Complex Powers and Logs (5A)

Young Won Lim
10/17/13

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Power and Taylor Series

Power Series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

$$= c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots$$

always converges if $|z-a| < R$

➡ can also be differentiated

Taylor Series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

$$= f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

only valid if the series converges

Power and Taylor Series

Power Series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

$$= c_0 + c_1 (z-a) + c_2 (z-a)^2 + \cdots$$

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

$$= f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(z-a)^2 + \cdots$$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n$$

only valid if the series converges

$$\frac{1}{z} = \frac{1}{a + (z-a)} = \frac{1}{a} \frac{1}{1 + \left(\frac{z-a}{a}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{a} \left(\frac{z-a}{a}\right)^n$$

$$f(z) = e^z \qquad f^{(n)}(0) = 1$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

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Complex Log : ln z

Power Series

$$\frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{a}\left(\frac{z-a}{a}\right)^n$$

converges if $|z-a| < |a|$

for any $a \neq 0$

for a sufficiently far away from 0
the size of the disk can be made big

$$\frac{d}{dz}(\ln z) = \frac{1}{z}$$

$\ln z$: integral of the power series expansion of $\frac{1}{z}$ at $z = 1$

Taylor Series

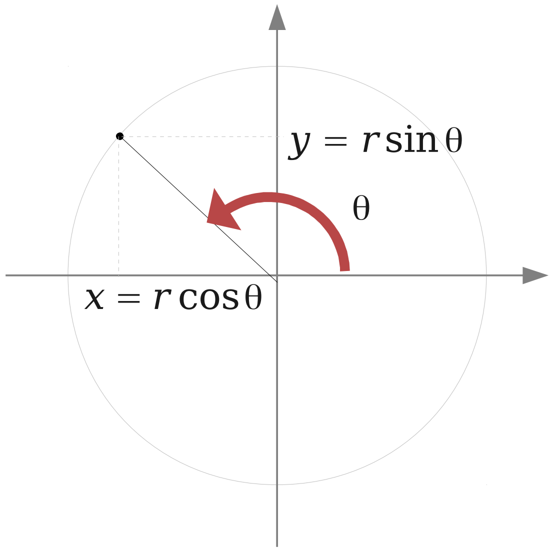
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges for all z

$e^w = z \qquad w = \ln z$

$\ln z$: inverse of e^z

Polar Representation



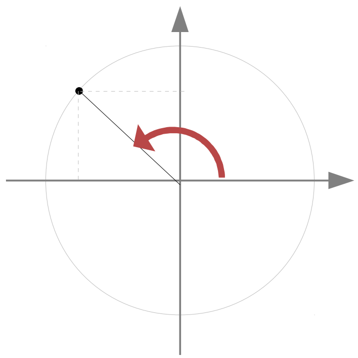
$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + i r\sin\theta \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \\ &= |z|e^{i\,arg(z)} \end{aligned}$$

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2} \end{aligned}$$

$$\begin{aligned} z_1 z_2 &= r_1 e^{i\theta_1} r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Principal Argument



$$z = x + iy$$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$
$$= |z|e^{i\,arg(z)}$$

$arg(z)$: to be a function of z,
it needs to be uniquely defined
for every z, by choosing its range

ex) $0 \le arg(z) < 2\pi$

free to choose a more convenient definition
for a particular problem.

