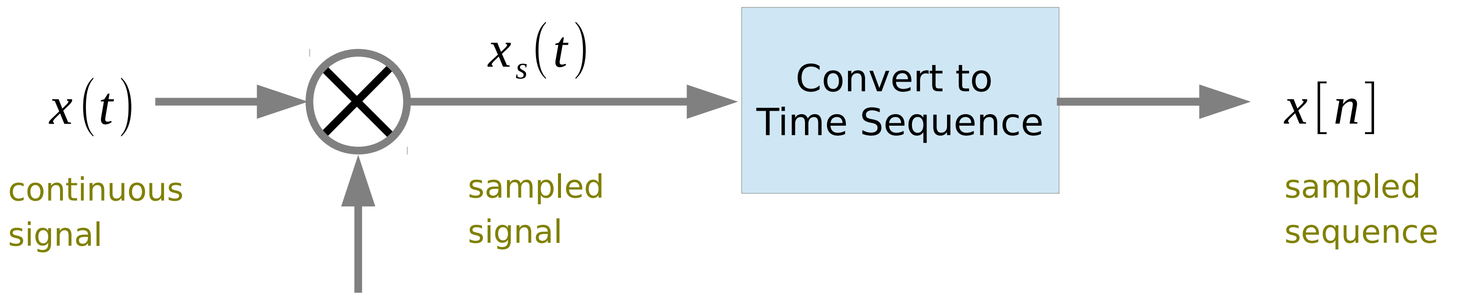
$$N_0 \to \infty, \quad \hat{\omega}_0 \to d\hat{\omega} \quad \left(\frac{2\pi}{N_0} \to 0\right), \quad k\hat{\omega}_0 \to \omega \quad \Rightarrow \quad x_{N_0}[n] \to x[n], \quad \gamma_k N_0 \to X(e^{j\hat{\omega}})$$

Discrete Time Fourier Transform

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

Sampling and Reconstruction

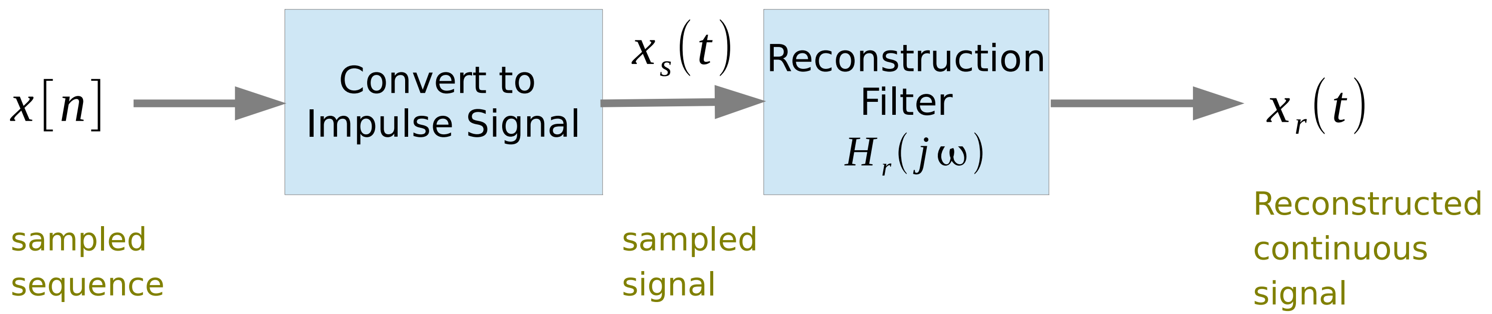
Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

T_s **Sampling Period**

Ideal Reconstruction



CTFS of an Impulse Train (1)

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

CTFS =

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

Fourier Series Expansion

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{+jk\omega_s t}$$

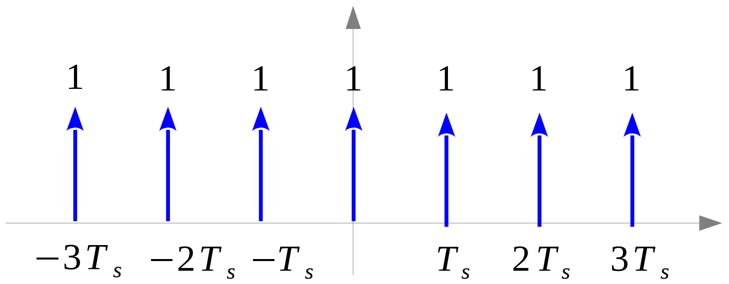
$$\omega_s = \frac{2\pi}{T_s}$$

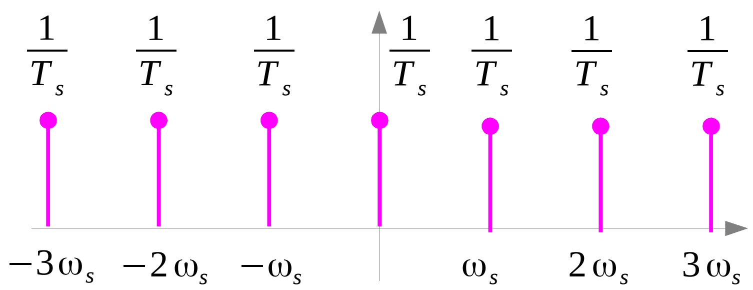
Fourier Series Coefficients

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, e^{-jk\omega_s 0}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{+T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$





CTFS of an Impulse Train (2)

$$p(t) = \sum_{k=-\infty}^{+\infty} a_k \, e^{+jk\omega_s t}$$

CTFT ⇒

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \, \delta(\omega - k\omega_s)$$

