Laurent Series and z-Transform - Geometric Series Time Shift A

20180815 Wed

Causal Signal

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^1 + \left(\frac{1}{2}\right)^2 z^2 + \cdots = \frac{1}{1 - \left(\frac{z}{2}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^0 z^0 + \left(\frac{1}{2}\right)^1 z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \cdots = \frac{1}{1 - \left(\frac{1}{2z}\right)}$$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - \left(\frac{z}{2}\right)} \rightarrow \frac{2}{2 - z} \quad (|z| < 2)$$

$$X(z) = \frac{1}{1 - \left(\frac{1}{2z}\right)} \rightarrow \frac{z}{z - 0.5} \quad (|z| > 0.5)$$

Anti-Causal Signal

$$a_n = \left(\frac{1}{2}\right)^n \qquad (n < 0)$$

$$f(z) = \left(\frac{1}{2}\right)^{-1} z^{-1} + \left(\frac{1}{2}\right)^{-2} z^{-2} + \left(\frac{1}{2}\right)^{-3} z^{-3} + \cdots = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)}$$

$$X(z) = \left(\frac{1}{2}\right)^{-1} z^{1} + \left(\frac{1}{2}\right)^{-2} z^{2} + \left(\frac{1}{2}\right)^{-3} z^{3} + \cdots = \frac{(2z)}{1 - (2z)}$$

$$a_n = \left(\frac{1}{2}\right)^n \qquad (n < 0)$$

$$f(z) = \frac{\left(\frac{2}{z}\right)}{1 - \left(\frac{2}{z}\right)} \rightarrow \frac{2}{z - 2} \qquad (|z| > 2)$$

$$X(z) = \frac{(2z)}{1 - (2z)} \rightarrow \frac{z}{0.5 - z} \qquad (|z| < 0.5)$$

Inverse z

Causal

$$f(z) = \frac{2}{2-z} \quad (|z|<2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z|>0.5)$$

anti-Causal

$$f(z^{-1}) = \frac{2}{2-z^{-1}} \quad (|z^{-1}|<2) \rightarrow f(z^{-1}) = X(z) = \frac{z}{z-0.5} \quad (|z|>0.5)$$

$$X(z^{-1}) = \frac{z^{-1}}{z^{-1}-0.5} \quad (|z^{-1}|>0.5) \rightarrow X(z^{-1}) = f(z) = \frac{2}{2-z} \quad (|z|<2)$$

$f(z^{-1}) = X(z)$ Laurent Series (anti-causal signal)
with the same formula as causal X(z)

$X(z^{-1}) = f(z)$ z-Transform (anti-causal signal)
with the same formula as causal f(z)

Causal | anti-Causal

$$f(z) = \frac{2}{2-z} \quad (|z|<2) \qquad X(z^{-1}) = \frac{2}{2-z} \quad (|z|<2)$$

$$X(z) = \frac{z}{z-0.5} \quad (|z|>0.5) \qquad f(z^{-1}) = \frac{z}{z-0.5} \quad (|z|>0.5)$$

Inverse ROC

Causal | anti-Causal

$$f(z) = \frac{z}{2 - z} \quad (|z| < 2) \quad \longrightarrow \quad -f(z) = -\frac{z}{2 - z} \quad (|z| > 2)$$

$$X(z) = \frac{z}{z - 0.5} \quad (|z| > 0.5) \quad \longrightarrow \quad -X(z) = -\frac{z}{z - 0.5} \quad (|z| < 0.5)$$

p=2

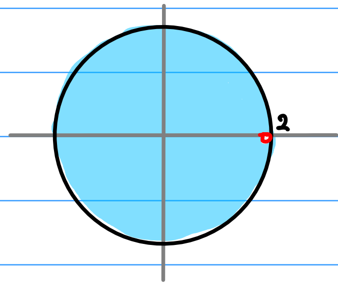
causal	anti-causal
$a_n = (\frac{1}{2})^n \quad (n \geqslant 0)$	$a_n = (\frac{1}{2})^n \quad (n < 0)$
$f(z) = \dfrac{1}{1-(\frac{z}{2})} \quad (\lvert z \rvert < 2)$ $X(z) = \dfrac{1}{1-(\frac{1}{2z})} \quad (\lvert z \rvert > 0.5)$	$f(z) = \dfrac{(\frac{2}{z})}{1-(\frac{2}{z})} \quad (\lvert z \rvert > 2)$ $X(z) = \dfrac{(2z)}{1-(2z)} \quad (\lvert z \rvert < 0.5)$
$f(z) = \dfrac{2}{2-z} \quad (\lvert z \rvert < 2)$ $X(z) = \dfrac{z}{z-0.5} \quad (\lvert z \rvert > 0.5)$	$f(z) = \dfrac{2}{z-2} \quad (\lvert z \rvert > 2)$ $X(z) = \dfrac{z}{0.5-z} \quad (\lvert z \rvert < 0.5)$

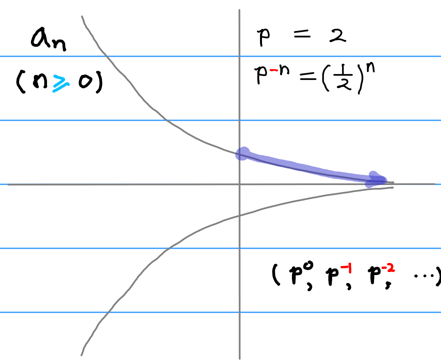
p=1/2

causal	anti-causal
$a_n = (2)^n \quad (n \geqslant 0)$	$a_n = (2)^n \quad (n < 0)$
$f(z) = \dfrac{1}{1-(2z)} \quad (\lvert z \rvert < 0.5)$ $X(z) = \dfrac{1}{1-(\frac{2}{z})} \quad (\lvert z \rvert > 2)$	$f(z) = \dfrac{(\frac{1}{2z})}{1-(\frac{1}{2z})} \quad (\lvert z \rvert > 0.5)$ $X(z) = \dfrac{(\frac{z}{2})}{1-(\frac{z}{2})} \quad (\lvert z \rvert < 2)$
$f(z) = \dfrac{0.5}{0.5-z} \quad (\lvert z \rvert < 0.5)$ $X(z) = \dfrac{z}{z-2} \quad (\lvert z \rvert > 2)$	$f(z) = \dfrac{0.5}{z-0.5} \quad (\lvert z \rvert > 0.5)$ $X(z) = \dfrac{z}{2-z} \quad (\lvert z \rvert < 2)$