by specifying its range
an argument can be uniquely defined

$\theta = arg(z)$

unique angles $0 \le arg(z) < 2\pi$

The special argument

Principal Argument $Arg(z)$

unique angles $-\pi \le Arg(z) < +\pi$

Argument $\theta = arg(z)$

many angles $= Arg(z) + 2k\pi$

Complex Log

$$z = e^{w}$$

$$= e^{u+iv}$$

$$= \left|e^{u+iv}\right| e^{i\,arg(e^{u+iv})}$$

$$= \left|e^{u}\cdot e^{iv}\right| e^{i\,arg(e^{u}\cdot e^{iv})}$$

$$= |z|\, e^{i\,arg(z)} \qquad = e^{u}\, e^{i\,arg(e^{iv})}$$

$$w = u + i\,v$$

$$\left|e^{u}\right| = 1, \qquad arg(e^{u}) = 0$$

$$z = |z|\, e^{i\,arg(z)} \qquad = e^{u}\, e^{i\,arg(e^{iv})}$$

$$\Rightarrow \quad z = e^{\ln|z|}\, e^{i(arg(z)+2k\pi)}$$

$$|z| = e^{u}$$

$$u = \ln|z|$$

$$arg(z) = arg(e^{iv})$$

$$v = arg(z) + 2k\pi$$

$2k\pi$ is necessary
assuming *arg* is uniquely defined
with its range specified

Complex Log : ln z

$$z = |z|\, e^{i\,arg(z)} = e^{u}\, e^{i\,arg(e^{iv})}$$

$$z = e^{\ln|z|}\, e^{i(arg(z) + 2k\pi)}$$

many angles

$$z = e^{\ln|z| + i(arg(z) + 2k\pi)}$$

$$\ln z = \ln|z| + i(arg(z) + 2k\pi)$$

$2k\pi$ is needed :
assuming *arg* is uniquely defined
with its range specified

unique argument

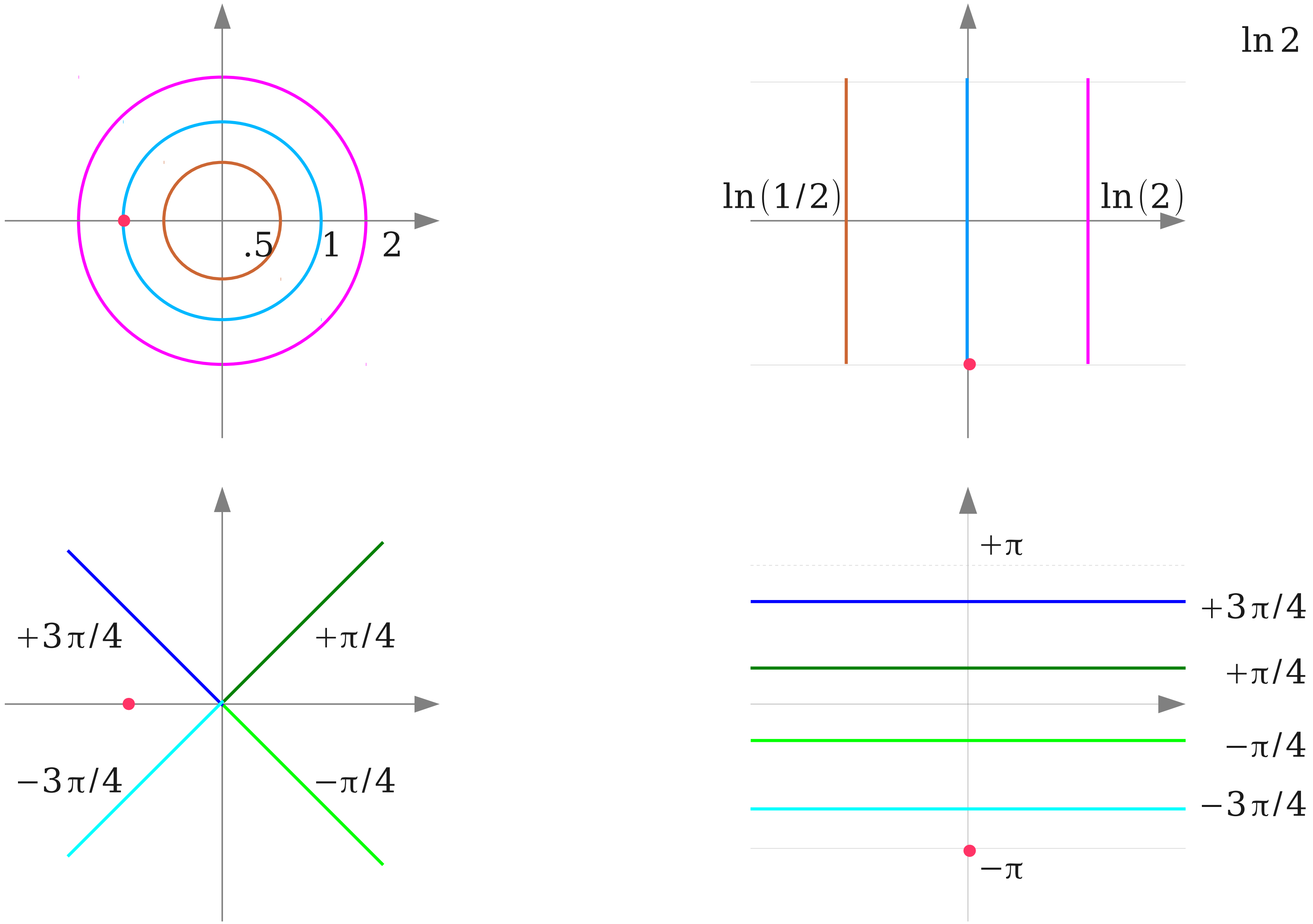
$$z = e^{\ln|z| + i\,arg(z)}$$

$$\ln z = \ln|z| + i\,arg(z)$$

$$z = |z|\, e^{i\,arg(z)}$$

$$\ln z = \ln|z| + i\,arg(z)$$

A mapping of ln z



$$\ln 2 = 0.693$$

Complex Powers

$z = e^w \Leftrightarrow w = \ln z$

$z = e^w = e^{\ln z}$

$a^b = e^w \Leftrightarrow a^b = \ln z$

$a^b = e^w = e^{\ln a^b} = e^{b \ln a}$

$$z = e^{\ln|z|} e^{i\,arg(z)}$$

$$z^b = e^{b \ln|z|} e^{i\,b\,arg(z)}$$

$$a^z = e^{z \ln|a|} e^{i\,z\,arg(a)}$$

$z^b = e^w = e^{\ln z^b} = \boxed{e^{b \ln z}}$

$z^b = e^{b(\ln|z| + i\,arg(z))} = \boxed{e^{b \ln|z|} e^{i\,b\,arg(z)}}$

$a^z = e^w = e^{\ln a^z} = \boxed{e^{z \ln a}}$

$a^z = e^{z(\ln|a| + i\,arg(a))} = \boxed{e^{z \ln|a|} e^{i\,z\,arg(a)}}$

$e^{\ln|z|}$ *real*

$\left\{ \begin{array}{l} e^{b \ln|z|} \\ e^{z \ln|a|} \end{array} \right.$ *in general not real*

Complex Logs Example

free to choose more convenient definition
for a problem

$$(-1) = e^{-i\pi + i2k\pi}$$

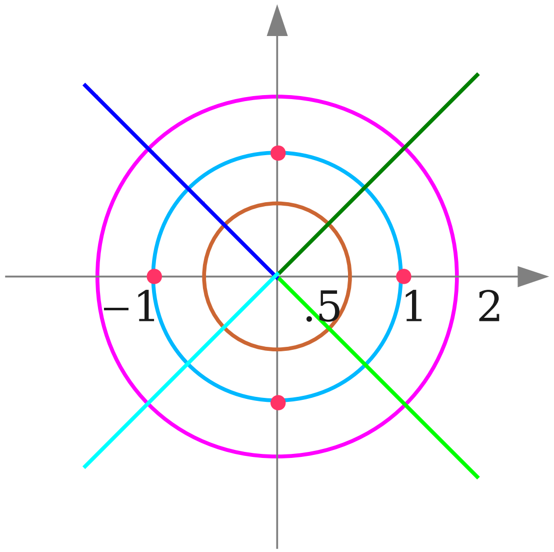
$$\ln(-1) = \ln|-1| + i\,arg(-1)$$

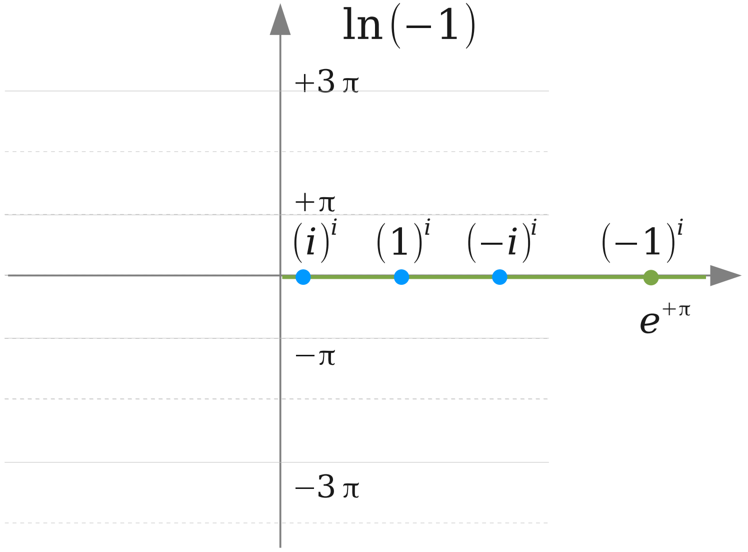
$$\begin{cases} -i(\pi - 2k\pi) \\ -i\pi \end{cases}$$ (Principal Argument)

$$(-1)^i = e^{i\ln(-1)} = e^{i(-i(\pi-2k\pi))}$$

$$\begin{cases} e^{\pi-2k\pi} = e^{+\pi}e^{-2k\pi} \\ e^{+\pi} \end{cases}$$ (Principal Argument)

$(+1)$	$= e^{0}$	$(+1)^{i}$	$= e^{0}$	
$(+i)$	$= e^{+i\pi/2}$	$(+i)^{i}$	$= e^{-\pi/2}$	0.207
(-1)	$= e^{-i\pi}$	$(-1)^{i}$	$= e^{+\pi}$	23.1
$(-i)$	$= e^{-i\pi/2}$	$(-i)^{i}$	$= e^{+\pi/2}$	4.81





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10/17/13

Complex Power Examples : z^i

$$(+1) = e^{+i(0-2k\pi)} \qquad (+1)^i = e^{i\ln(+1)} = e^{0} \; e^{+2k\pi} \qquad (+1)^{i} = e^{0}$$

$$(+i) = e^{+i(\pi/2-2k\pi)} \qquad (+i)^i = e^{i\ln(+i)} = e^{-\pi/2} e^{+2k\pi} \qquad (+i)^{i} = e^{-\pi/2}$$

$$(-1) = e^{-i(\pi-2k\pi)} \qquad (-1)^i = e^{i\ln(-1)} = e^{+\pi} \; e^{-2k\pi} \qquad (-1)^{i} = e^{+\pi}$$

$$(-i)^i = e^{-i(\pi/2-2k\pi)} \qquad (-i)^i = e^{i\ln(-i)} = e^{+\pi/2} e^{-2k\pi} \qquad (-i)^{i} = e^{+\pi/2}$$

$$(+2) = e^{\ln 2 + i(0-2k\pi)} \qquad (+2)^i = e^{i(\ln 2+i(0-2k\pi))} = e^{0} \; e^{+2k\pi} e^{i\ln 2} \qquad (+2)^{i} = e^{0} \; e^{i\ln 2}$$

$$(+2i) = e^{\ln 2 + i(\pi/2-2k\pi)} \qquad (+2i)^i = e^{i(\ln 2+i(\pi/2-2k\pi))} = e^{-\pi/2} e^{+2k\pi} e^{i\ln 2} \qquad (+2i)^{i} = e^{-\pi/2} e^{i\ln 2}$$

$$(-2) = e^{\ln 2 - i(\pi-2k\pi)} \qquad (-2)^i = e^{i(\ln 2-i(\pi-2k\pi))} = e^{+\pi} \; e^{-2k\pi} e^{i\ln 2} \qquad (-2)^{i} = e^{+\pi} \; e^{i\ln 2}$$

