# Example Random Processes 

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Based on
Probability, Random Variables and Random Signal Principles, P.Z. Peebles,Jr. and B. Shi

## Outline

(1) Gaussian Random Processes
(2) Poisson Random Process

## Gaussian Random Process

## $N$ Gaussian random variables

## Definition

$$
\begin{gathered}
f_{X}\left(x_{1}, \cdots, x_{N} ; t_{1}, \cdots, t_{N}\right)= \\
\frac{\exp \left\{-(1 / 2)[x-\bar{X}]^{t}\left[C_{X}\right]^{-1}[x-\bar{X}]\right\}}{\sqrt{(2 \pi)^{N} \mid\left[C_{X}\right]}} \\
x=\left[\begin{array}{c}
x_{1} \\
x_{2} \\
\vdots \\
x_{N}
\end{array}\right] \quad \bar{X}=\left[\begin{array}{c}
\bar{X}_{1} \\
\bar{X}_{2} \\
\vdots \\
\bar{X}_{N}
\end{array}\right] \quad[x-\bar{X}]=\left[\begin{array}{c}
x_{1}-\bar{X}_{1} \\
x_{2}-\bar{X}_{2} \\
\vdots \\
x_{N}-\bar{X}_{N}
\end{array}\right]
\end{gathered}
$$

## The Covariance Matrix (1)

## $N$ Gaussian random variables

## Definition

$$
\begin{gathered}
\bar{X}_{i}=E\left[X_{i}\right]=E\left[X\left(t_{i}\right)\right] \\
\bar{X}=\left[\begin{array}{c}
\bar{X}_{1} \\
\bar{X}_{2} \\
\vdots \\
\bar{X}_{N}
\end{array}\right]=\left[\begin{array}{c}
E\left[X_{1}\right] \\
E\left[X_{2}\right] \\
\vdots \\
E\left[X_{N}\right]
\end{array}\right]=\left[\begin{array}{c}
E\left[X\left(t_{1}\right)\right] \\
E\left[X\left(t_{2}\right)\right] \\
\vdots \\
E\left[X\left(t_{N}\right)\right]
\end{array}\right]
\end{gathered}
$$

## The Covariance Matrix (2)

## $N$ Gaussian random variables

## Definition

$$
\begin{aligned}
C_{i k} & =C_{X_{i} X_{k}}=E\left[\left(X_{i}-\bar{X}_{i}\right)\left(X_{k}-\bar{X}_{k}\right)\right] \\
& =E\left[\left(X\left(t_{i}\right)-E\left[X\left(t_{i}\right)\right]\right)\left(X\left(t_{k}\right)-E\left[X\left(t_{k}\right)\right]\right)\right] \\
& \\
& C_{i k}=C_{X_{i} X_{k}}=C_{X X}\left(t_{i}, t_{k}\right) \\
& =R_{X X}\left(t_{i}, t_{k}\right)-E\left[X\left(t_{i}\right)\right] E\left[X\left(t_{k}\right)\right]
\end{aligned}
$$

## Stationary Gaussian Process

$N$ Gaussian random variables

## Definition

$$
\begin{gathered}
\bar{X}_{i}=E\left[X_{i}\right]=E\left[X\left(t_{i}\right)\right]=\bar{X}=\text { const } \\
C_{X X}\left(t_{i}, t_{k}\right)=C_{X X}\left(t_{k}-t_{i}\right) \\
R_{X X}\left(t_{i}, t_{k}\right)=R_{X X}\left(t_{k}-t_{i}\right)
\end{gathered}
$$

## Jointly Gaussian Process <br> $N$ Gaussian random variables

## Definition

the two random processes $X(t)$ and $Y(t)$
are jointly Gaussian if the random variables
$X\left(t_{1}\right), \ldots, X\left(t_{N}\right)$ at times $t_{1}, \ldots, t_{N}$ for $X(t)$ and
$Y\left(t_{1}^{\prime}\right), \ldots, Y\left(t_{M}^{\prime}\right)$ at times $t_{1}^{\prime}, \ldots, t_{M}^{\prime}$ for $Y(t)$
are jointly gaussian for any $N, t_{1}, \ldots, t_{N}$, and $M, t_{1}^{\prime}, \ldots, t_{M}^{\prime}$

## Stationary Gaussian Markov Process

 $N$ Gaussian random variables
## Definition

$$
\begin{gathered}
C_{X X}(\tau)=\sigma^{2} e^{-\beta|\tau|} \\
C_{X X}[k]=\sigma^{2} a^{-|k|} \\
a=e^{\beta T_{s}}
\end{gathered}
$$

## Poisson Random Process

## $N$ Gaussian random variables

## Definition

$$
\begin{gathered}
p[X(t)=k]=\frac{(\lambda t)^{k} e^{-\lambda t}}{k!}, \quad k=0,1,2, \cdots \\
f_{X}(x)=\sum_{k=0}^{\infty} \frac{(\lambda t)^{k} e^{-\lambda t}}{k!} \delta(x-k)
\end{gathered}
$$

