

Laurent Series and z-Transform - Geometric Series Applications

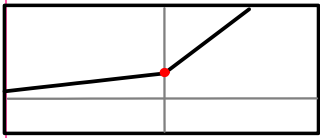


20200729 Wed

Copyright (c) 2016 - 2019 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

a^n



*a
/a

Shifting a sequence

$$\frac{1}{1-az} \quad |z| < a^{-1}$$

$a^n u(n)$

$$\frac{a}{1-az} \quad |z| < a^{-1}$$

$a^{n+1} u(n)$

$$\frac{az}{1-az} \quad |z| < a^{-1}$$

$a^n u(n-1)$

$$\frac{z}{1-az} \quad |z| < a^{-1}$$

$a^{n-1} u(n-1)$

$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$a^n u(-n-1)$

$$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

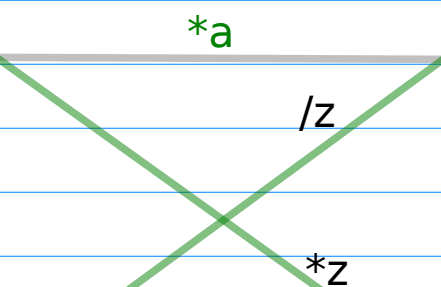
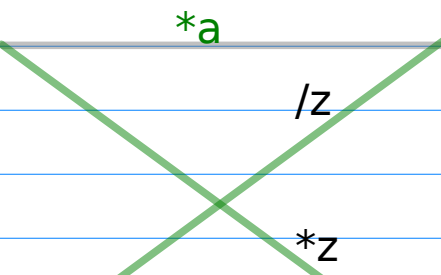
$a^{n+1} u(-n-1)$

$$-\frac{1}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

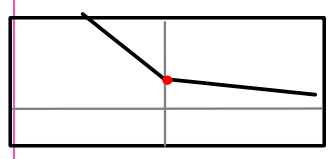
$a^n u(-n)$

$$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$a^{n-1} u(-n)$



$$a^{-n}$$



/a
*a

Shifting a sequence

$\frac{1}{1-a^{-1}z} \quad z < a$	/a /z	$\frac{a^{-1}}{1-a^{-1}z} \quad z < a$
-------------------------------------	----------	--

$$a^{-n} u(n)$$

$$a^{-n-1} u(n)$$

$\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$	*a	$\frac{z}{1-a^{-1}z} \quad z < a$
---	----	-------------------------------------

$$a^{-n} u(n-1)$$

$$a^{-n+1} u(n-1)$$

$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a$	/a /z	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a$
--	----------	--

$$a^{-n} u(-n-1)$$

$$a^{-n-1} u(-n-1)$$

$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a$	*a	$-\frac{a}{1-a^{-1}z^{-1}} \quad z > a$
---	----	---

$$a^{-n} u(-n)$$

$$a^{-n+1} u(-n)$$

2 formulas

Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

2 representations each

Geometric Series Form

$$\frac{1}{z - p} \begin{cases} \frac{p^{-1}}{1 - p^{-1}z} \triangleq f(z) = \chi(z^{-1}) \\ \frac{z^{-1}}{1 - pz^{-1}} \triangleq \gamma(z) = g(z^{-1}) \end{cases}$$

causal anti-causal
|| ||
causal anti-causal

$$\frac{1}{z^{-1} - p} \begin{cases} -\frac{p^{-1}}{1 - p^{-1}z^{-1}} \triangleq \chi(z) = f(z^{-1}) \\ \frac{z}{1 - pz} \triangleq g(z) = \gamma(z^{-1}) \end{cases}$$

causal anti-causal
|| ||
causal anti-causal

Simple Pole Form

Geometric Series Form

Geometric Series (1)

2 formulas

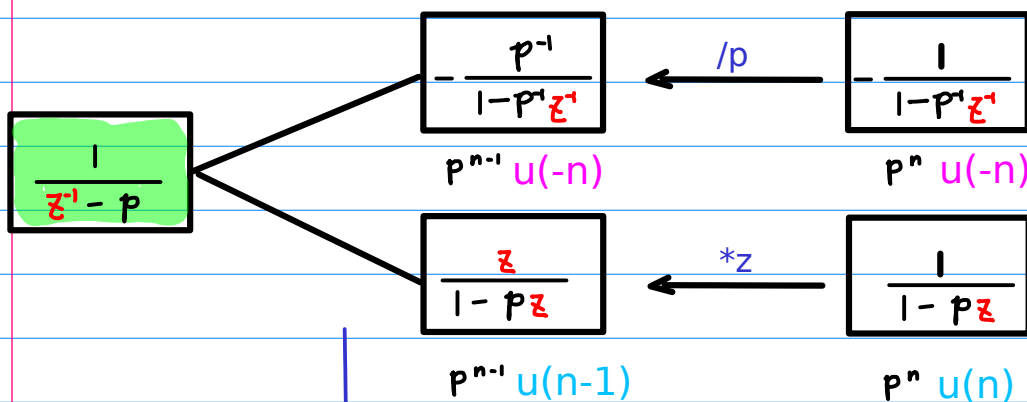
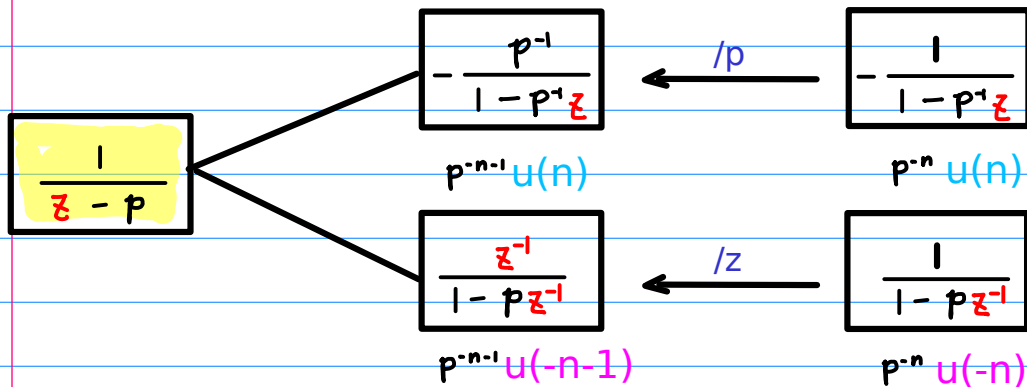
Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

2 representations each

Geometric Series Form



Simple Pole Form

Geometric Series Form

Geometric Series (2)

2 formulas

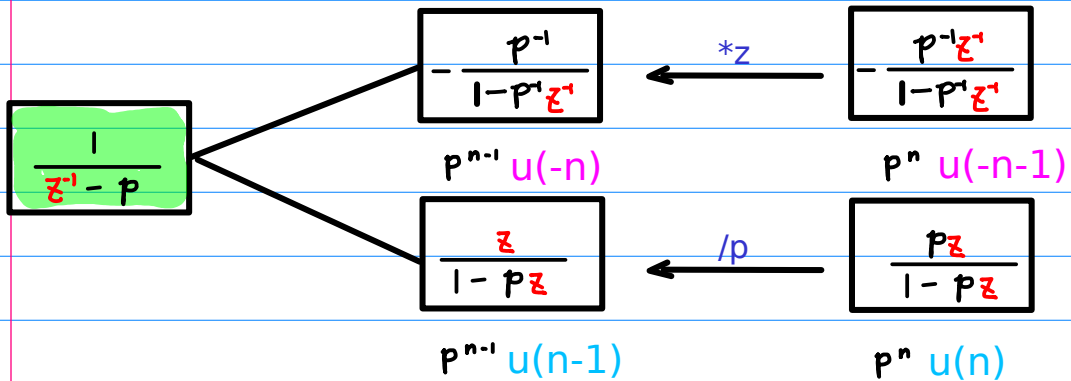
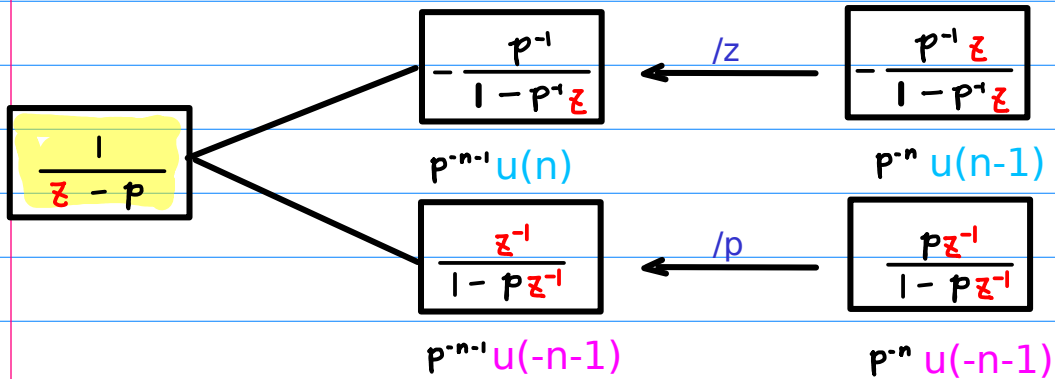
Simple Pole Form