$$(-2i) = e^{\ln 2 - i(\pi/2-2k\pi)} \qquad (-2i)^i = e^{i(\ln 2-i(\pi/2-2k\pi))} = e^{+\pi/2} e^{-2k\pi} e^{i\ln 2} \qquad (-2i)^{i} = e^{+\pi/2} e^{i\ln 2}$$

$$\ln 2 = 0.693 \qquad \cos(\ln 2) = 0.769 \qquad e^{\pi} = 20.140$$

$$\sin(\ln 2) = 0.639 \qquad e^{\pi/2} = 4.810$$

Complex Roots

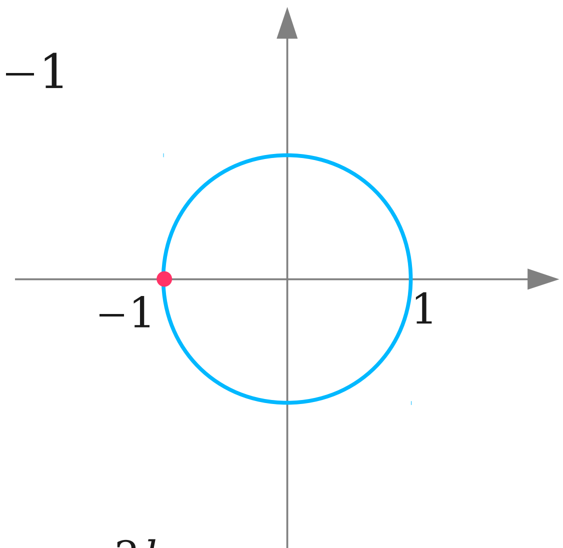
$$z^n - a = 0 \iff z = |a|^{1/n} e^{i(arg(a)/n + 2k\pi/n)}$$

$$z^n = r^n e^{in\theta} = a = |a| e^{i\,arg(a)}$$

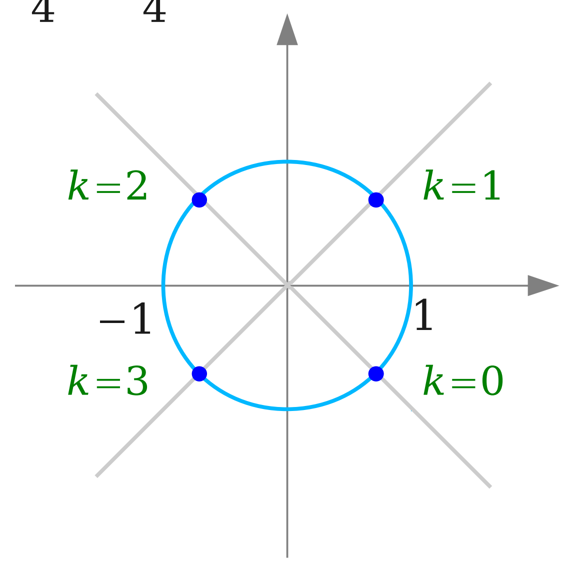
$r^n = |a|$ $\qquad$ $n\theta = arg(a) + 2k\pi$

$r = |a|^{\frac{1}{n}}$ $\qquad$ $\theta = \dfrac{arg(a)}{n} + \dfrac{2k\pi}{n}$

$$z^4 = -1$$

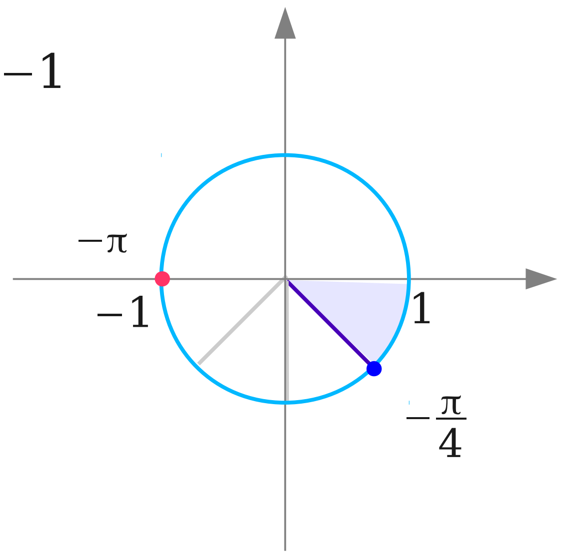


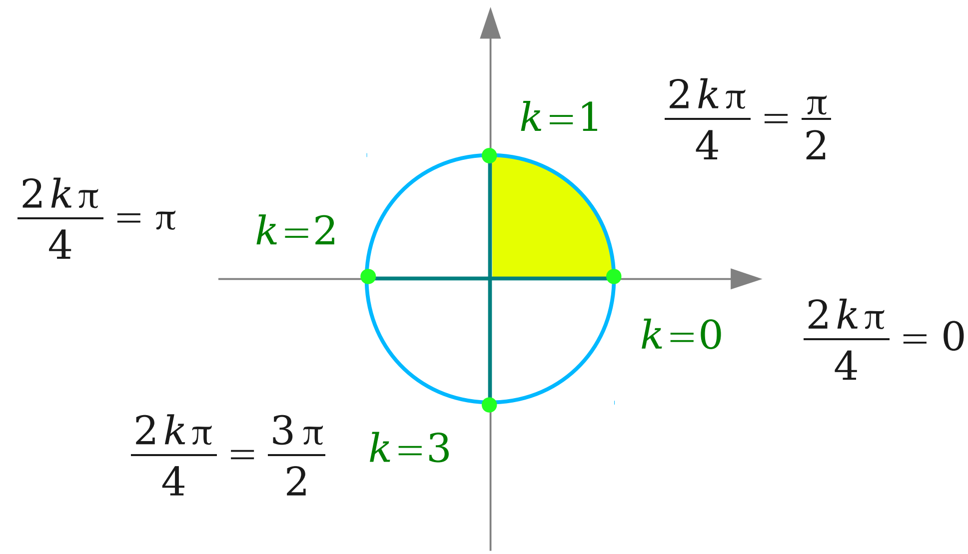
$$z = -\frac{\pi}{4} + \frac{2k\pi}{4}$$



