Eistein Notation (H.1)

20160623

In mathematics, especially in applications of linear algebra to physics, the **Einstein notation** or **Einstein summation convention** is a notational convention that implies summation over a set of indexed terms in a formula, thus achieving notational brevity. As part of mathematics it is a notational subset of Ricci calculus; however, it is often used in applications in physics that do not distinguish between tangent and cotangent spaces. It was introduced to physics by Albert Einstein in 1916.[1]

Upper Indices

Statement of convention [edit]

According to this convention, when an index variable appears twice in a single term and is not otherwise defined (see free and bound variables), it implies summation of that term over all the values of the index. So where the indices can range over the set $\{1, 2, 3\}$,

$$y = \sum_{i=1}^{3} c_i x^i = c_1 x^1 + c_2 x^2 + c_3 x^3$$

is simplified by the convention to:

$$\boxed{y = c_i x^i.}$$

The upper indices are not exponents but are indices of coordinates, coefficients or basis vectors. That is, in this context x^2 should be understood as the second component of $\mathbf{x}$ rather than the square of $\mathbf{x}$ (this can occasionally lead to ambiguity). Typically (x^1, x^2, x^3) would be equivalent to the traditional (x, y, z).

Tensor Product & Duality

The value of the Einstein convention is that it applies to other vector spaces built from V using the tensor product and duality. For example, $V \otimes V$, the tensor product of V with itself, has a basis consisting of tensors of the form $\boxed{\mathbf{e}_{ij} = \mathbf{e}_i \otimes \mathbf{e}_j}$. Any tensor $\mathbf{T}$ in $V \otimes V$ can be written as:

$$\boxed{\mathbf{T} = T^{ij}\mathbf{e}_{ij}.}$$

V^*, the dual of V, has a basis $\mathbf{e}^1, \mathbf{e}^2, \ldots, \mathbf{e}^n$ which obeys the rule

$$\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j.$$

where δ is the Kronecker delta. As

$$\mathrm{Hom}(V, W) = V^* \otimes W$$

the row-column coordinates on a matrix correspond to the upper-lower indices on the tensor product.

① $\boxed{V \otimes V}$ the tensor product of V with V

$e_i \otimes e_j$ a basis of $V \otimes V$

any tensor $T \in V \otimes V$

$$T = T^{ij} e_{ij} = T^{ij}(e_i \otimes e_j)$$

② $\boxed{V^*}$ the dual of V

e^i a basis of V^*

$$e^i e_j = \delta^i_j$$

$\mathrm{Hom}(V, W)$

the collection of all morphisms from V to W

V^* the dual space of V

$V^* \otimes W$

Examples

In Einstein notation, the usual element reference A_{mn} for the mth row and nth column of matrix $\mathbf{A}$ becomes $A^m{}_n$. We can then write the following operations in Einstein notation as follows.

1 **Inner product (hence also vector dot product)**

Using an orthogonal basis, the inner product is the sum of corresponding components multiplied together:

$$\mathbf{u} \cdot \mathbf{v} = u^j v_j$$

This can also be calculated by multiplying the covector on the vector.

2 **Vector cross product**

Again using an orthogonal basis (in 3d) the cross product intrinsically involves summations over permutations of components:

$$\mathbf{u} \times \mathbf{v} = \epsilon^i{}_{jk} u^j v^k \mathbf{e}_i$$

where

$$\epsilon^i{}_{jk} = \delta^{il} \epsilon_{ljk}$$

and ϵ_{ijk} is the Levi-Civita symbol. Based on this definition of ϵ, there is no difference between $\epsilon^i{}_{jk}$ and ϵ_{ijk} but the position of indices.

Examples

3 Matrix multiplication

The matrix product of two matrices A_{ij} and B_{jk} is:

$$\mathbf{C}_{ik} = (\mathbf{AB})_{ik} = \sum_{j=1}^{N} A_{ij} B_{jk}$$

equivalent to

$$C^i{}_k = A^i{}_j B^j{}_k$$

4 Trace

For a square matrix $A^i{}_j$, the trace is the sum of the diagonal elements, hence the sum over a common index $A^i{}_i$.

5 Outer product

The outer product of the column vector u^i by the row vector v_j yields an $m \times n$ matrix $\mathbf{A}$:

$$A^i{}_j = u^i v_j = (uv)^i{}_j$$

Since i and j represent two *different* indices, there is no summation and the indices are not eliminated by the multiplication.