$$\frac{1}{z - p}$$

$$\frac{1}{z^{-1} - p}$$

2 representations each

Geometric Series Form



Simple Pole Form

Geometric Series Form

Geometric Series Form Combinations with a unit start term

$$(1) \quad 1 + \frac{1}{1 - az}$$

$a^n u(n)$

$$(2) \quad 1 + \frac{1}{1 - a^{-1}z}$$

$a^{-n} u(n)$

$$(3) \quad -\frac{1}{1 - a^{-1}z^{-1}}$$

$-a^n u(-n)$

$$(4) \quad -\frac{1}{1 - az^{-1}}$$

$-a^{-n} u(-n)$

$$(1)' \quad 1 + \frac{1}{1 - a^{-1}z}$$

$a^n u(n)$

$$(2)' \quad 1 + \frac{1}{1 - az}$$

$a^{-n} u(n)$

$$(3)' \quad -\frac{1}{1 - az^{-1}}$$

$-a^{-n} u(-n)$

$$(4)' \quad -\frac{1}{1 - a^{-1}z^{-1}}$$

$-a^n u(-n)$

Geometric Series Form Combinations with a common-ratio start term

$$(5) \quad \frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \quad -a^n u(-n-1)$$

$$(6) \quad \frac{az^{-1}}{1 - az^{-1}} \quad -a^{-n}u(-n-1)$$

$$(7) \quad \frac{az}{1 - az} \quad a^n u(n-1)$$

$$(8) \quad \frac{a^{-1}z}{1 - a^{-1}z} \quad a^{-n}u(n-1)$$

$$(5)' \quad \frac{az^{-1}}{1 - az^{-1}} \quad -a^{-n}u(-n-1)$$

$$(6)' \quad \frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}} \quad -a^n u(-n-1)$$

$$(7)' \quad \frac{a^{-1}z}{1 - a^{-1}z} \quad a^{-n}u(n-1)$$

$$(8)' \quad \frac{az}{1 - az} \quad a^n u(n-1)$$

Geometric Series - a unit start term

Laurent Series

Geometric Series - a unit start term

z-Transform

Geometric Series - a unit start term

Laurent Series vs. z-Transform

a^n

$\blacksquare^{-1}\blacksquare$

a^{-n}

(1) $a z$



(1') $a^{-1}z$

(2) $a^{-1}z$



(2') $a z$

(3) $a^{-1}z^{-1}$



(3') $a z^{-1}$

(4) $a z^{-1}$



(4') $a^{-1}z^{-1}$

(5) $a^{-1}z^{-1}$



(5') $a z^{-1}$

(6) $a z^{-1}$



(6') $a^{-1}z^{-1}$

(7) $a z$



(7') $a^{-1}z$

(8) $a^{-1}z$



(8') $a z$

Geometric Series - a unit start term

Laurent Series

(1) $\frac{1}{1 - az}$ $|z| < a^{-1}$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$a^n u(n) \quad (n \geq 0)$$

(2) $\frac{1}{1 - a^{-1}z}$ $|z| < a$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$\left(\frac{1}{a}\right)^n u(n) \quad (n \geq 0)$$

(3) $\frac{1}{1 - a^{-1}z^{-1}}$ $|z| > a^{-1}$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$-a^n u(-n) \quad (n < 1)$$

(4) $\frac{1}{1 - az^{-1}}$ $|z| > a$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$-\left(\frac{1}{a}\right)^n u(-n) \quad (n < 1)$$

(1') $\frac{1}{1 - a^{-1}z}$ $|z| < a$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$\left(\frac{1}{a}\right)^n u(n) \quad (n \geq 0)$$

(2') $\frac{1}{1 - az}$ $|z| < a^{-1}$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$a^n u(n) \quad (n \geq 0)$$

(3') $\frac{1}{1 - az^{-1}}$ $|z| > a$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$-\left(\frac{1}{a}\right)^n u(-n) \quad (n < 1)$$

(4') $\frac{1}{1 - a^{-1}z^{-1}}$ $|z| > a^{-1}$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$-a^n u(-n) \quad (n < 1)$$

Geometric Series - a unit start term

z-Transform ($n \rightarrow -n$)

(1)
$$+ \frac{1}{1 - az}$$
 $|z| < a^{-1}$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$a^n u(-n)$	$(-n \geq 0)$
$\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$

(2)
$$+ \frac{1}{1 - a^{-1}z}$$
 $|z| < a$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$\left(\frac{1}{a}\right)^{-n} u(-n)$	$(-n \geq 0)$
$a^n u(-n)$	$(n < 1)$

(3)
$$- \frac{1}{1 - a^{-1}z^{-1}}$$
 $|z| > a^{-1}$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$- a^n u(-(-n))$	$(-n < 1)$
$-\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$

(4)
$$- \frac{1}{1 - az^{-1}}$$
 $|z| > a$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$-\left(\frac{1}{a}\right)^n u(-(-n))$	$(-n < 1)$
$- a^n u(n)$	$(n \geq 0)$

(1')
$$+ \frac{1}{1 - a^{-1}z}$$
 $|z| < a$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$\left(\frac{1}{a}\right)^{-n} u(-n)$	$(-n \geq 0)$
$a^n u(-n)$	$(n < 1)$

(2')
$$+ \frac{1}{1 - az}$$
 $|z| < a^{-1}$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$a^n u(-n)$	$(-n \geq 0)$
$\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$

(3')
$$- \frac{1}{1 - az^{-1}}$$
 $|z| > a$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$-\left(\frac{1}{a}\right)^n u(-(-n))$	$(-n < 1)$
$- a^n u(n)$	$(n \geq 0)$

(4')
$$- \frac{1}{1 - a^{-1}z^{-1}}$$
 $|z| > a^{-1}$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots$$

$- a^n u(-(-n))$	$(-n < 1)$
$-\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$

Geometric Series - a unit start term

Laurent Series vs. z-Transform ($n \rightarrow -n$)

(1) $\boxed{+ \frac{1}{1 - az}}$ $\boxed{|z| < a^{-1}}$ $\boxed{+ \frac{1}{1 - a^{-1}z}}$ $\boxed{|z| < a}$ (2)

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

Laurent	$a^n u(n)$	$(n \geq 0)$
z-Trans	$\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$

Laurent	$\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$
z-Trans	$a^n u(-n)$	$(n < 1)$

(3) $\boxed{- \frac{1}{1 - a^{-1}z^{-1}}}$ $\boxed{|z| > a^{-1}}$ $\boxed{- \frac{1}{1 - az^{-1}}}$ $\boxed{|z| > a}$ (4)

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

Laurent	$-a^n u(-n)$	$(n < 1)$
z-Trans	$-\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$

Laurent	$-\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$
z-Trans	$-a^n u(n)$	$(n \geq 0)$

(1') $\boxed{+ \frac{1}{1 - a^{-1}z}}$ $\boxed{|z| < a}$ $\boxed{+ \frac{1}{1 - az}}$ $\boxed{|z| < a^{-1}}$ (2')

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$(a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$\left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

Laurent	$\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$
z-Trans	$a^n u(-n)$	$(n < 1)$

Laurent	$a^n u(n)$	$(n \geq 0)$
z-Trans	$\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$

(3') $\boxed{- \frac{1}{1 - az^{-1}}}$ $\boxed{|z| > a}$ $\boxed{- \frac{1}{1 - a^{-1}z^{-1}}}$ $\boxed{|z| > a^{-1}}$ (4')

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- (a^0 z^0 + a^1 z^1 + a^2 z^2 + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

$$- \left(\left(\frac{1}{a}\right)^0 z^0 + \left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \dots\right)$$

Laurent	$-\left(\frac{1}{a}\right)^n u(-n)$	$(n < 1)$
z-Trans	$-a^n u(n)$	$(n \geq 0)$

Laurent	$-a^n u(-n)$	$(n < 1)$
z-Trans	$-\left(\frac{1}{a}\right)^n u(n)$	$(n \geq 0)$

Geometric Series - a non-unit start term

Laurent Series

Geometric Series - a non-unit start term

z-Transform

Geometric Series - a non-unit start term

Laurent Series vs. z-Transform

a^n	$\begin{matrix} \blacksquare & -1 & \blacksquare \end{matrix}$	a^{-n}
(1) $a z$	\longleftrightarrow	(1') $a^{-1}z$
(2) $a^{-1}z$	\longleftrightarrow	(2') $a z$
(3) $a^{-1}z^{-1}$	\longleftrightarrow	(3') $a z^{-1}$
(4) $a z^{-1}$	\longleftrightarrow	(4') $a^{-1}z^{-1}$
(5) $a^{-1}z^{-1}$	\longleftrightarrow	(5') $a z^{-1}$
(6) $a z^{-1}$	\longleftrightarrow	(6') $a^{-1}z^{-1}$
(7) $a z$	\longleftrightarrow	(7') $a^{-1}z$
(8) $a^{-1}z$	\longleftrightarrow	(8') $a z$

Geometric Series - a non-unit start term

Laurent Series

(5)
$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \dots)$

$-a^n u(-n-1) \quad (n < 0)$

(6)
$$-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a$$

$-(a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \dots)$

$-\left(\left(\frac{1}{a}\right)^1z^{-1} + \left(\frac{1}{a}\right)^2z^{-2} + \left(\frac{1}{a}\right)^3z^{-3} + \dots\right)$

$-\left(\frac{1}{a}\right)^n u(-n-1) \quad (n < 0)$

(7)
$$+\frac{az}{1-az} \quad |z| < a^{-1}$$

$(a^1z^1 + a^2z^2 + a^3z^3 + \dots)$

$a^n u(n-1) \quad (n \geq 1)$

(8)
$$+\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$$

$(a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \dots)$

$\left(\left(\frac{1}{a}\right)^1z^1 + \left(\frac{1}{a}\right)^2z^2 + \left(\frac{1}{a}\right)^3z^3 + \dots\right)$

$\left(\frac{1}{a}\right)^n u(n-1) \quad (n \geq 1)$

(5')
$$-\frac{az^{-1}}{1-az^{-1}} \quad |z| > a$$

$-(a^1z^{-1} + a^2z^{-2} + a^3z^{-3} + \dots)$

$-\left(\left(\frac{1}{a}\right)^1z^{-1} + \left(\frac{1}{a}\right)^2z^{-2} + \left(\frac{1}{a}\right)^3z^{-3} + \dots\right)$

