Planar Graph (7A)

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Planar Graph

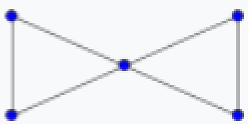
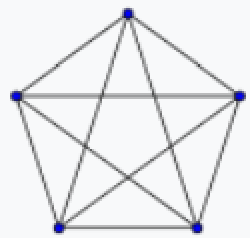
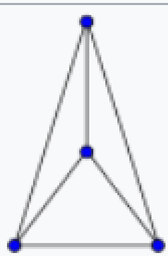
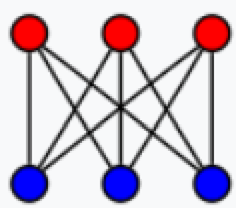
a planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

it can be drawn in such a way that no edges cross each other. Such a drawing is called a **plane graph** or **planar embedding** of the graph. (**planar representation**)

A **plane graph** can be defined as a planar graph with a mapping from every node to a point on a plane, and from every edge to a plane curve on that plane,
such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points.

https://en.wikipedia.org/wiki/Planar_graph

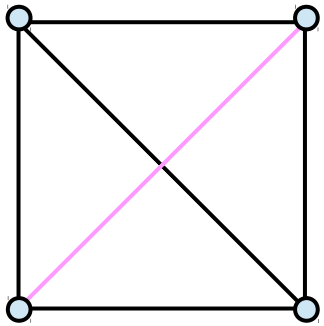
Planar Graph Examples

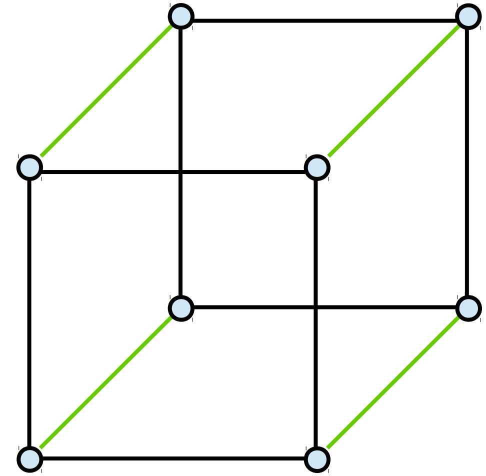
Example graphs	
Planar	**Nonplanar**
 Butterfly graph	 Complete graph K_5
 Complete graph K_4	 Utility graph $K_{3,3}$

