Vector Calculus (H.1)
Surface Integrals

20151214

References

[1] Paul's online math note

Parametric Surfaces

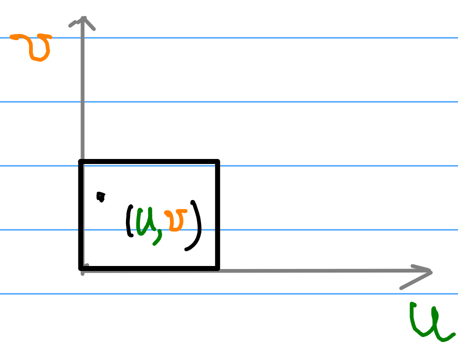
parametric curve

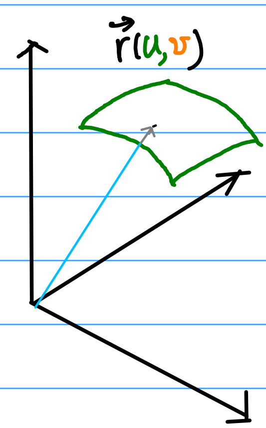
$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$



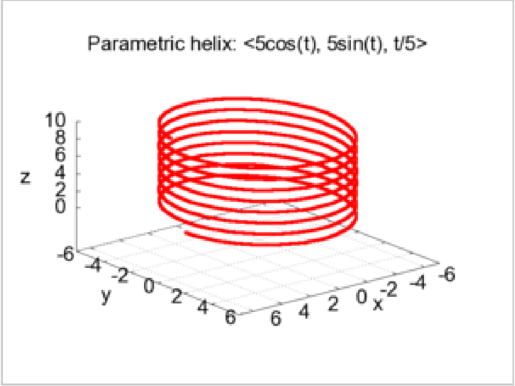


Helix [edit]

Parametric equations are convenient for describing curves in higher-dimensional spaces. For example:

$$x = a\cos(t)$$
$$y = a\sin(t)$$
$$z = bt$$

describes a three-dimensional curve, the helix, with a radius of *a* and rising by $2\pi b$ units per turn. Note that the equations are identical in the plane to those for a circle. Such expressions as the one above are commonly written as



Parametric helix

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (a\cos(t), a\sin(t), bt),$$

where **r** is a three-dimensional vector.

https://en.wikipedia.org/wiki/Parametric_equation

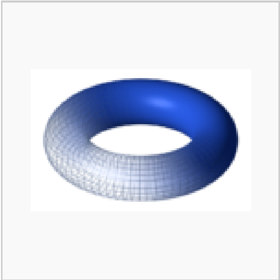
Parametric surfaces [edit]

Main article: Parametric surface

A torus with major radius *R* and minor radius *r* may be defined parametrically as

$$x = \cos[t]\,[R + r\cos(u)]\,,$$
$$y = \sin[t]\,[R + r\cos(u)]\,,$$
$$z = r\sin[u].$$

where the two parameters t and u both vary between 0 and 2π.



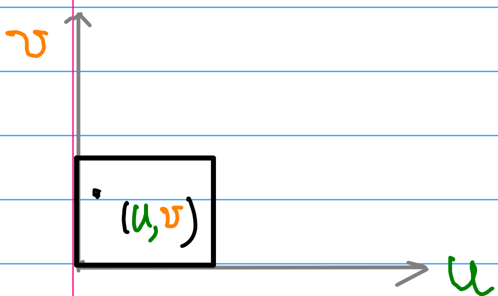
R=2, r=1/2

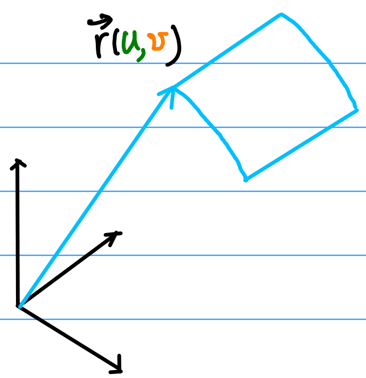
As u varies from 0 to 2π the point on the surface moves about a short circle passing through the hole in the torus. As t varies from 0 to 2π the point on the surface moves about a long circle around the hole in the torus.

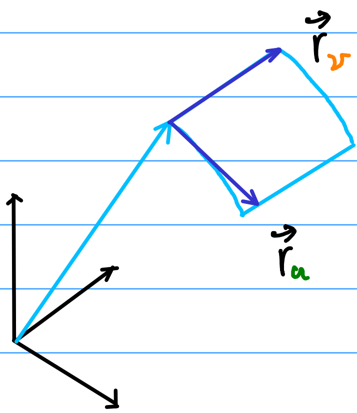
https://en.wikipedia.org/wiki/Parametric_equation parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