$-\left(\frac{1}{a}\right)^n u(-n-1) \quad (n < 0)$

(6')
$$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad |z| > a^{-1}$$

$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \dots)$

$-a^n u(-n-1) \quad (n < 0)$

(7')
$$+\frac{a^{-1}z}{1-a^{-1}z} \quad |z| < a$$

$(a^{-1}z^1 + a^{-2}z^2 + a^{-3}z^3 + \dots)$

$\left(\left(\frac{1}{a}\right)^1z^1 + \left(\frac{1}{a}\right)^2z^2 + \left(\frac{1}{a}\right)^3z^3 + \dots\right)$

$\left(\frac{1}{a}\right)^n u(n-1) \quad (n \geq 1)$

(8')
$$+\frac{az}{1-az} \quad |z| < a^{-1}$$

$(a^1z^1 + a^2z^2 + a^3z^3 + \dots)$

$a^n u(n-1) \quad (n \geq 1)$

Geometric Series - a non-unit start term

z-Transform ($n \rightarrow -n$)

(5)

$$\frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}}$$

$$|z| > a^{-1}$$

$$- (a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \left(\frac{1}{a}\right)^3 z^{-3} + \dots \right)$$

$$- a^{-n} u(-(-n)-1) \quad (n < 0)$$

$$- \left(\frac{1}{a}\right)^n u(n-1) \quad (n \geq 1)$$

(6)

$$\frac{az^{-1}}{1 - az^{-1}}$$

$$|z| > a$$

$$- (a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \left(\frac{1}{a}\right)^3 z^{-3} + \dots \right)$$

$$- \left(\frac{1}{a}\right)^n u(-(-n)-1) \quad (n < 0)$$

$$- a^n u(n-1) \quad (n \geq 1)$$

(7)

$$+ \frac{az}{1 - az}$$

$$|z| < a^{-1}$$

$$(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \left(\frac{1}{a}\right)^3 z^3 + \dots \right)$$

$$a^n u((-n)-1) \quad (n \geq 1)$$

$$\left(\frac{1}{a}\right)^n u(-n-1) \quad (n < 0)$$

(8)

$$+ \frac{a^{-1}z}{1 - a^{-1}z}$$

$$|z| < a$$

$$(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \left(\frac{1}{a}\right)^3 z^3 + \dots \right)$$

$$\left(\frac{1}{a}\right)^n u((-n)-1) \quad (n \geq 1)$$

$$a^n u(-n-1) \quad (n < 0)$$

(5')

$$\frac{az^{-1}}{1 - az^{-1}}$$

$$|z| > a$$

$$- (a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \left(\frac{1}{a}\right)^3 z^{-3} + \dots \right)$$

$$- \left(\frac{1}{a}\right)^n u(-(-n)-1) \quad (n < 0)$$

$$- a^n u(n-1) \quad (n \geq 1)$$

(6')

$$\frac{a^{-1}z^{-1}}{1 - a^{-1}z^{-1}}$$

$$|z| > a^{-1}$$

$$- (a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$- \left(\left(\frac{1}{a}\right)^1 z^{-1} + \left(\frac{1}{a}\right)^2 z^{-2} + \left(\frac{1}{a}\right)^3 z^{-3} + \dots \right)$$

$$- a^{-n} u(-(-n)-1) \quad (n < 0)$$

$$- \left(\frac{1}{a}\right)^n u(n-1) \quad (n \geq 1)$$

(7')

$$+ \frac{a^{-1}z}{1 - a^{-1}z}$$

$$|z| < a$$

$$(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \left(\frac{1}{a}\right)^3 z^3 + \dots \right)$$

$$\left(\frac{1}{a}\right)^n u((-n)-1) \quad (n \geq 1)$$

$$a^n u(-n-1) \quad (n < 0)$$

(8')

$$+ \frac{az}{1 - az}$$

$$|z| < a^{-1}$$

$$(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

$$\left(\left(\frac{1}{a}\right)^1 z^1 + \left(\frac{1}{a}\right)^2 z^2 + \left(\frac{1}{a}\right)^3 z^3 + \dots \right)$$

$$a^n u((-n)-1) \quad (n \geq 1)$$

$$\left(\frac{1}{a}\right)^n u(-n-1) \quad (n < 0)$$

Geometric Series - a non-unit start term

Laurent Series vs. z-Transform ($n \rightarrow -n$)

(5) $\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z| > a^{-1}$ $\frac{az^{-1}}{1-az^{-1}}$ $|z| > a$ (6)

$$-(a^{-1}z^{-1} + a^{-2}z^{-2} + a^{-3}z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1}z^{-1} + (\frac{1}{a})^{-2}z^{-2} + (\frac{1}{a})^{-3}z^{-3} + \dots)$$

$$-(a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$-((\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + (\frac{1}{a})^3 z^{-3} + \dots)$$

Laurent	$-a^n u(-n-1)$	$(n < 0)$
z-Trans	$-(\frac{1}{a})^n u(n-1)$	$(n \geq 1)$

Laurent	$-(\frac{1}{a})^n u(-n-1)$	$(n < 0)$
z-Trans	$-a^n u(n-1)$	$(n \geq 1)$

(7) $+\frac{az}{1-az}$ $|z| < a^{-1}$ $+\frac{a^{-1}z}{1-a^{-1}z}$ $|z| < a$ (8)

$$(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

$$((\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + (\frac{1}{a})^3 z^3 + \dots)$$

$$(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

$$((\frac{1}{a})^{-1} z^1 + (\frac{1}{a})^{-2} z^2 + (\frac{1}{a})^{-3} z^3 + \dots)$$

Laurent	$a^n u(n-1)$	$(n \geq 1)$
z-Trans	$(\frac{1}{a})^n u(-n-1)$	$(n < 0)$

Laurent	$(\frac{1}{a})^n u(n-1)$	$(n \geq 1)$
z-Trans	$a^n u(-n-1)$	$(n < 0)$

(5') $\frac{az^{-1}}{1-az^{-1}}$ $|z| > a$ $\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $|z| > a^{-1}$ (6')

$$-(a^1 z^{-1} + a^2 z^{-2} + a^3 z^{-3} + \dots)$$

$$-((\frac{1}{a})^1 z^{-1} + (\frac{1}{a})^2 z^{-2} + (\frac{1}{a})^3 z^{-3} + \dots)$$

$$-(a^{-1} z^{-1} + a^{-2} z^{-2} + a^{-3} z^{-3} + \dots)$$

$$-((\frac{1}{a})^{-1} z^{-1} + (\frac{1}{a})^{-2} z^{-2} + (\frac{1}{a})^{-3} z^{-3} + \dots)$$

Laurent	$-(\frac{1}{a})^n u(-n-1)$	$(n < 0)$
z-Trans	$-a^n u(n-1)$	$(n \geq 1)$

Laurent	$-a^n u(-n-1)$	$(n < 0)$
z-Trans	$-(\frac{1}{a})^n u(n-1)$	$(n \geq 1)$

(7') $+\frac{a^{-1}z}{1-a^{-1}z}$ $|z| < a$ $+\frac{az}{1-az}$ $|z| < a^{-1}$ (8')

$$(a^{-1} z^1 + a^{-2} z^2 + a^{-3} z^3 + \dots)$$

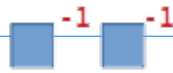
$$((\frac{1}{a})^{-1} z^1 + (\frac{1}{a})^{-2} z^2 + (\frac{1}{a})^{-3} z^3 + \dots)$$

$$(a^1 z^1 + a^2 z^2 + a^3 z^3 + \dots)$$

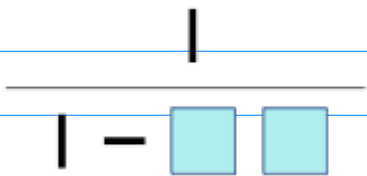
$$((\frac{1}{a})^1 z^1 + (\frac{1}{a})^2 z^2 + (\frac{1}{a})^3 z^3 + \dots)$$

Laurent	$(\frac{1}{a})^n u(n-1)$	$(n \geq 1)$
z-Trans	$a^n u(-n-1)$	$(n < 0)$

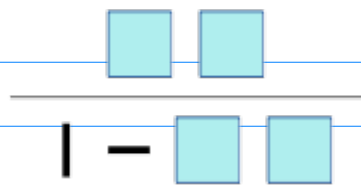
Laurent	$a^n u(n-1)$	$(n \geq 1)$
z-Trans	$(\frac{1}{a})^n u(-n-1)$	$(n < 0)$



(1) $a z$	\longleftrightarrow	(5) $a^{-1} z^{-1}$	a^n
(2) $a^{-1} z$	\longleftrightarrow	(6) $a z^{-1}$	
(3) $a^{-1} z^{-1}$	\longleftrightarrow	(7) $a z$	
(4) $a z^{-1}$	\longleftrightarrow	(8) $a^{-1} z$	
(1') $a^{-1} z$	\longleftrightarrow	(5') $a z^{-1}$	a^{-n}
(2') $a z$	\longleftrightarrow	(6') $a^{-1} z^{-1}$	
(3') $a z^{-1}$	\longleftrightarrow	(7') $a^{-1} z$	
(4') $a^{-1} z^{-1}$	\longleftrightarrow	(8') $a z$	



a unit start term



a non-unit start term

Complement ROC Pairs - Original Geometric Series Form Combinations

(1) / (5)

unit	$\frac{1}{1-az}$ $ z < a^{-1}$	$a^n u(n)$
non-unit	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^n u(-n-1)$

