Residue Integrals (4A)

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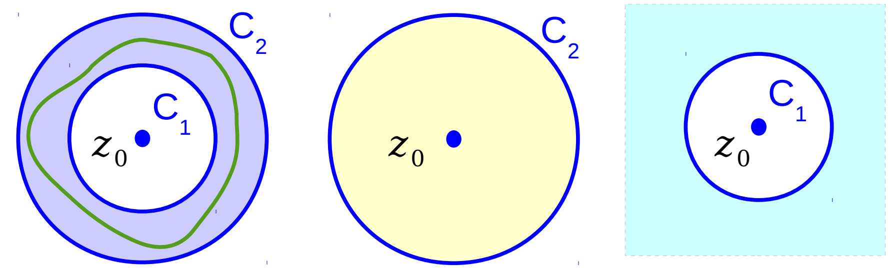
Laurent's Theorem and Coefficients

$f(z)$: **analytic** in the annular domain D

between concentric circles C_1 and C_2

centered at z_0 $\quad r < |z - z_0| < R$



$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$
$$+ b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots$$

convergent in the domain D

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}\,dz$$

any simple closed path C in D

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

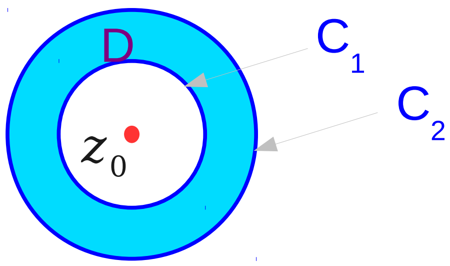
$$a_k = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

What is a Residue?

$f(z)$: **analytic** in the domain D

between circles C_1, C_2

centered at z_0



$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0) + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

: **convergent** in the domain D

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots$$

Principal part

coefficient b_1 of $\frac{1}{(z-z_0)}$

$b_1 = Res(f(z), z_0)$

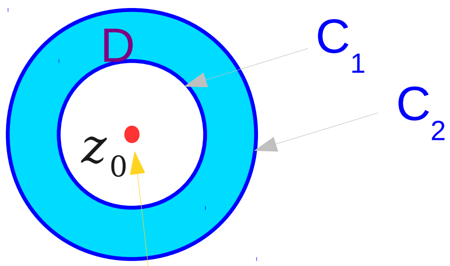
: the **Residue** of the function $f(z)$ at the isolated singularity z_0

What is the use of a Residue?

$f(z)$: **analytic** in the domain D

between circles C_1, C_2

centered at z_0



: the **Residue** of the function $f(z)$ at the isolated singularity z_0

coefficient b_1 of $\dfrac{1}{(z-z_0)}$

$$b_1 = Res(f(z), z_0)$$

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k$$

$$a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{k+1}}$$

$$\oint_C f(z)\,dz = 2\pi i\, Res(f(z), z_0)$$

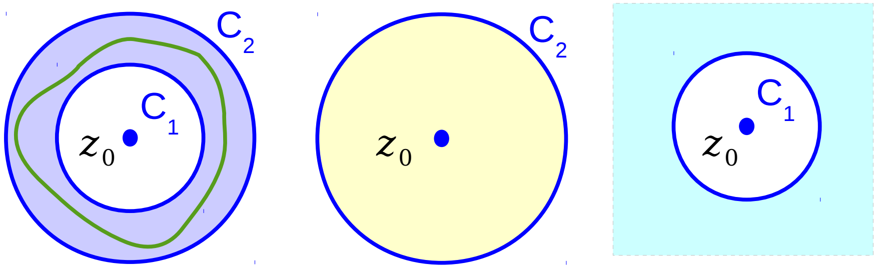
$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\,dz$$

Contour integration

$f(z)$: **analytic** in the annular domain D

between concentric circles C_1 and C_2

centered at z_0 $\quad r < |z - z_0| < R$



Laurent Theorem

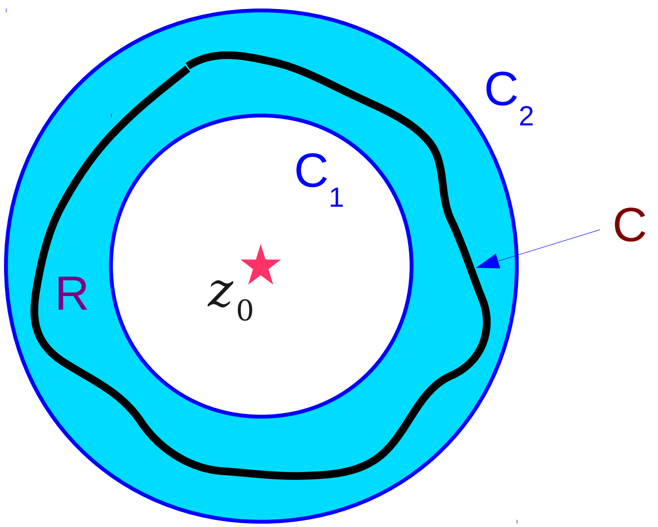
$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k \qquad a_k = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{k+1}}$$

$$\oint_C f(z)\,dz = 2\pi i \, Res(f(z), z_0)$$

at the isolated singularity z_0

$$\oint_C f(z)\,dz = 0$$

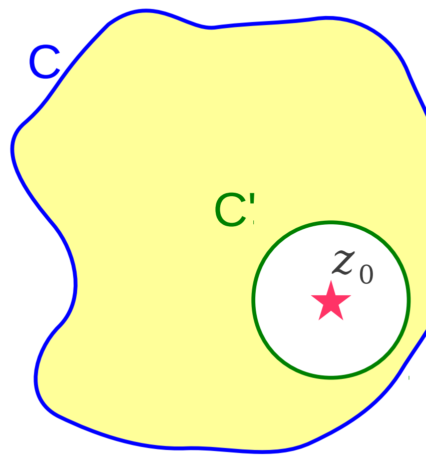
at the regular point z_0



Residue Integration (1)