https://en.wikipedia.org/wiki/Planar_graph

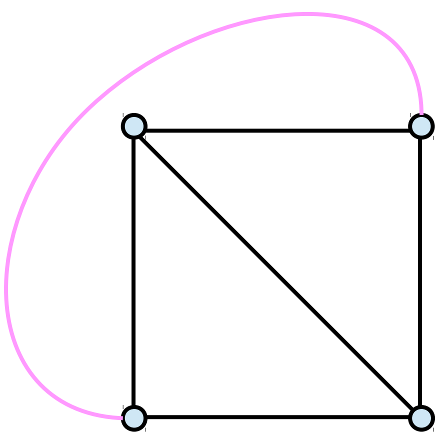
Planar Representation

K_4



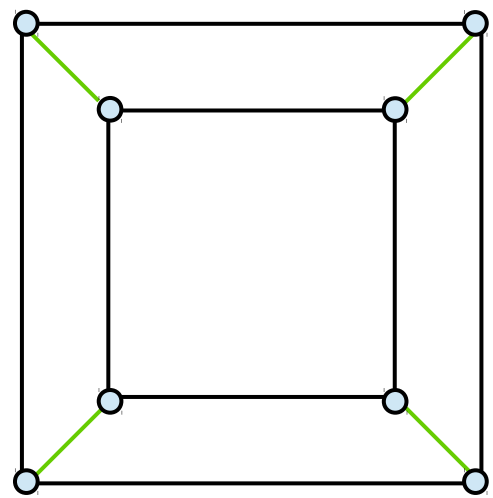


Q_3



No crossing

K_4 Planar

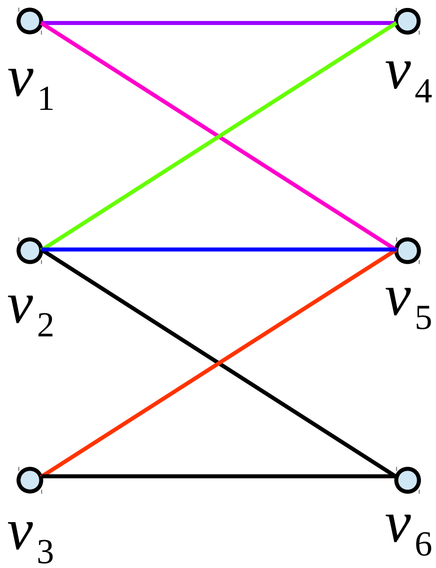


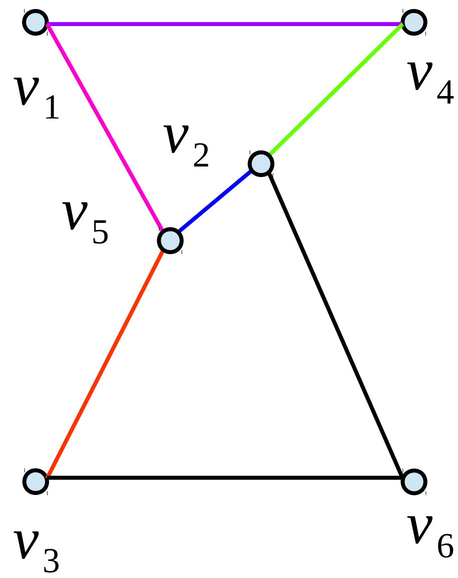
No crossing

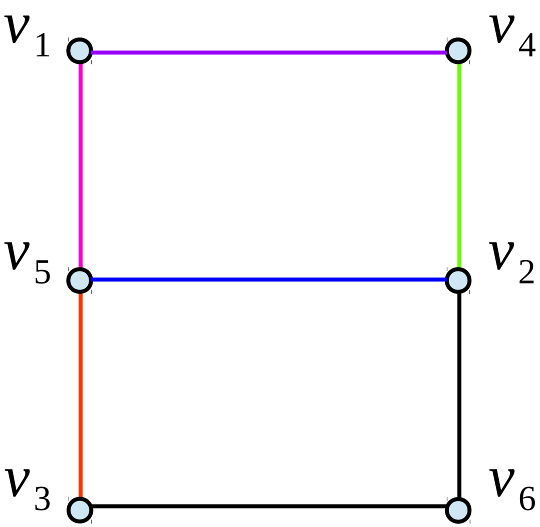
Q_3 Planar

Discrete Mathematics, Rosen

A planar bipartite graph



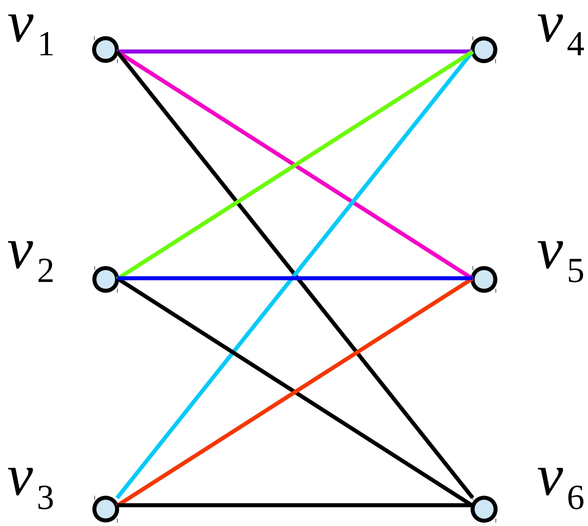


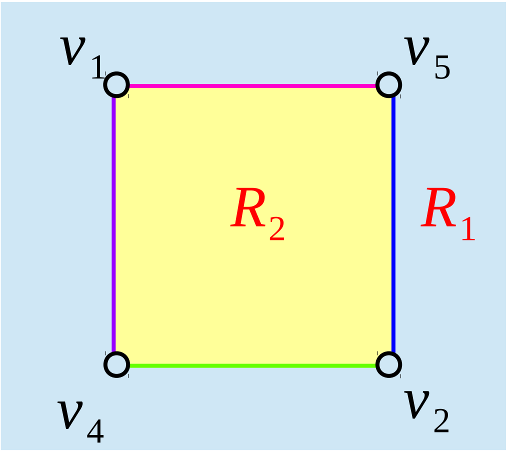


Bipartite graph
but <u>not</u> complete
bipartite graph
$K_{3,3}$

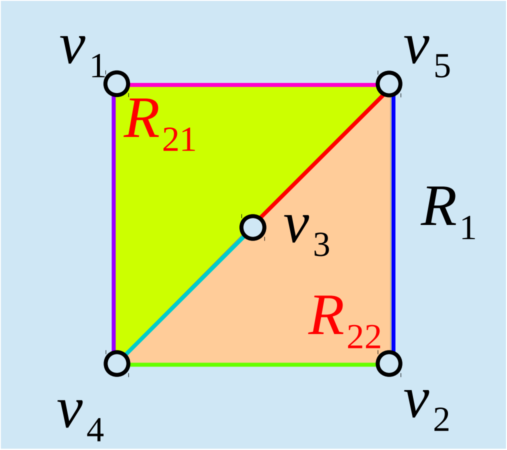
Planar Graph

Non-planar Graph $K_{3,3}$



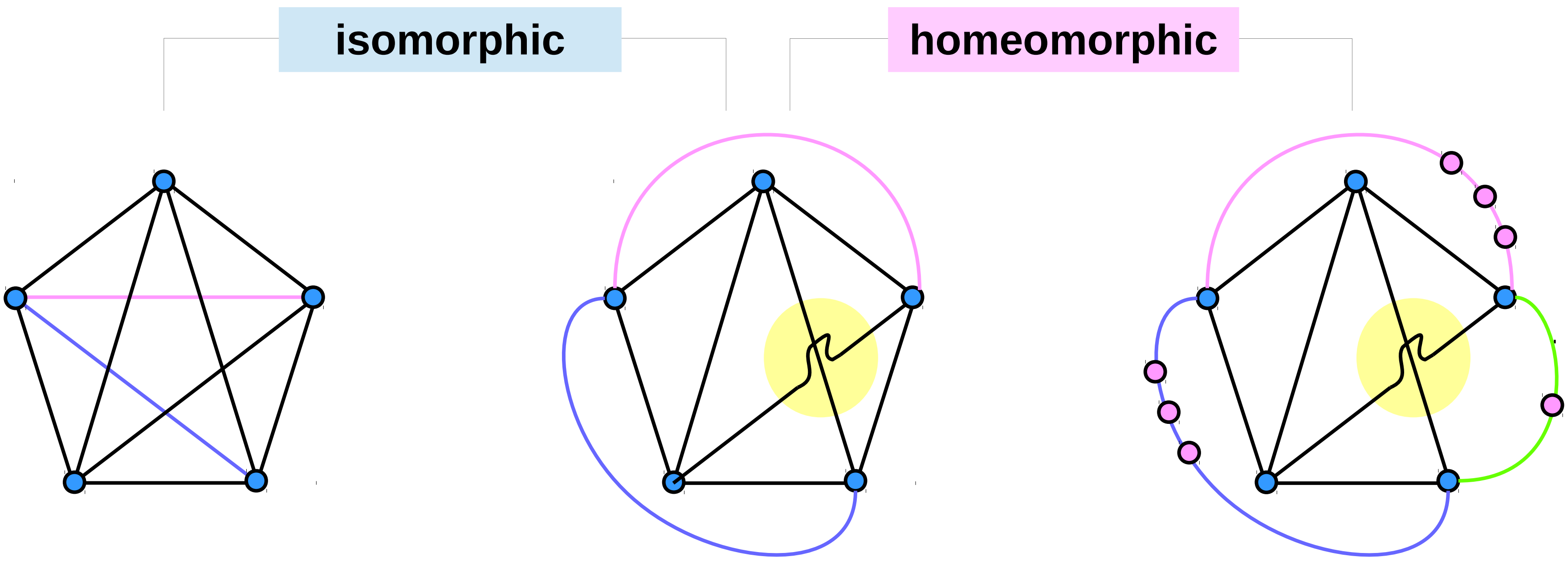


no where v_6



Non-planar

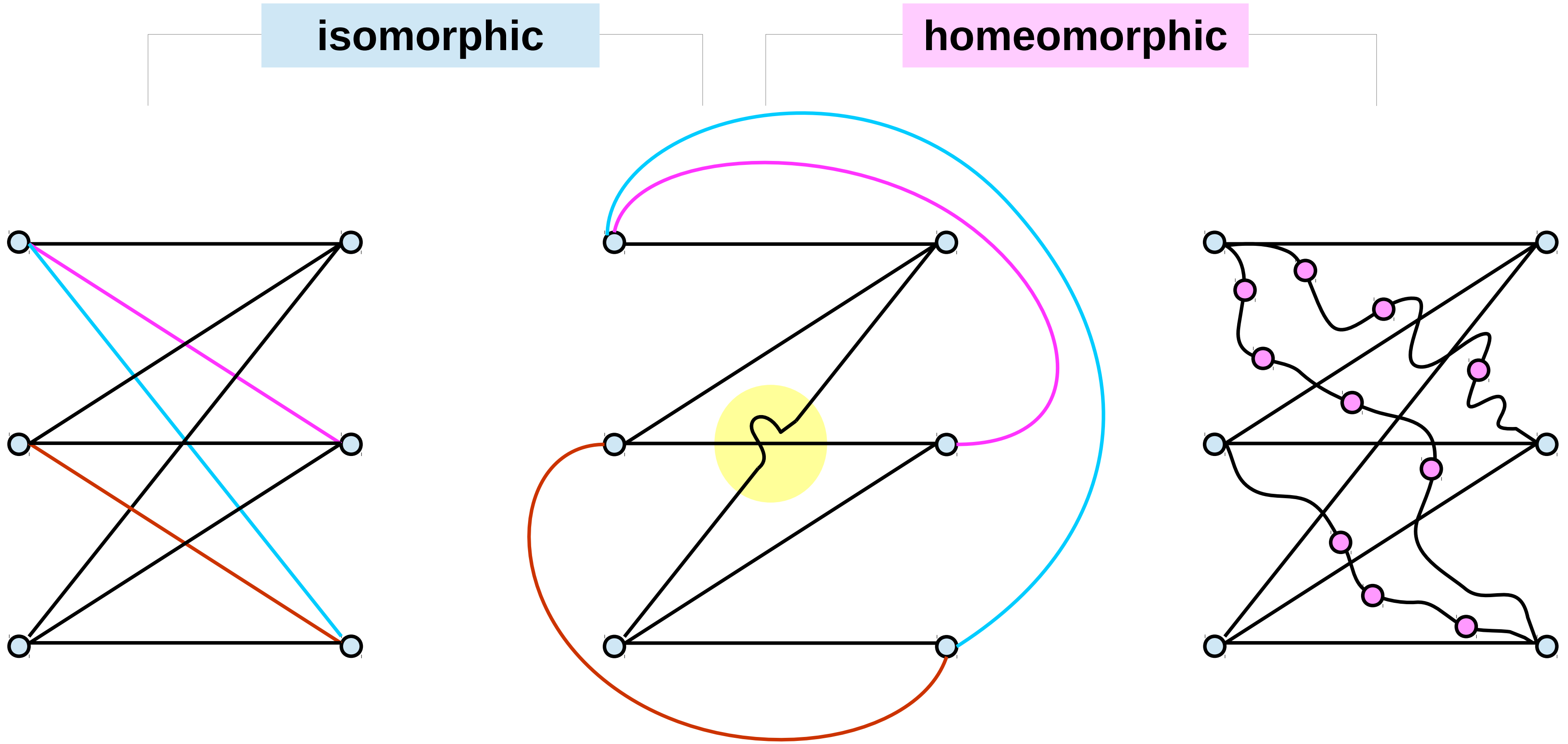
Discrete Mathematics, Rosen



Non-planar **Non-planar** **Non-planar**

All these graphs are <u>similar</u> in determining whether they are planar or not