$n \in [0, \infty)$ $\quad |z| < 2$

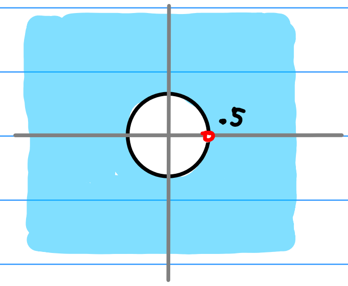
a_n | causal 0.5^n | $\dfrac{2}{2 - z}$

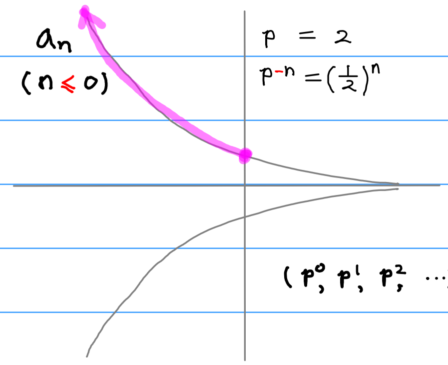




$|z| > 0.5$ $\quad n \in (-\infty, 0]$

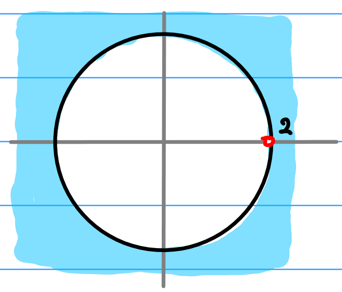
$\dfrac{z}{z - 0.5}$ | anti-causal 0.5^n



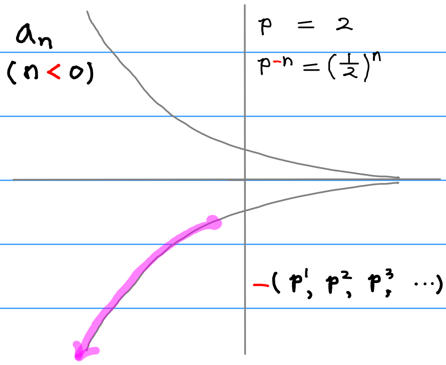


b_n | 0.5^n anti-causal | $-\dfrac{2}{2 - z}$

$n \in (-\infty, -1]$ $\quad |z| > 2$

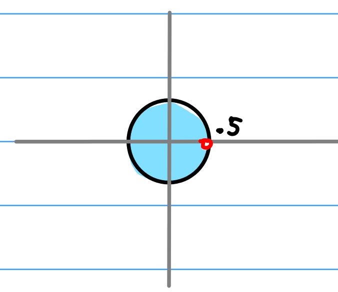


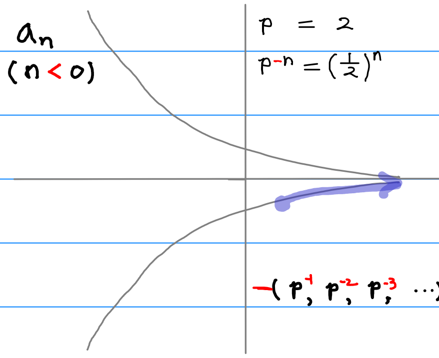
anti- causal $g(z^{-1})$



$-\dfrac{z}{z - 0.5}$ | 0.5^n causal

$|z| < 0.5$ $\quad n \in [1, \infty)$





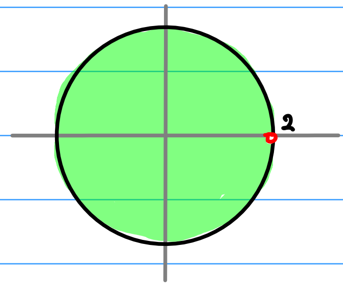
$n < 1$ | $|z| < 2$

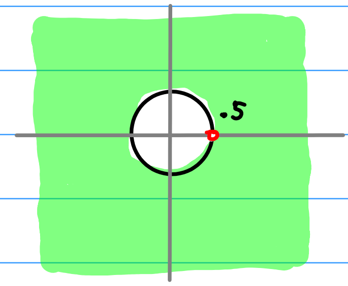
x_{-n} | anti-causal $-(p^{-1}, p^{-2}, p^{-3}, \dots)$ | $-\dfrac{0.5}{1 - 0.5z}$

z^{-1}

$|z| > 0.5$ | $n \geq 0$

$-\dfrac{0.5}{1 - 0.5z^{-1}}$ | causal $-(p^{-1}, p^{-2}, p^{-3}, \dots)$ | x_n

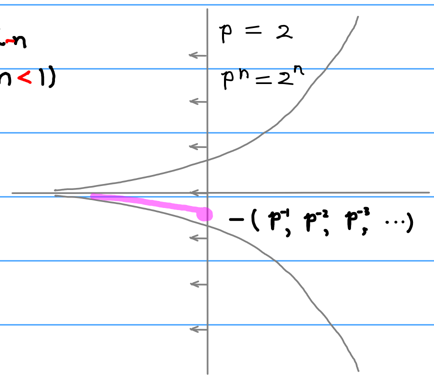




anti-causal $X(z^{-1})$

x_{-n}

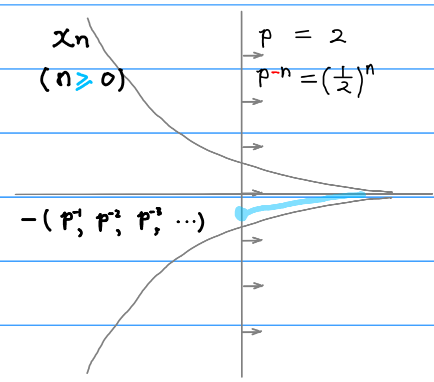
$(n < 1)$



causal $X(z)$

x_n

$(n \geq 0)$



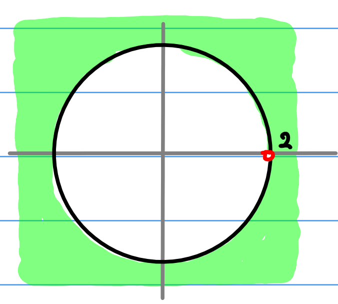
$n \geq 1$ | $|z| > 2$

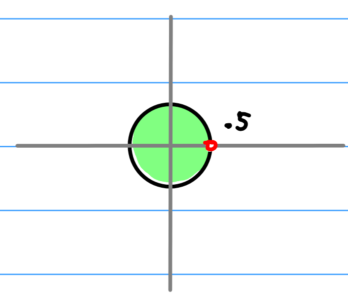
y_n | $(p^0, p^1, p^2, \dots)$ causal | $\dfrac{z^{-1}}{1 - 2z^{-1}}$