## Poisson Random Process - mean and 2nd moment

## $N$ Gaussian random variables

## Definition

$$
\begin{aligned}
E[X(t)] & =\int_{-\infty}^{\infty} x f_{X}(x) d x=\int_{-\infty}^{\infty} x \sum_{k=0}^{\infty} \frac{(\lambda t)^{k} e^{-\lambda t}}{k!} \delta(x-k) d x \\
& =\sum_{k=0}^{\infty} \frac{k(\lambda t)^{k} e^{-\lambda t}}{k!}=\lambda t \\
E\left[X^{2}(t)\right] & =\int_{-\infty}^{\infty} x^{2} f_{x}(x) d x=\int_{-\infty}^{\infty} x^{2} \sum_{k=0}^{\infty} \frac{(\lambda t)^{k} e^{-\lambda t}}{k!} \delta(x-k) d x \\
& =\sum_{k=0}^{\infty} \frac{k^{2}(\lambda t)^{k} e^{-\lambda t}}{k!}=\lambda t(1+\lambda t)
\end{aligned}
$$

## Poisson Random Process - joint probability density

## N Gaussian random variables

## Definition

$$
\begin{aligned}
& P\left[X\left(t_{1}\right)=k_{1}\right]=\frac{\left(\lambda t_{1}\right)^{k_{1}} e^{-\lambda t_{1}}}{k_{1}!} \quad k_{1}=0,1,2, \cdots \\
& \begin{aligned}
P\left[X\left(t_{2}\right)=\right. & \left.k_{2} \mid X\left(t_{1}\right)=k_{1}\right]=\frac{\left[\lambda\left(t_{2}-t_{1}\right)\right]^{k_{2}-k_{1}} e^{-\lambda\left(t_{2}-t_{1}\right)}}{\left(k_{2}-k_{1}\right)!} \\
P\left(k_{1}, k_{2}\right) & =P\left[X\left(t_{2}\right)=k_{2} \mid X\left(t_{1}\right)=k_{1}\right] \cdot P\left[X\left(t_{1}\right)=k_{1}\right] \\
& =\frac{\left(\lambda t_{1}\right)^{k_{1}}\left[\lambda\left(t_{2}-t_{1}\right)\right]^{k_{2}-k_{1}} e^{-\lambda t_{2}}}{k_{1}!\left(k_{2}-k_{1}\right)!} \\
f_{X}\left(x_{1}, x_{2}\right) & =\sum_{k_{1}=0}^{\infty} \sum_{k_{2}=k_{1}}^{\infty} P\left(k_{1}, k_{2}\right) \delta\left(x_{1}-k_{1}\right) \delta\left(x_{2}-k_{2}\right)
\end{aligned}
\end{aligned}
$$

## Bernoulli Random Process (1)

$N$ Gaussian random variables

## Definition

A Bernoulli process is a finite or infinite sequence of independent random variables $I[1], I[2], I[3], \ldots$, such that for each $n$, the value of $I[n]$ is either 0 or 1 ;
for all values of $n$, the probability $p$ that $l[n]=1$ is the same.
In other words, a Bernoulli process is a sequence of independent identically distributed Bernoulli trials.

Independence of the trials implies that the process is memoryless.

## Bernoulli Random Process (2)

## N Gaussian random variables

## Definition

the Bernoulli random process at sample index $n$ is $I[n]$ the number of events that have occurred after sample index 0 and up to $n$

$$
X[n]=\sum_{m=1}^{n} I[m]
$$

the binomial counting process is an example of what is called a sum process, since it can be obtained by summing the values of another random process

## Bernoulli Random Process (3)

## $N$ Gaussian random variables

## Definition

the density function for $X[n]$ is represented by a binomial density function

$$
\begin{gathered}
f_{X}(x)=\sum_{k=0}^{n} P(k) \delta(x-k) \\
P(k)=\binom{n}{k} p^{k}(1-p)^{n-k}
\end{gathered}
$$

the mean and the variance of the binomial counting process

$$
\begin{aligned}
E[X[n]] & =n p \\
\operatorname{Var}[X[n]] & =n p(1-p)
\end{aligned}
$$

## Binomial Counting Process

## $N$ Gaussian random variables

## Definition

$$
\begin{gathered}
f_{X}\left(x_{1}, x_{2}\right)=\sum_{k_{1}=0}^{n_{1}} \sum_{k_{2}=k_{1}}^{n_{2}} P\left(k_{1}, k_{2}\right) \delta\left(x_{1}-k_{1}\right) \delta\left(x_{2}-k_{2}\right) \\
P\left(k_{1}, k_{2}\right)=P\left[X\left[n_{1}\right]=k_{1}, X\left[n_{2}\right]=k_{2}\right] \\
=\binom{n_{2}-n_{1}}{k_{2}-k_{1}}\binom{n_{1}}{k_{1}} p^{k_{2}}(1-p)^{n_{2}-k_{2}} \\
P(k)=\frac{(n p)^{k} e^{-n p}}{k!}=\frac{(\lambda t)^{k} e^{-\lambda t}}{k!}
\end{gathered}
$$