Non-planar graph examples – $K_{3,3}$

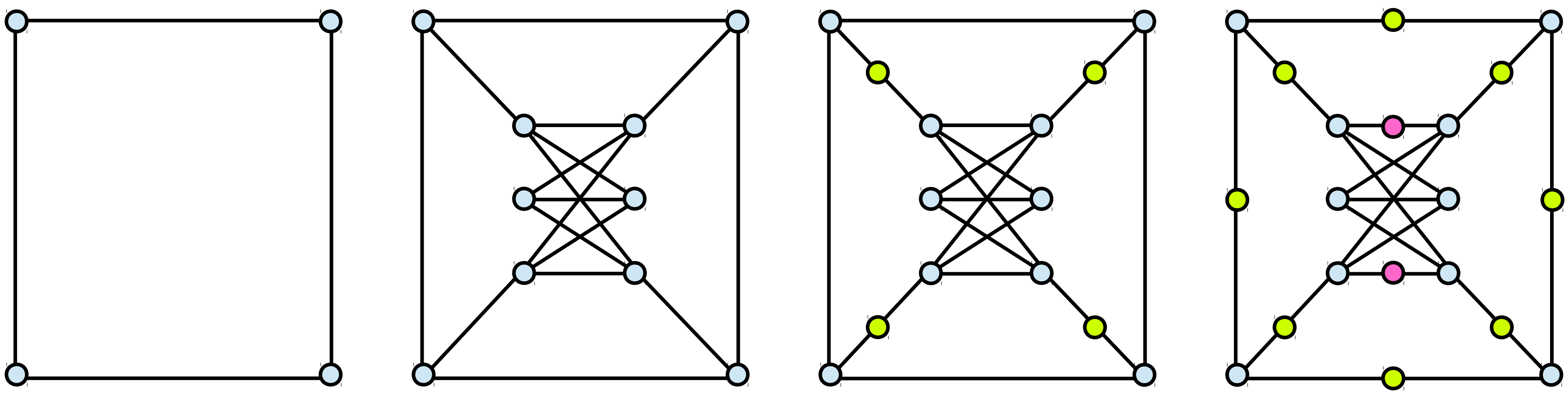


Non-planar **Non-planar** **Non-planar**

All these graphs are similar in determining whether they are planar or not

Non-planar graph examples – embedding $K_{3,3}$

Planar **Non-planar** **Non-planar** **Non-planar**



contains $K_{3,3}$

contains $K_{3,3}$

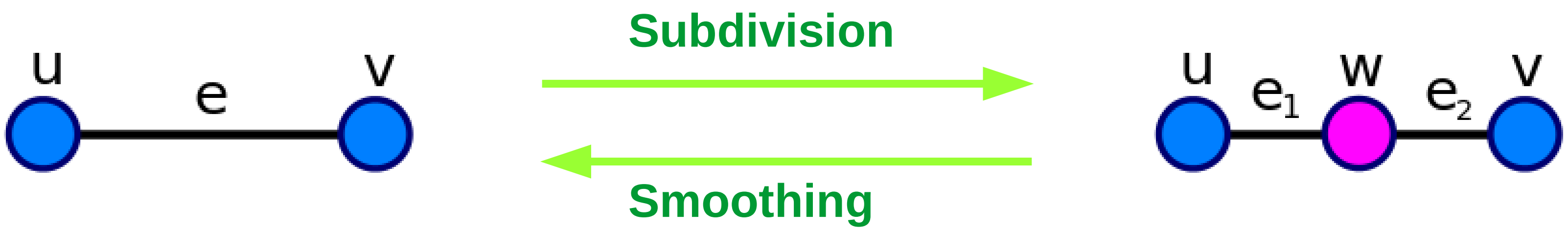
contains a <u>subdivision</u> of $K_{3,3}$

non-planar subgraph

non-planar subgraph

non-planar subgraph

Subdivision and Smoothing



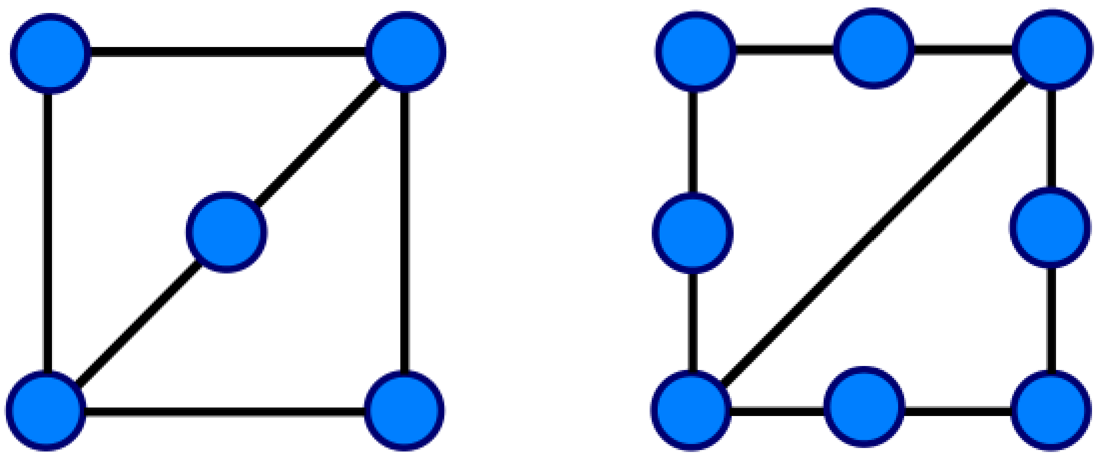
https://en.wikipedia.org/wiki/Planar_graph

Homeomorphism

two graphs G_1 and G_2 are **homeomorphic**
if there is a graph **isomorphism**
from some **subdivision** of G_1
to some **subdivision** of G_2

homeo (identity, sameness)

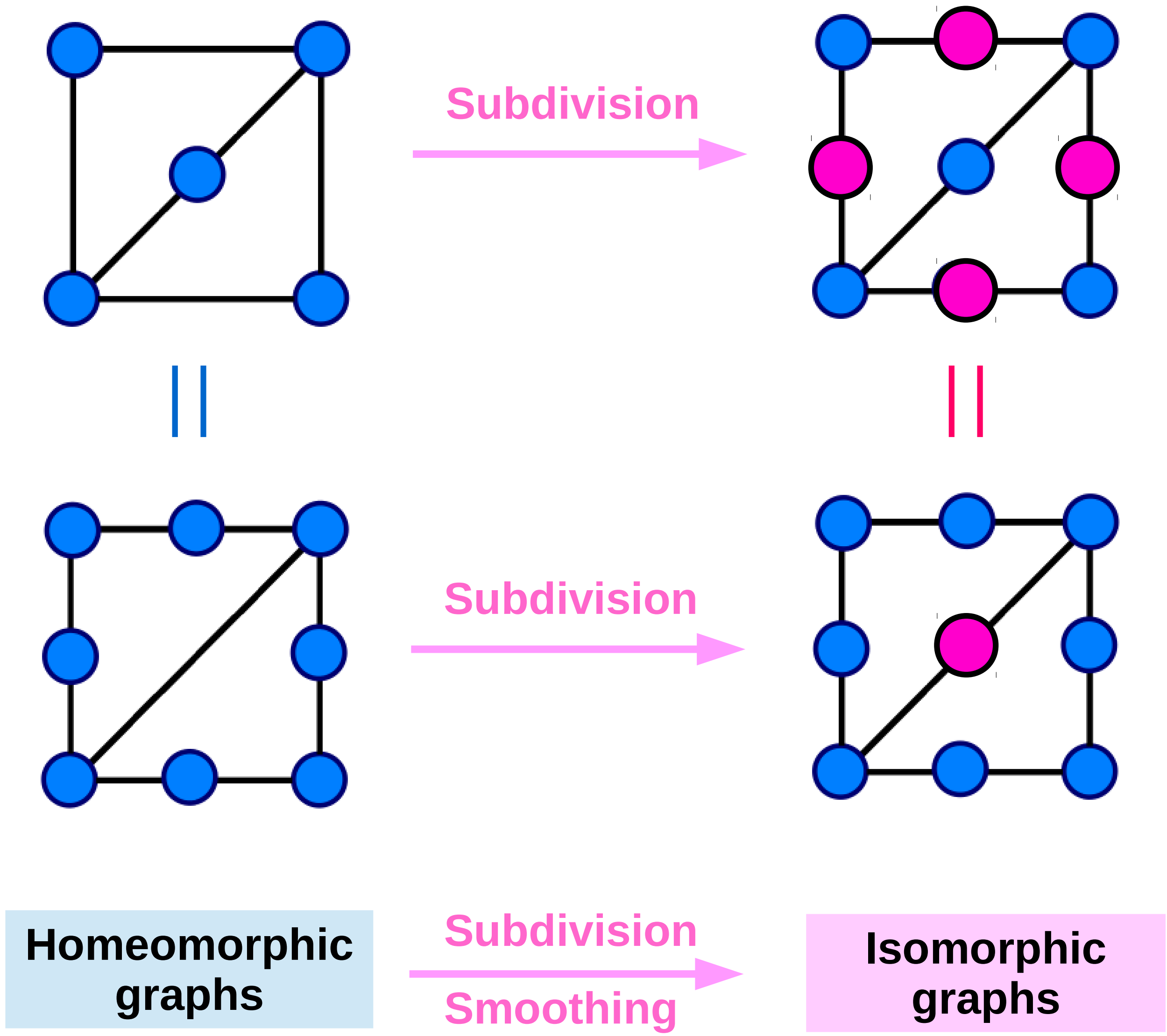
iso (equal)



homeomorphic graphs

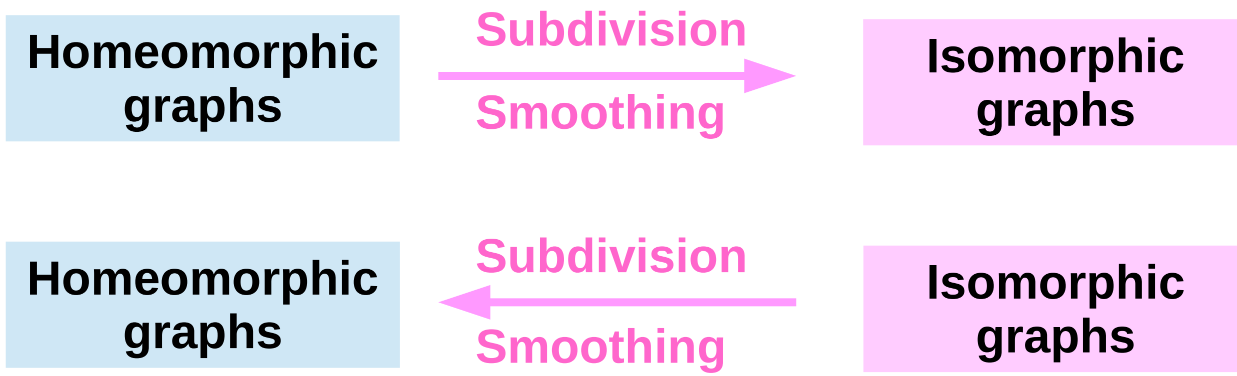
https://en.wikipedia.org/wiki/Planar_graph

