The Nyquist Channel

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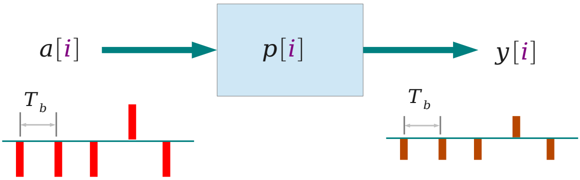
The ISI Problem

- the overall pulse spectrum $P(f)$
- the optimum solution for the pulse shaping
  - zero intersymbol interference
  - minimum transmission bandwidth possible

$$y_i = a_i p_0 + \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p_{i-k} \qquad i = 0, \pm 1, \pm 2, \cdots$$

Discrete LTI System

$$y[i] = \sum_k a[k]\, p[i-k]$$

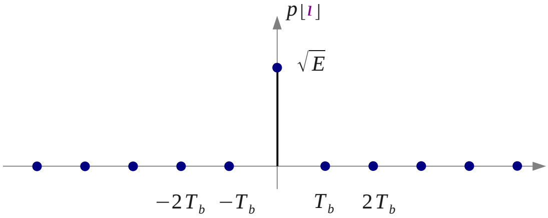


Zero ISI Condition

- $y_i = a_i p_0 + \boxed{\sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p_{i-k}} \qquad i = 0, \pm 1, \pm 2, \cdots$

- $y_i = a_i p_0$ for all $i \qquad \Longleftrightarrow$

  - $p_0 = \sqrt{E} \qquad$ for $i = 0$,
  - $p_i = 0 \qquad$ for all $i \neq 0$



To Find the Optimum Pulse Shape $p_{opt}(t)$

- $y_i = \sqrt{E} a_i$ for all i $\Longleftrightarrow$

  - $p_0 = \sqrt{E}$ for $i = 0$,
  - $p_i = 0$ for all $i \neq 0$

- $p_i = p(i T_b)$ $\Longrightarrow$ $p_{opt}(t) = ?$
  - the sampling rate is equal to the bit rate $R = 1/T_b$
  - the bandlimited pulse $p(t)$ for the interval $-B_0 < f < +B_0$
- interpolate $p_i = p(i T_b)$ keeping the bandwidth B_0 as small as possible
- But the Nyquist sampling theorem gives
  - the minimum sampling rate in terms of a signal's bandwidth

Interpolation

Interpolation Formula: strictly bandlimited signal $g(t)$, bandwidth W

$$g(t) = \sum_{n=-\infty}^{\infty} g\left(\frac{n}{2W}\right) sinc(2Wt - n)$$

reconstructing the original signal $g(t)$
from the sequence of sample values $\{g(n/2W)\}$
the sinc function $sinc(2Wt)$: an interpolation function

- $p_i = p(iT_b)$: sampling $p(t)$ at a uniform bit rate $R = 1/T_b$
- the pulse shape $p(t)$ in terms of its sample values

$$p(t) = \sum_{i=-\infty}^{\infty} p\left(\frac{i}{2B_0}\right) sinc(2B_0 t - i)$$

The Optimum Pulse Shape $p_{opt}(t)$ - Zero ISI

- The interpolated pulse shape

$$p(t) = \sum_{i=-\infty}^{\infty} p\left(\frac{i}{2B_0}\right) sinc(2B_0 t - i)$$

- Substitute the following equations for zero ISI
  - $p_0 = \sqrt{E}$ for $i = 0$,
  - $p_i = 0$ for all $i \neq 0$

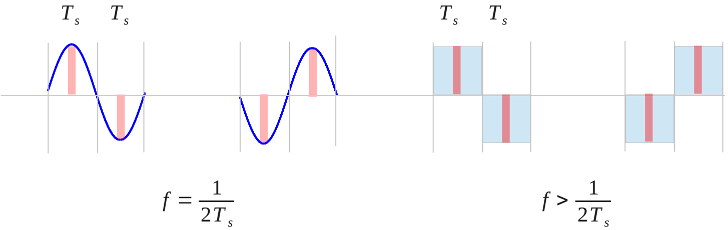
$$p(t) = p\left(\frac{0}{2B_0}\right) sinc(2B_0 t - 0)$$

$$p_{opt}(t) = \sqrt{E} sinc(2B_0 t) = \frac{\sqrt{E} sin(2\pi B_0 t)}{2\pi B_0 T}$$

The Optimum Pulse Shape $p_{opt}(t)$ - Minimum Bandwidth

- $p_{opt}(t) = \sqrt{E} sinc(2B_0 t) = \frac{\sqrt{E} sin(2\pi B_0 t)}{2\pi B_0 T}$