Fourier Transform of impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

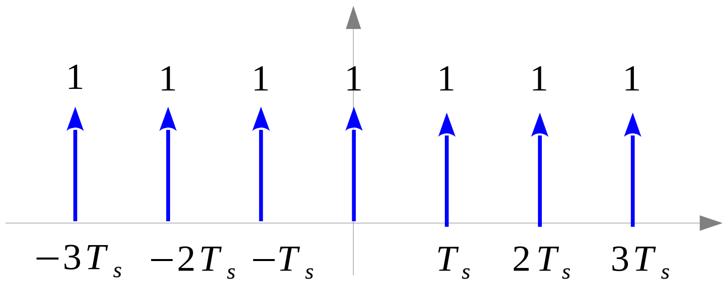
CTFS =

$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

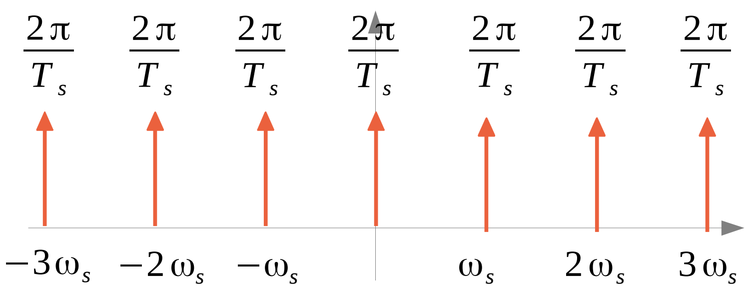
$$p(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{+jk\omega_s t}$$

CTFT ⇒

$$P(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

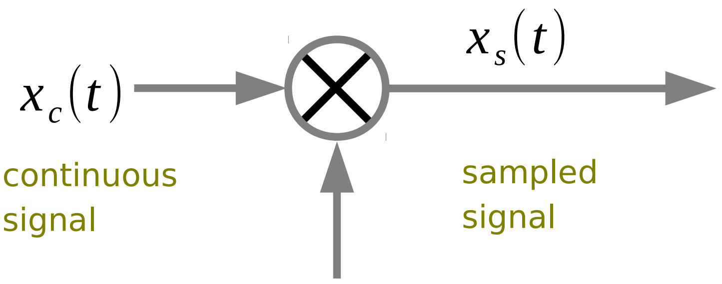


$$\omega_s = \frac{2\pi}{T_s}$$



Sampled Signal

Ideal Sampling



$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

sampled signal

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$= \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFS

$$x_s(t) = x_c(t) \sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

sampled signal

CTFT Frequency Shift Property

Frequency Shift Property

		CTFT	
continuous signal	$x_c(t)$	$\Leftrightarrow$	$X_c(j\omega)$
continuous signal	$x_c(t)\, e^{jk\omega_s t}$	$\Leftrightarrow$	$X_c(j(\omega - k\omega_s))$

CTFT

$$x_s(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(t)\, e^{jk\omega_s t} \quad \Leftrightarrow \quad X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s} \qquad \omega_s = \frac{2\pi}{T_s}$$

CTFT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \Leftrightarrow \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

CTFT Time Shift Property

Fourier Transform of an Impulse

$$\delta(t - t_d) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad e^{j\omega t_d}$$

$$\delta(t - nT_s) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad e^{-j\omega n T_s}$$

$$x_s(t) = x_c(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s) \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \quad \overset{\text{CTFT}}{\Longleftrightarrow} \quad X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Sampling and Replication

sampled signal

$$x_s(t) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(t)\, e^{jk\omega_s t}$$

CTFT

replicated spectrum

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

sampled sequence

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

sampled signal

replicated spectrum

sampled sequence

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

DTFT

replicated spectrum

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

CTFT of a Sampled Signal

Continuous Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

CTFT ⟺

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x_s(t) = x_c(t)\sum_{n=-\infty}^{+\infty} \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\,\delta(t - nT_s)$$

CTFT ⟺

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega nT_s}$$

$$X_s(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega nT_s}$$

CTFS ||

$$\hat{\omega} = \omega T_s$$

DTFT ||

$$x_s(t) = x_c(t)\sum_{k=-\infty}^{+\infty} \frac{1}{T_s} e^{jk\omega_s t}$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT ⟺

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

z-Transform of a Sampled Signal

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

CTFT of a sampled signal

$$\sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s)) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - \frac{2\pi k}{T_s}))$$

Z-Transform of a sampled signal

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad x[n] = x_c(nT_s)$$

$$X(z)\Big|_{z = e^{j\omega T_s}} = X(e^{j\omega T_s})$$

evaluated at $z = e^{j\omega T_s}$

z-Transform and Normalized Frequency

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

=

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(z)\Big|_{z=e^{j\omega T_s}} = X(e^{j\omega T_s}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

z-Transform

$$\hat{\omega} = \omega T_s$$

Normalized Frequency

$$X(z)\Big|_{z=e^{j\hat{\omega}}} = X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform

DTFT and CTFT

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$\Leftrightarrow$ =

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} x_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

DTFT of a sampled signal

$$X(e^{j\hat{\omega}}) \qquad \hat{\omega} = \omega T_s$$

$$= X(e^{j\omega T_s}) = \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$= \frac{1}{T_s}\sum_{k=-\infty}^{+\infty} X_c\left(j\left(\omega - \frac{2\pi k}{T_s}\right)\right)$$

CTFT of a sampled signal

DTFT and CTFT

Continuous Time Fourier Transform **CTFT**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier Transform **DTFT**

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X(e^{j\hat{\omega}})\, e^{+j\hat{\omega} n}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{+\infty} x_c(nT_s)\, e^{-j\omega n T_s}$$

$$= \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\omega n T_s}$$

$=$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} X_c(j(\omega - k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M.J. Roberts, Fundamentals of Signals and Systems
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[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings