

Power Spectral Density - Continuous Time

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Energy and average power in time domain

power spectral density for continuous time signals

Energy, Average Power – deterministic, time domain

a deterministic signal $x(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

the energy

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \int_{-\infty}^{+\infty} x_T^2(t) dt$$

the average power

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2T} \int_{-\infty}^{+\infty} x_T^2(t) dt$$

Fourier transform

power spectral density for continuous time signals

Fourier Transform Pair $x(t) \iff X(\omega)$

Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

a deterministic signal $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Fourier transform of $x_T(t)$

for continuous time signals

bounded duration, bounded variation

for a finite T , $x_T(t)$ is assumed to have bounded variation

$$\int_{-T}^{+T} |x(t)| dt < \infty$$

the Fourier transform of $x_T(t)$

$$\begin{aligned} X_T(\omega) &= \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt \\ &= \int_{-T}^{+T} x(t) e^{-j\omega t} dt \end{aligned}$$

Fourier transforms of $x_T(t)$ and $X_T(t)$

power spectral density for continuous time signals

deterministic $X_T(\omega)$ v.s. random $X_T(\omega)$

a **deterministic** sample signal $x_T(t)$

$$x_T(t) \iff X_T(\omega)$$

a **random process** signal $X_T(t)$

$$X_T(t) \iff X_T(\omega)$$

Parseval's theorem (I)

power spectral density for continuous time signals

for a deterministic $x_T(t)$

a **deterministic** sample signal $x_T(t)$

$$\int_{-\infty}^{+\infty} x_T(\tau)x_T^*(\tau)d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega)X_T^*(\omega)d\omega$$

$$\int_{-\infty}^{+\infty} |x_T(\tau)|^2 d\tau = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

Parseval's theorem (II)

power spectral density for continuous time signals

for a deterministic $x_T(t)$ v.s. a random $X_T(t)$

- a **deterministic** signal $x_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

- a **random** signal $X_T(t) \iff X_T(\omega)$

$$\int_{-\infty}^{+\infty} E \left[|X_T(t)|^2 \right] dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E \left[|X_T(\omega)|^2 \right] d\omega$$

Energy and average power in frequency domain

power spectral density for continuous time signals

Energy, Average Power – Parseval's theorem applied

a deterministic signal $x_T(t)$

$$x_T(t) = \begin{cases} x(t) & -T < t < T \\ 0 & \text{otherwise} \end{cases} \quad x_T(t) \iff X_T(\omega)$$

the **energy** by Parseval's theorem

$$E(T) = \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** by Parseval's theorem

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

$E(T)$ and $P(T)$ in frequency domain – deterministic case

power spectral density for continuous time signals

deterministic $x_T(t) \iff X_T(\omega)$

the **energy** for the **deterministic** $X_T(\omega)$ in $x_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X_T(\omega)|^2 d\omega$$

the **average power** for the **deterministic** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{|X_T(\omega)|^2}{2T} d\omega$$

the **power spectral density** for the **deterministic** $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{|X_T(\omega)|^2}{2T}$$

$E(T)$ and $P(T)$ in frequency domain – random case

power spectral density for continuous time signals

random $X_T(t) \iff X_T(\omega)$

the **energy** for the **random** $X_T(\omega)$ in $X_T(t) \iff X_T(\omega)$

$$E(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E[|X_T(\omega)|^2] d\omega$$

the **average power** for the **random** $X_T(\omega)$

$$P(T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

the **power spectral density** for the **random** $X_T(\omega)$

$$\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

Average power $P(T)$ – bounded duration $(-T, +T)$

power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- not the average power in a **random process**
only the power in **one** sample function
 - to obtain the **average power** over all possible realizations,
replace $x(t)$ by $X(t)$
take the **expected value** of $x^2(t)$, that is $TE [X^2(t)]$
 - then, the **average power** is a **random variable**
with respect to the **random process** $X(t)$
- not the average power in an **entire** sample function
 - take $T \rightarrow \infty$ to include all power in the **ensemble** member