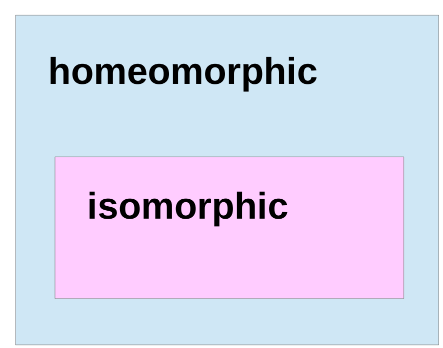
Homeomorphism Examples



https://en.wikipedia.org/wiki/Planar_graph

Homeomorphism and Isomorphism





https://en.wikipedia.org/wiki/Planar_graph

Embedding on a surface

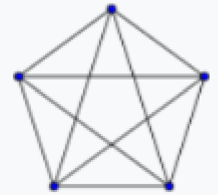
subdividing a graph preserves **planarity**.

Kuratowski's theorem states that

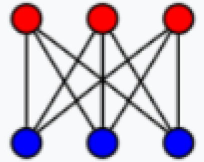
> a finite graph is **planar** if and only if
> it contains **no** subgraph **homeomorphic**
> to $\mathbf{K_5}$ (complete graph on five vertices) or
> $\mathbf{K_{3,3}}$ (complete bipartite graph on six vertices,
> three of which connect to each other of the other three).

In fact, a graph **homeomorphic** to $\mathbf{K_5}$ or $\mathbf{K_{3,3}}$
is called a **Kuratowski subgraph**.

Nonplanar



Complete graph K_5



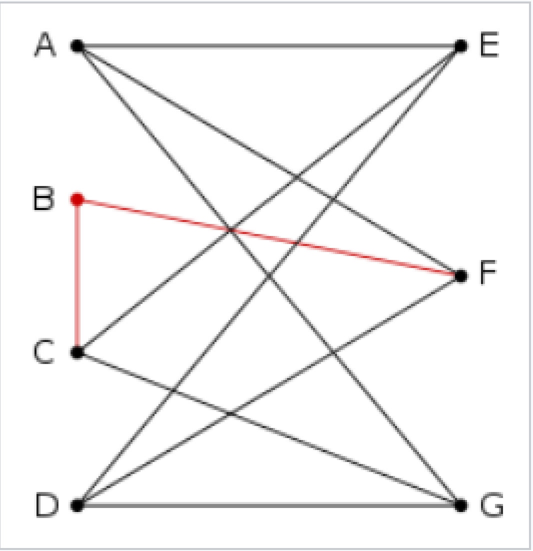
Utility graph $K_{3,3}$

https://en.wikipedia.org/wiki/Planar_graph

Kuratowski's Theorem

A finite graph is **planar** if and only if
it does <u>not</u> contain a **subgraph**
that is a **subdivision** of the complete graph $\mathbf{K_5}$ or
the complete bipartite graph $\mathbf{K_{3,3}}$ (utility graph).

A **subdivision** of a graph results
from inserting vertices into edges
(changing an edge •——• to •—•—•)
<u>zero</u> or <u>more</u> <u>times</u>.

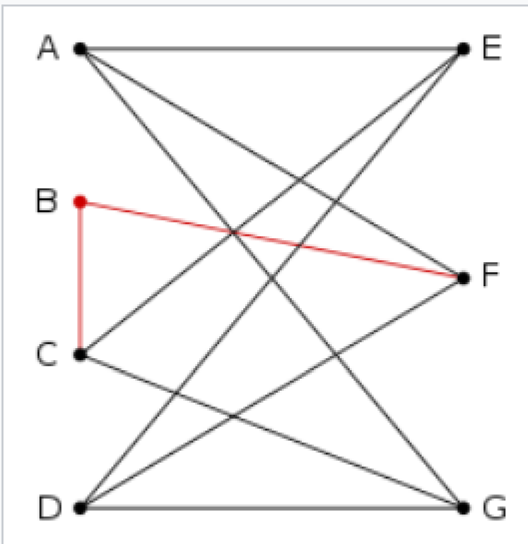


An example of a graph with no K_5 or $K_{3,3}$ subgraph. However, it contains a subdivision of $K_{3,3}$ and is therefore non-planar.

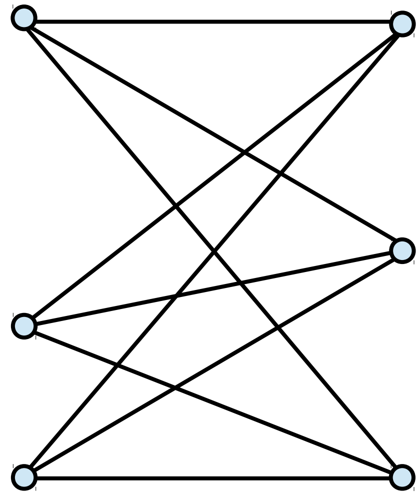
https://en.wikipedia.org/wiki/Planar_graph

Kuratowski's Theorem

homeomorphic



An example of a graph with no K_5 or $K_{3,3}$ subgraph. However, it contains a subdivision of $K_{3,3}$ and is therefore non-planar.



planar

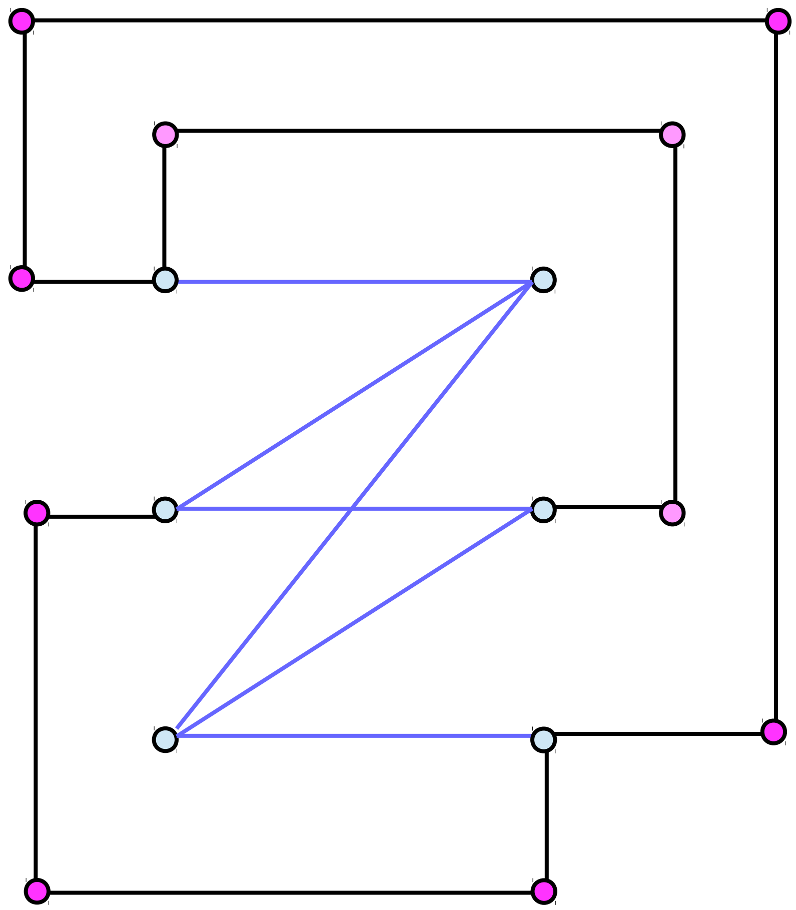
no subgraph **homeomorphic** to $\mathbf{K_5}$ or $\mathbf{K_{3,3}}$