z^{-1}

$|z| < 0.5$ | $n < 0$

$\dfrac{z}{1 - 2z}$ | $(p^0, p^1, p^2, \dots)$ anti-causal | y_{-n}

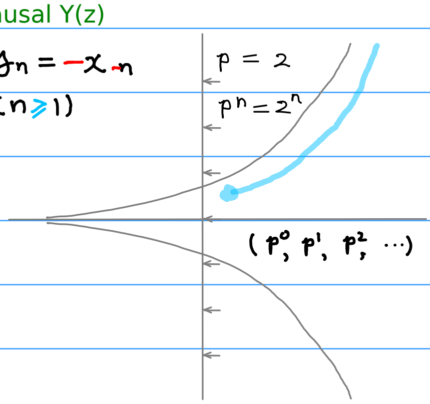




causal $Y(z)$

$y_n = -x_{-n}$

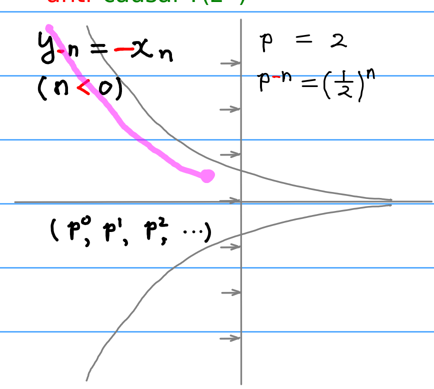
$(n \geq 1)$



anti-causal $Y(z^{-1})$

$y_{-n} = -x_n$

$(n < 0)$



2 formulas

$$\boxed{\frac{1}{z-p}} \qquad \boxed{\frac{1}{z^{-1}-p}}$$

2 representations each

$$
\begin{array}{ccccc}
 & & \text{causal} & & \text{anti- causal} \\
 & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z}} \triangleq & f(z) & = & X(z^{-1}) \\
\boxed{\dfrac{1}{z-p}} & & \| & & \| \\
 & \boxed{\dfrac{z^{-1}}{1-pz^{-1}}} \triangleq & Y(z) & = & g(z^{-1}) \\
 & & \text{causal} & & \text{anti- causal}
\end{array}
$$

$$
\begin{array}{ccccc}
 & & \text{causal} & & \text{anti- causal} \\
 & \boxed{\dfrac{z}{1-pz}} \triangleq & g(z) & = & Y(z^{-1}) \\
\boxed{\dfrac{1}{z^{-1}-p}} & & \| & & \| \\
 & \boxed{-\dfrac{p^{-1}}{1-p^{-1}z^{-1}}} \triangleq & X(z) & = & f(z^{-1}) \\
 & & \text{causal} & & \text{anti- causal}
\end{array}
$$

Time Shift

$p = 2$

① $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

③ $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

⑤ $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

⑦ $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

⑨ $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

⑪ $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

Time Shift

$p = \frac{1}{2}$

2 $(n \geq 0)$ $a_n = (2)^n$ $\quad f(z) = \frac{0.5}{0.5 - z}$ $\quad X(z) = \frac{z}{z-2}$

4 $(n < 0)$ $a_n = (2)^n$ $\quad f(z) = -\frac{0.5}{0.5 - z}$ $\quad X(z) = -\frac{z}{z-2}$

6 $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$ $\quad f(z) = \frac{0.5z}{0.5 - z}$ $\quad X(z) = \frac{1}{z-2}$

8 $(n < 1)$ $a_{n-1} = (2)^{n-1}$ $\quad f(z) = -\frac{0.5z}{0.5 - z}$ $\quad X(z) = -\frac{1}{z-2}$

10 $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $\quad f(z) = \frac{0.5}{(0.5 - z)z}$ $\quad X(z) = \frac{z^2}{z-2}$

12 $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $\quad f(z) = -\frac{0.5}{(0.5 - z)z}$ $\quad X(z) = -\frac{z^2}{z-2}$

Time Shift

$2 \leftrightarrow \frac{1}{2}$

1. $(n \geq 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = \frac{2}{2-z} \quad X(z) = \frac{z}{z-0.5}$

2. $(n \geq 0) \quad a_n = (2)^n \quad f(z) = \frac{0.5}{0.5-z} \quad X(z) = \frac{z}{z-2}$

3. $(n < 0) \quad a_n = \left(\frac{1}{2}\right)^n \quad f(z) = -\frac{2}{2-z} \quad X(z) = -\frac{z}{z-0.5}$

4. $(n < 0) \quad a_n = (2)^n \quad f(z) = -\frac{0.5}{0.5-z} \quad X(z) = -\frac{z}{z-2}$

5. $(n \geq 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \quad f(z) = \frac{2z}{2-z} \quad X(z) = \frac{1}{z-0.5}$

6. $(n \geq 1) \quad a_{n-1} = (2)^{n-1} \quad f(z) = \frac{0.5z}{0.5-z} \quad X(z) = \frac{1}{z-2}$

7. $(n < 1) \quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1} \quad f(z) = -\frac{2z}{2-z} \quad X(z) = -\frac{1}{z-0.5}$

8. $(n < 1) \quad a_{n-1} = (2)^{n-1} \quad f(z) = -\frac{0.5z}{0.5-z} \quad X(z) = -\frac{1}{z-2}$

9. $(n \geq -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = \frac{2}{(2-z)z} \quad X(z) = \frac{z^2}{z-0.5}$

10. $(n \geq -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = \frac{0.5}{(0.5-z)z} \quad X(z) = \frac{z^2}{z-2}$

11. $(n < -1) \quad a_{n+1} = \left(\frac{1}{2}\right)^{n+1} \quad f(z) = -\frac{2}{(2-z)z} \quad X(z) = -\frac{z^2}{z-0.5}$

12. $(n < -1) \quad a_{n+1} = (2)^{n+1} \quad f(z) = -\frac{0.5}{(0.5-z)z} \quad X(z) = -\frac{z^2}{z-2}$

$2 \longleftrightarrow \frac{1}{2}$

1 $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = \frac{2}{2-z}$ $X(z) = \frac{z}{z-0.5}$

3 $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $f(z) = -\frac{2}{2-z}$ $X(z) = -\frac{z}{z-0.5}$

5 $(n \geq 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = \frac{2z}{2-z}$ $X(z) = \frac{1}{z-0.5}$

7 $(n < 1)$ $a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$ $f(z) = -\frac{2z}{2-z}$ $X(z) = -\frac{1}{z-0.5}$

9 $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = \frac{2}{(2-z)z}$ $X(z) = \frac{z^2}{z-0.5}$

11 $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $f(z) = -\frac{2}{(2-z)z}$ $X(z) = -\frac{z^2}{z-0.5}$