Examples

6 Raising and lowering indices

Given a tensor, one can raise an index or lower an index by contracting the tensor with the metric tensor, $g_{\mu\nu}$. For example, take the tensor $T^{\alpha}{}_{\beta}$, one can raise an index:

$$T^{\mu\alpha} = g^{\mu\sigma} T_{\sigma}{}^{\alpha}$$

Or one can lower an index:

$$T_{\mu\beta} = g_{\mu\sigma} T^{\sigma}{}_{\beta}$$

Dual Space

$\boxed{V}$ any vector space

$\boxed{V^*}$ dual vector space

Consiting of all linear functionals on V
With the induced linear structure

- algebraic dual space
  defined for all vector spaces
- continuous dual space
  defined for <u>topological</u> vector space

Algebraic Dual Space

$$\varphi : V \longrightarrow F$$

Vector Space Scalar field

the dual space V^* : a vector space over F

$$(\varphi + \psi)(x) = \varphi(x) + \psi(x)$$
$$(a\varphi)(x) = a\,(\varphi(x))$$

$\varphi \in V^*$ $x \in V$

$\psi \in V^*$ $a \in F$

the pairing of a functional φ in the dual space V^* and an element x of V

$$\varphi(x) = [\varphi, x]$$
$$\varphi(x) = \langle \varphi, x \rangle$$

The pairing defines a non-degenerate bilinear mapping

$$V^* \times V \longrightarrow F$$

Linear Functional

linear functional linear form one form covector

$$V \longrightarrow k$$

$f \in \mathrm{Hom}_k(V, k)$

$g \in \mathrm{Hom}_k(V, k)$

$f+g \in \mathrm{Hom}_k(V, k)$

$cf \in \mathrm{Hom}_k(V, k)$

$\vec{v} \in V$ $\quad \vec{v}$ col vector $\quad$ Vectors

$\varphi \in V^*$ $\quad \varphi$ row vector $\quad$ linear functionals

a linear functional f

a linear function from V to k

$f(\vec{v}+\vec{w}) = f(\vec{v}) + f(\vec{w}) \qquad \vec{v} \in V, \ \vec{w} \in V$

$f(a\vec{v}) = a f(\vec{v}) \qquad a \in k$

the set of all linear functionals from V to k

$\triangleq \mathrm{Hom}_k(V, k)$: Vector space

$$\boxed{\begin{array}{l} f \in \mathrm{Hom}_k(V, k) \\ g \in \mathrm{Hom}_k(V, k) \end{array}} \qquad \boxed{\begin{array}{l} f+g \in \mathrm{Hom}_k(V, k) \\ cf \in \mathrm{Hom}_k(V, k) \end{array}}$$

Dual space : Linear Vector space

Linear Function

Linear Map Linear Transformation Linear Function	$V \to W$
Linear Operator Endomorphism	$V \to V$
linear functional linear form one form covector	$V \to k$