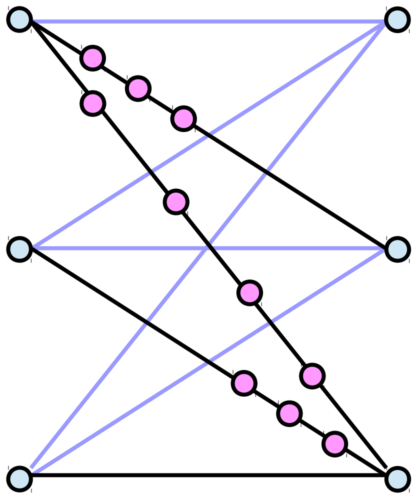
Non-planar

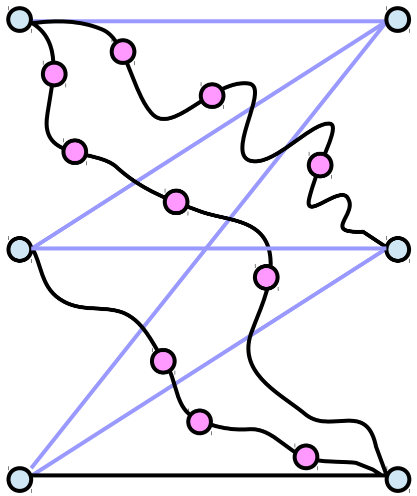
subgraph **homeomorphic** to $\mathbf{K_5}$ or $\mathbf{K_{3,3}}$

https://en.wikipedia.org/wiki/Planar_graph

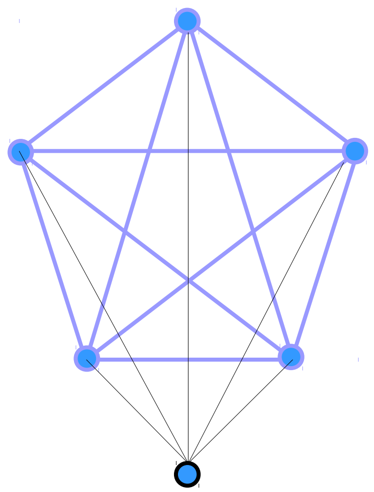
Homeomorphic to $K_{3,3}$

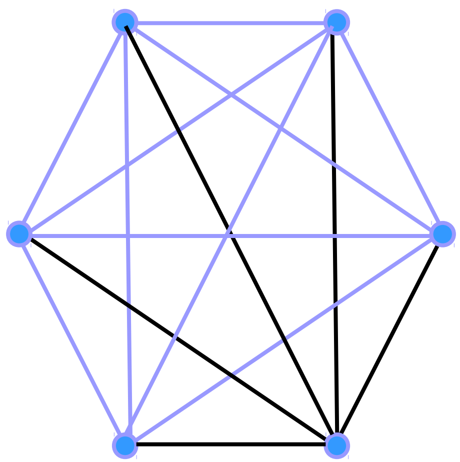


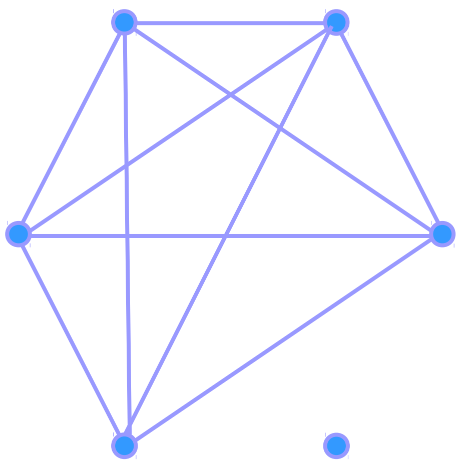


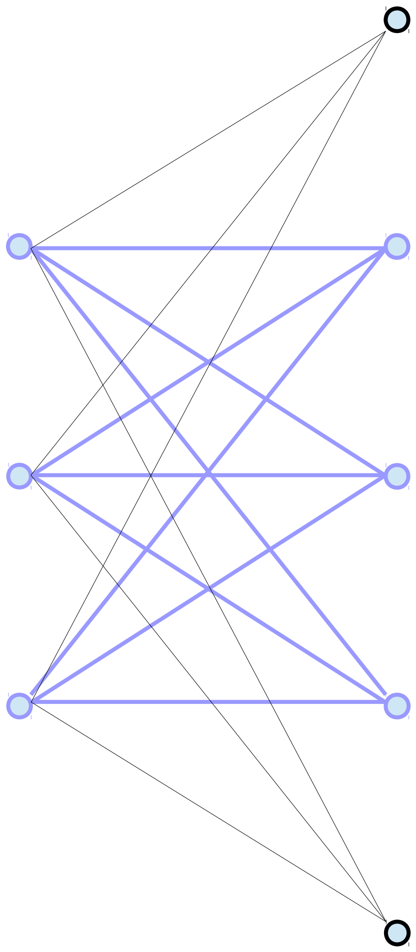


Non-planar graphs: K_6 and $K_{3,3}$









Euler’s Formula

Euler's formula states that if a **finite**, **connected**, **planar graph** is drawn in the plane without any edge intersections, and **v** is the number of **vertices**, **e** is the number of **edges** and **f** is the number of **faces** (regions bounded by edges, including the outer, infinitely large region), then

$$v - e + f = 2$$

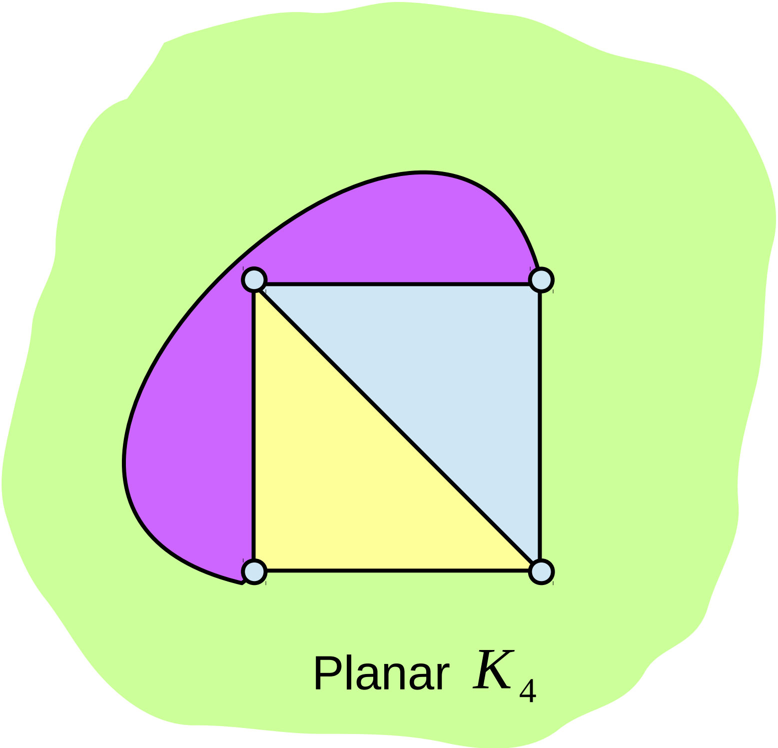
https://en.wikipedia.org/wiki/Planar_graph

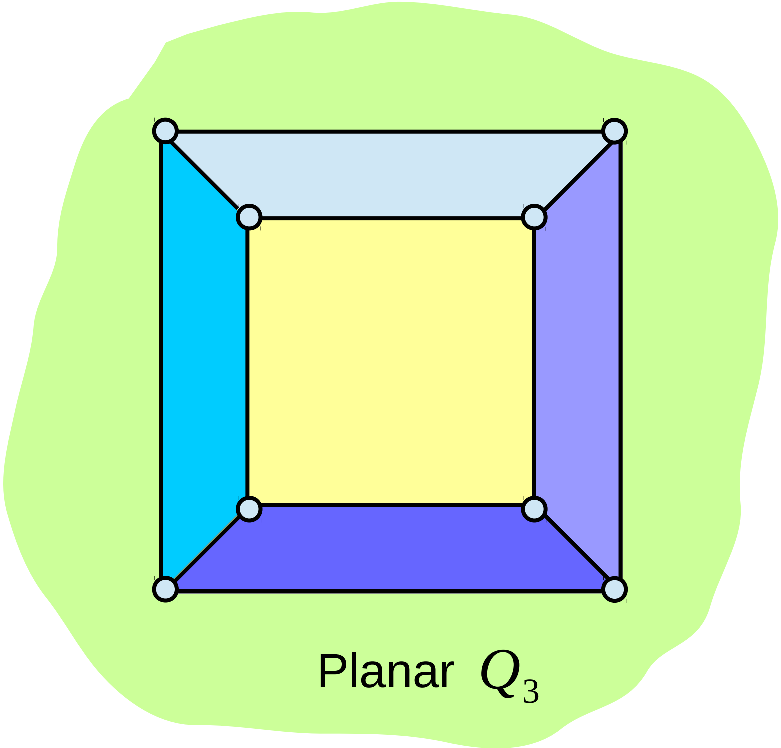
Euler's Formula Examples

v = 4
e = 6
f = 4

v − e + f = 2

v = 8
e = 12
f = 6





https://en.wikipedia.org/wiki/Planar_graph

Corollary 1

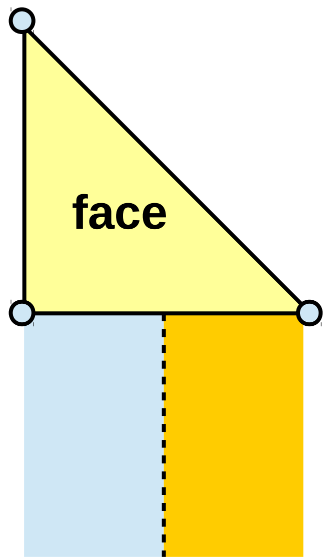
In a **finite**, **connected**, **simple**, **planar graph**,

