The Conditional Distribution and Density Functions

Young W Lim

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

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Conditional Probability

Definition

two events A and B where $P(B) \neq 0$
the conditional probability of A given B had occurred

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Conditional Distribution

Definition

Let A denote the event $\{X \leq x\}$ in $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$
the conditional distribution function of X is defined as

$$F_X(x \mid B) = P\{X \leq x \mid B\} = \frac{P\{X \leq x \cap B\}}{P(B)}$$

where $\{X \leq x \cap B\}$ means the joint event $\{X \leq x\} \cap B$
consisting all outcomes s such that $X(s) \leq x$ and $x \in B$

Properties of Conditional Distribution

$$F_X(x = -\infty \mid B) = 0$$

$$F_X(x = +\infty \mid B) = 1$$

$$0 \le F_X(x \mid B) \le 1 \quad (x_1 < x_2) \Longrightarrow F_X(x_1 \mid B) \le F_X(x_2 \mid B)$$

$$F_X(x_2 \mid B) - F_X(x_1 \mid B) = P\{x_1 < X \le x_2 \mid B\}$$

$$F_X(x^+ \mid B) = F_X(x \mid B)$$

Conditional Density

Definition

$$f_X(x \mid B) = \frac{dF_x(x \mid B)}{dx}$$

the density function of the random variable X
the derivative of the distribution function $F_x(x \mid B)$

Properties of Conditional Density

$$0 \leq f_X(x \mid B) forallx$$

$$\int_{-\infty}^{+\infty} f_X(x \mid B) dx = 1$$

$$F_X(x \mid B) = \int_{-\infty}^{x} f_X(\xi \mid B) d\xi$$

$$\int_{x_1}^{x_2} f_X(x \mid B) dx = P\{x_1 < X \leq x_2 \mid B\}$$

Outline

1. Conditonal Distribution and Density Functions
2. Defining Conditioning Events

Methods of defining conditioning events 1

- event B is defined in terms of the random variable X
  - let $B = \{X \leq b\}$, where b is a real number $-\infty < b < \infty$, then

$$F_X(x \mid X \leq b) = P\{X \leq x \mid X \leq b\} = \frac{P\{X \leq x \cap X \leq b\}}{P\{X \leq b\}}$$

  - for all event $\{X \leq b\}$ for which $P\{X \leq b\} \neq 0$
  - two cases
    - $b \leq x$
    - $x < b$
- event B may depend on some random variable other than X

Methods of defining conditioning events 2

- case 1 : $b \leq x$
  - then the event $\{X \leq b\}$is a subset of $\{X \leq x\}$
  - $\{X \leq x\} \cap \{X \leq b\} = \{X \leq b\}$

$$F_X(x \mid X \leq b) = P\{X \leq x \mid X \leq b\} = \frac{P\{X \leq b\}}{P\{X \leq b\}} = 1$$

- case 2 : $x < b$
  - then the event $\{X \leq x\}$ is a subset of $\{X \leq b\}$
  - $\{X \leq x\} \cap \{X \leq b\} = \{X \leq x\}$

$$F_X(x \mid X \leq b) = P\{X \leq x \mid X \leq b\} = \frac{P\{X \leq x\}}{P\{X \leq b\}} = \frac{F_X(x)}{F_X(b)}$$

Methods of defining conditioning events 3

$$F_X(x \mid X \le b) = \begin{cases} \dfrac{F_X(x)}{F_X(b)} & x < b \\ 1 & b \le x \end{cases}$$

$$f_X(x \mid X \le b) = \begin{cases} \dfrac{f_X(x)}{F_X(b)} = \dfrac{f_X(x)}{\int_{-\infty}^{b} f_X(x)dx} & x < b \\ 0 & b \le x \end{cases}$$