module

V

linear map
linear function
linear transform

$\longrightarrow$

module

W

matrix rep

addition
scalar multiplication

module
vector space

V

linear operator

$\longrightarrow$

module
vector space

$W = V$

matrix rep

addition
scalar multiplication

vector space
over a field F

V

linear form
linear functional

$\longrightarrow$

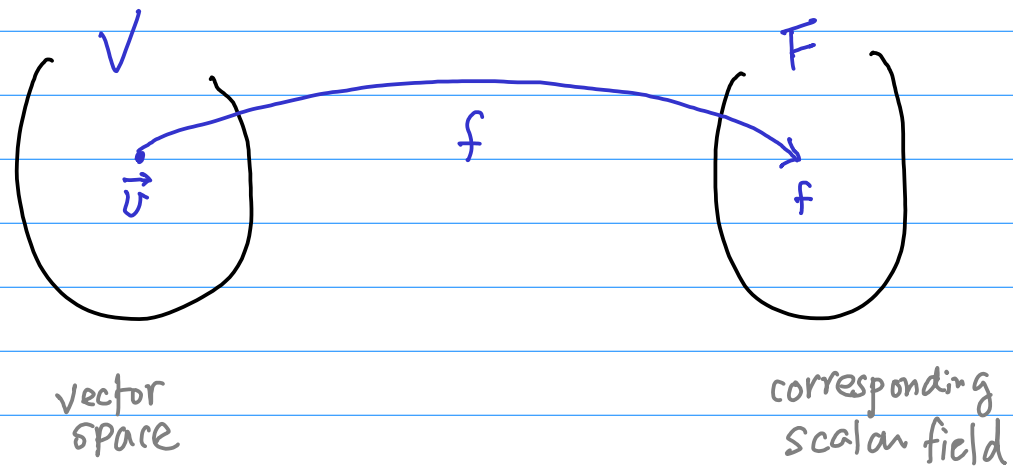
field

F

matrix rep

addition
scalar multiplication

functional



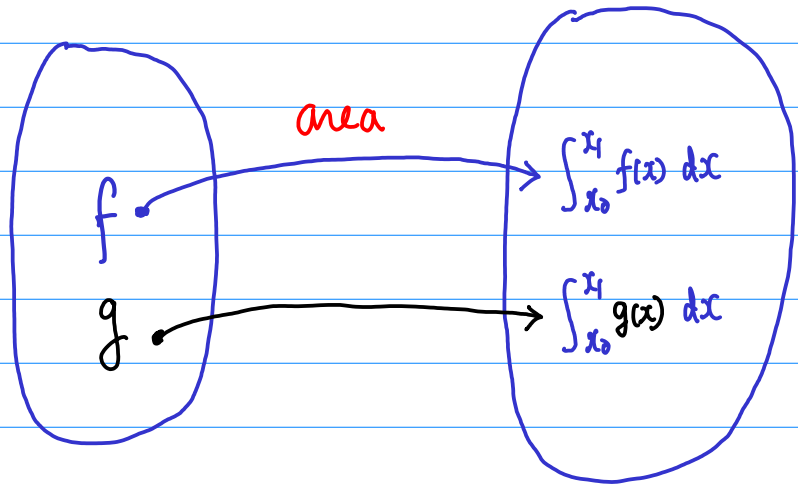
Vetor space
= a space of functions

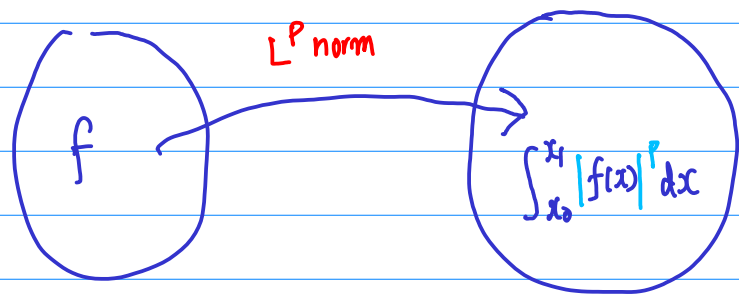
function space → Vector space
commonly

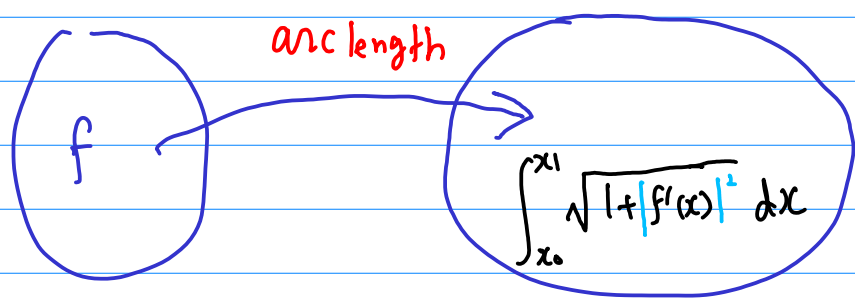
functional = function of functions
higher order function

a set of real function → real number set

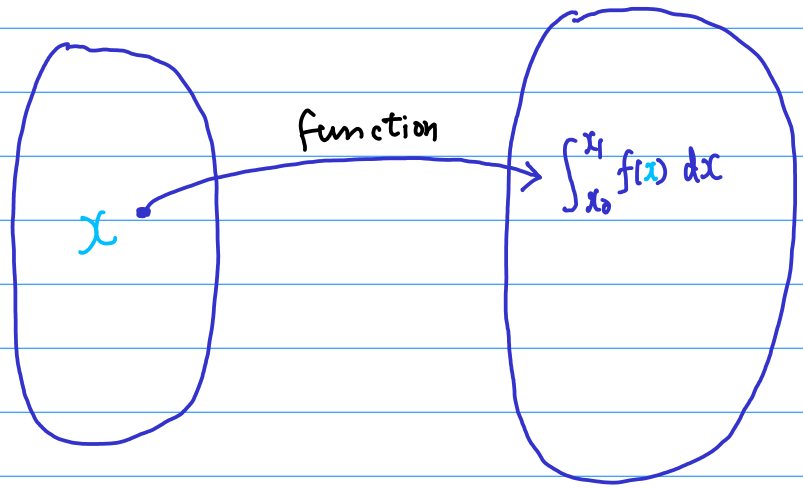
functional : function of functions



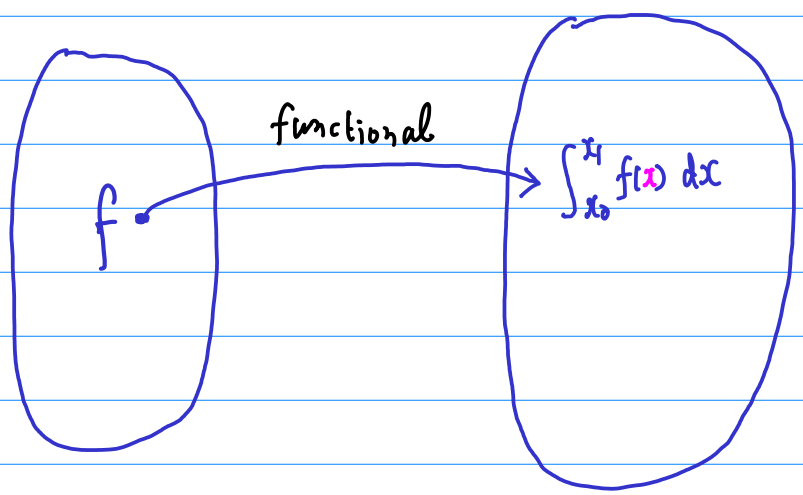




Duality



x: argument



x: parameter