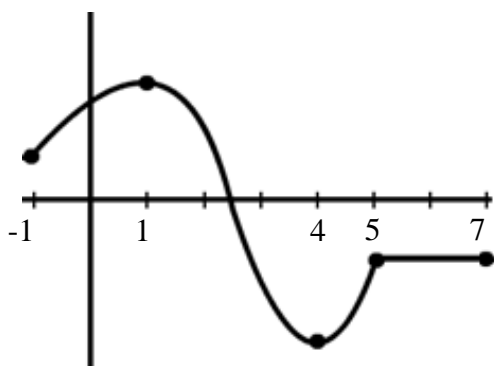## 4.5 Functions; Increasing and Decreasing

Prior to Calculus we are often introduced to the ideas of what it means for a function to be increasing or decreasing. However, these prior notions were often loosely defined and relied on a very intuitive approach to understanding these pieces of terminology.

General Idea:
Identify the intervals on which the function pictured below is increasing, decreasing, and constant.

This function is increasing on the intervals $(-1,1)$ & $(4,5)$.

This function is decreasing on the interval $(1,4)$.

This function is constant on the interval $(5,7)$.

Note: It would also be correct to say the function is increasing on $[-1,1] \cup [4,5]$, decreasing on $[1,4]$, and constant on $[5,7]$. We will have a class agreement that we will always use open intervals when describing these properties.

We notice that at points where the function is decreasing the slopes of the tangent lines to those points are negative. Similarly, at points where the function is increasing the slopes of the tangent lines to those points are positive. We can also see this in the diagram below, and this observation is summarized by the following theorem.

<u>Decreasing = negative slopes</u>
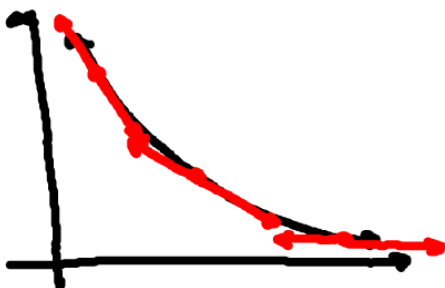
<u>Increasing = positive slopes</u>
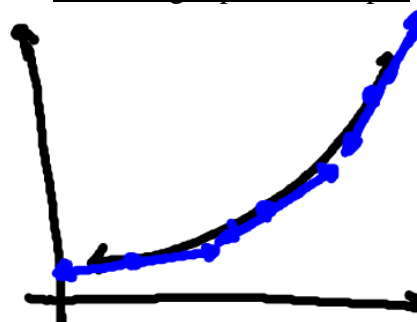
Theorem: Let f be a function that is continuous $[a,b]$ and differentiable on (a,b).

   (a) If $f'(x) > 0$ for every value of x in (a,b), then f is increasing on (a,b).

   (b) If $f'(x) < 0$ for every value of x in (a,b), then f is decreasing on (a,b).

Note: A proof of this theorem requires the Mean Value Theorem. A proof is provided in the lecture notes.

*Proof of (i): (A proof of ii is similar to the following)* (NOT DONE IN CLASS)

Suppose we know that f is continuous on $[a,b]$, is differentiable on (a,b), and $f'(x)>0$ for every x in (a,b). Let's assume that f is not an increasing function on (a,b). Then, there exists x-values j and k in (a,b) such that $j<k$, but $f(j)\geq f(k)$. Since f satisfies the conditions of the MVT we know that there must exist a c in (j,k) such that $f'(c)=\dfrac{f(k)-f(j)}{k-j}$. However, since $j<k$ & $f(j)\geq f(k)$ this means that $k-j>0$ & $f(k)-f(j)\leq0$ which implies that $f'(c)=\dfrac{f(k)-f(j)}{k-j}\leq0$. This is a contradiction because $f'(x)>0$ for every x in (a,b). Therefore, f must be increasing on (a,b).

$\blacksquare$

Example 1: Determine on what intervals $f(x)=x^3+3x^2$ is increasing and decreasing.

We calculate that $f'(x)=3x^2+6x$. To make it easier to see when $f'(x)>0$ & $f'(x)<0$ we will find when $f'(x)=0$.

Clearly $f'(x)=0 \iff 3x^2+6x=0 \iff 3x(x+2)=0 \iff x=0 \text{ or } x=-2$.
We set up the following number line and using test values we find when the derivative is positive and negative.
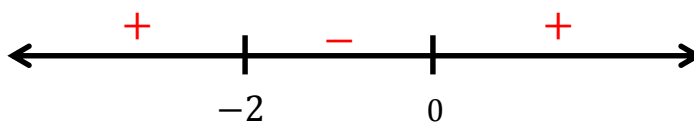
Based on the number line above we conclude that:
   $f(x)$ is increasing on $(-\infty,-2)\cup(0,\infty)$
   $f(x)$ is decreasing on $(-2,0)$

Question: In the example above what would you expect is happening on the graph of the function when $x=-2$ and $x=0$?

Answer: We would probably expect these x-values to be locations of a local maximum and local minimum values. The next theorem makes this idea precise.

Theorem (The First Derivative Test): Suppose that c is a critical number of a continuous function f.
   (a) If $f'(x)$ changes from positive to negative at c, then f has a local maximum at c.
   (b) If $f'(x)$ changes from negative to positive at c, then f has a local minimum at c.
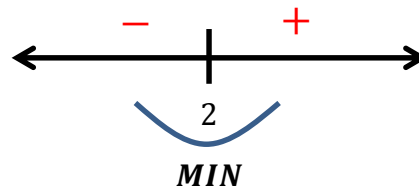   (c) If $f'(x)$ does not change (positive on both sides of c or negative on both sides of c) at c, then f has no local maximum or minimum at c.