$2 \longleftrightarrow \frac{1}{2}$

② $(n \geq 0)$ $a_n = (2)^n$

$f(z) = \frac{0.5}{0.5 - z}$ $X(z) = \frac{z}{z-2}$

$|z|<0.5$ $\frac{1}{1-2z}$ $|z|<0.5$ $\frac{1}{1-2z^{-1}}$

④ $(n < 0)$ $a_n = (2)^n$

$f(z) = -\frac{0.5}{0.5 - z}$ $X(z) = -\frac{z}{z-2}$

$|z|>0.5$ $\frac{-1}{1-2z}$ $|z|>0.5$ $\frac{-1}{1-2z^{-1}}$

⑥ $(n \geq 1)$ $a_{n-1} = (2)^{n-1}$

$f(z) = \frac{0.5z}{0.5 - z}$ $X(z) = \frac{1}{z-2}$

$|z|>0.5$ $\frac{z}{1-2z}$ $|z|>0.5$ $\frac{z^{-1}}{1-2z^{-1}}$

⑧ $(n < 1)$ $a_{n-1} = (2)^{n-1}$

$f(z) = -\frac{0.5z}{0.5 - z}$ $X(z) = -\frac{1}{z-2}$

$|z|>0.5$ $\frac{-z}{1-2z}$ $|z|>0.5$ $\frac{-z^{-1}}{1-2z^{-1}}$

⑩ $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$

$f(z) = \frac{0.5}{(0.5 - z)z}$ $X(z) = \frac{z^2}{z-2}$

$|z|<0.5$ $\frac{1}{(1-2z)z}$ $|z|<0.5$ $\frac{1}{(1-2z^{-1})z^{-1}}$

⑫ $(n < -1)$ $a_{n+1} = (2)^{n+1}$

$f(z) = -\frac{0.5}{(0.5 - z)z}$ $X(z) = -\frac{z^2}{z-2}$

$|z|<0.5$ $\frac{-1}{(1-2z)z}$ $|z|<0.5$ $\frac{-1}{(1-2z^{-1})z^{-1}}$

Causality

$$f(z) \quad (|z| < p) \quad \longleftrightarrow \quad a_n \quad (n \geq 0) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$X(z^{-1}) \quad (|z| < p) \quad \longleftrightarrow \quad x_{-n} \quad (n < 1) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z^{-1}) \quad (|z| > p^{-1}) \quad \longleftrightarrow \quad a_{-n} \quad (n < 1) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$X(z) \quad (|z| > p^{-1}) \quad \longleftrightarrow \quad x_n \quad (n \geq 0) \quad -(p^{-1}, p^{-2}, p^{-3}, \cdots)$$

$$f(z) \quad (|z| > p) \quad \longleftrightarrow \quad -a_n \quad (n < 0) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$X(z^{-1}) \quad (|z| > p) \quad \longleftrightarrow \quad -x_{-n} \quad (n \geq 1) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$f(z^{-1}) \quad (|z| < p^{-1}) \quad \longleftrightarrow \quad -a_{-n} \quad (n \geq 1) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$X(z) \quad (|z| < p^{-1}) \quad \longleftrightarrow \quad -x_n \quad (n < 0) \quad (p^{0}, p^{1}, p^{2}, \cdots)$$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ g(z^{-1}) & g(z) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ Y(z) & Y(z^{-1}) \end{array} \qquad \begin{array}{cc} a_n & a_{-n} \\ b_{-n} & b_n \end{array} \qquad \begin{array}{cc} x_{-n} & x_n \\ y_n & y_{-n} \end{array}$$

$$\begin{array}{cc} f(z) & f(z^{-1}) \\ f(z) & f(z^{-1}) \end{array} \qquad \begin{array}{cc} X(z^{-1}) & X(z) \\ X(z^{-1}) & X(z) \end{array}$$

$$\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array} \qquad \begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \dots) & -(p^{-1}, p^{-2}, p^{-3}, \dots) \\ (p^{0}, p^{1}, p^{2}, \dots) & (p^{0}, p^{1}, p^{2}, \dots) \end{array}$$

$$\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array} \qquad \begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}$$

$f(z)$ $g(z)$ $f(z)$ $g(z)$	$Y(z)$ $X(z)$ $Y(z)$ $X(z)$	a_n a_{-n} $-a_n$ $-a_{-n}$	x_{-n} x_n $-x_{-n}$ $-x_n$
$\lvert z \rvert < p$ $\lvert z \rvert > p^{-1}$ $\lvert z \rvert > p$ $\lvert z \rvert < p^{-1}$	$\lvert z \rvert < p$ $\lvert z \rvert > p^{-1}$ $\lvert z \rvert > p$ $\lvert z \rvert < p^{-1}$		
$[0, \infty)$ $(-\infty, 0]$ $(-\infty, -1]$ $[1, \infty)$	$(-\infty, 0]$ $[0, \infty)$ $[1, \infty)$ $(-\infty, -1]$		
$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^0, p^1, p^2, \cdots)$	$-(p^{-1}, p^{-2}, p^{-3}, \cdots)$ $(p^0, p^1, p^2, \cdots)$		

$$\boxed{\begin{array}{cc} a_n & a_{-n} \\ -a_n & -a_{-n} \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad a_n = -2^{-n}$$

$$\boxed{\begin{array}{cc} x_{-n} & x_n \\ -x_{-n} & -x_n \end{array}} \qquad \boxed{\begin{array}{cc} 2^{-n} & 2^{n} \\ -2^{-n} & -2^{n} \end{array}} \qquad x_n = -2^{n}$$

$$\boxed{\begin{array}{cc} -(p^{-1}, p^{-2}, p^{-3}, \cdots) & -(p^{-1}, p^{-2}, p^{-3}, \cdots) \\ (p^{0}, p^{1}, p^{2}, \cdots) & (p^{0}, p^{1}, p^{2}, \cdots) \end{array}} \qquad \boxed{\begin{array}{cc} -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) & -(-2^{-1}, -2^{-2}, -2^{-3}, \cdots) \\ (-2^{0}, -2^{1}, -2^{2}, \cdots) & (-2^{0}, -2^{1}, -2^{2}, \cdots) \end{array}}$$

$$\boxed{\begin{array}{cc} -\dfrac{p^{-1}}{1-p^{-1}z} & -\dfrac{p^{-1}}{1-p^{-1}z^{-1}} \\ \dfrac{z^{-1}}{1-pz^{-1}} & \dfrac{z}{1-pz} \end{array}} \qquad \boxed{\begin{array}{cc} \dfrac{2^{-1}}{1-2^{-1}z} & \dfrac{2^{-1}}{1-2^{-1}z^{-1}} \\ -\dfrac{z^{-1}}{1-2z^{-1}} & -\dfrac{z}{1-2z} \end{array}} \qquad \begin{array}{cc} \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{z}{2}\right)} & \dfrac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2z}\right)} \\ -\dfrac{\left(\frac{1}{z}\right)}{1-\left(\frac{2}{z}\right)} & -\dfrac{z}{1-2z} \end{array}$$

$$\boxed{\begin{array}{cc} |z| < p & |z| > p^{-1} \\ |z| > p & |z| < p^{-1} \end{array}} \qquad \boxed{\begin{array}{cc} |z| < 2 & |z| > 2^{-1} \\ |z| > 2 & |z| < 2^{-1} \end{array}}$$

$$\boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}} \qquad \boxed{\begin{array}{cc} [0, \infty) & (-\infty, 0] \\ (-\infty, -1] & [1, \infty) \end{array}}$$

Shift to the right

shift to the right $\rightarrow$ delete a_0		$\times z$	$\times z^{-1}$
1	$(n \geq 0)\quad a_n = \left(\frac{1}{2}\right)^n$	$f(z) = \frac{2}{2-z}$	$X(z) = \frac{z}{z-0.5}$
5	$(n \geq 1)\quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = \frac{2z}{2-z}$	$X(z) = \frac{1}{z-0.5}$
2	$(n \geq 0)\quad a_n = (2)^n$	$f(z) = \frac{0.5}{0.5-z}$	$X(z) = \frac{z}{z-2}$
6	$(n \geq 1)\quad a_{n-1} = (2)^{n-1}$	$f(z) = \frac{0.5z}{0.5-z}$	$X(z) = \frac{1}{z-2}$

shift to the right $\rightarrow$ insert a_0		$\times z$	$\times z^{-1}$
3	$(n < 0)\quad a_n = \left(\frac{1}{2}\right)^n$	$f(z) = -\frac{2}{2-z}$	$X(z) = -\frac{z}{z-0.5}$
7	$(n < 1)\quad a_{n-1} = \left(\frac{1}{2}\right)^{n-1}$	$f(z) = -\frac{2z}{2-z}$	$X(z) = -\frac{1}{z-0.5}$
4	$(n < 0)\quad a_n = (2)^n$	$f(z) = -\frac{0.5}{0.5-z}$	$X(z) = -\frac{z}{z-2}$
8	$(n < 1)\quad a_{n-1} = (2)^{n-1}$	$f(z) = -\frac{0.5z}{0.5-z}$	$X(z) = -\frac{1}{z-2}$