(2) / (6)

unit	$\frac{1}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^n u(n)$
non-unit	$-\frac{az^{-1}}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^n u(-n-1)$

(3) / (7)

unit	$-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^n u(-n)$
non-unit	$\frac{az}{1-az}$ $ z < a^{-1}$	$a^n u(n-1)$ ⁽¹⁾

(4) / (8)

unit	$-\frac{1}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^n u(-n)$
non-unit	$\frac{a^{-1}z}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^n u(n-1)$

(1') / (5')

unit	$\frac{1}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^n u(n)$
non-unit	$-\frac{az^{-1}}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^n u(-n-1)$

(2') / (6')

unit	$\frac{1}{1-az}$ $ z < a^{-1}$	$a^n u(n)$
non-unit	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^n u(-n-1)$

(3') / (7')

unit	$-\frac{1}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^n u(-n)$
non-unit	$\frac{a^{-1}z}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^n u(n-1)$

(4') / (8')

unit	$-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^n u(-n)$
non-unit	$\frac{az}{1-az}$ $ z < a^{-1}$	$a^n u(n-1)$

start term

Complement ROC Pairs - Shifted Geometric Series Form Combinations

(S1) / (S5)

$\frac{a}{1-az}$ $ z < a^{-1}$	$a^{n+1} u(n)$
$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^{n+1} u(-n-1)$

(S2) / (S6)

$\frac{z}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^{n-1} u(n-1)$
$-\frac{a}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n-1} u(-n)$

(S3) / (S7)

$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^{n+1} u(-n-1)$
$\frac{a}{1-az}$ $ z < a^{-1}$	$a^{n+1} u(n)$

(S4) / (S8)

$-\frac{a}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n-1} u(-n)$
$\frac{z}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^{n-1} u(n-1)$

(S1') / (S5')

$\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^{n+1} u(n)$
$-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n+1} u(-n-1)$

(S2') / (S6')

$\frac{z}{1-az}$ $ z < a^{-1}$	$a^{n-1} u(n-1)$
$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^{n-1} u(-n)$

(S3') / (S7')

$-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n+1} u(-n-1)$
$\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$	$(\frac{1}{a})^{n+1} u(n)$

(S24') / (S8')

$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-a^{n-1} u(-n)$
$\frac{z}{1-az}$ $ z < a^{-1}$	$a^{n-1} u(n-1)$

Complement ROC Pairs - Reduced Shifted Geometric Series Form Combinations

$\frac{a}{1-a^2z}$ $ z < a^{-1}$	$a^{n+1} u(n)$	$-\frac{a}{1-a^2z^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n-1} u(-n)$
$-\frac{z^{-1}}{1-a^2z^{-1}}$ $ z > a^{-1}$	$-a^{n+1} u(-n-1)$	$\frac{z}{1-a^2z}$ $ z < a$	$(\frac{1}{a})^{n-1} u(n-1)$

$\frac{a^{-1}}{1-a^2z}$ $ z < a$	$(\frac{1}{a})^{n+1} u(n)$	$-\frac{a^{-1}}{1-a^2z^{-1}}$ $ z > a^{-1}$	$-a^{n-1} u(-n)$
$-\frac{z^{-1}}{1-a^2z^{-1}}$ $ z > a$	$-(\frac{1}{a})^{n+1} u(-n-1)$	$\frac{z}{1-a^2z}$ $ z < a^{-1}$	$a^{n-1} u(n-1)$

$$2z$$

$$2z^{-1}$$

$$2^{-1}z^{-1}$$

$$2^{-1}z$$

$$|z| < 0.5$$

$$|z| > 2$$

$$|z| > 0.5$$

$$|z| < 2$$

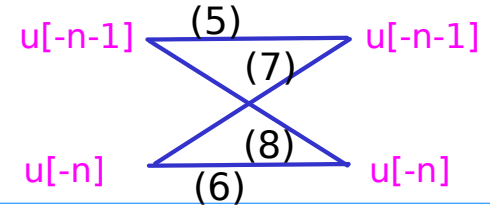
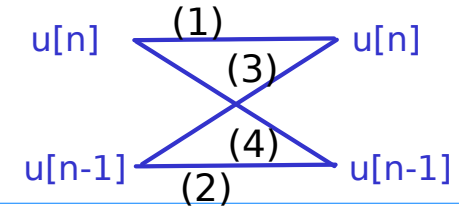
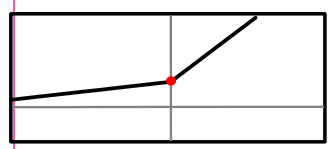
$$- \frac{2}{1-2z} \xleftrightarrow{z^{-1}} - \frac{2}{1-2z^{-1}}$$

$$\cdot \frac{(2z)^{-1}}{(2z)^{-1}} \cdot \frac{(2z)}{(2z)}$$

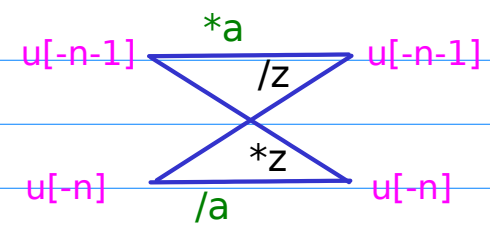
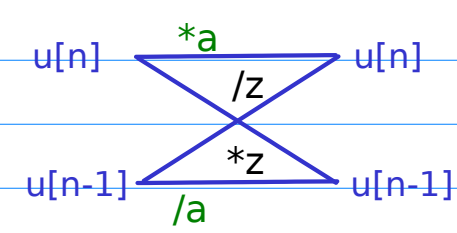
$$\cdot \frac{(2z^{-1})^{-1}}{(2z^{-1})^{-1}} \cdot \frac{(2z^{-1})}{(2z^{-1})}$$

$$+ \frac{z^{-1}}{1-0.5z^{-1}} \xleftrightarrow{z^{-1}} + \frac{z}{1-0.5z}$$

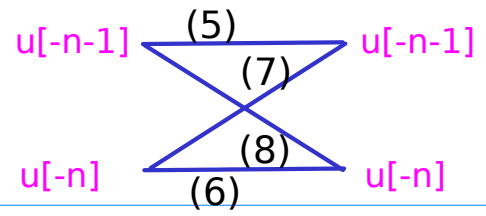
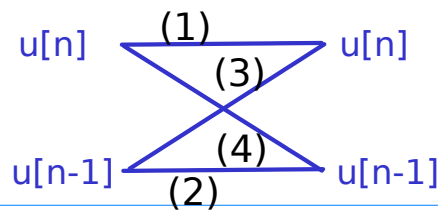
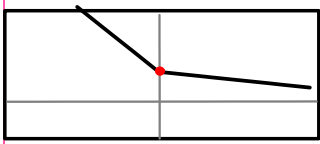
a^n



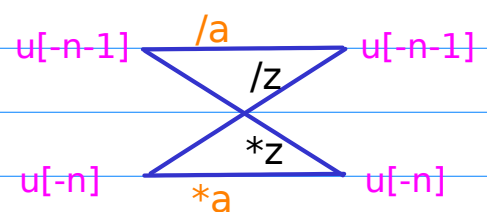
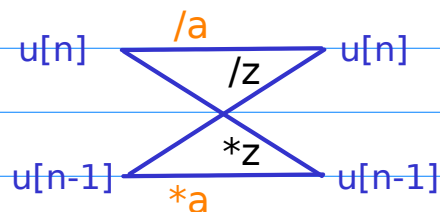
(1)	$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$a^n u(n)$ (a^0, a^1, a^2, \dots)	$a^{n+1} u(n)$ (a^1, a^2, a^3, \dots)
(2)	$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$	$a^n u(n-1)$ (a^1, a^2, a^3, \dots)	$a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots)
(3)	$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$a^n u(n-1)$ (a^1, a^2, a^3, \dots)	$a^{n+1} u(n)$ (a^1, a^2, a^3, \dots)
(4)	$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$	$a^n u(n)$ (a^0, a^1, a^2, \dots)	$a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots)
(5)	$-\frac{a^i z^i}{1-a^i z^i} \quad z > a^{-1}$	$-\frac{z^i}{1-a^i z^i} \quad z > a^{-1}$	$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$
(6)	$-\frac{1}{1-a^i z^i} \quad z > a^{-1}$	$-\frac{a^i}{1-a^i z^i} \quad z > a^{-1}$	$-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$
(7)	$-\frac{1}{1-a^i z^i} \quad z > a^{-1}$	$-\frac{z^i}{1-a^i z^i} \quad z > a^{-1}$	$-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$
(8)	$-\frac{a^i z^i}{1-a^i z^i} \quad z > a^{-1}$	$-\frac{a^i}{1-a^i z^i} \quad z > a^{-1}$	$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^3}, \frac{1}{a^4}, \frac{1}{a^5})$