Complex Roots: $z^4 = -1$

$$z^4 = -1$$





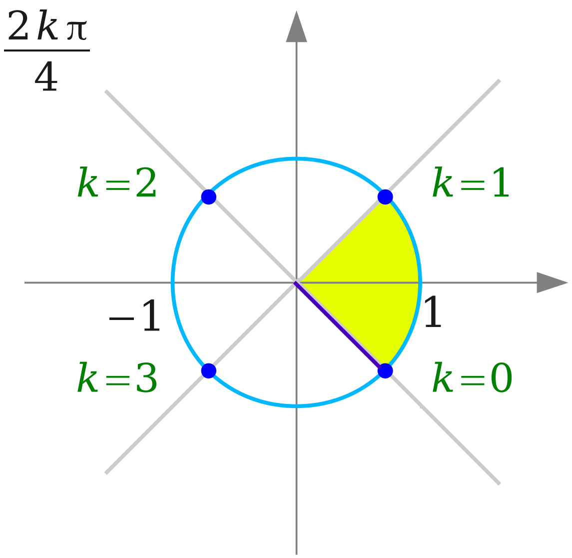
$$z^n - a = 0$$

$$z = |a|^{1/n} e^{i(arg(a)/n + 2k\pi/n)}$$

$$r = |a|^{\frac{1}{n}}$$

$$\theta = \frac{arg(a)}{n} + \frac{2k\pi}{n}$$

$$z = -\frac{\pi}{4} + \frac{2k\pi}{4}$$



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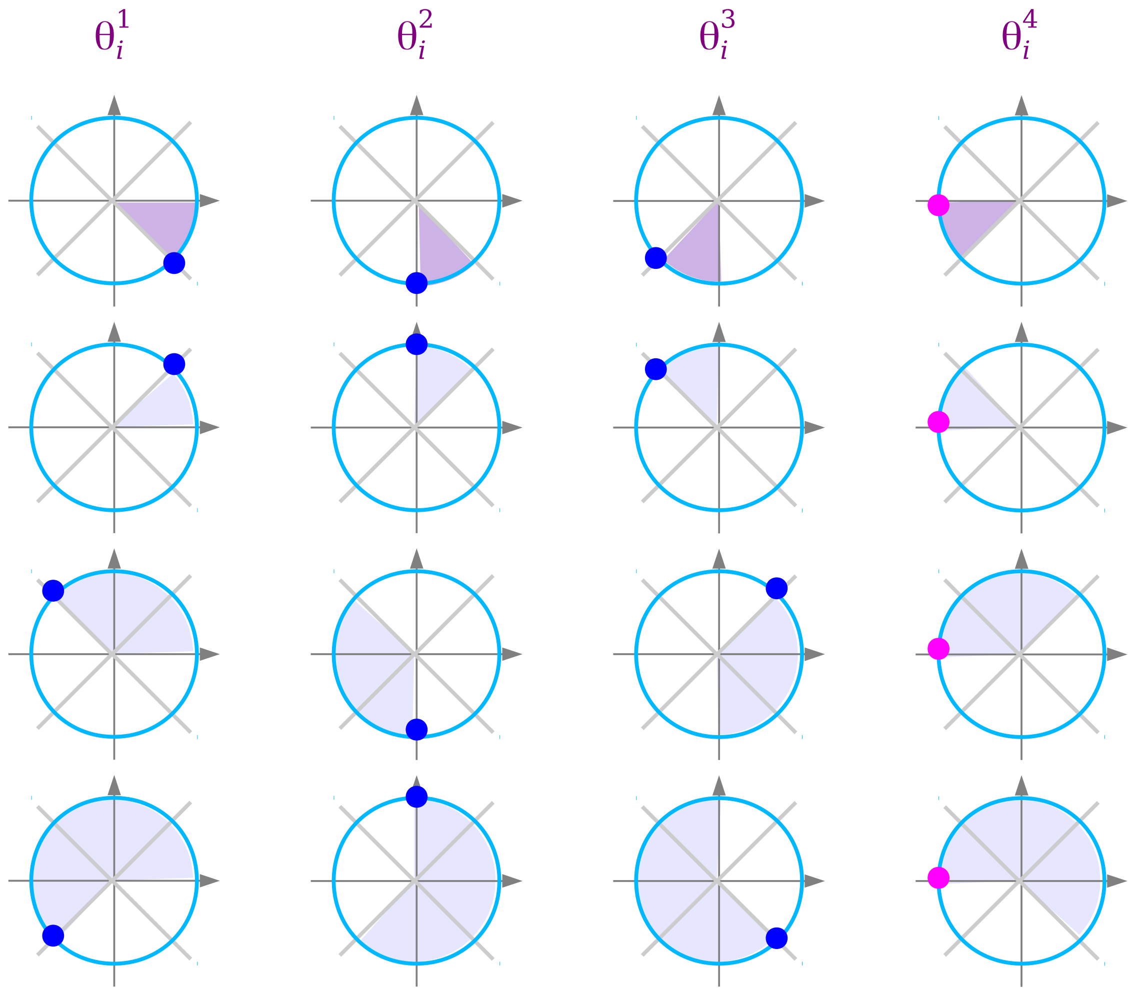
10/17/13

Complex Roots : $z^4 = 1$

$r = |a|^{\frac{1}{4}}$ $\theta = \frac{1}{4}(-\pi + 2k\pi)$

$z^4 = -1$



$0 \le arg(z) < 2\pi$: $-\pi$

$\theta_0 = -\frac{1}{4}\pi$ ➡ $\theta_0^4 = -\pi$

$2\pi \le arg(z) < 4\pi$: $+\pi$

$\theta_1 = +\frac{1}{4}\pi$ ➡ $\theta_1^4 = -\pi$

$4\pi \le arg(z) < 6\pi$: $+3\pi$

$\theta_2 = +\frac{3}{4}\pi$ ➡ $\theta_2^4 = -\pi$

$6\pi \le arg(z) < 8\pi$: 5π

$\theta_3 = +\frac{5}{4}\pi$ ➡ $\theta_3^4 = -\pi$

Complex Powers: $z^{1/2}$

$$\oint_{|z|=1} \sqrt{z}\, dz = \int_{\theta_0}^{\theta_0+2\pi} e^{i\theta/2}\, i e^{i\theta}\, d\theta$$

$$= \int_{\theta_0}^{\theta_0+2\pi} i e^{i3\theta/2}\, d\theta = \left[\frac{2}{3} e^{i3\theta/2}\right]_{\theta_0}^{\theta_0+2\pi}$$

$$= \frac{2}{3}\left(e^{i3(\theta_0+2\pi)/2} - e^{i3\theta_0/2}\right)$$

$$= \frac{2}{3} e^{i3\theta_0/2}\left(e^{i3\pi} - 1\right) = -\frac{4}{3} e^{i3\theta_0/2}$$

$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

$$\theta_0 \le arg(z) < \theta_0 + 2\pi$$

$$\theta_0 < arg(z) \le \theta_0 + 2\pi$$

the definition of $\sqrt{z}$ also depends on θ_0

the circular integration depends on where the closed circle begins θ_0

implicitly assumed definition of $\sqrt{z}$ depends on θ_0

Complex Powers: $z^{1/2}$

real $x \ge 0$

$$y^2 = x$$

$$y = +\sqrt{x}$$

$$y = -\sqrt{x}$$

$$\sqrt{x} \ge 0$$

complex z

$$w^2 = z$$

$$w = +\sqrt{z}$$

$$w = -\sqrt{z}$$

$$\Re\{\sqrt{z}\} \ge 0$$

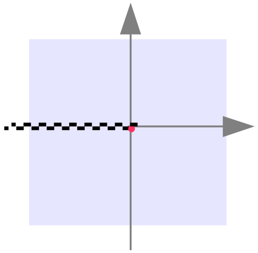
if we would want this, then

$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

$$-\pi < arg(z) < +\pi$$

$$-\pi/2 < arg(z)/2 < +\pi/2$$

but we would have excluded all the pure imaginary numbers also.



The MATLAB's defintion

$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

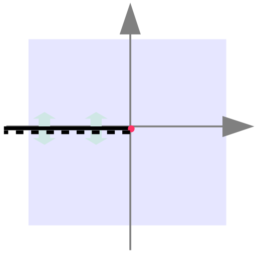
$$-\pi < arg(z) \le +\pi$$

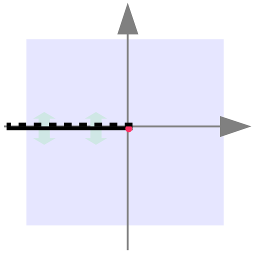
a discontinuity in $\sqrt{z}$ when the negative real axis