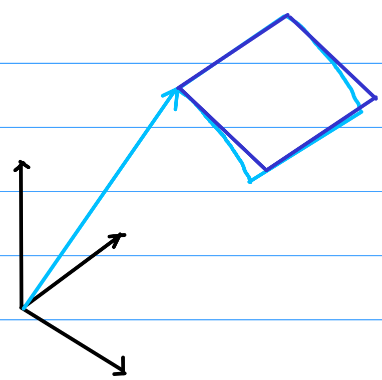
$$\boxed{\vec{r}_u = \frac{\partial \vec{r}}{\partial u}} \qquad \boxed{\vec{r}_v = \frac{\partial \vec{r}}{\partial v}}$$







tangent vectors



tangent plane

area: $\|\vec{r}_u \times \vec{r}_v\|$

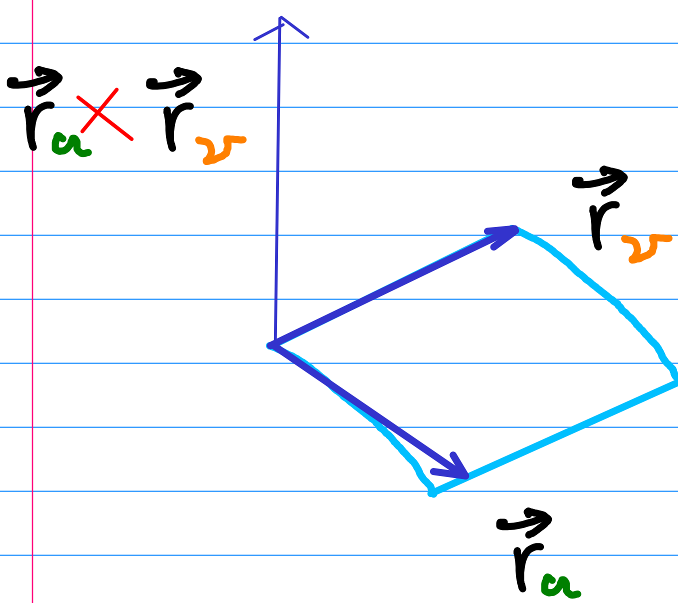
Regular Parametrization

$\{\vec{r}_u, \vec{r}_v\}$ linearly independent

any tangent vector can be uniquely decomposed into a linear combination of $\vec{r}_u$ & $\vec{r}_v$

a normal vector $\qquad \vec{r}_u \times \vec{r}_v$

a unit normal vector $\qquad \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$



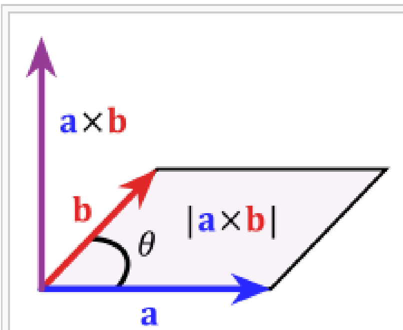


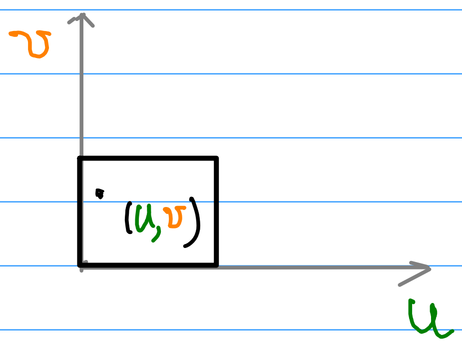
Figure 1. The area of a parallelogram as the magnitude of a cross product

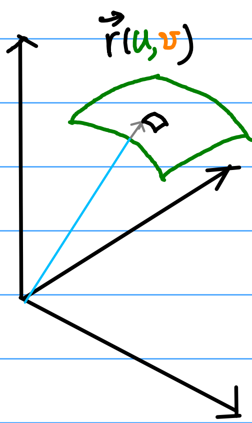
parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

Surface Area

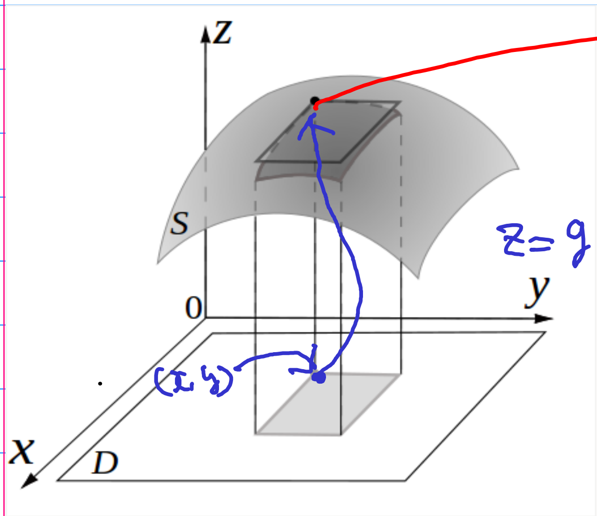
$$A(D) = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$





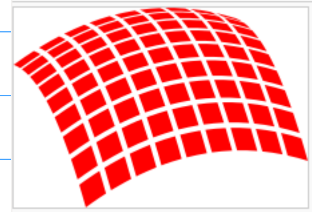
$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

Surface Integrals



$f(x, y, g(x,y))$

$z = g(x,y)$



The definition of surface integral relies on splitting the surface into small surface elements.

dS

https://en.wikipedia.org/wiki/Surface_integral

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, du\, dv$$

$$= \iint_D f(\vec{r}(u,v))\, \|\vec{r}_u \times \vec{r}_v\|\, dA$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$