any **face** (except possibly the outer one)
is bounded by at least three **edges** and

every **edge** touches at most two **faces**;

using Euler's formula, one can then show
that these graphs are **sparse** in the sense that if $v \geq 3$:

$$\mathbf{e \leq 3v - 6}$$

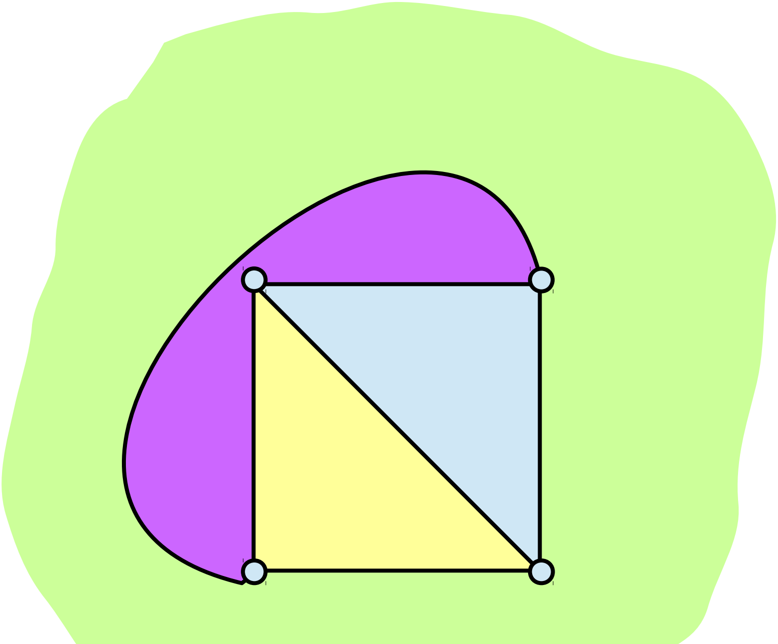


https://en.wikipedia.org/wiki/Planar_graph

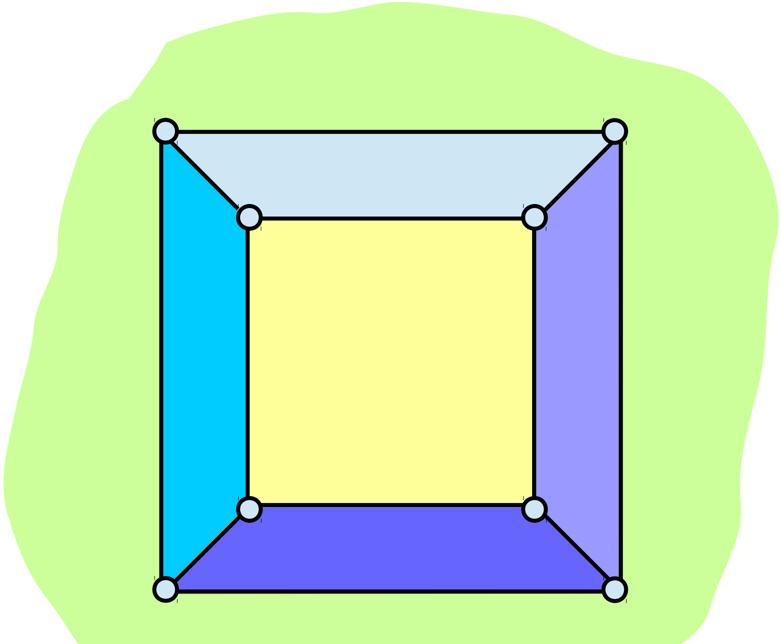
Corollary 1 Examples

v = 4 **e ≤ 3 v − 6**
e = 6 **6 ≤ 3·4 − 6**
f = 4

v = 8 **e ≤ 3 v − 6**
e = 12 **12 ≤ 3·8 − 6**
f = 6



Planar K_4



Planar Q_3

https://en.wikipedia.org/wiki/Planar_graph

Euler’s Formula : Corollary 2

In a **finite**, **connected**, **simple**, **planar graph**,

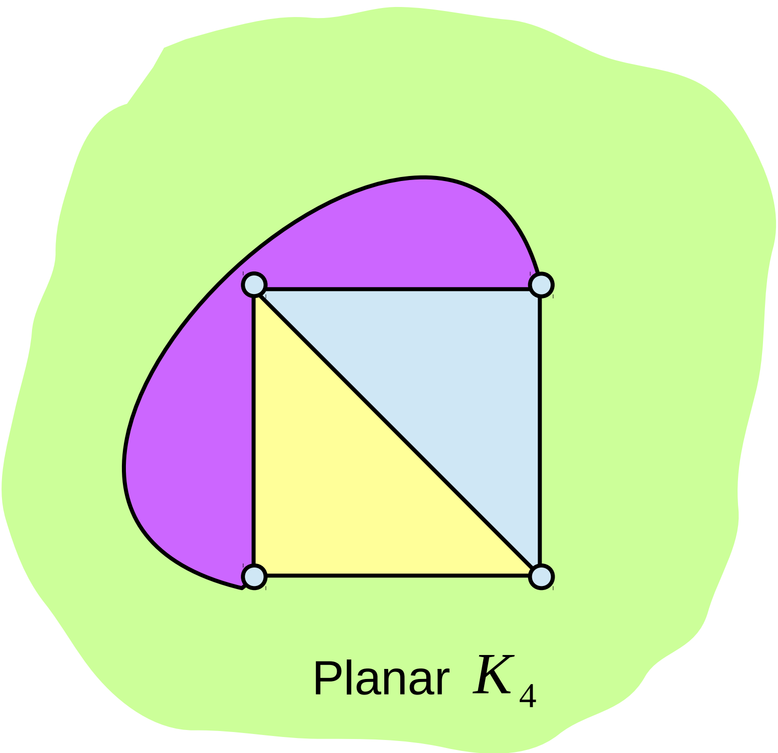
Every vertex has a **degree** not exceeding **5**.

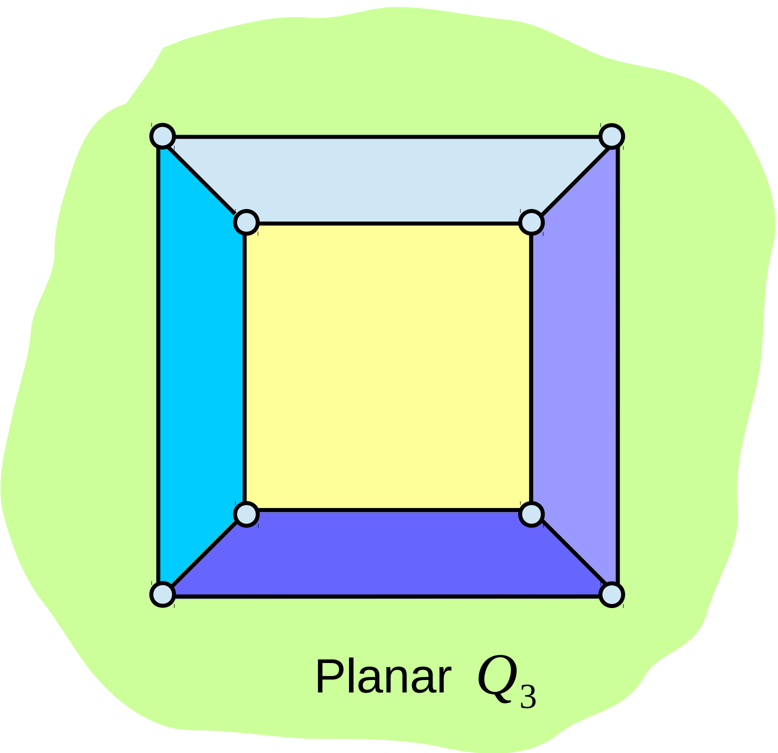
$$\deg(v) \leq 5$$

https://en.wikipedia.org/wiki/Planar_graph

degree: 3 **deg(v) ≤ 5**

degree: 3 **deg(v) ≤ 5**





https://en.wikipedia.org/wiki/Planar_graph

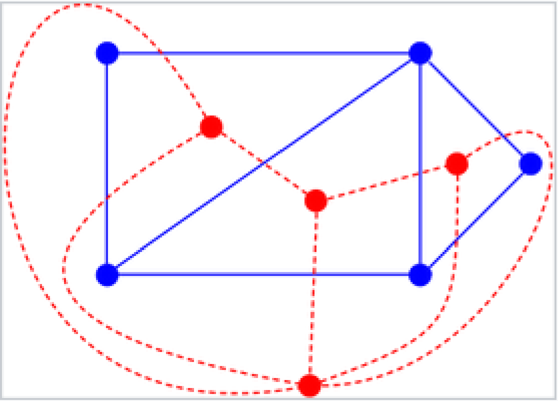
Dual Graph

the dual graph of a plane graph G is a graph that has a **vertex** for each **face** of G.