$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

$$-\pi \le arg(z) < +\pi$$

a discontinuity in $\sqrt{z}$ when the positive real axis

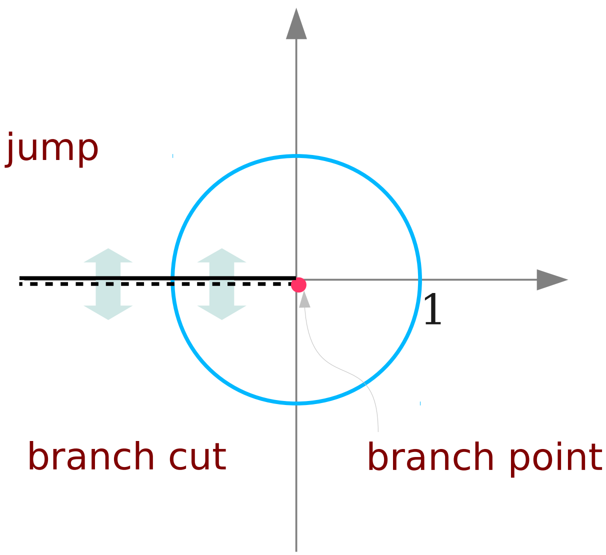




Branch Cut

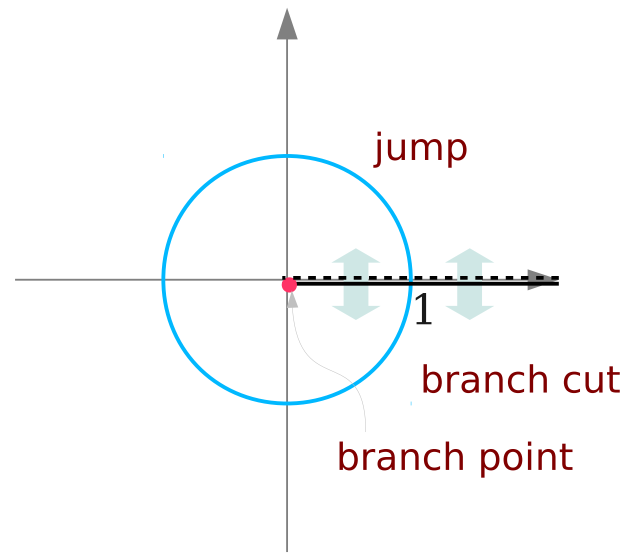
$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

$$-\pi < arg(z) \leq +\pi$$



$$\sqrt{z} = |z|^{1/2} e^{+i\,arg(z)/2}$$

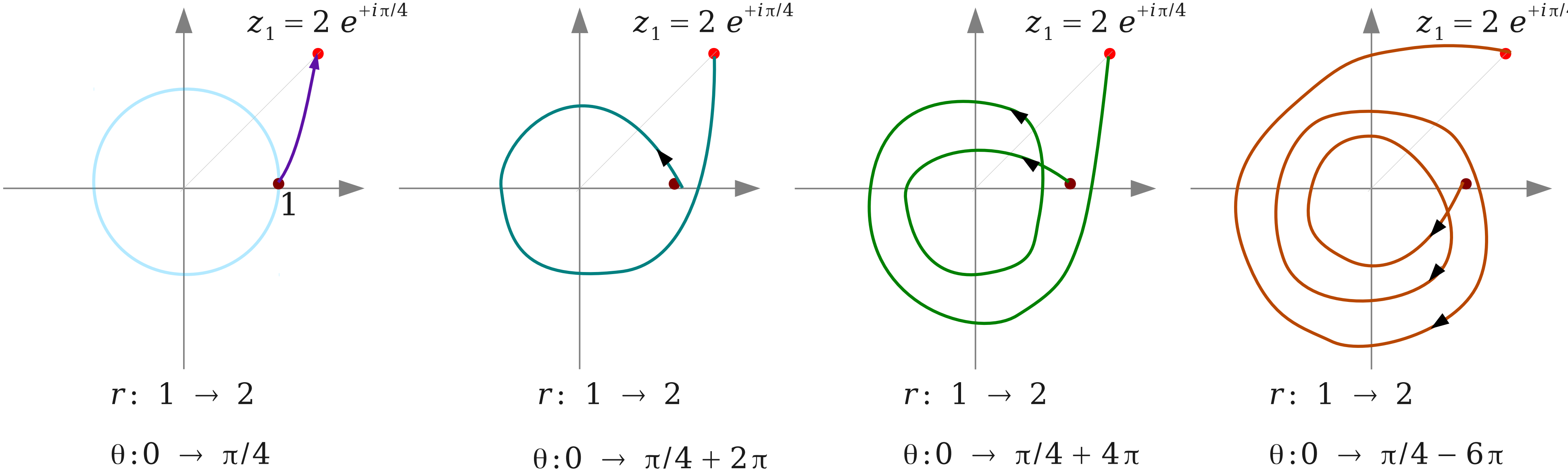
$$0 < arg(z) \leq +2\pi$$



if we go around a branch point, then there is always a discontinuity somewhere

Winding Number ($z^{½}$)

Consider computing the square root of points along the different paths that are continuous



two classes of paths

$$\sqrt{z} = \sqrt{r\, e^{+i\theta}} = \sqrt{r}\, e^{+i\theta/2} \qquad \sqrt{z_1} = +\sqrt{2}\, e^{+i\pi/8}$$ even number of rounding

$$\sqrt{z} = \sqrt{r\, e^{+i(\theta+2\pi)}} = \sqrt{r}\, e^{+i\theta/2}\, e^{i\pi} \qquad \sqrt{z_1} = -\sqrt{2}\, e^{+i\pi/8}$$ odd number of rounding

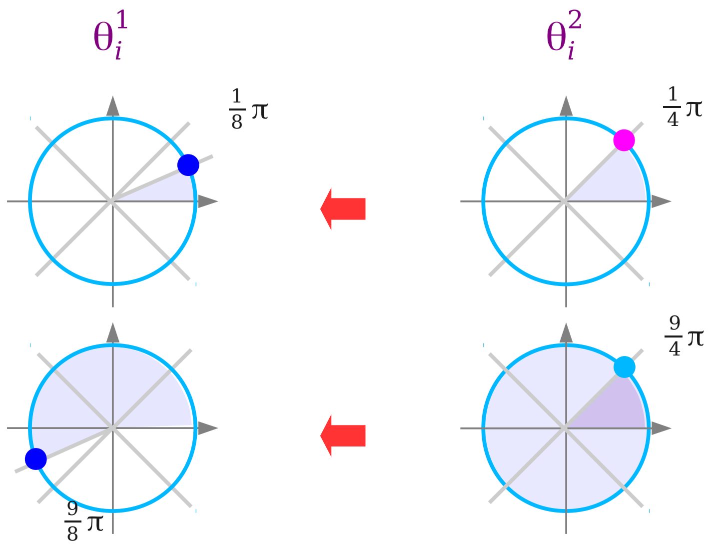
Range of Arguments ($z^{½}$)

$$r = |2|^{\frac{1}{2}}$$

$$\theta = \frac{1}{2}\left(\frac{\pi}{4} + 2k\pi\right)$$

$\Leftarrow$

$$z^2 = 2 \cdot e^{+j\pi/4}$$

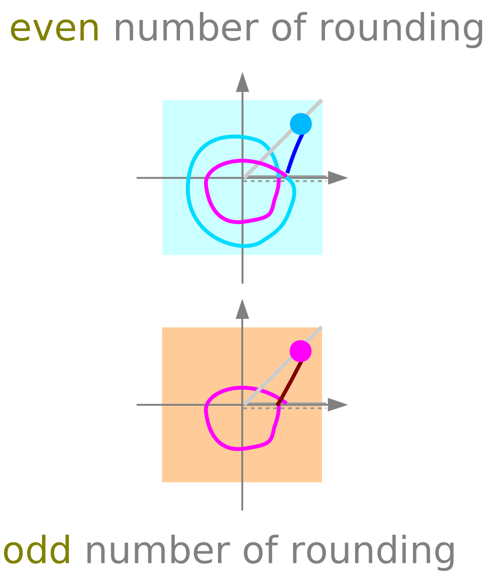


$0 \le arg(z) < 2\pi \quad : \frac{1}{4}\pi$

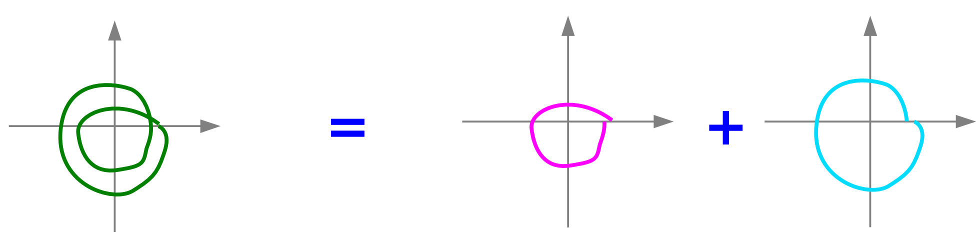
$\theta_0 = +\frac{1}{2}(\frac{1}{4}\pi) \Leftarrow \theta_0^2 = \frac{1}{4}\pi$