$f(z)$: **analytic on** and **inside** C except z_0

z_0 Isolated singular point



Converging Laurent series

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \cdots$$

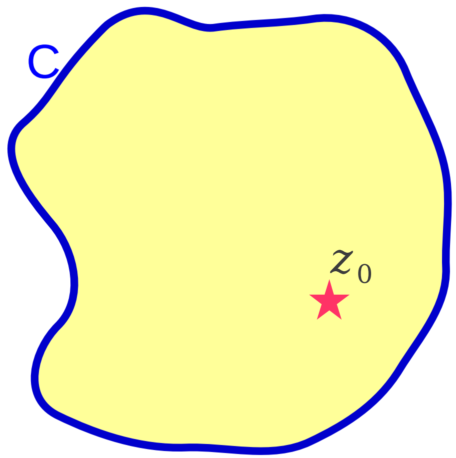
$$\oint_C f(z)\,dz = \oint_{C'} a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\,dz + \oint_{C'} \frac{b_1}{(z-z_0)}\,dz + \oint_{C'} \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots\,dz$$

0 **0**

$$\oint_C f(z)\,dz = 2\pi i\, b_1$$

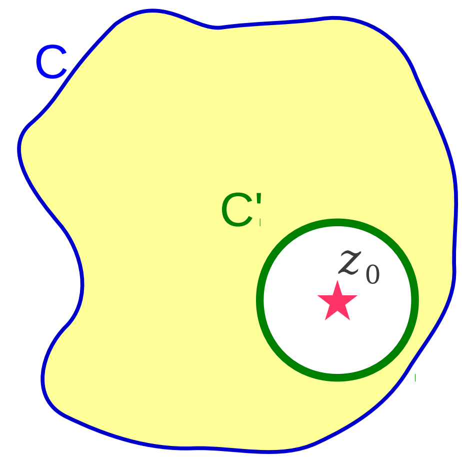
Residue Integration (2)

$$\oint_C f(z)\ dz$$



$$\oint_C f(z)\ dz = \oint_{C'} f(z)\ dz$$

along C'



$$z = z_0 + \rho e^{i\theta} \qquad dz = i\rho e^{i\theta} d\theta$$

$(k \geq 0)$ $$\oint_{C'} a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\ dz = 0$$

$(k < -1)$ $$\oint_{C'} \frac{b_2}{(z-z_0)^2} + \frac{b_3}{(z-z_0)^3} + \cdots\ dz = 0$$

⬅ **analytic** on and inside C

$$\int_0^{2\pi} e^{ik\theta} d\theta = \left[\frac{e^{ik\theta}}{ik}\right]_0^{2\pi} = 0$$

$(k = -1)$ $$\oint_{C'} \frac{b_1}{(z-z_0)} dz = b_1 \int_0^{2\pi} \frac{\rho i e^{i\theta} d\theta}{\rho e^{i\theta}} = b_1 \int_0^{2\pi} i\, d\theta = 2\pi i \cdot b_1$$

the only remnant

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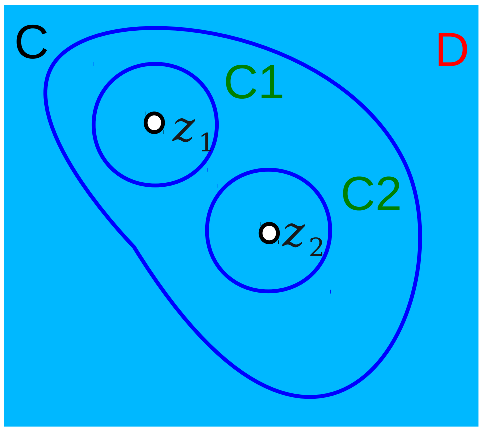
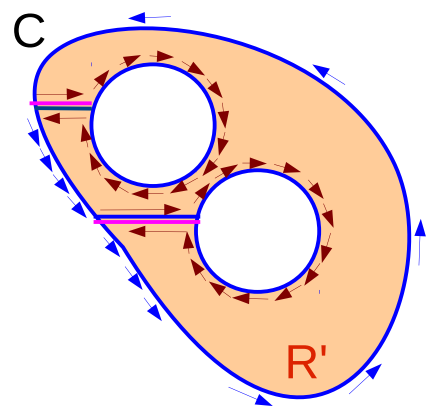
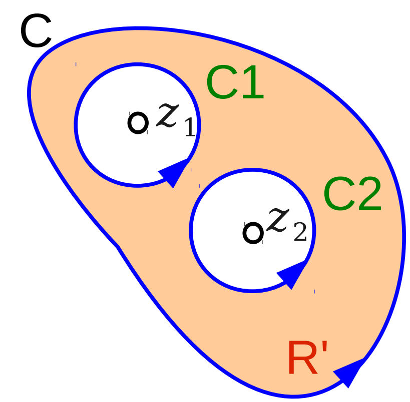
Cauchy-Goursat Theorem

triply connected domain D ➡ **simply connected region** R'