The dual graph has an **edge** whenever two **faces** of G are separated from each other by an **edge**,

and a **self-loop** when the same **face** appears on both sides of an **edge**.

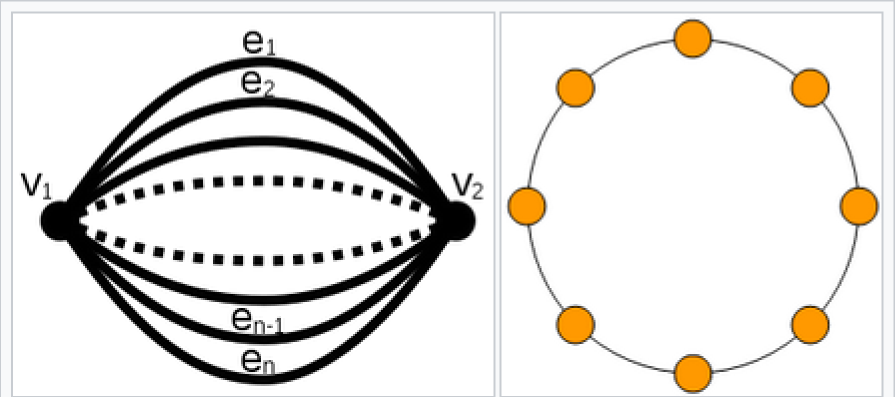
each **edge e** of G has a corresponding **dual edge**, whose endpoints are the **dual vertices** corresponding to the **faces** on either side of **e**.



The red graph is the dual graph of the blue graph, and *vice versa*.

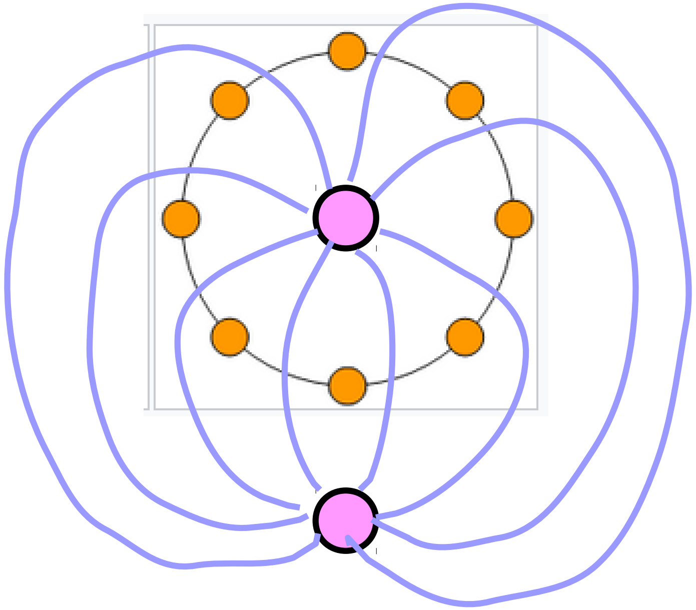
https://en.wikipedia.org/wiki/Dual_graph

Dipoles and Cycles



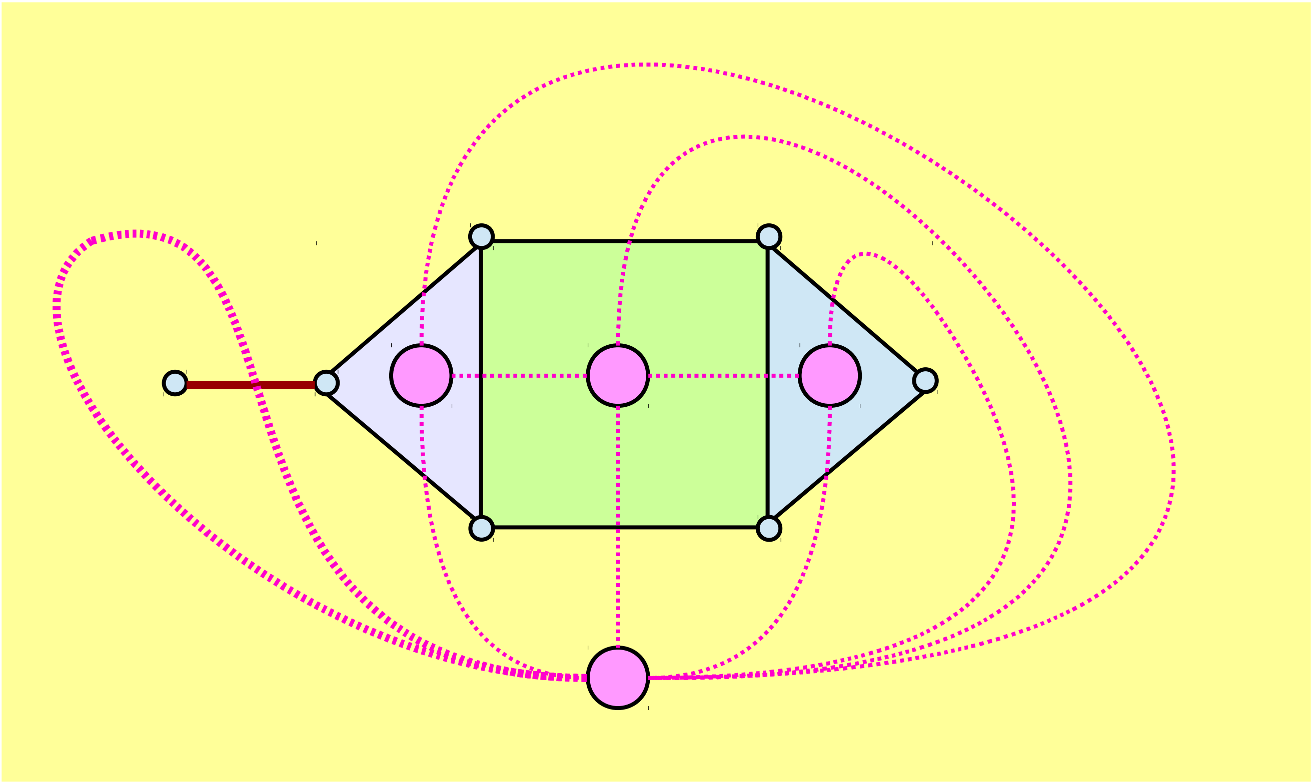
A dipole graph

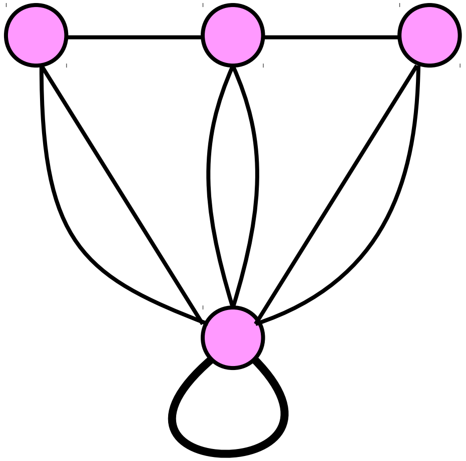
A cycle graph



https://en.wikipedia.org/wiki/Dual_graph

Self-loop in a dual graph





a **self-loop** when the <u>same</u> **face** appears on <u>both</u> <u>sides</u> of an **edge**.

https://www.math.hmc.edu/~kindred/cuc-only/math104/lectures/lect17-slides-handout.pdf

Correspondence between G and G*

Vertices of G*	Faces of G
Edges of G*	Edges of G
Multigraph	Dual of a plane graph
Loops of G*	Cut edge of G
Multiple edges of G*	distinct faces of G with multiple common boundary edges

https://en.wikipedia.org/wiki/Hamiltonian_path

Cut

a **cut** is a **partition** of the **vertices** of a graph
into two disjoint **subsets**.

