

# Integration (2A)

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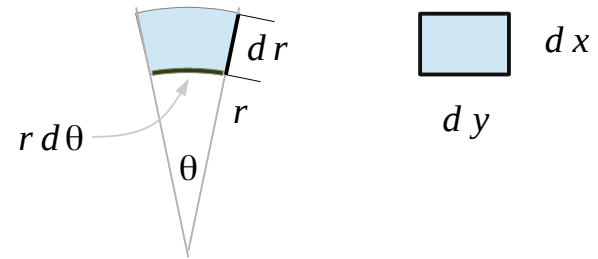
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# Integrating Gaussian Function

$$\begin{aligned} \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 &= \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) \\ &= \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{+\infty} e^{-y^2} dy \right) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} dx dy \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r dr d\theta \\ &= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{+\infty} e^{-r^2} r dr \right) \\ &= 2\pi \left( \int_0^{+\infty} e^{-r^2} r dr \right) \end{aligned}$$



$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

# Integrating Guassian Function

$$\begin{aligned}\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r dr d\theta \\ &= 2\pi \left(\int_0^{+\infty} e^{-r^2} r dr\right) \quad s = -r^2 \quad ds = -2r dr \\ &= \pi \left(\int_{-\infty}^0 e^s ds\right) \\ &= \pi [e^s]_{-\infty}^0\end{aligned}$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \pi [e^0 - e^{-\infty}] = \pi$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right) = \sqrt{\pi}$$

# Integrating Guassian Function

A standard way to compute the Gaussian integral, the idea of which goes back to Poisson,<sup>[2]</sup> is

- consider the function  $e^{-(x^2 + y^2)} = e^{-r^2}$  on the plane  $\mathbf{R}^2$ , and compute its integral two ways:

1. on the one hand, by **double integration** in the **Cartesian coordinate system**, its integral is a square:

$$\left( \int e^{-x^2} dx \right)^2 ;$$

2. on the other hand, by **shell integration** (a case of double integration in **polar coordinates**), its integral is computed to be  $\pi$ .

Comparing these two computations yields the integral, though one should take care about the improper integrals involved.

<http://en.wikipedia.org/wiki/Derivative>

$$z = f(x, y)$$

# Integrating Guassian Function

On the other hand,

$$\begin{aligned}\iint_{\mathbf{R}^2} e^{-(x^2+y^2)} d(x, y) &= \int_0^{2\pi} \int_0^\infty e^{-r^2} r dr d\theta \\ &= 2\pi \int_0^\infty r e^{-r^2} dr \\ &= 2\pi \int_{-\infty}^0 \frac{1}{2} e^s ds && s = -r^2 \\ &= \pi \int_{-\infty}^0 e^s ds \\ &= \pi(e^0 - e^{-\infty}) \\ &= \pi,\end{aligned}$$

where the factor of  $r$  comes from [the transform to polar coordinates](#) ( $r dr d\theta$  is the standard measure on the plane, expressed in polar coordinates [1]), and the substitution involves taking  $s = -r^2$ , so  $ds = -2r dr$ .

Combining these yields

$$\left( \int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \pi,$$

so

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

<http://en.wikipedia.org/wiki/Derivative>

# Integrating Guassian Function

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$y = xs$$

$$dy = x ds$$

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = 2 \int_0^{+\infty} e^{-x^2} dx$$

$$\begin{aligned} I^2 &= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dy dx \\ &= 4 \int_0^{\infty} \left( \int_0^{\infty} e^{-(x^2+y^2)} dy \right) dx \\ &= 4 \int_0^{\infty} \left( \int_0^{\infty} e^{-x^2(1+s^2)} x ds \right) dx && : f(x) \\ &= 4 \int_0^{\infty} \left( \int_0^{\infty} e^{-x^2(1+s^2)} x dx \right) ds \\ &= 4 \int_0^{\infty} \left[ \frac{1}{-2(1+s^2)} e^{-x^2(1+s^2)} \right]_{x=0}^{x=\infty} ds && : f(x) \\ &= 4 \left( \frac{1}{2} \int_0^{\infty} \frac{ds}{1+s^2} \right) \\ &= 2 \left[ \arctan s \right]_0^{\infty} \\ &= \pi \end{aligned}$$

<http://en.wikipedia.org/wiki/Derivative>

$$z = f(x, y)$$

## References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
- [3] E. Kreyszig, “Advanced Engineering Mathematics”
- [4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
- [5] [www.chem.arizona.edu/~salzmanr/480a](http://www.chem.arizona.edu/~salzmanr/480a)