Shift to the left

Shift to the left $\leftarrow$ | $\times z^{-1}$ | $\times z$

delete a_0

(1) $(n \geq 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $\quad f(z) = \frac{2}{2-z}$ $\quad X(z) = \frac{z}{z-0.5}$

(9) $(n \geq -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $\quad f(z) = \frac{2}{(2-z)z}$ $\quad X(z) = \frac{z}{z-0.5}$

(2) $(n \geq 0)$ $a_n = (2)^n$ $\quad f(z) = \frac{0.5}{0.5-z}$ $\quad X(z) = \frac{z}{z-2}$

(10) $(n \geq -1)$ $a_{n+1} = (2)^{n+1}$ $\quad f(z) = \frac{0.5}{(0.5-z)z}$ $\quad X(z) = \frac{z}{z-2}$

Shift to the left $\leftarrow$ | $\times z^{-1}$ | $\times z$

insert a_0

(3) $(n < 0)$ $a_n = \left(\frac{1}{2}\right)^n$ $\quad f(z) = -\frac{2}{2-z}$ $\quad X(z) = -\frac{z}{z-0.5}$

(11) $(n < -1)$ $a_{n+1} = \left(\frac{1}{2}\right)^{n+1}$ $\quad f(z) = -\frac{2}{(2-z)z}$ $\quad X(z) = -\frac{z}{z-0.5}$

(4) $(n < 0)$ $a_n = (2)^n$ $\quad f(z) = -\frac{0.5}{0.5-z}$ $\quad X(z) = -\frac{z}{z-2}$

(12) $(n < -1)$ $a_{n+1} = (2)^{n+1}$ $\quad f(z) = -\frac{0.5}{(0.5-z)z}$ $\quad X(z) = -\frac{z}{z-2}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$

b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$

b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$

b^{n-1} $n=1,2,3,\cdots$

$$1 \longleftrightarrow \frac{1}{1}$$

1. $(n \geqslant 0)$ $\quad a_n = (1)^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

2. $(n \geqslant 0)$ $\quad a_n = (1^{-1})^n \qquad f(z) = \frac{1}{1-z} \qquad X(z) = \frac{z}{z-1}$

3. $(n < 0)$ $\quad a_n = (1)^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

4. $(n < 0)$ $\quad a_n = (1^{-1})^n \qquad f(z) = -\frac{1}{1-z} \qquad X(z) = -\frac{z}{z-1}$

5. $(n \geqslant 1)$ $\quad a_{n-1} = (1)^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

6. $(n \geqslant 1)$ $\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \frac{z}{1-z} \qquad X(z) = \frac{1}{z-1}$

7. $(n < 1)$ $\quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

8. $(n < 1)$ $\quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\frac{z}{1-z} \qquad X(z) = -\frac{1}{z-1}$

9. $(n \geqslant -1)$ $\quad a_{n+1} = (1)^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

10. $(n \geqslant -1)$ $\quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = \frac{1}{(1-z)z} \qquad X(z) = \frac{z}{z-1}$

11. $(n < -1)$ $\quad a_{n+1} = (1)^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

12. $(n < -1)$ $\quad a_{n+1} = (1^{-1})^{n+1} \qquad f(z) = -\frac{1}{(1-z)z} \qquad X(z) = -\frac{z}{z-1}$

① $(n \geq 0) \quad a_n = (1)^n \qquad f(z) = \dfrac{1}{1-z} \qquad X(z) = \dfrac{z}{z-1}$

② $(n \geq 0) \quad a_n = (1^{-1})^n \qquad f(z) = \dfrac{1}{1-z} \qquad X(z) = \dfrac{z}{z-1}$

shift to the right $\rightarrow$ $\qquad \times z \qquad \times z^{-1}$

delete a_0

⑤ $(n \geq 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = \dfrac{z}{1-z} \qquad X(z) = \dfrac{1}{z-1}$

⑥ $(n \geq 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = \dfrac{z}{1-z} \qquad X(z) = \dfrac{1}{z-1}$

③ $(n < 0) \quad a_n = (1)^n \qquad f(z) = -\dfrac{1}{1-z} \qquad X(z) = -\dfrac{z}{z-1}$

④ $(n < 0) \quad a_n = (1^{-1})^n \qquad f(z) = -\dfrac{1}{1-z} \qquad X(z) = -\dfrac{z}{z-1}$

shift to the right $\rightarrow$ $\qquad \times z \qquad \times z^{-1}$

insert a_0

⑦ $(n < 1) \quad a_{n-1} = (1)^{n-1} \qquad f(z) = -\dfrac{z}{1-z} \qquad X(z) = -\dfrac{1}{z-1}$

⑧ $(n < 1) \quad a_{n-1} = (1^{-1})^{n-1} \qquad f(z) = -\dfrac{z}{1-z} \qquad X(z) = -\dfrac{1}{z-1}$

1 $(n \geq 0)$ $a_n = (1)^n$ $\qquad f(z) = \dfrac{1}{1-z}$ $\qquad X(z) = \dfrac{z}{z-1}$

2 $(n \geq 0)$ $a_n = (1^{-1})^n$ $\qquad f(z) = \dfrac{1}{1-z}$ $\qquad X(z) = \dfrac{z}{z-1}$

shift to the left $\leftarrow$ $\qquad \times z^{-1}$ $\qquad \times z$

delete a_0

9 $(n \geq -1)$ $a_{n+1} = (1)^{n+1}$ $\qquad f(z) = \dfrac{1}{(1-z)z}$ $\qquad X(z) = \dfrac{z}{z-1}$

10 $(n \geq -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $\qquad f(z) = \dfrac{1}{(1-z)z}$ $\qquad X(z) = \dfrac{z}{z-1}$

3 $(n < 0)$ $a_n = (1)^n$ $\qquad f(z) = -\dfrac{1}{1-z}$ $\qquad X(z) = -\dfrac{z}{z-1}$

4 $(n < 0)$ $a_n = (1^{-1})^n$ $\qquad f(z) = -\dfrac{1}{1-z}$ $\qquad X(z) = -\dfrac{z}{z-1}$

shift to the left $\leftarrow$ $\qquad \times z^{-1}$ $\qquad \times z$

insert a_0

11 $(n < -1)$ $a_{n+1} = (1)^{n+1}$ $\qquad f(z) = -\dfrac{1}{(1-z)z}$ $\qquad X(z) = -\dfrac{z}{z-1}$