$2\pi \le arg(z) < 4\pi \quad : \frac{9}{4}\pi$

$\theta_1 = +\frac{1}{2}(\frac{9}{4}\pi) \Leftarrow \theta_1^2 = \frac{9}{4}\pi$



a closed loop in a
Riemann surface



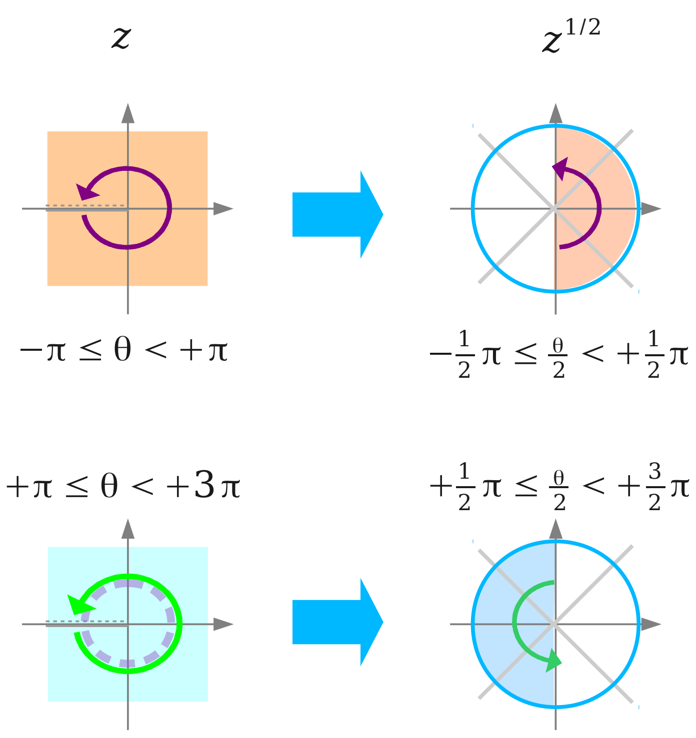
Domain & Range Complex Planes ($z^{1/2}$)

$$\sqrt{z} = \sqrt{r\, e^{+i\theta}} = \sqrt{r}\, e^{+i\theta/2} \qquad +\sqrt{r}\cos\frac{\theta}{2} + i\,\sqrt{r}\sin\frac{\theta}{2}$$

$$\sqrt{z} = \sqrt{r\, e^{+i\theta+2\pi}} = \sqrt{r}\, e^{+i\theta/2}\, e^{i\pi} \qquad -\sqrt{r}\cos\frac{\theta}{2} - i\,\sqrt{r}\sin\frac{\theta}{2}$$

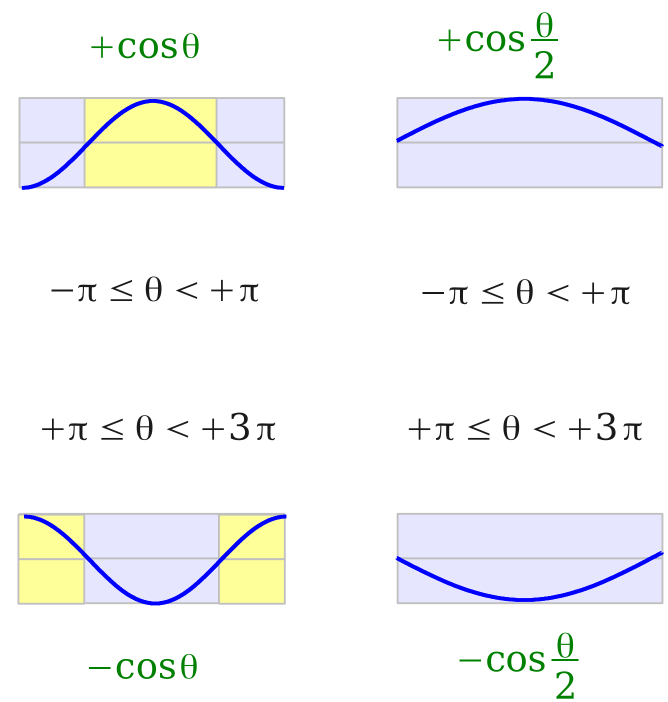
$$\Re\{\sqrt{z}\} = +\sqrt{r}\cos\frac{\theta}{2} \qquad \Im\{\sqrt{z}\} = +\sqrt{r}\sin\frac{\theta}{2}$$

$$\Re\{\sqrt{z}\} = -\sqrt{r}\cos\frac{\theta}{2} \qquad \Im\{\sqrt{z}\} = -\sqrt{r}\sin\frac{\theta}{2}$$

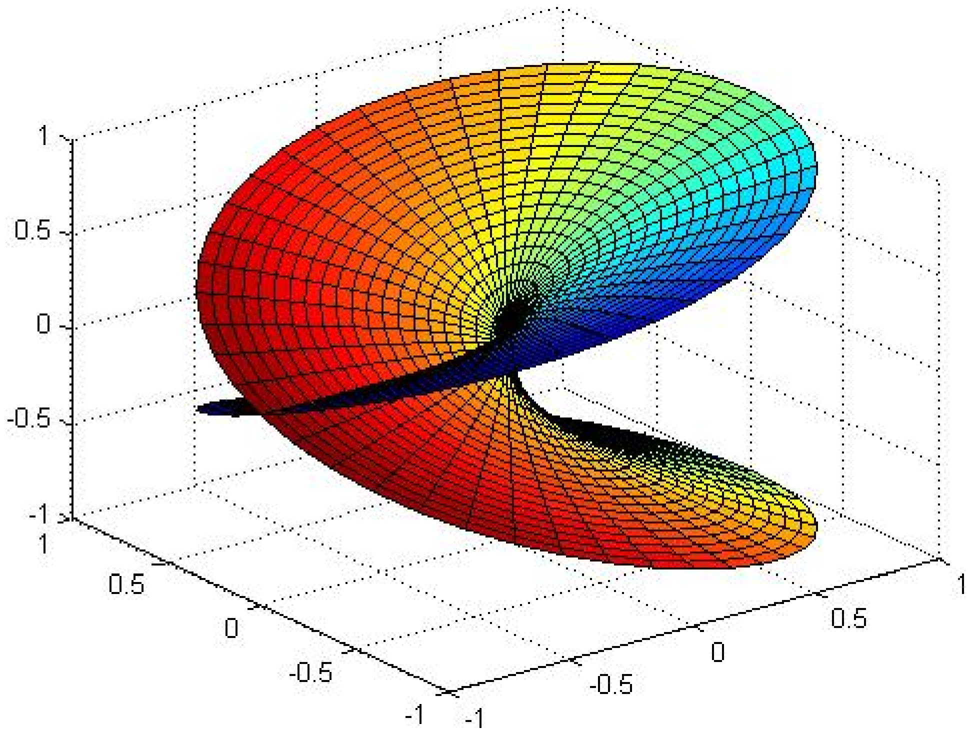


Domain Complex Plane

Range Complex Plane

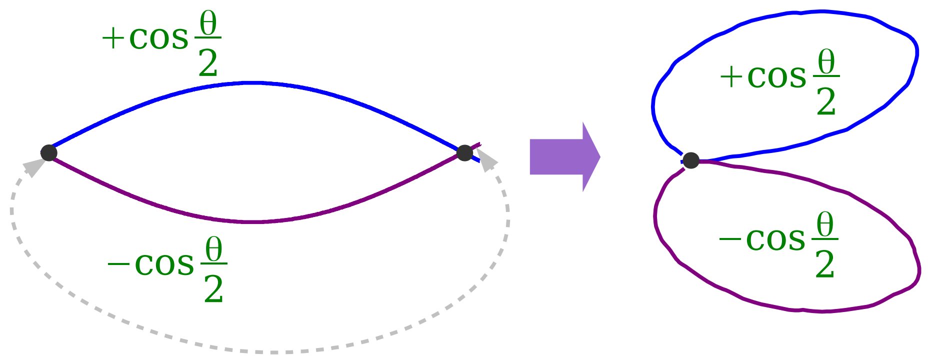


Riemann Surface ($z^{½}$) - (1)