Example 2: Determine on what intervals $f(x) = x^2 - 4x + 3$ is increasing and decreasing and state all local extrema of the graph.

We calculate that $f'(x) = 2x - 4$ and want to find critical values.
$f'(x) = 0 \iff 2x - 4 = 0 \iff x = 2$
$f'(x)$ is never undefined.

Based on the number line we conclude that:

$f(x)$ is increasing on $(2, \infty)$.

$f(x)$ is decreasing on $(-\infty, 2)$.

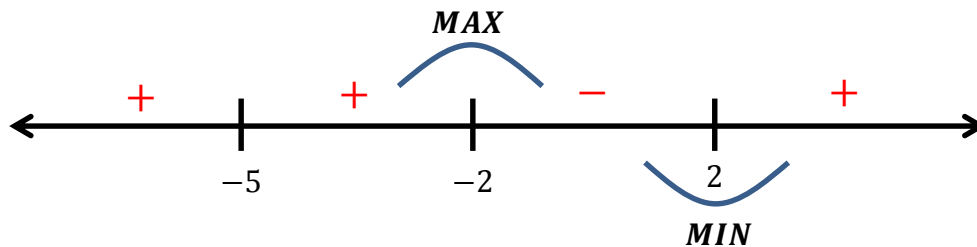$f(x)$ has a local minimum of $f(2) = -1$.

$f(x)$ has no local maximum.

Example 3: Suppose $f(x)$ is a function where $f'(x) = e^x(x+5)^2(x^2-4)$. Find the intervals on which f is increasing and decreasing. Find the x-value(s) at where all local extrema occur.

We need to find all critical values.
$f'(x) = 0 \iff e^x(x+5)^2(x^2-4) = 0 \iff x = -5, -2, 2$
$f'(x)$ is never undefined.

Based on the number line we conclude that:

$f(x)$ is increasing on $(-\infty, -5) \cup (-5, -2) \cup (2, \infty)$.

$f(x)$ is decreasing on $(-2, 2)$.

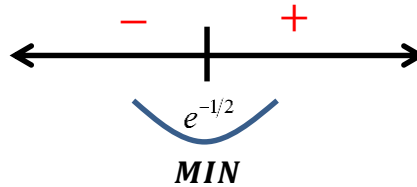$f(x)$ has a local minimum at $x = 2$.

$f(x)$ has a local maximum at $x = -2$

Example 4 (you try): Determine on what intervals $f(x) = x^2 \ln(x)$ is increasing and decreasing and state all local extrema of the graph.

We calculate that $f'(x) = 2x \ln(x) + x^2 \left(\dfrac{1}{x}\right) = 2x \ln(x) + x = x\left(2\ln(x) + 1\right)$ and want to find critical values.

$f'(x) = 0 \iff x\left(2\ln(x) + 1\right) = 0 \iff x = 0 \text{ or } 2\ln(x) + 1 = 0 \iff x = 0 \text{ or } \ln(x) = -\dfrac{1}{2} \iff x = 0 \text{ or } x = e^{-1/2}$

$f'(x)$ is undefined for $x \le 0$.

Of all of these values only $x = e^{-1/2}$ is in the domain of $f(x)$, so $x = e^{-1/2}$ is our only critical value.

$$
\begin{array}{c}
\quad - \qquad\quad + \\
\longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\Big|\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\longrightarrow \\
e^{-1/2} \\
\textbf{\textit{MIN}}
\end{array}
$$

Based on the number line we conclude that:

$f(x)$ is increasing on $\left(e^{-1/2}, \infty\right)$. $f(x)$ has a local minimum of $f\left(e^{-1/2}\right) = -\dfrac{1}{2}e^{-1} = -\dfrac{1}{2e}$.

$f(x)$ is decreasing on $\left(0, e^{-1/2}\right)$. $f(x)$ has no local maximum.
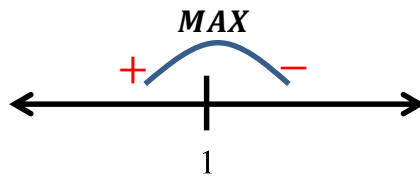
Example 5 (you try): A drug company is testing a drug that is supposed to help alleviate nausea symptoms. The drug is quite potent and the company wants to fully understand the effects experienced by the drug after it is ingested. The drug enters the blood and dissipates from it at a according the function $Q(t) = \dfrac{t}{t^2 + 1}$ where Q measures the quantity of the drug in the system and t is the number of hours since the drug was ingested. Determine at what point in time at which the total amount of drug in the system begins to decrease.

We calculate that $Q'(t) = \dfrac{\left(t^2 + 1\right) - t(2t)}{\left(t^2 + 1\right)^2} = \dfrac{1 - t^2}{\left(t^2 + 1\right)^2} = \dfrac{(1 - t)(1 + t)}{\left(t^2 + 1\right)^2}$ and want to find critical values.

$Q'(t) = 0 \iff (1 - t)(1 + t) = 0 \iff t = 1 \text{ or } t = -1$

$Q'(t)$ is never undefined.

We really only need to consider $t = 1$ since $t = -1$ is not a valid value for time.

$$
\begin{array}{c}
\textbf{\textit{MAX}} \\
+ \qquad\quad - \\
\longleftarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\Big|\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\longrightarrow \\
1
\end{array}
$$

Based on the number line we conclude that the amount of drug in the bloodstream begins to decrease after 1 hour.