

Example Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 Averages and Ergodicity
- 2 Mean Ergodic Processes
- 3 Correlation Ergodic Processes

Average

N Gaussian random variables

Definition

$$\bar{m}_x = \frac{1}{N} \sum_{i=1}^N X_i(t)$$

—

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

—

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

Expectation of Time-Autocorrelation Function

N Gaussian random variables

Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

$$E[\bar{X}_T] = E[A_T[x(t)]] = \bar{X}$$

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

$$E[R_T(\tau)] = E[A_T[x(t)x(t+\tau)]] = R_{XX}(\tau)$$

Ergodicity Theorem

N Gaussian random variables

Definition

$$\lim_{n \rightarrow \infty} E [(X_n - X)^2] = 0$$

$$A[\bullet] = \lim_{n \rightarrow \infty} A_T[\bullet]$$

Conditions

N Gaussian random variables

- 1 $X(t)$ has a finite constant mean \bar{X} for all t
- 2 $X(t)$ is bounded $x(t) < \infty$ for all t and all $x(t)$
- 3 Bounded time average of $E[|X(t)|]$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[|X(t)|] dt$$

- 4 $X(t)$ is a regular process

$$E \left[|X(t)|^2 \right] = R_{XX}(t, t) < \infty$$

Mean Ergodic

N Gaussian random variables

Definition

A wide sense stationary process $X(t)$ with a constant mean value \bar{X} is called mean-ergodic if $\bar{x}_T = A_T[x(t)]$ converges to \bar{X} as $T \rightarrow \infty$

$$\lim_{T \rightarrow \infty} E [(\bar{x}_T - \bar{X})^2] = 0$$

$$\lim_{T \rightarrow \infty} \sigma_{\bar{x}_T} = 0$$

Variance of \bar{X}_T (1) N Gaussian random variables

Definition

$$\begin{aligned}\sigma_{\bar{X}_T} &= E \left[\left\{ \frac{1}{2T} \int_{-T}^T (X(t) - \bar{X}) dt \right\}^2 \right] \\ &= E \left[\left(\frac{1}{2T} \right)^2 \left\{ \int_{-T}^T (X(t) - \bar{X}) dt \right\} \left\{ \int_{-T}^T (X(t_1) - \bar{X}) dt_1 \right\} \right] \\ &= E \left[\left(\frac{1}{2T} \right)^2 \int_{-T}^T (X(t) - \bar{X}) (X(t_1) - \bar{X}) dt dt_1 \right] \\ &= \left(\frac{1}{2T} \right)^2 \int_{-T}^T \int_{-T}^T E [(X(t) - \bar{X}) (X(t_1) - \bar{X})] dt dt_1 \\ &= \left(\frac{1}{2T} \right)^2 \int_{-T}^T C_{XX}(t, t_1) dt dt_1\end{aligned}$$

Variance of \bar{x}_T (2)

N Gaussian random variables

Definition

$$\sigma_{\bar{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{-T}^T C_{XX}(t, t_1) dt dt_1$$

$$C_{XX}(t, t_1) = C_{XX}(T), \quad \tau = t_1 - t, \quad dt_1 = dT$$

$$\sigma_{\bar{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^T \int_{\tau=-T-t}^T C_{XX}(\tau) dt d\tau$$