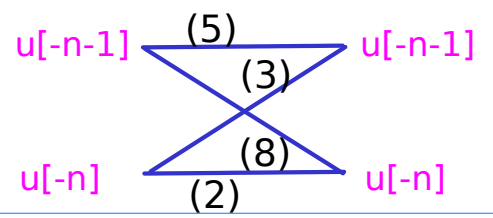
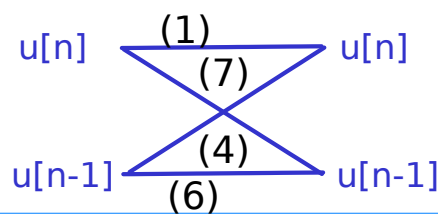


a^{-n}



(1)	$\frac{1}{1-a^1z} \quad z < a$	$\frac{a^1}{1-a^1z} \quad z < a$	$(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
(2)	$\frac{a^1z}{1-a^1z} \quad z < a$	$\frac{z}{1-a^1z} \quad z < a$	$(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$
(3)	$\frac{a^1z}{1-a^1z} \quad z < a$	$\frac{a^1}{1-a^1z} \quad z < a$	$(\frac{1}{a})^n u(n-1)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$	$(\frac{1}{a})^{n+1} u(n)$ $(\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots)$
(4)	$\frac{1}{1-a^1z} \quad z < a$	$\frac{z}{1-a^1z} \quad z < a$	$(\frac{1}{a})^n u(n)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$	$(\frac{1}{a})^{n-1} u(n-1)$ $(\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots)$
(5)	$-\frac{az^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad z > a$	$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$
(6)	$-\frac{1}{1-az^{-1}} \quad z > a$	$-\frac{a}{1-az^{-1}} \quad z > a$	$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^1, a^0, a^{-1})$
(7)	$-\frac{1}{1-az^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad z > a$	$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$
(8)	$-\frac{az^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{a}{1-az^{-1}} \quad z > a$	$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$





(1) / (5)

scale(a)

(2) / (6)

scale(z)

	$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$\frac{1}{1-a^{-1}z} \quad z < a$	$\frac{z}{1-a^{-1}z} \quad z < a$
Comp.ROC	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a$	$-\frac{a}{1-a^{-1}z^{-1}} \quad z > a$

(3) / (7)

scale(1/z)

(4) / (8)

scale(a)

	$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a$	$-\frac{a}{1-a^{-1}z^{-1}} \quad z > a$
Comp.ROC	$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$	$\frac{z}{1-a^{-1}z} \quad z < a$

(1') / (5')

scale(1/a)

(2') / (6')

scale(z)

	$\frac{1}{1-a^{-1}z} \quad z < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad z < a$	$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$
Comp.ROC	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$

(3') / (7')

scale(1/z)

(4') / (8')

scale(1/a)

	$-\frac{1}{1-az^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$
Comp.ROC	$\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad z < a$	$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$

(1) / (5)

scale(a)

(2) / (6)

scale(z)

Comp.ROC

$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$\frac{1}{1-a^{-1}z} \quad z < a$	$\frac{z}{1-a^{-1}z} \quad z < a$
$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{az^{-1}}{1-a^{-1}z^{-1}} \quad z > a$	$-\frac{a}{1-a^{-1}z^{-1}} \quad z > a$

(3) / (7)

scale(1/z)

(4) / (8)

scale(a)

Comp.ROC

$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{1}{1-az^{-1}} \quad z > a$	$-\frac{a}{1-az^{-1}} \quad z > a$
$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{a}{1-az} \quad z < a^{-1}$	$\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$	$\frac{z}{1-a^{-1}z} \quad z < a$

(1') / (5')

scale(1/a)

(2') / (6')

scale(z)

Comp.ROC

$\frac{1}{1-a^{-1}z} \quad z < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad z < a$	$\frac{1}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$
$-\frac{az^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$

(3') / (7')

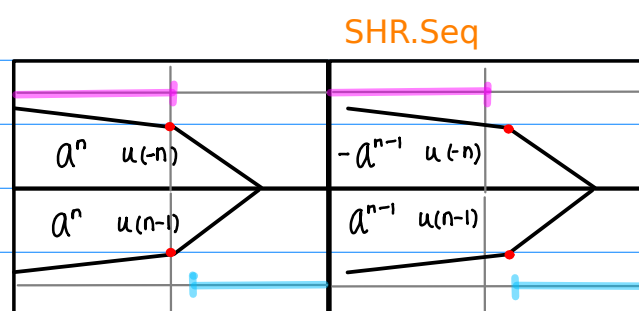
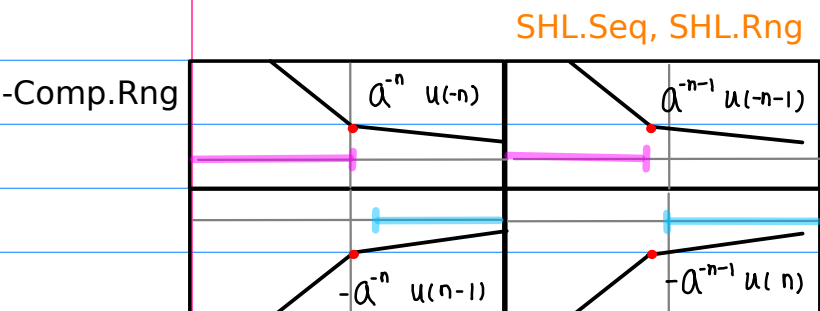
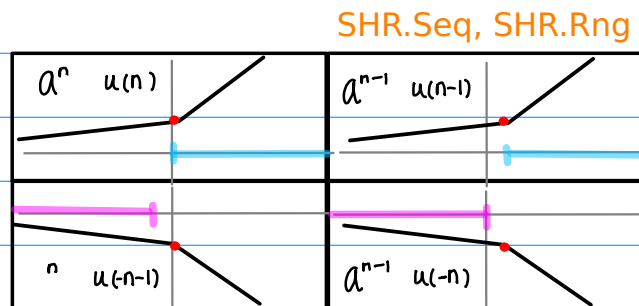
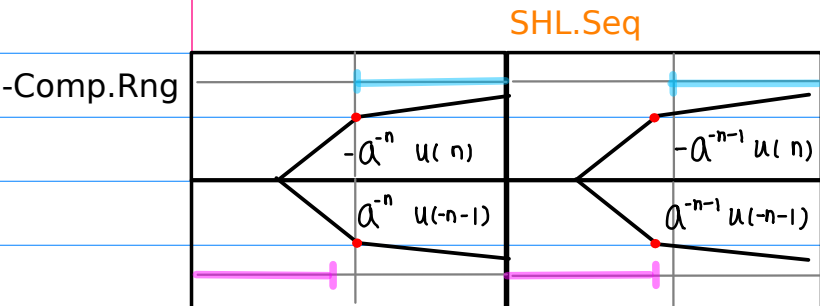
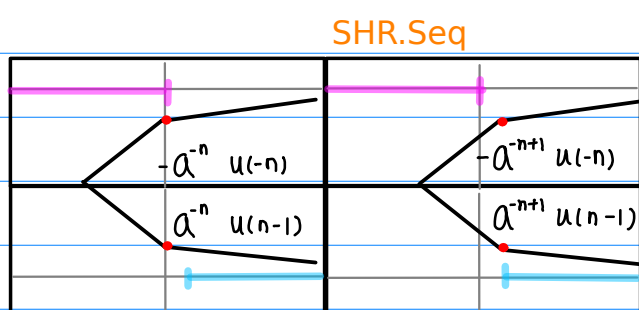
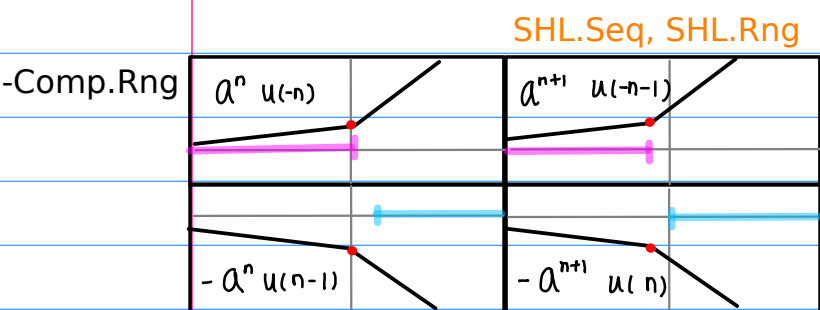
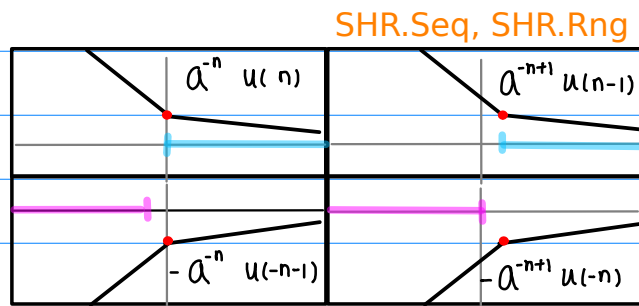
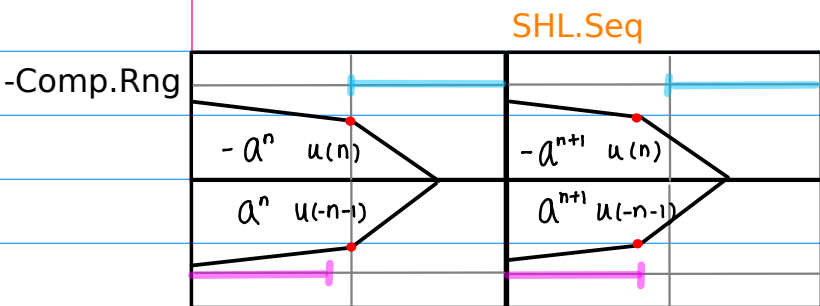
scale(1/z)

(4') / (8')

scale(1/a)

Comp.ROC

$-\frac{1}{1-az^{-1}} \quad z > a$	$-\frac{z^{-1}}{1-az^{-1}} \quad z > a$	$-\frac{1}{1-a^{-1}z^{-1}} \quad z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}} \quad z > a^{-1}$
$\frac{a^{-1}z}{1-a^{-1}z} \quad z < a$	$\frac{a^{-1}}{1-a^{-1}z} \quad z < a$	$\frac{az}{1-az} \quad z < a^{-1}$	$\frac{z}{1-az} \quad z < a^{-1}$