$$\Re\{\sqrt{z}\} = +\sqrt{r}\cos\frac{\theta}{2}$$

$$\Re\{\sqrt{z}\} = -\sqrt{r}\cos\frac{\theta}{2}$$

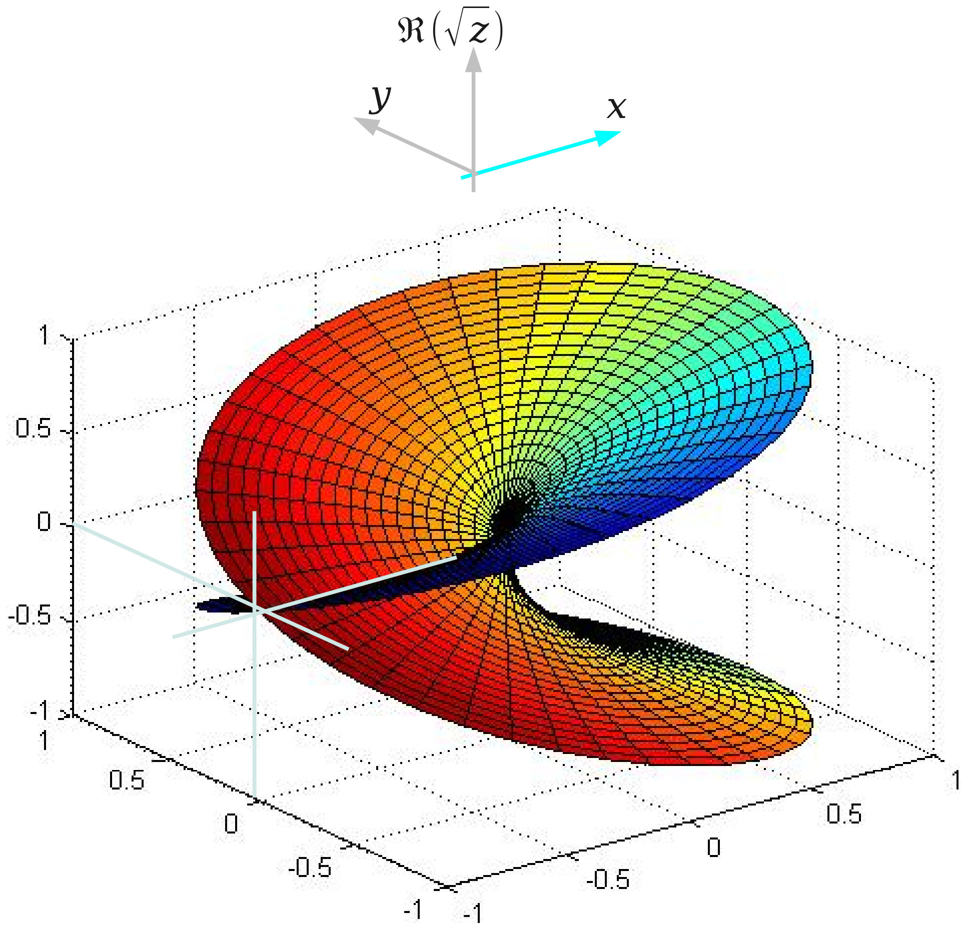


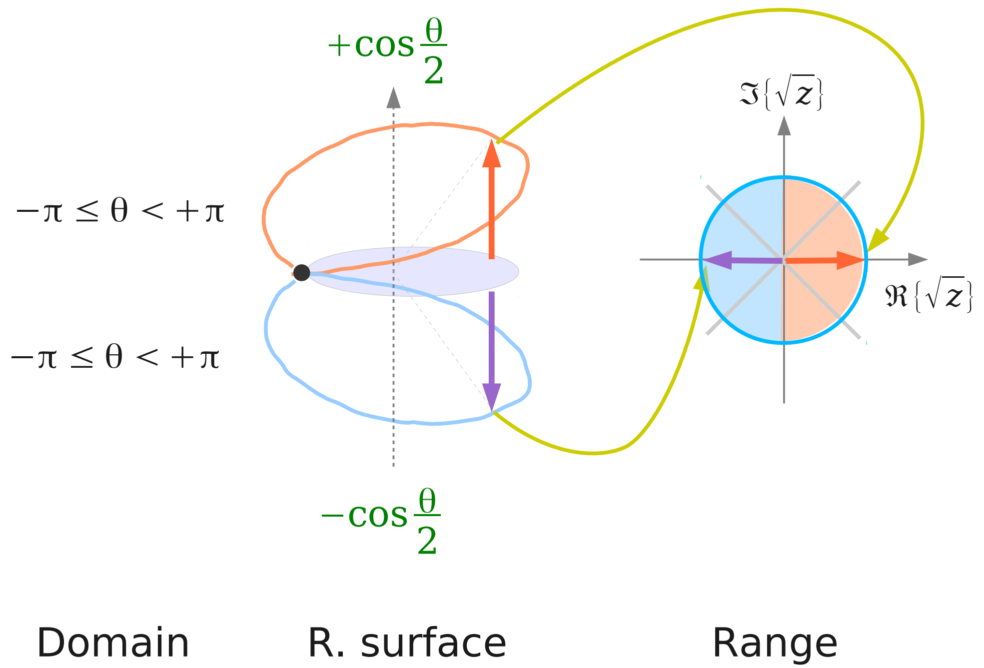
Riemann surface for the function ƒ(z) = √z. The two horizontal axes represent the real and imaginary parts of z, while the vertical axis represents the real part of √z. For the imaginary part of √z, rotate the plot 180° around the vertical axis. [wikipedia.org]

For visualizing a multivalued function.

a geometric construction that permits surfaces to be the domain or range of a multivalued function.

Riemann Surface $(z^{1/2})$ - (2)

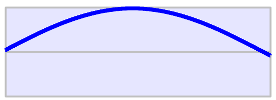


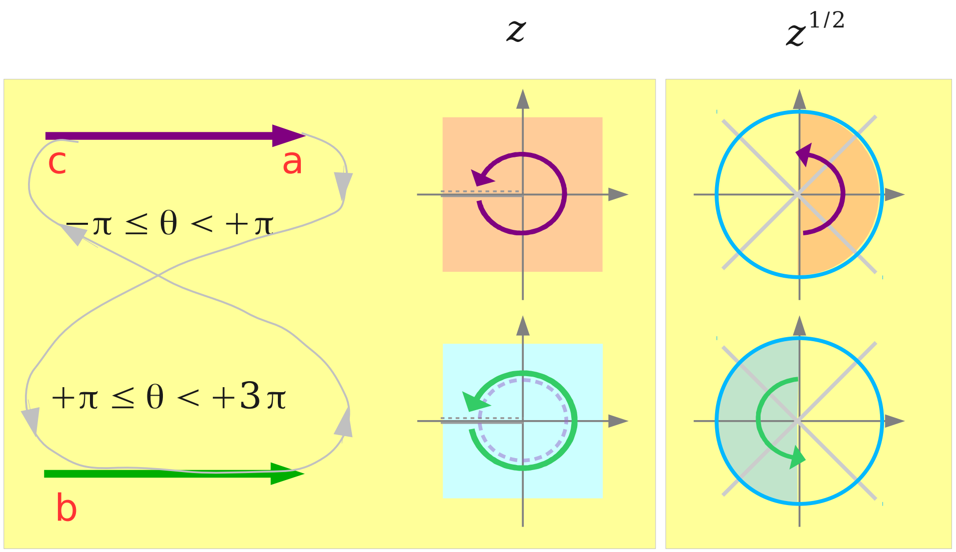


http://scipp.ucsc.edu/~haber/ph116A/ComplexFunBranchTheory.pdf

Riemann Surface ($z^{1/2}$) - (3)