cuts





$$\oint_{C1} f(z)\,dz = Res(f(z), z_1)$$

$$\oint_{C2} f(z)\,dz = Res(f(z), z_2)$$

$$\oint_{ccw\ C} f(z)\,dz + \oint_{cw\,C1} f(z)\,dz + \oint_{cw\,C2} f(z)\,dz = 0$$

$$\oint_{C} f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz$$

Cauchy's Residue Theorem

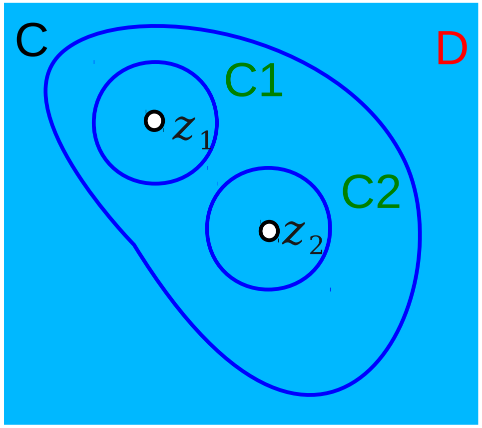
A **simply connected** domain D

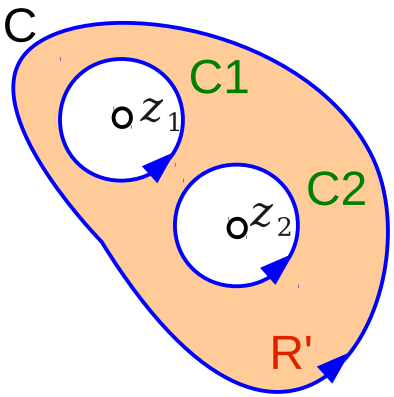
A **simple closed contour** C lying entirely in D

$f(z)$ **: Analytic on an within C**

Except at a finite number of singular points $z_1, z_2, \cdots, z_n$

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$



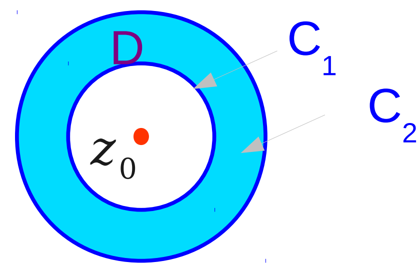


$$\oint_C f(z)\,dz = \oint_{C1} f(z)\,dz + \oint_{C2} f(z)\,dz$$

$$= 2\pi i\{Res(f(z), z_1) + Res(f(z), z_2)\}$$

Types of Isolated Singularities

$f(z)$: **analytic** in the region R between concentric circles C_1, C_2 centered at z_0



residue of $f(z)$ at z_0 (isolated singular point)

$$b_1 = a_{-1} = Res(f(z), z_0)$$

$b_k = 0$

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

removable singularity z_0

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

simple pole z_0

$b_1 = a_{-1}$

$$+ \frac{b_1}{(z-z_0)}$$

⇐ one term

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

pole of order n z_0

$b_1 = a_{-1}$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n}$$

⇐ n terms

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

essential singularity z_0

$b_1 = a_{-1}$

$$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

⇐ infinite terms

Laurent Expansion at a regular point

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$z = +2$ Not an isolated singular point

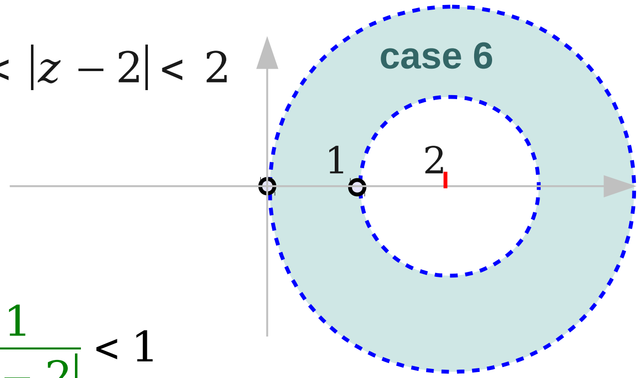
$$\frac{1}{z-1} = \frac{1}{1+z-2} = \frac{1}{(z-2)\left(1+\frac{1}{z-2}\right)}$$

$$= \frac{1}{z-2}\left[1 - \frac{1}{(z-2)} + \frac{1}{(z-2)^2} - \frac{1}{(z-2)^3} \cdots\right]$$

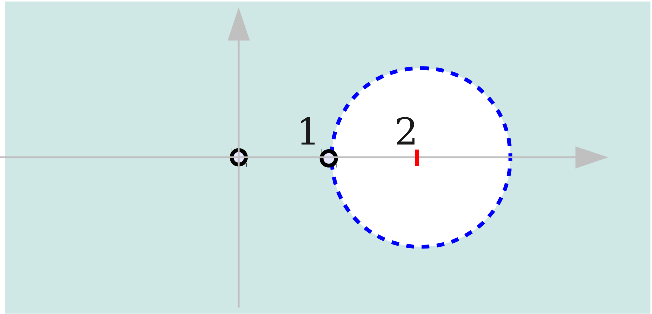
$$-\frac{1}{z} = -\frac{1}{2+z-2} = -\frac{1}{2\left(1+\frac{z-2}{2}\right)}$$

$$= -\frac{1}{2}\left[1 - \frac{(z-2)}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} \cdots\right]$$