Left Shifted
Sequence

Right Shifted
Sequence

(1) / (5)

scale(a)

(2) / (6)

scale(z)

Comp.ROC

$\frac{1}{1-az}$ $ z < a^{-1}$	$\frac{a}{1-az}$ $ z < a^{-1}$
$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$

$\frac{1}{1-a^{-1}z}$ $ z < a$	$\frac{z}{1-a^{-1}z}$ $ z < a$
$-\frac{az^{-1}}{1-az^{-1}}$ $ z > a$	$-\frac{a}{1-az^{-1}}$ $ z > a$

(3) / (7)

scale(1/z)

(4) / (8)

scale(a)

Comp.ROC

$-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-\frac{z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$
$\frac{az}{1-az}$ $ z < a^{-1}$	$\frac{a}{1-az}$ $ z < a^{-1}$

$-\frac{1}{1-az^{-1}}$ $ z > a$	$-\frac{a}{1-az^{-1}}$ $ z > a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $ z < a$	$\frac{z}{1-a^{-1}z}$ $ z < a$

(1') / (5')

scale(1/a)

(2') / (6')

scale(z)

Comp.ROC

$\frac{1}{1-a^{-1}z}$ $ z < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$
$-\frac{az^{-1}}{1-az^{-1}}$ $ z > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$

$\frac{1}{1-az}$ $ z < a^{-1}$	$\frac{z}{1-az}$ $ z < a^{-1}$
$-\frac{a^{-1}z^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$

(3') / (7')

scale(1/z)

(4') / (8')

scale(1/a)

Comp.ROC

$-\frac{1}{1-az^{-1}}$ $ z > a$	$-\frac{z^{-1}}{1-az^{-1}}$ $ z > a$
$\frac{a^{-1}z}{1-a^{-1}z}$ $ z < a$	$\frac{a^{-1}}{1-a^{-1}z}$ $ z < a$

$-\frac{1}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$	$-\frac{a^{-1}}{1-a^{-1}z^{-1}}$ $ z > a^{-1}$
$\frac{az}{1-az}$ $ z < a^{-1}$	$\frac{z}{1-az}$ $ z < a^{-1}$

SHL.Seq

$a^n u(n)$ (a^0, a^1, a^2, \dots)	$a^{n+1} u(n)$ (a^1, a^2, a^3, \dots)
$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$

SHR.Seq, SHR.Rng

$(\frac{1}{a})^n u(n)$ ($\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots$)	$(\frac{1}{a})^{n-1} u(n-1)$ ($\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots$)
$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$

-Comp.Rng

SHL.Seq, SHL.Rng

$-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n+1} u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$
$a^n u(n-1)$ (a^1, a^2, a^3, \dots)	$a^{n+1} u(n)$ (a^1, a^2, a^3, \dots)

SHR.Seq

$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n-1} u(-n)$ $-(\dots, a^3, a^2, a^1)$
$(\frac{1}{a})^n u(n-1)$ ($\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots$)	$(\frac{1}{a})^{n-1} u(n-1)$ ($\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots$)

-Comp.Rng

SHL.Seq

$(\frac{1}{a})^n u(n)$ ($\frac{1}{a^0}, \frac{1}{a^1}, \frac{1}{a^2}, \dots$)	$(\frac{1}{a})^{n+1} u(n)$ ($\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots$)
$-(\frac{1}{a})^n u(-n-1)$ $-(\dots, a^3, a^2, a^1)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$

SHR.Seq, SHR.Rng

$a^n u(n)$ (a^0, a^1, a^2, \dots)	$a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots)
$-a^n u(-n-1)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$

-Comp.Rng

SHL.Seq, SHL.Rng

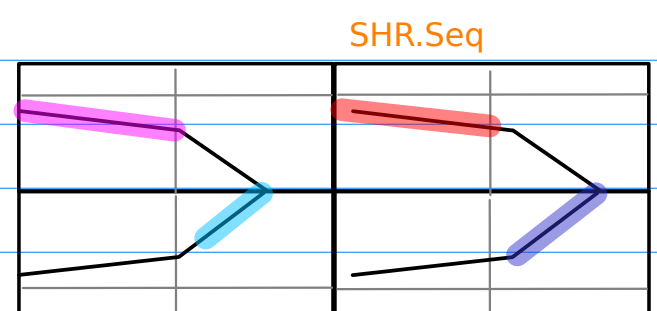
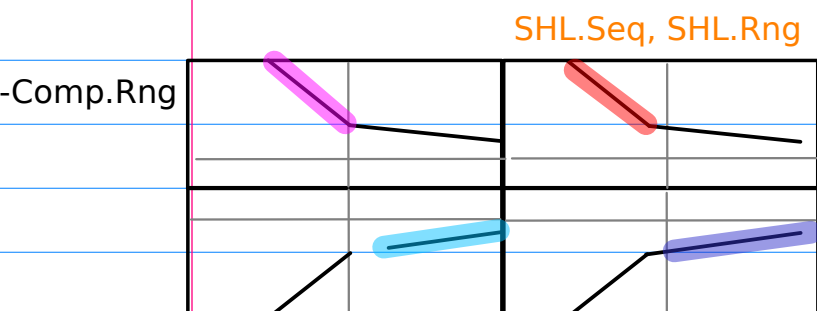
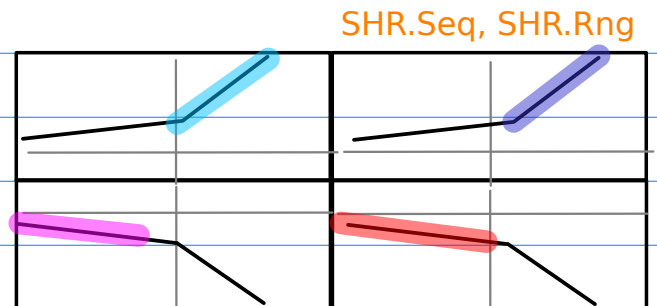
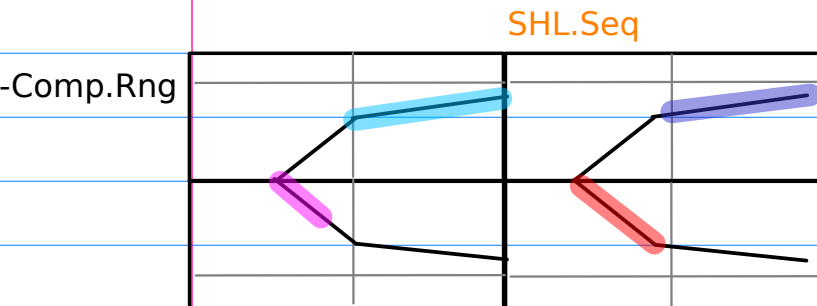
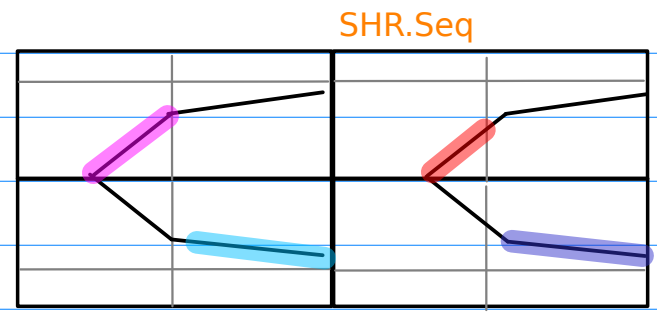
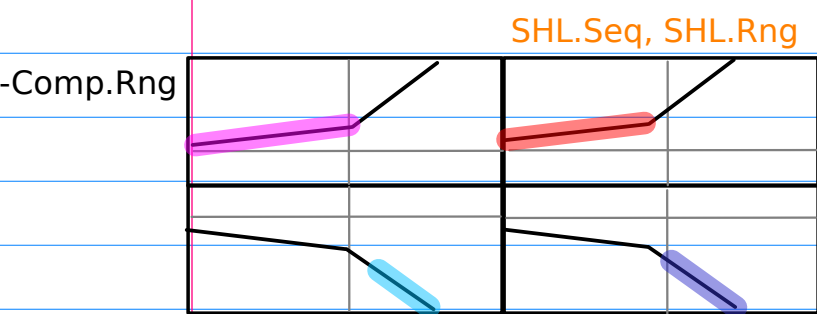
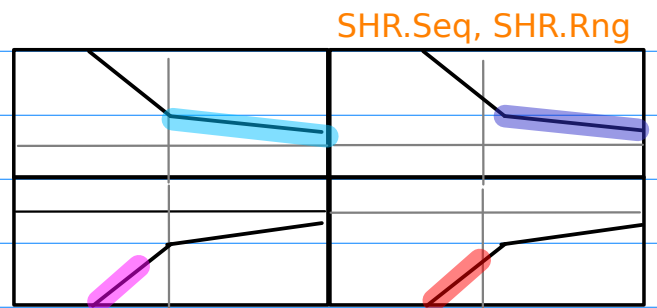
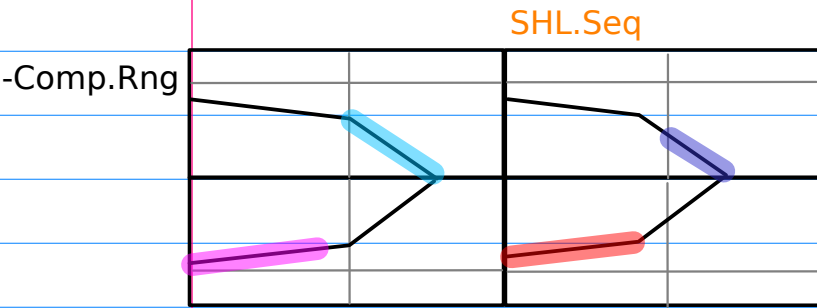
$-(\frac{1}{a})^n u(-n)$ $-(\dots, a^2, a^1, a^0)$	$-(\frac{1}{a})^{n+1} u(-n-1)$ $-(\dots, a^2, a^1, a^0)$
$(\frac{1}{a})^n u(n-1)$ ($\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots$)	$(\frac{1}{a})^{n+1} u(n)$ ($\frac{1}{a^1}, \frac{1}{a^2}, \frac{1}{a^3}, \dots$)

