Gaussian Random Variables

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Gaussian Density Function

Definition

A Gaussian random variable X
if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_X)^2/2\sigma_X^2}$$

where σ_X and a_X are real constants.

Gaussian Distribution Function

Definition

A Gaussian distribution function has the form

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2} \, d\xi$$

where σ_X and a_X are real constants.

Normalized Gaussian Distribution Function

Definition

A normalized Gaussian distribution function has the form

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$

where $\sigma_X = 1$ and $a_X = 0$

Let the function $F(x) = F_X(x)$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi \qquad (= \Phi(x))$$

Normalized Gaussian Distribution Function Property

Theorem

Assume $F(x)$ is a normalized (standard) Gaussian distribution function, then $F(-x) = 1 - F(x)$

$$F(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-\xi^2/2} d\xi$$

$$= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$

$$= 1 - F(x)$$

Computing General Gaussian Distribution Function

Definition

A general Gaussian distribution function
via a variable change $u = (x - a_X)/\sigma_X$

$$x \sim \mathcal{N}(a_X, \sigma_X) \implies u \sim \mathcal{N}(0,1)$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2} d\xi$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$

$$F_X(x) = F\left(\frac{x - a_x}{\sigma_X}\right) = F(u)$$

Change of variables $u = (x - a_X)/\sigma_X$

Definition

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2} d\xi$$

let $y = \xi - a_x$

$$\frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x - a_X} e^{-y^2/2\sigma_X^2} dy$$

let $z = y/\sigma_X$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x - a_X)/\sigma_X} e^{-z^2/2} dz = F\left(\frac{x - a_x}{\sigma_X}\right)$$

therefore $u = (x - a_X)/\sigma_X \implies F_X(x) = F\left(\frac{x - a_x}{\sigma_X}\right)$

Q Function

Definition

Q function is defined as

$$F(x) = 1 - Q(x)$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\xi^2/2} d\xi$$

$$Q(x) \approx \left[\frac{1}{(1-a)x + a\sqrt{x^2+b}} \right] \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}$$

$$x \geqslant 0$$

Outline

Gaussian Function

Definition

A Gaussian functon has the form

$$f(x) = ae^{-(x-b)^2/2c^2}$$

- a is the height of the curve's peak,
- b is the position of the center of the peak
- c (the standard deviation) controls the width of the "bell".

https://en.wikipedia.org/wiki/Gaussian_function

Gaussian PDF

Definition

- Gaussian functions
- used to represent the pdf of a normally distributed r.v.
- with expected value $\mu = b$ and variance $\sigma^2 = c^2$

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

https://en.wikipedia.org/wiki/Gaussian_function

Gaussian Integral

Definition

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} ae^{-(x-b)^2/2c^2}\,dx = ac\cdot\sqrt{2\pi}$$

- This integral is 1 if and only if $a = \frac{1}{c\sqrt{2\pi}}$
- in this case, the Gaussian is the p.d.f
  - of a normally distributed r.v.
  - with expected value $\mu = b$ and variance $\sigma^2 = c^2$:

https://en.wikipedia.org/wiki/Gaussian_integral

Standard Gaussian Integral

Definition

$$\int_{-\infty}^{\infty} ae^{-(x-b)^2/2c^2}\, dx$$

let $y = x - b$

$$a\int_{-\infty}^{\infty} e^{-y^2/2c^2}\, dy$$

let $z = y/\sqrt{2c^2}$

$$a\sqrt{2c^2}\int_{-\infty}^{\infty} e^{-z^2}\, dz$$

using $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\int_{-\infty}^{\infty} ae^{-(x-b)^2/2c^2}\, dx = a \cdot \sqrt{2\pi c^2}$$

https://en.wikipedia.org/wiki/Gaussian_integral

Standard Gaussian Density Function

Definition

- the standard normal distribution.
    - when $\mu = 0$ and $\sigma = 1$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

https://en.wikipedia.org/wiki/Gaussian_integral

Standard Gaussian Distribution Function

Definition

the standard Gaussian density function:
when $\mu = 0$ and $\sigma = 1$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

the standard Gaussian distribution function:
when $\mu = 0$ and $\sigma = 1$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

https://en.wikipedia.org/wiki/Gaussian_integral

General Gaussian Density Function

Definition

- Every normal distribution is
  - a version of the standard normal distribution
  - whose domain has been stretched
  - by a factor σ (the standard deviation)
  - then translated by μ (the mean value):

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma}} e^{-(x-\mu)^2/2\sigma^2}$$

https://en.wikipedia.org/wiki/Gaussian_integral

General Gaussian Distribution Function

Definition

The general Gaussian density function

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

The general Gaussian distribution function

$$F(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx$$

https://en.wikipedia.org/wiki/Gaussian_integral

Error Function (1)

Definition

the Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

the error function

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-x^2} dx$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^2} dx$$

https://en.wikipedia.org/wiki/Error_function

Error Function (2)

Fact

- *erf* (x) *is a sigmoid function*
- *for nonnegative values of* x
- *for a r.v.* Y *that is normally distributed*
- *with mean* 0 *and variance* $1/2$
- *erf* (x) *is the probability that* Y *falls in the range* $[-x, x]$.
- *the complementary error function*

$$erfc(x) = 1 - erf(x)$$

https://en.wikipedia.org/wiki/Error_function

Q Function (1)

Definition

- the Q-function is
  - the tail distribution function
  - of the standard normal distribution.
- $Q(x)$ is the probability
  - that a normal r.v. will obtain a value
  - larger than x standard deviations.

https://en.wikipedia.org/wiki/Q-function

Q Function (2)

Theorem

- *If Y is a Gaussian random variable*
  - *with mean μ and variance σ^2,*
  - *then $X = \dfrac{Y-\mu}{\sigma}$ is standard normal*
  - *and $x = \dfrac{y-\mu}{\sigma}$*

https://en.wikipedia.org/wiki/Q-function

Q Function (3)

Definition

$$P(Y > y) = P(X > x) = Q(x)$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du$$

$$Q(x) = 1 - Q(-x) = 1 - \Phi(x)$$

https://en.wikipedia.org/wiki/Q-function