12 $(n < -1)$ $a_{n+1} = (1^{-1})^{n+1}$ $\qquad f(z) = -\dfrac{1}{(1-z)z}$ $\qquad X(z) = -\dfrac{z}{z-1}$

$n=-4$	$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n+1} $n=-2,-3,-4,\cdots$

b^{n+1} $n=-1,0,1,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$	
b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n} $n=-1,-2,-3,\cdots$

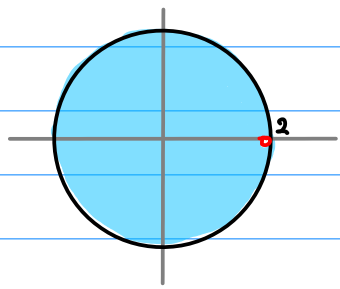
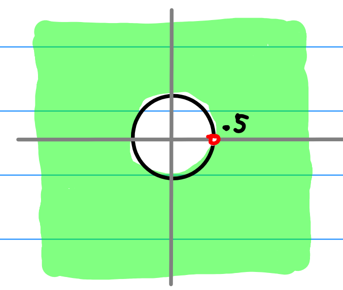
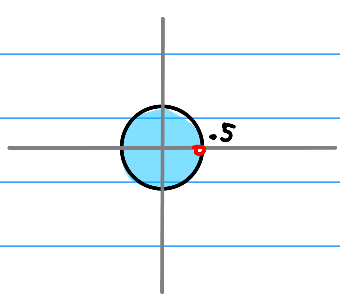
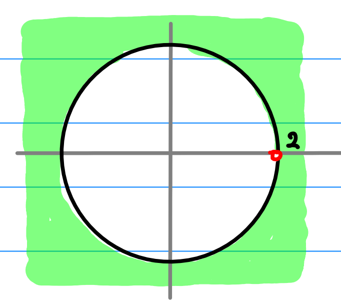
b^{n} $n=0,1,2,\cdots$

$n=-3$	$n=-2$	$n=-1$	$n=0$	$n=1$	$n=2$	$n=3$		
	b^{-3}	b^{-2}	b^{-1}	b^{0}	b^{1}	b^{2}	b^{3}	

b^{n-1} $n=0,-1,-2,\cdots$

b^{n-1} $n=1,2,3,\cdots$

Causal $(n \geq 0)$ $\left(\frac{1}{2}\right)^n$, $(2)^n$

	$f(z)$		$X(z)$
$\left(\frac{1}{2}\right)^n$	$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$		$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$
	$f(z) = \dfrac{1}{1-\frac{z}{2}} \quad \lvert z\rvert < 2$	$\overset{z^{-1}}{\longleftrightarrow}$	$X(z) = \dfrac{1}{1-\frac{1}{2z}} \quad \lvert z\rvert > 0.5$
	$= \dfrac{z^{-1}}{z^{-1}-0.5} = \dfrac{2}{2-z}$		$= \dfrac{z}{z-0.5}$
			
	$\updownarrow p^{-1}$		$\updownarrow p^{-1}$
$(2)^n$	$a_n = (2)^n \quad (n \geq 0)$		$a_n = (2)^n \quad (n \geq 0)$
	$f(z) = \dfrac{1}{1-2z} \quad \lvert z\rvert < 0.5$	$\overset{z^{-1}}{\longleftrightarrow}$	$X(z) = \dfrac{1}{1-\frac{2}{z}} \quad \lvert z\rvert > 2$
	$= \dfrac{z^{-1}}{z^{-1}-2} = \dfrac{0.5}{0.5-z}$		$= \dfrac{z}{z-2}$
			

Anti-causal $(n<0)$ $\left(\frac{1}{2}\right)^n$, $(2)^n$

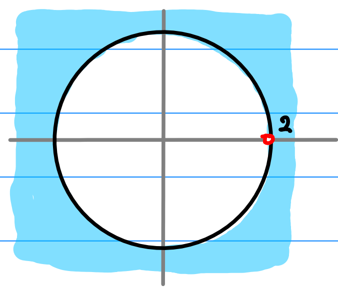
$f(z)$ | $X(z)$

$\left(\frac{1}{2}\right)^n$

$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$

$$f(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z|>2$$

$$= -\frac{2}{2-z}$$

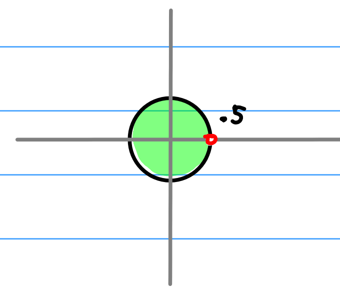


z^{-1}

$a_n = \left(\frac{1}{2}\right)^n \quad (n<0)$

$$X(z) = \frac{2z}{1-2z} \qquad |z| < 0.5$$

$$= -\frac{z}{z-0.5}$$



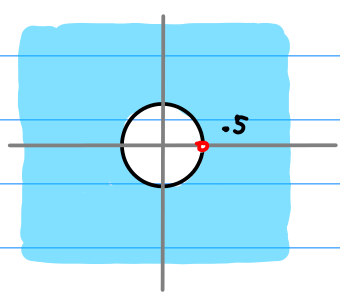
p^{-1} p^{-1}

$(2)^n$

$a_n = (2)^n \quad (n<0)$

$$f(z) = \frac{\frac{1}{2z}}{1-\frac{1}{2z}} \qquad |z| > 0.5$$

$$= -\frac{0.5}{0.5-z}$$

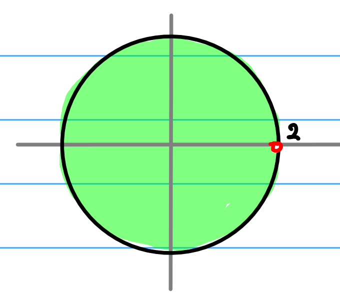


z^{-1}

$a_n = (2)^n \quad (n<0)$

$$X(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z| < 2$$

$$= -\frac{z}{z-2}$$



$f(z)$ Causal $(n \geq 0)$ Anti-causal $(n < 0)$

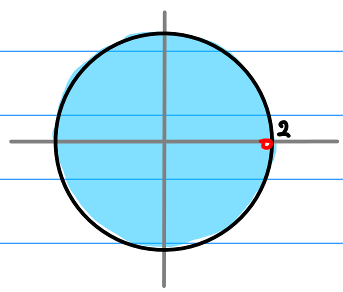
$\left(\frac{1}{2}\right)^n$

Causal

$$a_n = \left(\tfrac{1}{2}\right)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-\frac{z}{2}} \qquad |z| < 2$$