$1 < |z-2| < 2$



$$\frac{1}{|z-2|} < 1$$



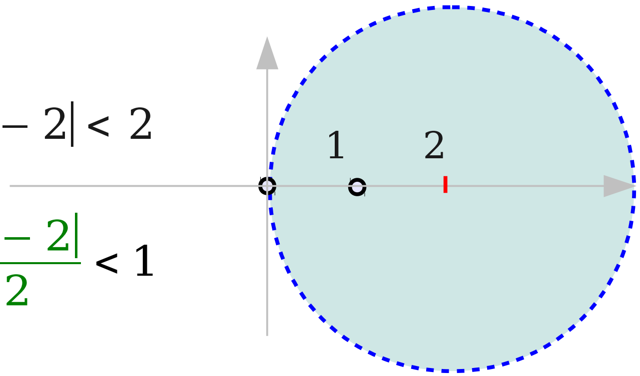
$$f(z) = \frac{1}{z(z-1)}$$

~~1~~ $= Res(f(z), z_0)$

$$= \cdots - \frac{1}{(z-2)^2} + \frac{1}{(z-2)} - \frac{1}{2} - \frac{(z-2)}{2^2} + \frac{(z-2)^2}{2^3} - \cdots$$

~~essential singularity~~

$|z-2| < 2$



$$\frac{|z-2|}{2} < 1$$

Finding Residues

Laurent Series The integral around C in the counterclockwise direction

$$f(z) = \sum_{n=-\infty}^{\infty} a_k (z - z_0)^k = \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Residue of *f*(*z*) at z_0: a_{-1} coefficient of $1/(z-z_0)$

$$Res(f(z), z_0) = a_{-1} = \left[\frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{k+1}} \, dz \right]_{k=-1}$$

$$Res(f(z), z_0) = \lim_{z \to z_0} (z-z_0) f(z)$$

a simple pole

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z)$$

an n-th order pole

$$Res(f(z), z_0) = \frac{g(z_0)}{h'(z_0)} \quad \Leftarrow \quad f(z) = \frac{g(z)}{h(z)}$$

$g(z_0) \neq 0$

$h(z_0) = 0 \quad h'(z_0) \neq 0$

a simple pole

Finding a Residue At a Simple Pole

$f(z)$ has a **simple pole** at $z = z_0$ ➡

$$Res(f(z), z_0) \;=\; \lim_{z \to z_0} (z - z_0)\, f(z)$$

A simple pole

$$f(z) \;=\; \frac{a_{-1}}{(z - z_0)} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + a_3(z - z_0)^3 + \cdots$$

$$(z - z_0) f(z) \;=\; a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + a_2(z - z_0)^3 + a_3(z - z_0)^4 + \cdots$$

$$\lim_{z \to z_0} (z - z_0) f(z) \;=\; a_{-1}$$

Finding a Residue At a Simple Pole – g(z)/f(z)

$f(z)$ has a **simple pole** at $z = z_0$ ➡

$$f(z) = \frac{g(z)}{h(z)}$$

$g(z)$ ← **analytic** at z_0 $\quad g(z_0) \neq 0$

$h(z)$ ← **analytic** at z_0 $\quad h(z_0) = 0 \quad h'(z_0) \neq 0$

→ a **zero of order 1** at z_0

$$Res(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}$$

A simple pole

$$(z-z_0) f(z) = a_{-1} + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + a_3(z-z_0)^4 + \cdots$$

$$\lim_{z \to z_0} (z-z_0) f(z) = a_{-1}$$

➡

$$\lim_{z \to z_0} (z-z_0) \frac{g(z)}{h(z)} = \lim_{z \to z_0} \frac{g(z)}{\dfrac{h(z) - h(z_0)}{z - z_0}} \qquad (h(z_0) = 0)$$

$$\lim_{z \to z_0} \frac{g(z)}{h'(z)} = a_{-1}$$

Finding a Residue At an n-th Order Pole

$f(z)$ has a **pole of order n** at $z = z_0$ ➡

$$Res(f(z), z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{d z^{n-1}} (z - z_0)^n f(z)$$

An n-th order pole

$$f(z) = \frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$(z-z_0)^n f(z) = a_{-n} + \cdots + a_{-1}(z-z_0)^{n-1} + a_0(z-z_0)^n + a_1(z-z_0)^{n+1} + \cdots$$

$$\frac{d^{n-1}}{d z^{n-1}} (z-z_0)^n f(z) = (n-1)!\, a_{-1} + (n)!\, a_0(z-z_0) + (n+1)!\, a_1(z-z_0)^2 + \cdots$$

$$\lim_{z \to z_0} \frac{d^{n-1}}{d z^{n-1}} (z-z_0)^n f(z) = (n-1)!\, a_{-1}$$

Improper Integral

An Improper Integral : (A) The limit of a definite integral as an endpoint of the intervals(s) of integration approaches either a specified real number or +infinity or -infinity

$$\int_a^{\infty} \frac{1}{1+x^2}\,dx$$

$$\lim_{b\to\infty} \int_a^b f(x)\,dx \qquad \lim_{a\to\infty} \int_a^b f(x)\,dx$$

$$\lim_{b\to c-} \int_a^b f(x)\,dx \qquad \lim_{a\to c+} \int_a^b f(x)\,dx$$

An Improper Integral : (B) When the **integrand is undefined** at an interior point of the domain of integration, or at multiple such points

$$\int_a^c \frac{e^x}{\sqrt{c-x}}\,dx$$

Calculation of Some Real Integrals

$$\int_0^{2\pi} F(\cos\theta,\ \sin\theta)\, d\theta$$

$$\oint_C F\left(\frac{z+z^{-1}}{2},\ \frac{z+z^{-1}}{2i}\right)\frac{dz}{iz}$$

$$\int_{-\infty}^{+\infty} f(x)\, dx$$

$$\lim_{R\to\infty}\int_{-R}^{+R} f(x)\, dx \Rightarrow 2\pi i \sum_k Res(f(z), z_k)$$

$$\int_{-\infty}^{+\infty} f(x)\cos(\alpha x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\sin(\alpha x)\, dx$$

$$\int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx = \int_{-\infty}^{+\infty} f(x)\cos(\alpha x)\, dx + \int_{-\infty}^{+\infty} f(x)\sin(\alpha x)\, dx$$

