

# Example Random Processes

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Based on  
Probability, Random Variables and Random Signal Principles,  
P.Z. Peebles,Jr. and B. Shi

# Outline

- 1 Averages and Ergodicity
- 2 Mean Ergodic Processes
- 3 Correlation Ergodic Processes

# Average

$N$  Gaussian random variables

## Definition

$$\overline{m}_x = \frac{1}{N} \sum_{i=1}^N X_i(t)$$

—

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^T [\bullet] dt$$

# Time-Autocorrelation Function

$N$  Gaussian random variables

## Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

—

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

# Expectation of Time-Autocorrelation Function

$N$  Gaussian random variables

## Definition

$$\bar{X}_T = A_T[x(t)] = \frac{1}{2T} \int_{-T}^T x(t) dt$$

$$E[\bar{X}_T] = E[A_T[x(t)]] = \bar{X}$$

$$R_T(\tau) = A_T[x(t)x(t+\tau)] = \frac{1}{2T} \int_{-T}^T x(t)x(t+\tau) dt$$

$$E[R_T(\tau)] = E[A_T[x(t)x(t+\tau)]] = R_{XX}(\tau)$$

# Ergodicity Theorem

$N$  Gaussian random variables

## Definition

$$\lim_{n \rightarrow \infty} E[(X_n - X)^2] = 0$$

$$A[\bullet] = \lim_{n \rightarrow \infty} A_T[\bullet]$$

# Conditions

$N$  Gaussian random variables

- ①  $X(t)$  has a finite constant mean  $\bar{X}$  for all  $t$
- ②  $X(t)$  is bounded  $x(t) < \infty$  for all  $t$  and all  $x(t)$
- ③ Bounded time average of  $E[|X(t)|]$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T E[|X(t)|] dt$$

- ④  $X(t)$  is a regular process

$$E[|X(t)|^2] = R_{XX}(t, t) < \infty$$



# Mean Ergodic

$N$  Gaussian random variables

## Definition

A wide sense stationary process  $X(t)$  with a constant mean value  $\bar{X}$  is called mean-ergodic if  $\bar{x}_T = A_T[x(t)]$  converges to  $\bar{X}$  as  $T \rightarrow \infty$

$$\lim_{T \rightarrow \infty} E [(\bar{x}_T - \bar{X})^2] = 0$$

$$\lim_{T \rightarrow \infty} \sigma_{\bar{x}_T} = 0$$

# Variance of $\bar{X}_T$ (1)

$N$  Gaussian random variables

## Definition

$$\begin{aligned}
 \sigma_{\bar{X}_T}^2 &= E \left[ \left\{ \frac{1}{2T} \int_{-T}^T (X(t) - \bar{X}) dt \right\}^2 \right] \\
 &= E \left[ \left( \frac{1}{2T} \right)^2 \left\{ \int_{-T}^T (X(t) - \bar{X}) dt \right\} \left\{ \int_{-T}^T (X(t_1) - \bar{X}) dt_1 \right\} \right] \\
 &= E \left[ \left( \frac{1}{2T} \right)^2 \int_{-T}^T (X(t) - \bar{X}) (X(t_1) - \bar{X}) dt dt_1 \right] \\
 &= \left( \frac{1}{2T} \right)^2 \int_{-T}^T \int_{-T}^T E [(X(t) - \bar{X}) (X(t_1) - \bar{X})] dt dt_1 \\
 &= \left( \frac{1}{2T} \right)^2 \int_{-T}^T \int_{-T}^T C_{XX}(t, t_1) dt dt_1
 \end{aligned}$$

# Variance of $\bar{x}_T$ (2)

$N$  Gaussian random variables

## Definition

$$\sigma_{\bar{x}_T}^2 = \left(\frac{1}{2T}\right)^2 \int_{-T}^T C_{XX}(t, t_1) dt dt_1$$

$$C_{XX}(t, t_1) = C_{XX}(T), \quad \tau = t_1 - t, \quad dt_1 = dT$$

$$\sigma_{\bar{x}_T}^2 = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^T \int_{\tau=-T-t}^T C_{XX}(\tau) dt d\tau$$

Variance of  $\bar{x}_T$  (3) $N$  Gaussian random variables

## Definition

using the symmetry  $C_{XX}(-\tau) = C_{XX}(\tau)$ 

$$\sigma_{\bar{x}_T}^2 = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau$$

a necessary and sufficient condition for a WSS process  $X(t)$  to be mean ergodic

$$\lim_{T \rightarrow \infty} \left\{ \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau \right\} = 0$$