Average power P_{XX} – unbounded duration $(-\infty, +\infty)$

power spectral density for continuous time signals

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

- replace $x(t)$ by the **random variable** $X(t)$
- take the **expected value** of $x^2(t)$, that is $E[X^2(t)]$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

- take $T \rightarrow \infty$ to include all power

$$P_{XX} = \lim_{T \rightarrow \infty} P(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt$$

Average power P_{XX} – time average $A[\bullet]$

power spectral density for continuous time signals

The time average

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt \quad A[\bullet] = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

time average and sample average operations

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} P(T) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= \lim_{T \rightarrow \infty} A_T[E[X^2(t)]] \\ &= A[E[X^2(t)]] \end{aligned}$$

Measuring average power

power spectral density for continuous time signals

for deterministic and random signals

the **average power** $P(T)$ for a deterministic signal $x(t)$

$$P(T) = \frac{1}{2T} \int_{-T}^{+T} x^2(t) dt$$

the **average power** P_{XX} for a random process $X(t)$

$$\begin{aligned} P_{XX} &= \lim_{T \rightarrow \infty} P(T) \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\ &= A[E[X^2(t)]] \end{aligned}$$

Power Spectral Density $S_{XX}(\omega)$

power spectral density for continuous time signals

the average power via power density

the average power P_{XX} for the random process $X_T(\omega)$

$$\begin{aligned} P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{S_{XX}(\omega)} d\omega \end{aligned}$$

the **Power Spectral Density (PSD)** $S_{XX}(\omega)$

$$\boxed{S_{XX}(\omega)} = \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}}$$

Properties of Power Spectral Density

power spectral density for continuous time signals

- $S_{XX}(\omega) \geq 0$
- $S_{XX}(-\omega) = S_{XX}(\omega)$ $X(t)$ real
- $S_{XX}(\omega)$ real
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$
- $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$
- $\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$
- $S_{XX}(\omega) = \int_{-\infty}^{+\infty} A [R_{XX}(t, t + \tau)] e^{-j\omega \tau} d\tau$

Equations involving $S_{XX}(\omega)$

power spectral density for continuous time signals

the average power P_{XX} and the inverse Fourier transform of $S_{XX}(\omega)$

the **average power** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

the **autocorrelation** related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A [R_{XX}(t, t + \tau)]$$

Average power related equation

power spectral density for continuous time signals

the average power P_{xx}

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = A [E [X^2(t)]]$$

- a random process $X(t)$ in time domain
- a random process $X(\omega)$ in frequency domain

$$X(t) = \lim_{T \rightarrow \infty} X_T(t) \quad X(\omega) = \lim_{T \rightarrow \infty} X_T(\omega)$$

- Parseval's theorem over $X_T(t) \iff X_T(\omega)$

Average power P_{XX} in time / frequency domain

power spectral density for continuous time signals

Average power P_{XX} using $X_T(t)$ and $X_T(\omega)$

- Using a random process $X_T(t)$ in time domain

$$\begin{aligned}P_{XX} &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} E[X^2(t)] dt \\&= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-\infty}^{+\infty} E[X_T^2(t)] dt \\&= \lim_{T \rightarrow \infty} A_T [E[X^2(t)]] = \boxed{A[E[X^2(t)]]}\end{aligned}$$

- Using a random process $X_T(\omega)$ in frequency domain

$$\begin{aligned}P_{XX} &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{\lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}} d\omega \\&= \boxed{\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}\end{aligned}$$

Autocorrelation related equation

power spectral density for continuous time signals

the Inverse Fourier transform of $S_{XX}(\omega)$

the average power related equation

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t + \tau)]$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \quad \Rightarrow R_{XX}(\tau)$$

- a random process $X(t)$ in time domain
- a random process $X(\omega)$ in frequency domain