Cauchy Principal Value

$f(x)$ **: continuous** on $(-\infty, +\infty)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx$$

- **Convergent**: Both limits exist
- **Divergent**: Any limit fails to exist

Cauchy Principal Value

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to+\infty}\int_0^b f(x)\,dx = \lim_{R\to-\infty}\int_{-R}^{+R} f(x)\,dx$$

In the case of **convergence** on **: always true**

But in the case of **divergence** **: not always true**

$$\int_{-\infty}^{+\infty} x\,dx = \lim_{a\to-\infty}\int_a^0 x\,dx + \lim_{b\to+\infty}\int_0^b x\,dx = \infty \neq \lim_{R\to-\infty}\int_{-R}^{+R} x\,dx = 0$$

For the converging $\int_{-\infty}^{+\infty} f(x)\,dx$

$f(x)$: **continuous** on $(-\infty, +\infty)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)\,dx + \lim_{b \to +\infty} \int_{0}^{b} f(x)\,dx$$

- **Convergent**: Both limits exist
- **Divergent**: Any fails to exist

For converging $\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{a \to -\infty} \int_{a}^{0} f(x)\,dx + \lim_{b \to +\infty} \int_{0}^{b} f(x)\,dx$

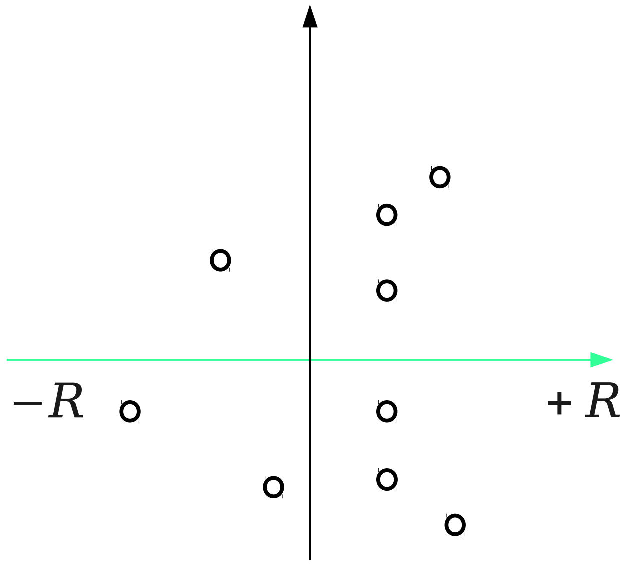
$$= \lim_{R \to -\infty} \int_{-R}^{+R} f(x)\,dx \equiv P.V. \int_{-\infty}^{+\infty} f(x)\,dx$$