SHR.Seq

$-a^n u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$	$-a^{n-1} u(-n)$ $-(\dots, \frac{1}{a^2}, \frac{1}{a^3}, \frac{1}{a^4})$
$a^n u(n-1)$ (a^1, a^2, a^3, \dots)	$a^{n-1} u(n-1)$ (a^0, a^1, a^2, \dots)

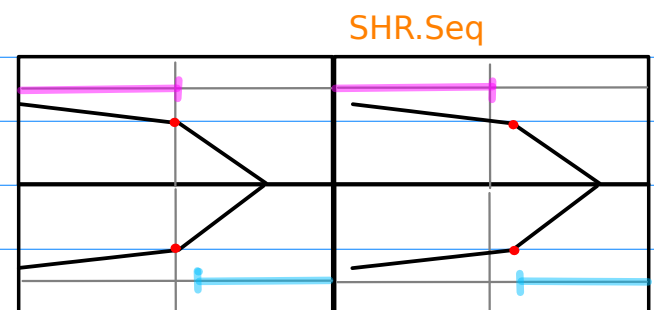
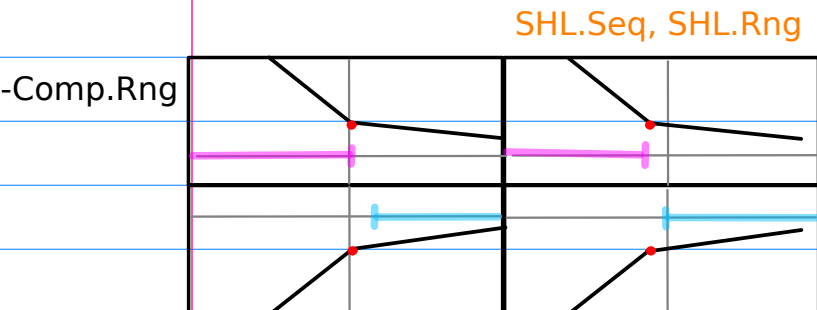
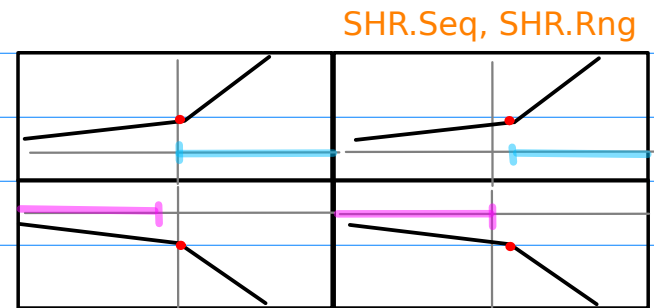
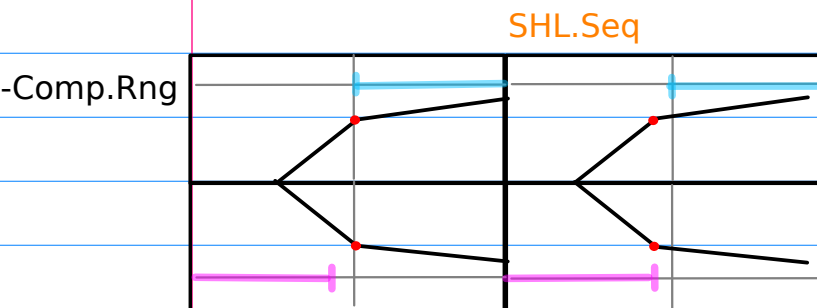
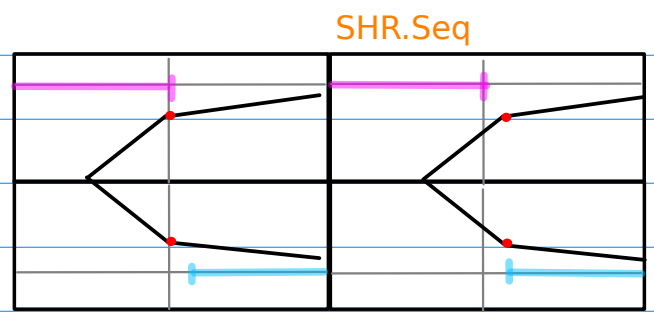
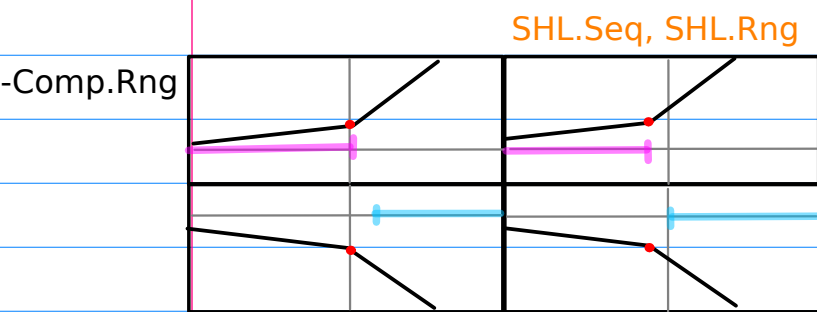
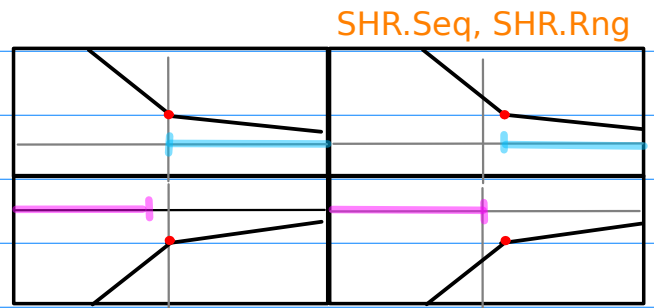
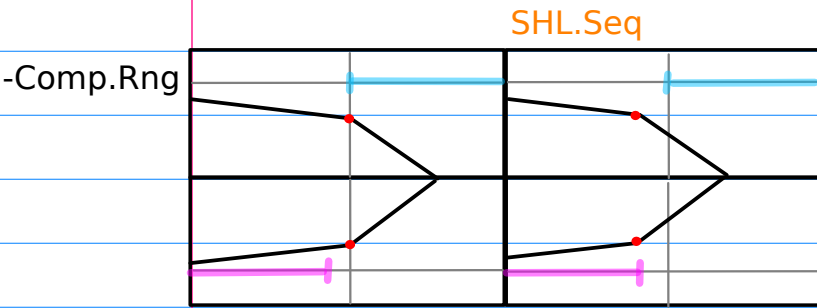
-Comp.Rng