$$= \frac{z^{-1}}{z^{-1}-0.5} = \frac{2}{2-z}$$



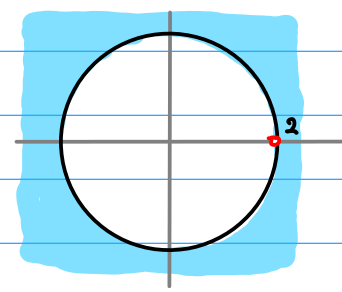
$\xleftrightarrow{-1}$

Anti-causal

$$a_n = \left(\tfrac{1}{2}\right)^n \quad (n < 0)$$

$$f(z) = \frac{\frac{2}{z}}{1-\frac{2}{z}} \qquad |z| > 2$$

$$= -\frac{2}{2-z}$$



$\updownarrow p^{-1}$ $\updownarrow p^{-1}$

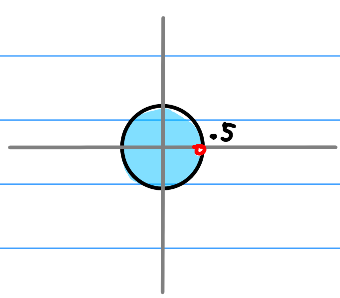
$(2)^n$

Causal

$$a_n = (2)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-2z} \qquad |z| < 0.5$$

$$= \frac{z^{-1}}{z^{-1}-2} = \frac{0.5}{0.5-z}$$



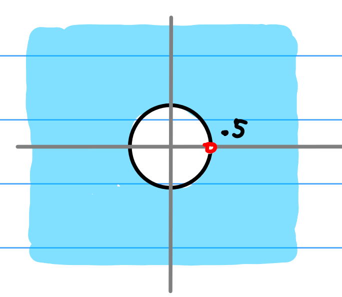
$\xleftrightarrow{-1}$

Anti-causal

$$a_n = (2)^n \quad (n < 0)$$

$$f(z) = \frac{\frac{1}{2z}}{1-\frac{1}{2z}} \qquad |z| > 0.5$$

$$= -\frac{0.5}{0.5-z}$$



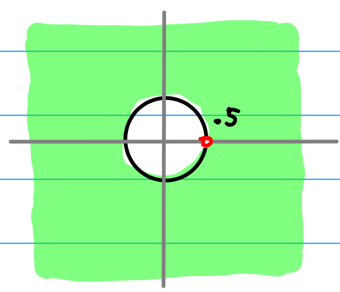
$X(z)$ | Causal $(n \geq 0)$ | Anti-causal $(n < 0)$

$\left(\frac{1}{2}\right)^n$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1 - \frac{1}{2z}} \quad |z| > 0.5$$

$$= \frac{z}{z - 0.5}$$

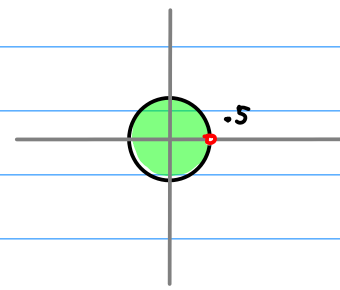


$\xleftrightarrow{-1}$

$$a_n = \left(\frac{1}{2}\right)^n \quad (n < 0)$$

$$X(z) = \frac{2z}{1 - 2z} \quad |z| < 0.5$$

$$= -\frac{z}{z - 0.5}$$



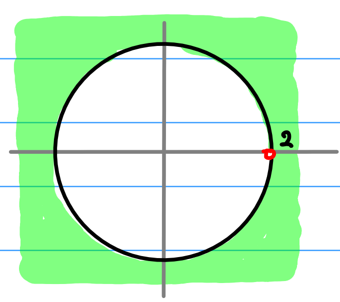
$\updownarrow p^{-1}$ $\qquad$ $\updownarrow p^{-1}$

$(2)^n$

$$a_n = (2)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1 - \frac{2}{z}} \quad |z| > 2$$

$$= \frac{z}{z - 2}$$

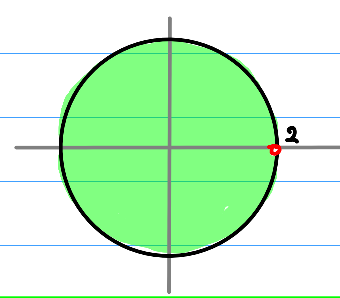


$\xleftrightarrow{-1}$

$$a_n = (2)^n \quad (n < 0)$$

$$X(z) = \frac{\frac{2}{z}}{1 - \frac{2}{z}} \quad |z| < 2$$

$$= -\frac{z}{z - 2}$$



Causal b^n | Anti-causal b^n

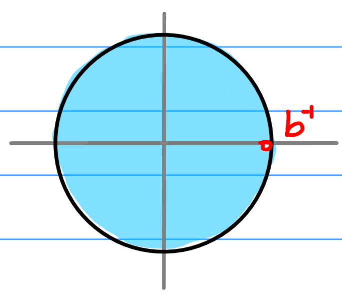
$f(z)$

Causal

$$a_n = (b)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - bz} \quad |z| < b^{-1}$$

$$= \frac{b^{-1}}{b^{-1} - z}$$

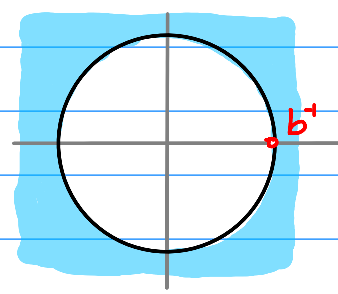


Anti-causal

$$a_n = (b)^n \quad (n < 0)$$

$$f(z) = \frac{b^{-1}z^{-1}}{1 - b^{-1}z^{-1}} \quad |z| > b^{-1}$$

$$= -\frac{b^{-1}}{b^{-1} - z}$$



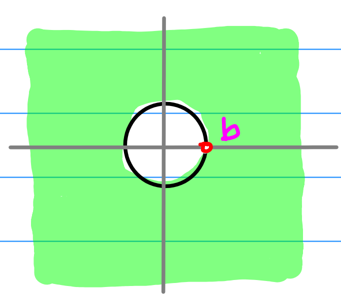
$X(z)$

Causal

$$a_n = (b)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1 - bz^{-1}} \quad |z| > b$$

$$= \frac{z}{z - b}$$

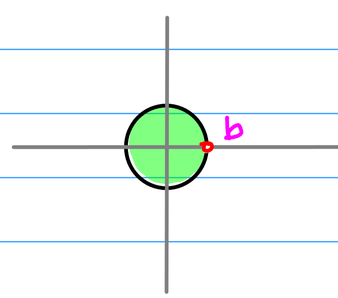


Anti-causal

$$a_n = (b)^n \quad (n < 0)$$

$$X(z) = \frac{b^{-1}z}{1 - b^{-1}z} \quad |z| < b$$

$$= -\frac{z}{z - b}$$



Causal $(n \geq 0)$ $\quad (1)^n, \ (1^{-1})^n$

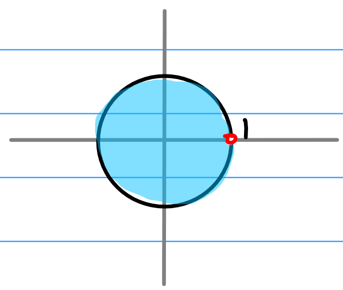
$f(z)$ $\qquad\qquad$ $X(z)$

$(1)^n$

$a_n = (1)^n \quad (n \geq 0)$

$$f(z) = \frac{1}{1-z} \qquad |z| < 1$$

$$= \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$$

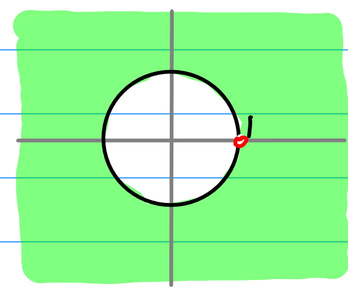


$\xleftrightarrow{z^{-1}}$

$a_n = (1)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1-\frac{1}{z}} \qquad |z| > 1$$