The **Cauchy Principal Value of** $\int_{-\infty}^{+\infty} f(x)\,dx$

$f(x)$ **: continuous** on $(-\infty, +\infty)$

➡ **no pole** on the x axis

$f(x) = \dfrac{P(x)}{Q(x)}$ $\int_{-\infty}^{+\infty} f(x)\,dx$ **= ?**



$\int_{-\infty}^{+\infty} f(x)\,dx$ real integral ➡

$\int_{-\infty}^{+\infty} f(z)\,dz$ complex integral

$$\text{poles of} \quad f(z) = \frac{P(z)}{Q(z)}$$

$x \rightarrow z \qquad Q(z) \rightarrow 0$

but **no pole** on the x axis

$f(x)$: **continuous** on $(-\infty, +\infty)$

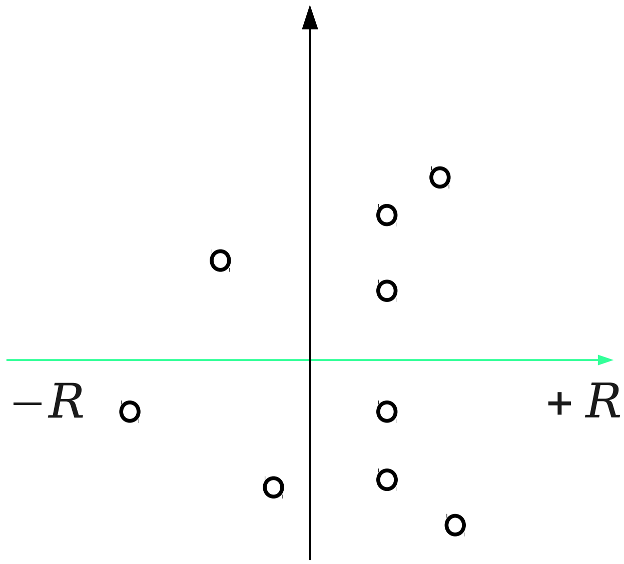
⇨ **no pole** on the x axis

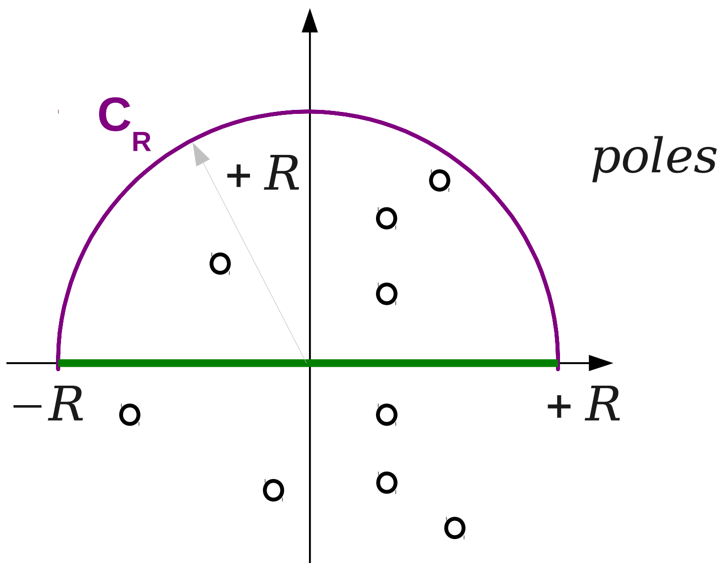
$$f(x) = \frac{P(x)}{Q(x)} \qquad \int_{-\infty}^{+\infty} f(x)\,dx = ?$$

Considering a half circle contour C_R large enough to include all the poles in the upper half plane z_k

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

$$= \int_{C_R} f(z)\,dz + \int_{-R}^{+R} f(x)\,dx$$





Improper Integral $\int_{-\infty}^{+\infty} f(x)\,dx$ (3)

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

$$\int_{C_R} f(z)\,dz + \int_{-R}^{+R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz + \lim_{R\to\infty}\int_{-R}^{+R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz \to 0$$

Under certain conditions

$$\lim_{R\to\infty}\int_{-R}^{+R} f(x)\,dx = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$

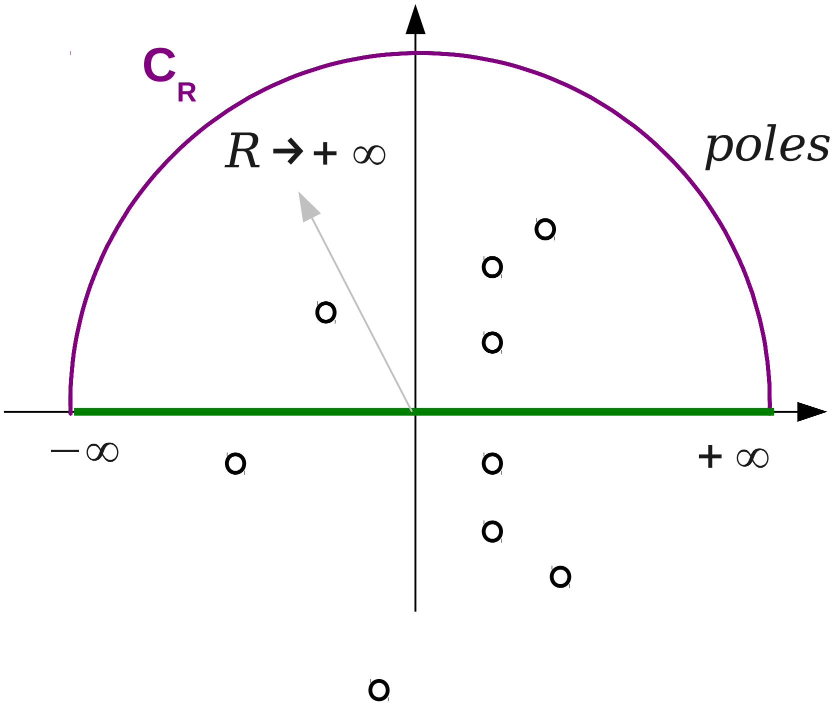
$$= P.V. \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx$$

For converging

$f(x)$ **: continuous** on $(-\infty, +\infty)$

➡ **no pole** on the x axis

$$f(x) = \frac{P(x)}{Q(x)} \qquad \int_{-\infty}^{+\infty} f(x)\,dx = ?$$



Under these conditions

$$f(x) = \frac{P(x)}{Q(x)}$$

$P(x)$ ← degree n

$Q(x)$ ← degree $m \geq n+2$

C_R a semicircular contour

$$z = Re^{j\theta} \qquad 0 \leq \theta \leq \pi$$

➡ $\lim_{R\to\infty} \int_{C_R} f(z)\,dz \to 0$ ➡ $\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$

Fourier Integrals

$$\int_{-\infty}^{+\infty} f(x)\cos\alpha x\,dx = \Re\left[\int_{-\infty}^{+\infty} f(x)\,e^{i\alpha x}\,dx\right]$$

$$\int_{-\infty}^{+\infty} f(x)\sin\alpha x\,dx = \Im\left[\int_{-\infty}^{+\infty} f(x)\,e^{i\alpha x}\,dx\right]$$

$(\alpha > 0)$

$$\int_{-\infty}^{+\infty} f(x)\,e^{i\alpha x}\,dx = \int_{-\infty}^{+\infty} f(x)\,(\cos\alpha x + \sin\alpha x)\,dx$$