Left Shifted
SequenceRight Shifted
Sequence



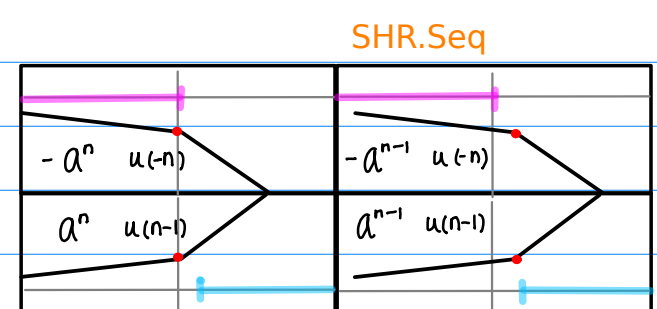
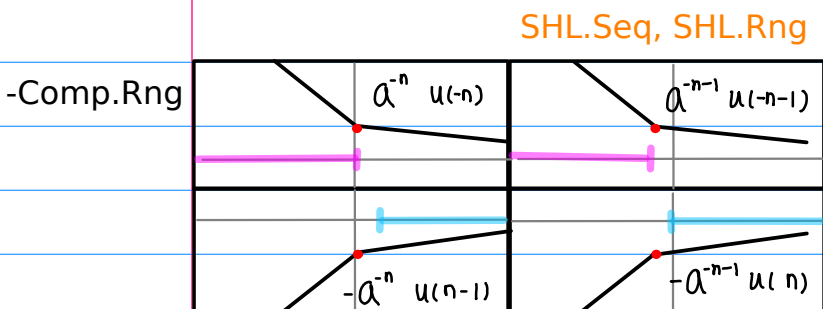
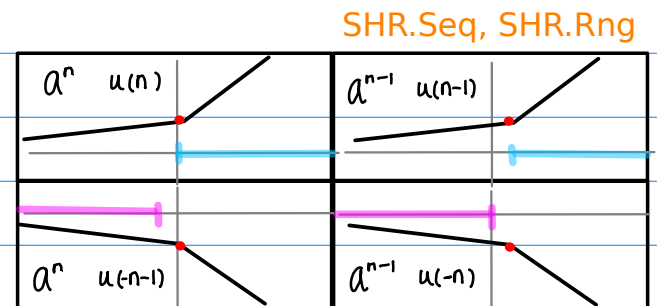
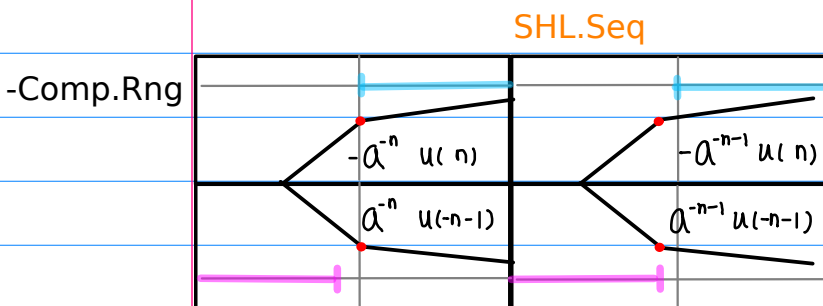
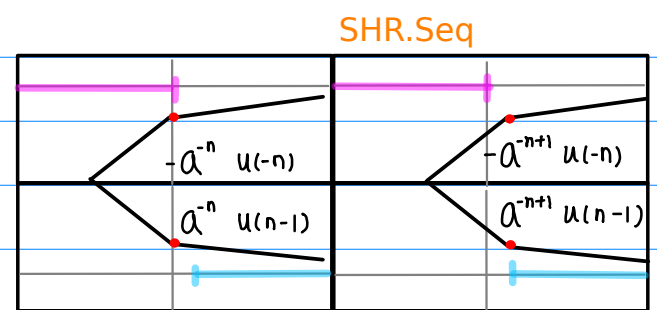
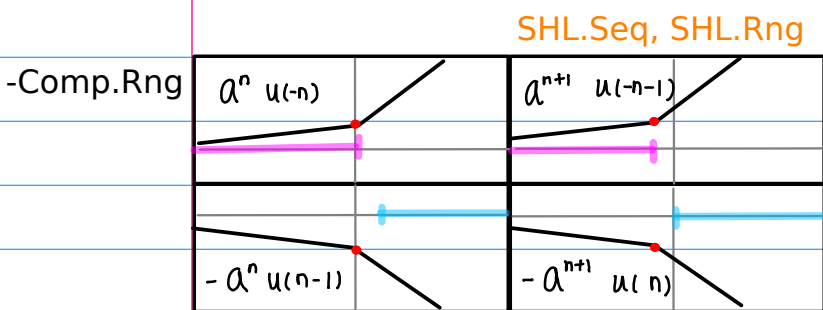
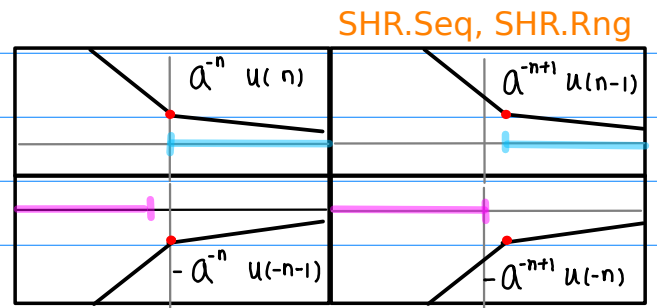
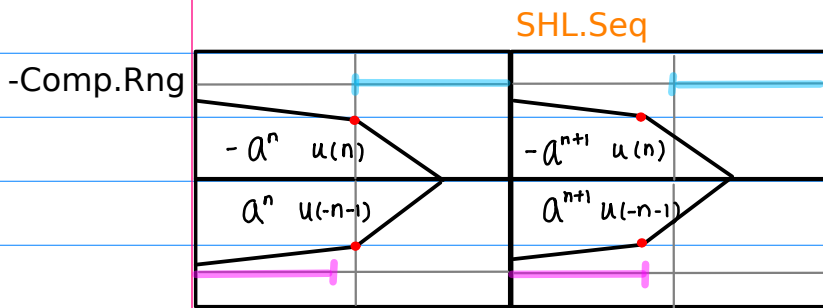
Left Shifted
Sequence

Right Shifted
Sequence



Left Shifted
Sequence

Right Shifted
Sequence

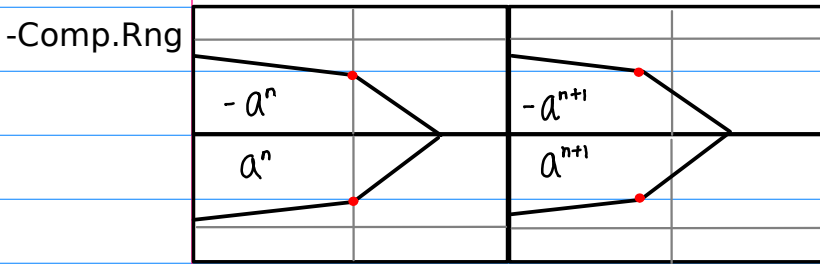


Left Shifted Sequence

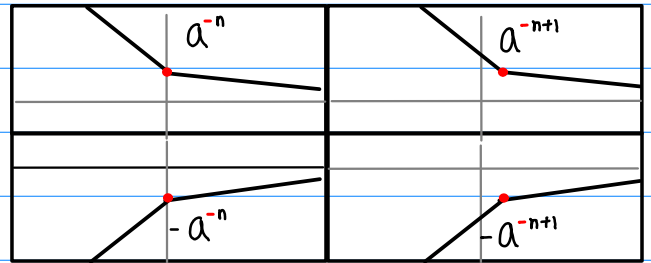
Right Shifted Sequence

a Sequence Function

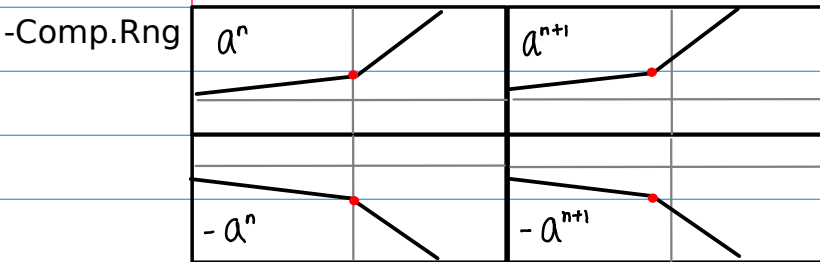
SHL.Seq



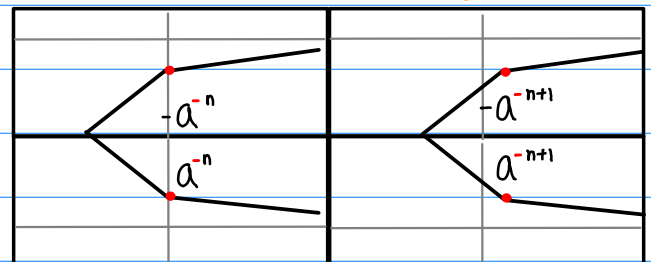
SHR.Seq, SHR.Rng



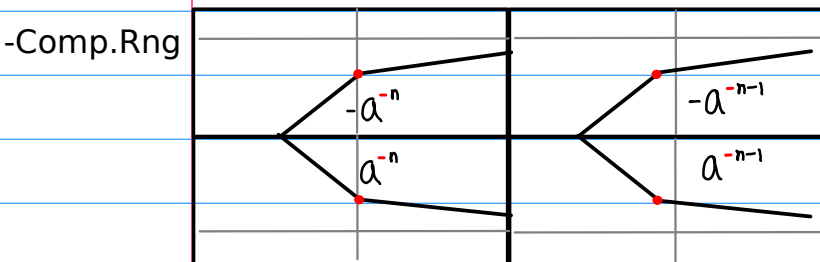
SHL.Seq, SHL.Rng



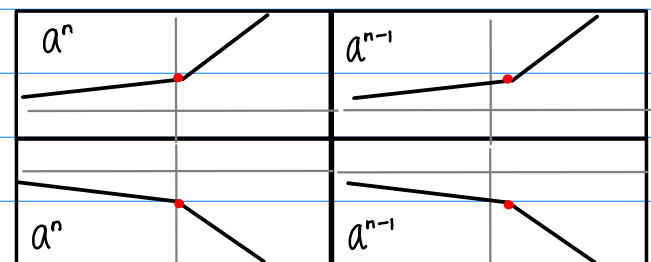
SHR.Seq



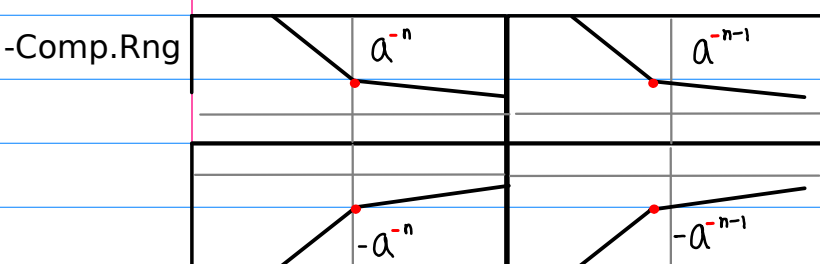
SHL.Seq



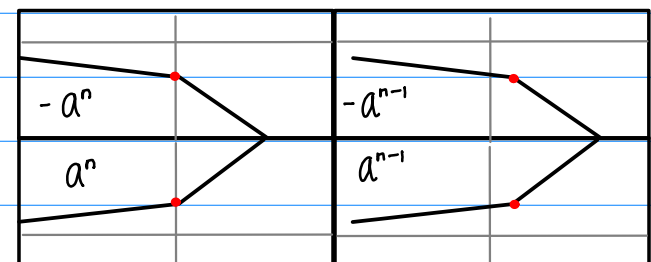
SHR.Seq, SHR.Rng



SHL.Seq, SHL.Rng



SHR.Seq



Left Shifted
Sequence

Right Shifted
Sequence

Range of a Sequence

