Integration (2A)

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Integrating Guassian Function

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)$$

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

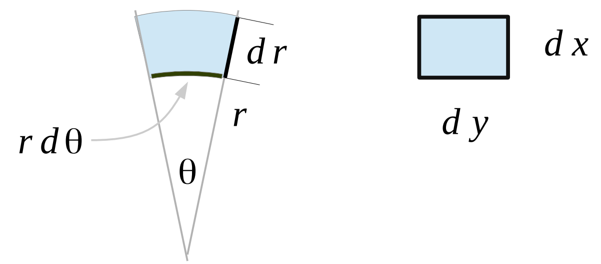
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \left(\int_{0}^{2\pi} d\theta\right)\left(\int_{0}^{+\infty} e^{-r^2}\,r\,dr\right)$$

$$= 2\pi\left(\int_{0}^{+\infty} e^{-r^2}\,r\,dr\right)$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

Integrating Guassian Function

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-r^2}\, r\, dr\, d\theta$$

$$= 2\pi \left(\int_0^{+\infty} e^{-r^2}\, r\, dr\right) \qquad s = -r^2 \qquad ds = -2r\,dr$$

$$= \pi \left(\int_{-\infty}^{0} e^{s}\, ds\right)$$

$$= \pi \left[e^{s}\right]_{-\infty}^{0}$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right)^2 = \pi \left[e^{0} - e^{-\infty}\right] = \pi$$

$$\left(\int_{-\infty}^{+\infty} e^{-x^2} dx\right) = \sqrt{\pi}$$

Integrating Guassian Function

A standard way to compute the Gaussian integral, the idea of which goes back to Poisson,[2] is

- consider the function $e^{-(x^2+y^2)} = e^{-r^2}$ on the plane $\mathbf{R}^2$, and compute its integral two ways:
  1. on the one hand, by double integration in the Cartesian coordinate system, its integral is a square:
     $$\left(\int e^{-x^2}\,dx\right)^2;$$
  2. on the other hand, by shell integration (a case of double integration in polar coordinates), its integral is computed to be π.

Comparing these two computations yields the integral, though one should take care about the improper integrals involved.

http://en.wikipedia.org/wiki/Derivative

$$z = f(x, y)$$

Integrating Guassian Function

On the other hand,

$$\iint_{\mathbf{R}^2} e^{-(x^2+y^2)}\,d(x,y) = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta$$
$$= 2\pi \int_0^{\infty} re^{-r^2}\,dr$$
$$= 2\pi \int_{-\infty}^{0} \tfrac{1}{2}e^{s}\,ds \qquad s = -r^2$$
$$= \pi \int_{-\infty}^{0} e^{s}\,ds$$
$$= \pi(e^0 - e^{-\infty})$$
$$= \pi,$$

where the factor of r comes from the transform to polar coordinates ($r\,dr\,d\theta$ is the standard measure on the plane, expressed in polar coordinates [1]), and the substitution involves taking $s = -r^2$, so $ds = -2r\,dr$.

Combining these yields

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi,$$

so

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

http://en.wikipedia.org/wiki/Derivative

Integrating Guassian Function

$$\int_{-\infty}^{+\infty} e^{x^2}\, dx \;=\; \sqrt{\pi}$$

$$y = x\,s$$

$$dy = x\,ds$$

$$\int_{-\infty}^{+\infty} e^{x^2}\, dx \;=\; 2\int_{0}^{+\infty} e^{x^2}\, dx$$

$$\begin{aligned}
I^2 &= 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dy\, dx \\
&= 4\int_0^\infty \left(\int_0^\infty e^{-(x^2+y^2)}\, dy\right)\, dx \\
&= 4\int_0^\infty \left(\int_0^\infty e^{-x^2(1+s^2)} x\, ds\right)\, dx \\
&= 4\int_0^\infty \left(\int_0^\infty e^{-x^2(1+s^2)} x\, dx\right)\, ds \\
&= 4\int_0^\infty \left[\frac{1}{-2(1+s^2)} e^{-x^2(1+s^2)}\right]_{x=0}^{x=\infty}\, ds \\
&= 4\left(\frac{1}{2}\int_0^\infty \frac{ds}{1+s^2}\right) \\
&= 2\Big[\arctan s\Big]_0^\infty \\
&= \pi
\end{aligned}$$

$f(x)$

$f(x)$

http://en.wikipedia.org/wiki/Derivative

$z = f(x, y)$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a