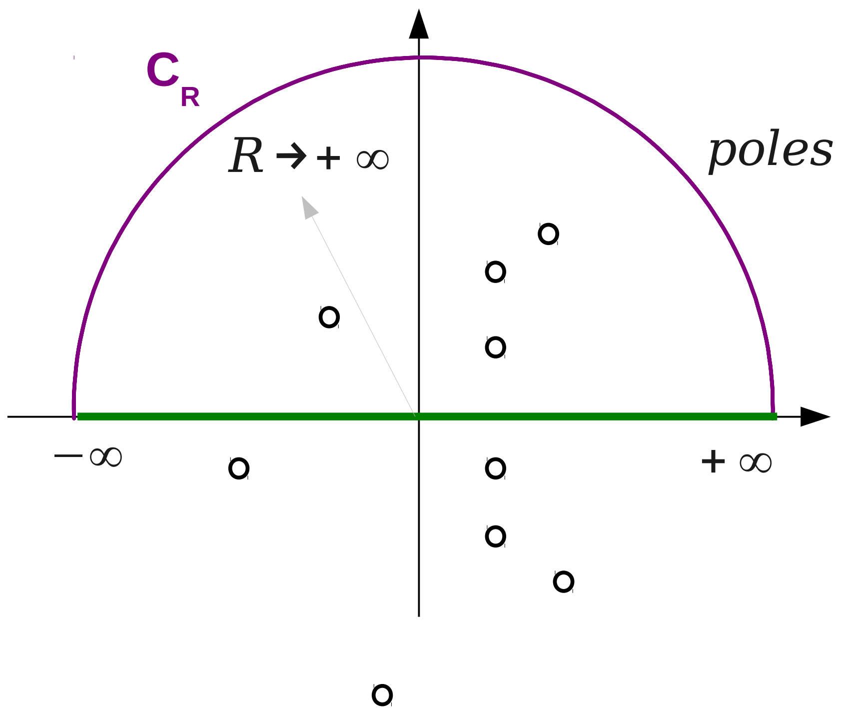
$$= \int_{-\infty}^{+\infty} f(x)\cos\alpha x\,dx + i\int_{-\infty}^{+\infty} f(x)\sin\alpha x\,dx$$

Improper Integral $\int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx$

$f(x)$: **continuous** on $(-\infty, +\infty)$

➡ **no pole** on the x axis

$$f(x) = \frac{P(x)}{Q(x)} \qquad \int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx = ?$$



Under these conditions

$$f(x) = \frac{P(x)}{Q(x)}$$

$P(x)$ ← degree n

$Q(x)$ ← degree $m \geq n+1$

C_R a semicircular contour

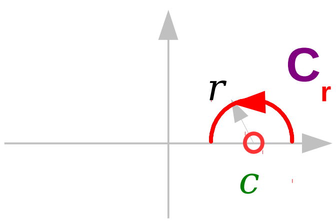
$$z = R e^{j\theta} \qquad 0 \leq \theta \leq \pi \qquad (\alpha > 0)$$

➡ $\lim_{R\to\infty} \int_{C_R} f(z)\, e^{i\alpha x}\, dz \to 0$ ➡ $\int_{-\infty}^{+\infty} f(x)\, e^{i\alpha x}\, dx = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$

Indented Contour (1)

$f(z)$: **continuous** on $(-\infty, +\infty)$

→ **poles** on the x axis



$$z = c + re^{i\theta}$$
$$0 \le \theta \le \pi$$

$$dz = ire^{i\theta} d\theta$$

$$f(z) = \frac{a_{-1}}{z-c} + g(z)$$ a pole at c

$$\oint_{C_r} f(z) = I_1 + I_2$$

$$= a_{-1}\int_0^{\pi} \frac{ire^{j\theta}}{re^{j\theta}} d\theta + \int_0^{\pi} g(c+re^{j\theta}) ire^{j\theta} d\theta$$

$$I_1 = a_{-1}\int_0^{\pi} i\, d\theta = \pi i a_{-1} = \pi i Res(f(z), c)$$

$g(z)$: **analytic** at c and thus continuous
bounded in a neighborhood of c
there is M > 0 such that $|g(c+re^{j\theta})| \le M$

$$|I_2| = \left|\int_0^{\pi} g(c+re^{j\theta}) ire^{j\theta} d\theta\right|$$

$$|I_2| \le r\int_0^{\pi} M\, d\theta = -\pi r M$$ (0) $$\lim_{r\to 0} I_2 = 0$$

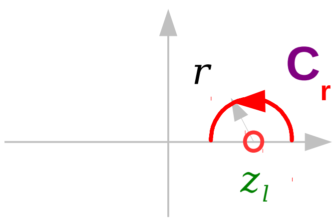
$$\lim_{r\to 0}\int_{C_r} f(z)\, dz = \pi i\, Res(f(z), c)$$

Indented Contour (2)

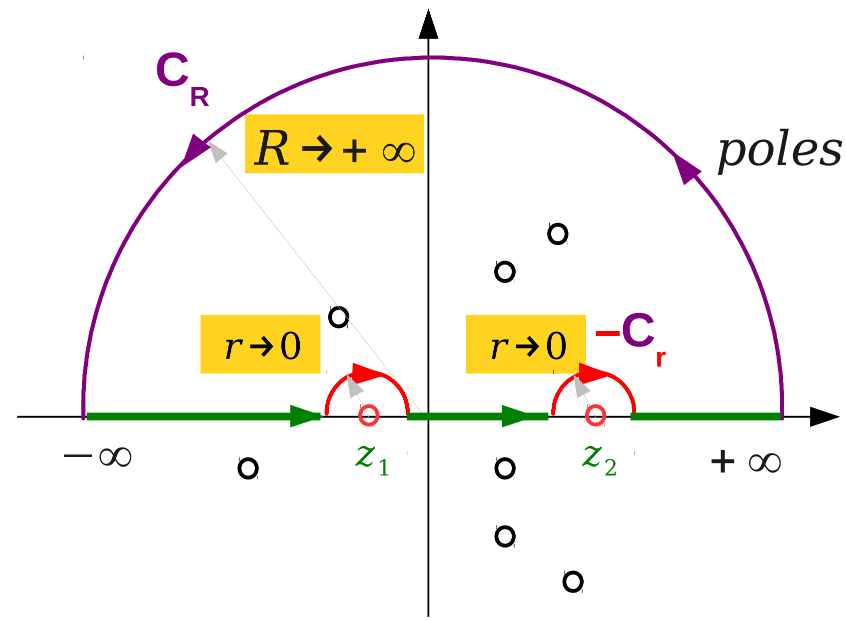
$f(x)$: ~~**continuous**~~ on $(-\infty, +\infty)$