$$= \frac{z}{z-1}$$



$\updownarrow p^{-1}$ $\qquad\qquad$ $\updownarrow p^{-1}$

$(1^{-1})^n$

Anti-causal $(n<0)$ $\quad (1)^n, \ (1^{-1})^n$

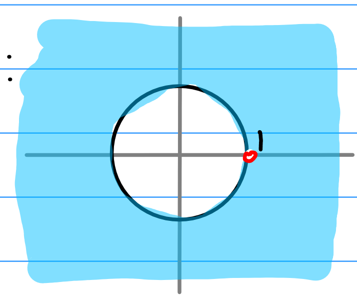
$f(z)$ $\qquad\qquad$ $X(z)$

$(1)^n$

$a_n = (1)^n \quad (n<0)$

$$f(z) = \frac{\frac{1}{z}}{1-\frac{1}{z}} \qquad |z|>1$$

$$= -\frac{1}{1-z}$$

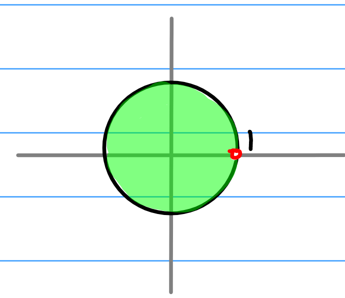


$\xleftrightarrow{z^{-1}}$

$a_n = (1)^n \quad (n<0)$

$$X(z) = \frac{z}{1-z} \qquad |z|<1$$

$$= -\frac{z}{z-1}$$



$\updownarrow p^{-1}$ $\qquad\qquad$ $\updownarrow p^{-1}$

$(1^{-1})^n$

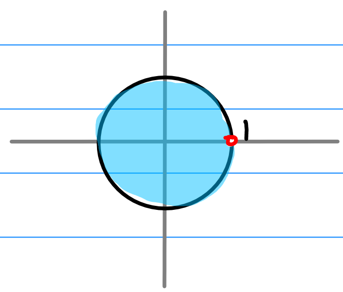
$f(z)$ Causal $(n \geq 0)$ Anti-causal $(n < 0)$

$(1)^n$

$$a_n = (1)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1-z} \quad |z| < 1$$

$$= \frac{z^{-1}}{z^{-1}-1} = \frac{1}{1-z}$$

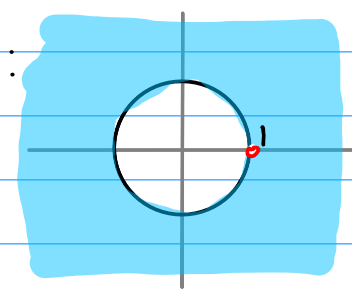


$\xleftrightarrow{-1}$

$$a_n = (1)^n \quad (n < 0)$$

$$f(z) = \frac{\frac{1}{z}}{1-\frac{1}{z}} \quad |z| > 1$$

$$= -\frac{1}{1-z}$$



$\updownarrow p^{-1}$ $\updownarrow p^{-1}$

$(1^{-1})^n$

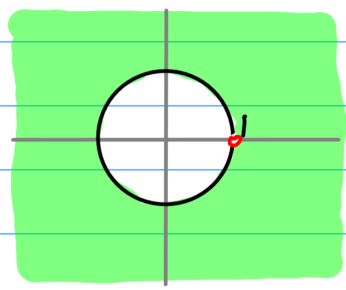
$X(z)$ Causal $(n \geq 0)$ Anti-causal $(n < 0)$

$(1)^n$

$a_n = (1)^n \quad (n \geq 0)$

$$X(z) = \frac{1}{1 - \frac{1}{z}} \qquad |z| > 1$$

$$= \frac{z}{z - 1}$$

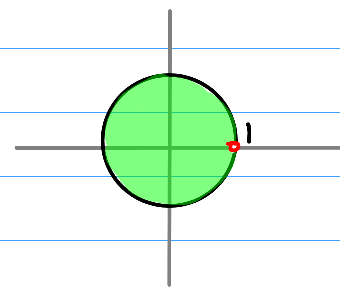


$\xleftrightarrow{-1}$

$a_n = (1)^n \quad (n < 0)$

$$X(z) = \frac{z}{1 - z} \qquad |z| < 1$$

$$= - \frac{z}{z - 1}$$



$\updownarrow p^{-1}$ $\updownarrow p^{-1}$

$(1^{-1})^n$

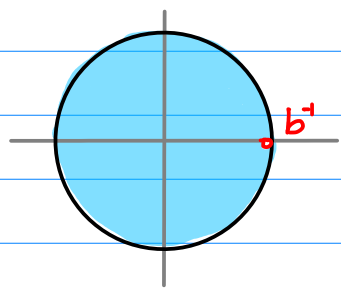
Causal b^n | Anti-causal b^n

$f(z)$

$$a_n = (b)^n \quad (n \geq 0)$$

$$f(z) = \frac{1}{1 - bz} \qquad |z| < b^{-1}$$

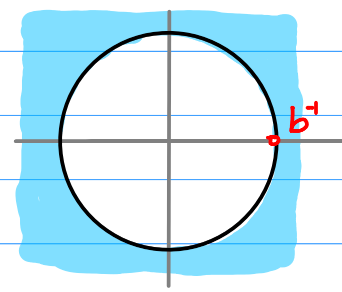
$$= \frac{b^{-1}}{b^{-1} - z}$$



$$a_n = (b)^n \quad (n < 0)$$

$$f(z) = \frac{b^{-1}z^{-1}}{1 - b^{-1}z^{-1}} \qquad |z| > b^{-1}$$

$$= -\frac{b^{-1}}{b^{-1} - z}$$

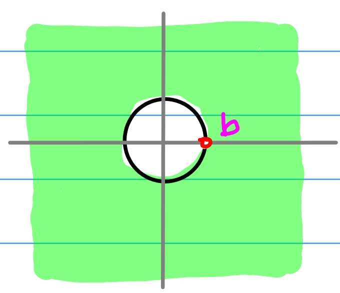


$X(z)$

$$a_n = (b)^n \quad (n \geq 0)$$

$$X(z) = \frac{1}{1 - bz^{-1}} \qquad |z| > b$$

$$= \frac{z}{z - b}$$



$$a_n = (b)^n \quad (n < 0)$$

$$X(z) = \frac{b^{-1}z}{1 - b^{-1}z} \qquad |z| < b$$

$$= -\frac{z}{z - b}$$