- when transmitting symbols via the bandlimited baseband channel
  - the maximum frequency that is allowed by the channel : B_0
  - the upper bound for the bit rate $R = 1/T_b$is obtained by
  - $\frac{1}{2}\frac{1}{T_b} \leq B_o$: the bandwidth, the half of the bit rate $1/T_b$
  - the lower bound for the required bandwidth
  - bandwidth $\geq$ *Nyquitst Bandwidth* $B_0 = \frac{1}{2}R$



The Optimum Pulse Spectrum

the optimum pulse shape function and spectrum

$$p_{opt}(t) = \sqrt{E}sinc(2B_0 t) = \frac{\sqrt{E}sin(2\pi B_0 t)}{2\pi B_0 T}$$

$P(f) = \frac{\sqrt{E}}{2B_0}$ for $-B_0 < f < +B_0$,

$P(f) = 0$ otherwise

- B_0 the minimum transmission bandwidth
  - a brick-wall function (a rectangular function) $\sqrt{E}rect(f/2B_0)$
  - no frequencies fo absolute value exceeding half the bit rate
  - bandwdith $\Longrightarrow B_0 = \frac{1}{2}R = \frac{1}{2T_b}$
  - if $rect(t/T_b)$ pulse is used, its spectrum becomes $T_b sinc(fT_b)$
  - the first zero crossing of $T_b sinc(fT_b)$: $1/T_b = R$
  - bandwdith $\Longrightarrow B = R = \frac{1}{T_b}$
- zero intersymbol interference

The Nyquist Channel

- the optimum pulse spectrum :$P_{opt}(f) = \frac{\sqrt{E}}{2B_0} rect(f/2B_0)$
- the Nyquist channel : the PAM system with $P_{opt}(f)$
  - the goal of reducing the requried system bandwidth
  - the channel with the minimum bandwidth
  - bandwidth $\geq$ *Nyquitst Bandwidth* $B_0 = \frac{1}{2T_b}$
- the optimum pluse shape : $p_{opt}(t) = \sqrt{E} sinc(2B_0 t) = \frac{\sqrt{E} sin(2\pi B_0 t)}{2\pi B_0 T}$
- the impulse response $p_{opt}(t)$ of the ideal low pass filter $P_{opt}(f)$
  - zero crossigns at $k/2B_0 = kT_b$
  - shifted pulses of $p_{opt}(t - kT_b)$ has no effect at zero crossings
  - zero ISI

The Problems in The Nyquist Channel

the optimum pulse spectrum : $P_{opt}(f) = \frac{\sqrt{E}}{2B_0} rect(f/2B_0)$

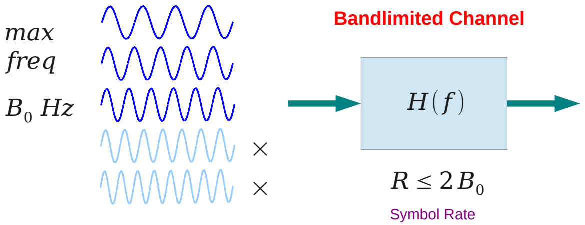
- $P(f)$ is flat from $-B_0$and $+B_0$and zero elsewhere
  physically unrealizable (the abrupt transitions at $\pm B_0$)

the optimum pluse shape : $p_{opt}(t) = \sqrt{E} sinc(2B_0 t) = \frac{\sqrt{E} sin(2\pi B_0 t)}{2\pi B_0 t}$

- $p(t)$ decreases as $1/|t|$ for large $|t|$
  relatively decays slowly (related to the abrupt transition at $\pm B_0$)
  no margin error in sampling times in the receiver

Nyquist Rate - Definition 1

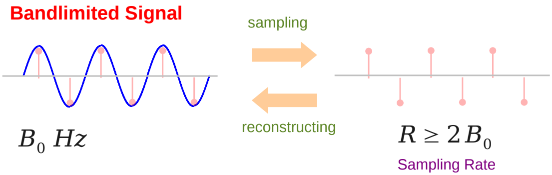
- upper bound for the symbol rate in the bandlimited channel
- given a bandwidth B_0



- bandwidth $\geq$ *Nyquitst Bandwidth* $B_0 = \frac{1}{2}R = \frac{1}{2T_b}$
- given a symbol rate R

Nyquist Rate - Definition 2

- lower bound for the sampling rate in the bandlimited signal
- given a bandwidth B_0



- sampling frequency $\geq$ *Nyquist Rate* $= 2B_0$
- sampling period $\leq$ *Nyquist Interval* $= 1/2B_0$

Reference

[1] S. Haykin, M Moher, “Introduction to Analog and Digital Communications”, 2ed