➡ **poles** on the x axis z_l

other **poles** z_k



$$z = z_l + r e^{i\theta} \qquad 0 \le \theta \le \pi$$



$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to-\infty}\int_{-R}^{+R} f(x)\,dx =$$

$$\lim_{R\to\infty}\int_{C_R} f(z)\,dz + \lim_{R\to-\infty}\int_{-R}^{+R} f(x)\,dx - \lim_{r\to 0}\int_{C_r} f(z)\,dz = 2\pi i\sum_k Res(f(z), z_k)$$

$$\lim_{R\to-\infty}\int_{-R}^{+R} f(x)\,dx = 2\pi i\sum_k Res(f(z), z_k) + \pi i\sum_l Res(f(z), z_l)$$

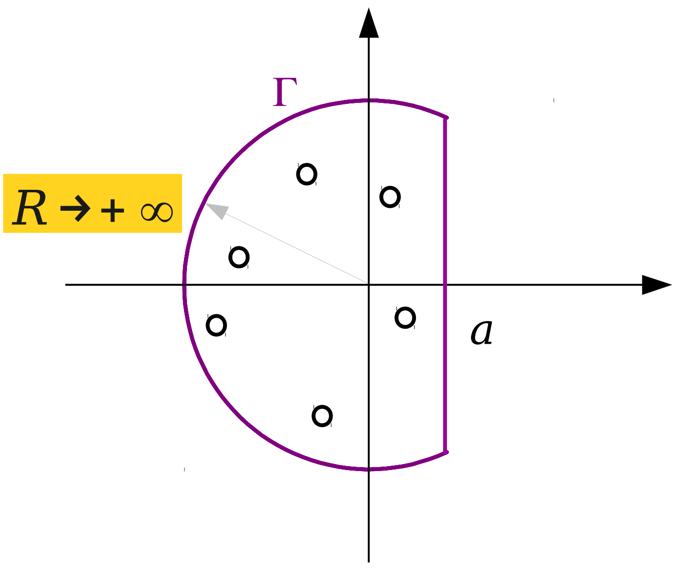
Inverse Laplace Transform

$$f(t) = L^{-1}\{F(s)\}$$

$$= \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}\,ds$$

$$= \frac{1}{2\pi j}\lim_{R\to\infty}\oint_{\Gamma} F(s)e^{st}\,ds$$

$$= 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$



$$F(s) = \frac{1}{s^2 + \omega^2}$$

$$f(t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} \frac{e^{st}}{(s+j\omega)(s-j\omega)}\,ds$$

$$Res\left((s-j\omega)\frac{e^{st}}{s^2+\omega^2},\quad j\omega\right) = \frac{e^{+j\omega t}}{2j\omega} \qquad Res\left((s+j\omega)\frac{e^{st}}{s^2+\omega^2},\quad j\omega\right) = -\frac{e^{-j\omega t}}{2j\omega}$$

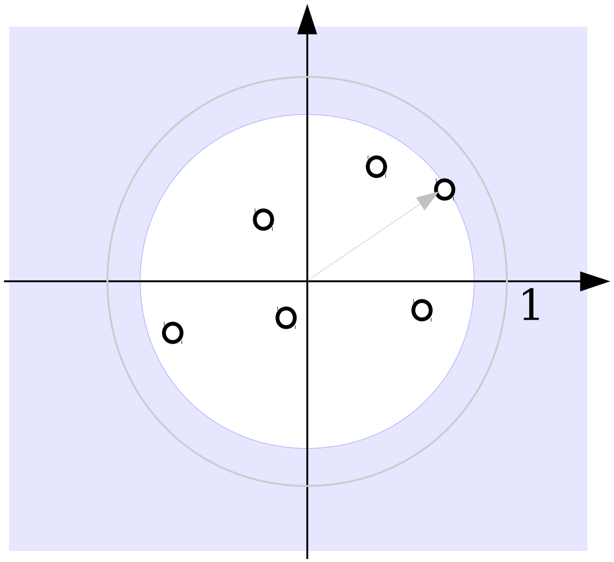
$$f(t) = \frac{e^{+j\omega t} - e^{-j\omega t}}{2j\omega} = \frac{\sin\omega t}{\omega}$$

Inverse z-Transform

$$f(kT) = L^{-1}\{F(z)\}$$

$$= \frac{1}{2\pi j} \oint F(z) z^{k-1}\, dz$$

$$= 2\pi i \sum_{k=1}^{n} Res(f(z), z_k)$$



$$F(z) = \frac{z}{z - 1/2}$$

$$f(t) = \frac{1}{2\pi j} \oint \frac{z}{z-1/2} z^{k-1}\, ds$$

$$Res\left(\frac{z^k}{z-1/2}, \frac{1}{2}\right) = \lim_{z \to 1/2} (z-1/2) \frac{z^k}{z-1/2} = \left(\frac{1}{2}\right)^k$$

$$f(t) = \frac{1}{2^k}$$

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