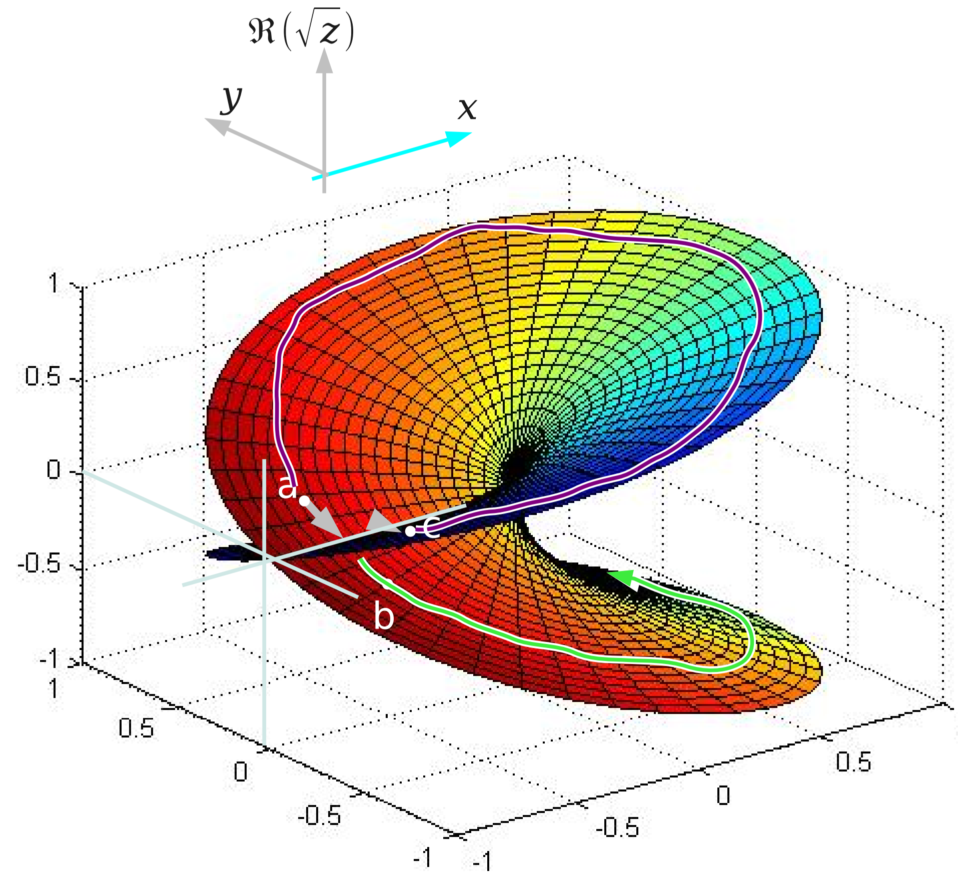
$\Re\{\sqrt{z}\} = +\cos\frac{\theta}{2}$

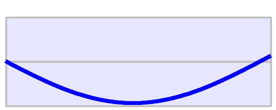




Domain

Range





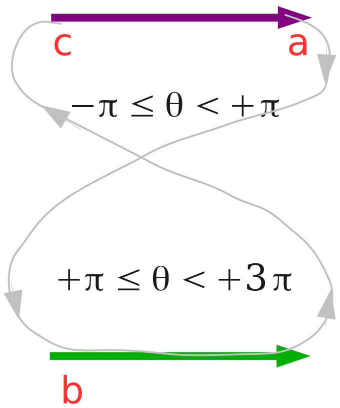
$\Re\{\sqrt{z}\} = -\cos\frac{\theta}{2}$

http://scipp.ucsc.edu/~haber/ph116A/ComplexFunBranchTheory.pdf

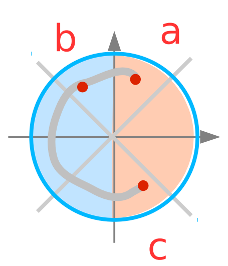
This Riemann surface makes "full square root function" continuous for all $z \neq 0$.

between the point a and the point b, the domain switches from the upper half-plane to the lower half-plane.

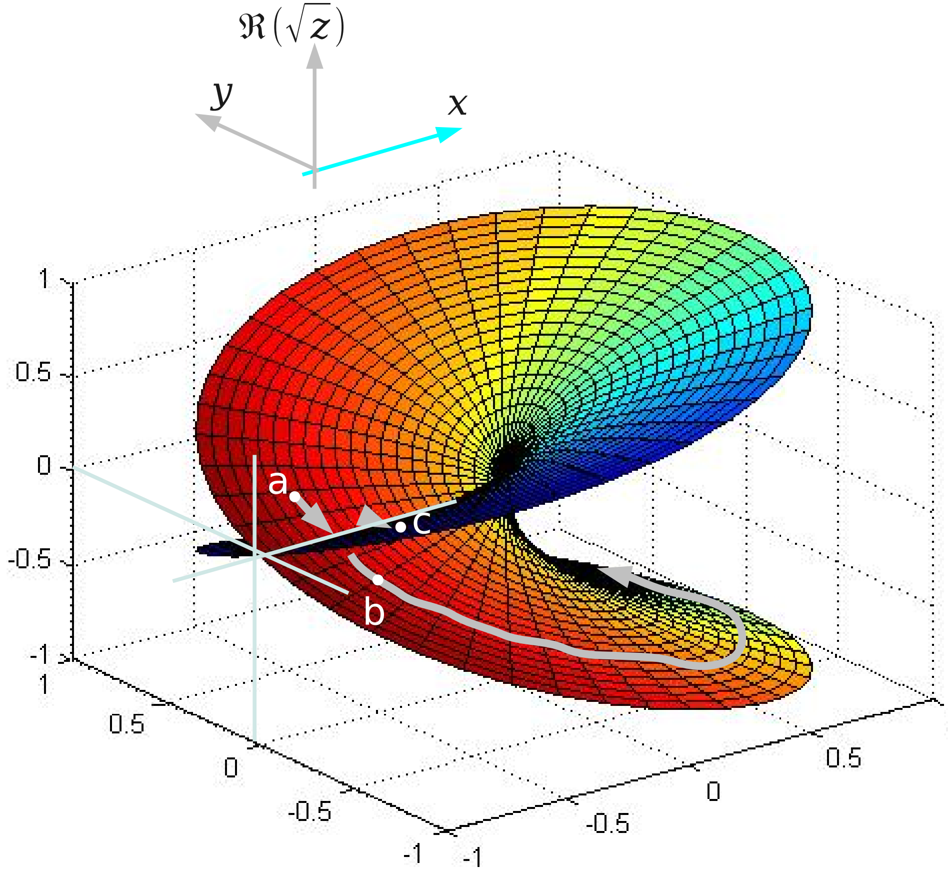
no other way to get from the point a to the point b except going counterclockwise.



Domain



Range



http://scipp.ucsc.edu/~haber/ph116A/ComplexFunBranchTheory.pdf

Riemann Surface

A one-dimensional complex manifold.
can be thought of as "deformed versions" of the complex plane: locally near every point they look like patches of the complex plane, but the global topology can be quite different. For example, they can look like a sphere or a torus or a couple of sheets glued together.

Holomorphic functions
(conformal maps, regular function)
Riemann surfaces are nowadays considered the natural setting for studying the global behavior of these functions, especially multi-valued functions such as the square root and other algebraic functions, or the logarithm.

The phrase "holomorphic at a point z0" means not just differentiable at z0, but differentiable everywhere within some neighborhood of z0.

The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series.
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References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] E. Kreyszig, “Advanced Engineering Mathematics”
[5] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[6] T. J. Cavicchi, “Digital Signal Processing”
[7] F. Waleffe, Math 321 Notes, UW 2012/12/11
[8] J. Nearing, University of Miami
[9] http://scipp.ucsc.edu/~haber/ph116A/ComplexFunBranchTheory.pdf