PSD and Auto-correlation

power spectral density for continuous time signals

Fourier transform pairs

- $A[R_{XX}(t, t + \tau)] \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} A[R_{XX}(t, t + \tau)] e^{-j\omega\tau} d\tau$$

$$A[R_{XX}(t, t + \tau)] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

- $R_{XX}(\tau) \iff S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

for a WSS $X(t)$, $A[R_{XX}(t, t + \tau)] = R_{XX}(\tau)$

PSD and Auto-Correlation

power spectral density for continuous time signals

$S_{XX}(\omega)$ and $R_{XX}(\tau)$

the power spectral density

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

the auto-correlation function

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{+j\omega\tau} d\omega$$

Fourier transform of a derivative function

power spectral density for continuous time signals

Fourier transform of $\frac{d^n}{dt^n}x(t)$

$$x(t) \iff X(\omega)$$

$$\frac{d^n}{dt^n}x(t) \iff (j\omega)^n X(\omega)$$

$$Y(t) = \frac{d}{dt}X_T(t) \iff (j\omega)X_T(\omega) = Y(\omega)$$

PSD and Auto-Correlation of a Derivative Function (1)

power spectral density for continuous time signals

$S_{\dot{X}\dot{X}}(\omega)$ and $S_{XX}(\omega)$

$$S_{XX}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$X_T(t) \iff X_T(\omega)$$

$$S_{YY}(\omega) = \lim_{T \rightarrow \infty} \frac{E[|Y_T(\omega)|^2]}{2T}$$

$$Y_T(t) \iff Y_T(\omega)$$

$$= \lim_{T \rightarrow \infty} \frac{\omega^2 E[|X_T(\omega)|^2]}{2T}$$

$$Y_T(t) = \dot{X}_T(t)$$

$$= \omega^2 S_{XX}(\omega)$$

$$Y_T(\omega) = \omega^2 X_T(\omega)$$

PSD and Auto-Correlation of a Derivative Function (2)

power spectral density for continuous time signals

$$E[X_T(\omega)] \text{ and } E[\dot{X}_T(\omega)]$$

$$\begin{aligned} Y(t) &= \dot{X}_T(t) = \frac{d}{dt} X_T(t) \\ |\dot{X}(\omega)|^2 &= \dot{X}_T(\omega) \dot{X}_T^*(\omega) \\ &= (j\omega) X_T(\omega) (-j\omega) X_T^*(\omega) \\ &= \omega^2 |X(\omega)|^2 \end{aligned}$$

$$\begin{aligned} S_{\dot{X}\dot{X}}(\omega) &= \lim_{T \rightarrow \infty} \frac{E[|\dot{X}_T(\omega)|^2]}{2T} \\ &= \lim_{T \rightarrow \infty} \frac{\omega^2 E[|X(\omega)|^2]}{2T} \\ &= \omega^2 S_{XX}(\omega) \end{aligned}$$

Fourier transforms of autocorrelation functions

power spectral density for continuous time signals

Definition

Fourier transform of an autocorrelation functions

$$S_{\dot{X}\dot{X}}(\omega) = \int_{-\infty}^{+\infty} R_{\dot{X}\dot{X}}(\tau) e^{-j\omega\tau} d\tau$$
$$\omega^2 S_{XX}(\omega) = \int_{-\infty}^{+\infty} \omega^2 R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

- auto-correlation function

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)] \Rightarrow R_{XX}(\tau)$$

$$R_{\dot{X}\dot{X}}(t, t + \tau) = E[\dot{X}(t)\dot{X}(t + \tau)] \Rightarrow R_{\dot{X}\dot{X}}(\tau)$$

- a random process $X(t)$ in time domain
- $\dot{X}(t) = \frac{d}{dt}X(t)$: the derivative of $X(t)$

RMS Bandwidth

power spectral density for continuous time signals

Definition

the standard deviation is

a measure of the spread in a density function.

the analogous quantity for the normalized power spectral density is a measure of its spread that we call the rms bandwidth (root-mean-square)

$$W_{rms}^2 = \frac{\int_{-\infty}^{+\infty} \omega^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

RMS Bandwidth and Mean Frequency

power spectral density for continuous time signals

Definition

the mean frequency $\bar{\omega}_0$

$$\bar{\omega}_0 = \frac{\int_{-\infty}^{+\infty} \omega S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

the rms bandwidth

$$W_{rms}^2 = \frac{4 \int_{-\infty}^{+\infty} (\omega - \bar{\omega}_0)^2 S_{XX}(\omega) d\omega}{\int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega}$$