Any **cut** determines a **cut-set**
the **set** of **edges** that have one endpoint
in each subset of the partition.

These edges are said to **cross** the cut.

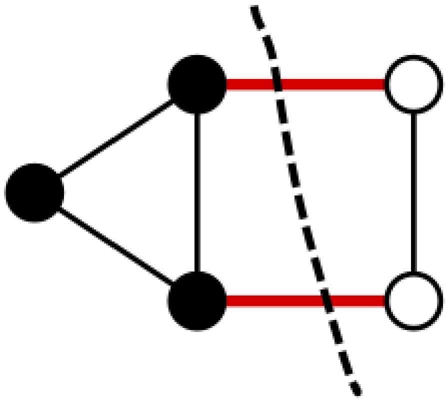
In a connected graph, each **cut-set** determines a unique **cut**,
and in some cases cuts are identified with their **cut**-**sets** rather than
with their **vertex** partitions.

https://en.wikipedia.org/wiki/Cut_(graph_theory)

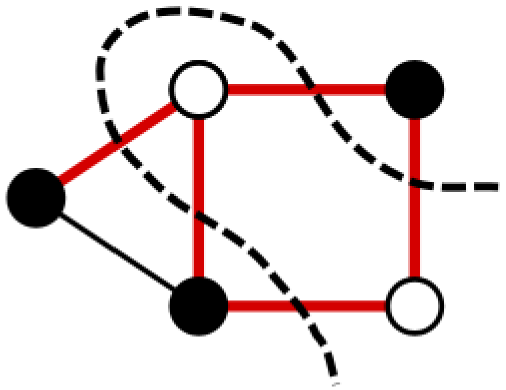
Minimum Cut

A cut is minimum if the size or weight of the cut
is not larger than the size of any other cut.

the size of this cut is 2,
and there is no cut of size 1
because the graph is bridgeless.



https://en.wikipedia.org/wiki/Cut_(graph_theory)



A cut is maximum if the size of the cut
is not smaller than the size of any other cut.

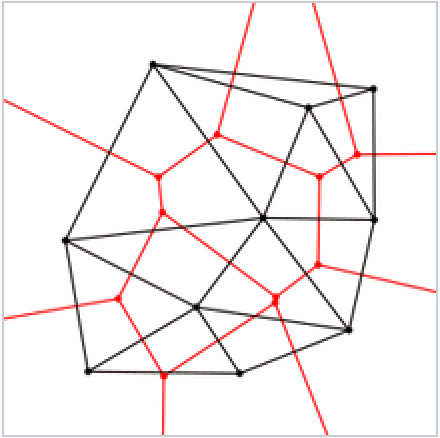
the size of the cut is equal to 5,
and there is no cut of size 6,
or |E| (the number of edges),
because the graph is not bipartite
(there is an odd cycle).

https://en.wikipedia.org/wiki/Cut_(graph_theory)

Infinite Graphs and Tessellations

The concept of duality applies as well
to **infinite graphs** embedded in the plane
as it does to **finite graphs**.

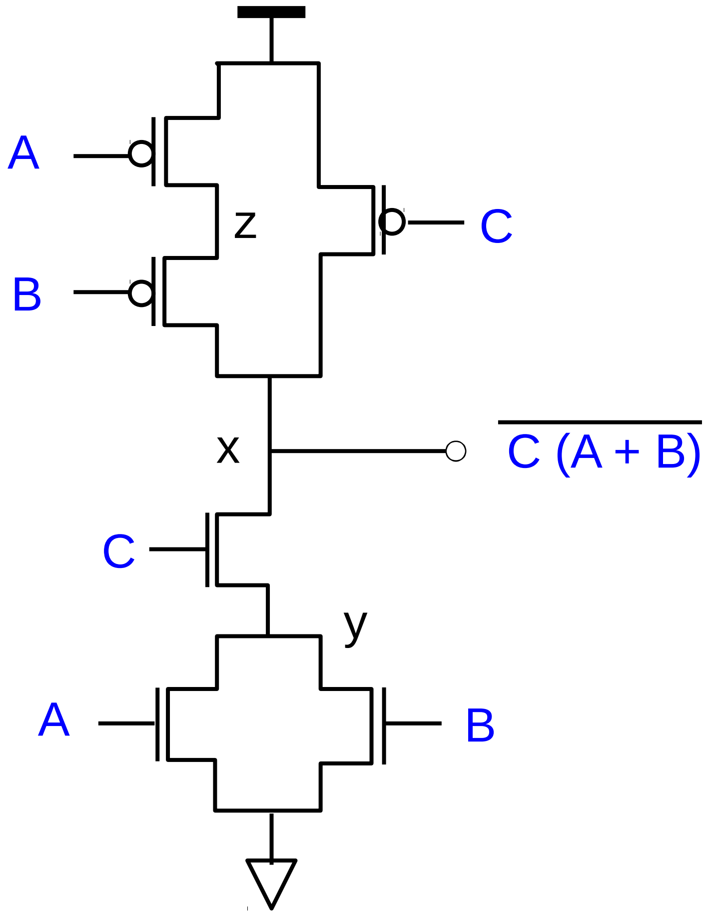
When all faces are bounded regions
surrounded by a cycle of the graph,
an **infinite planar** graph embedding
can also be viewed as a **tessellation** of the plane,
a covering of the plane by closed disks
(the **tiles** of the **tessellation**) whose interiors
(the **faces** of the **embedding**) are disjoint open disks.

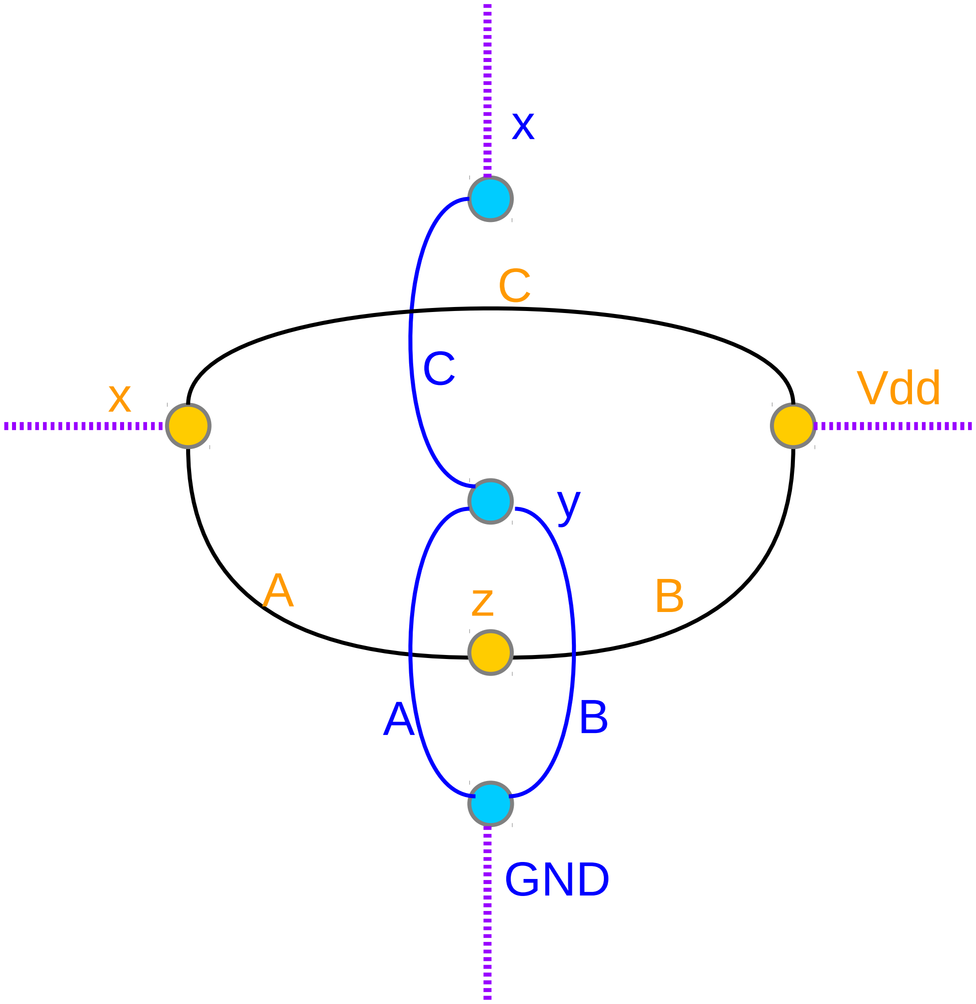


A Voronoi diagram (red) and Delaunay triangulation (black) of a finite point set (the black points)

https://en.wikipedia.org/wiki/Dual_graph

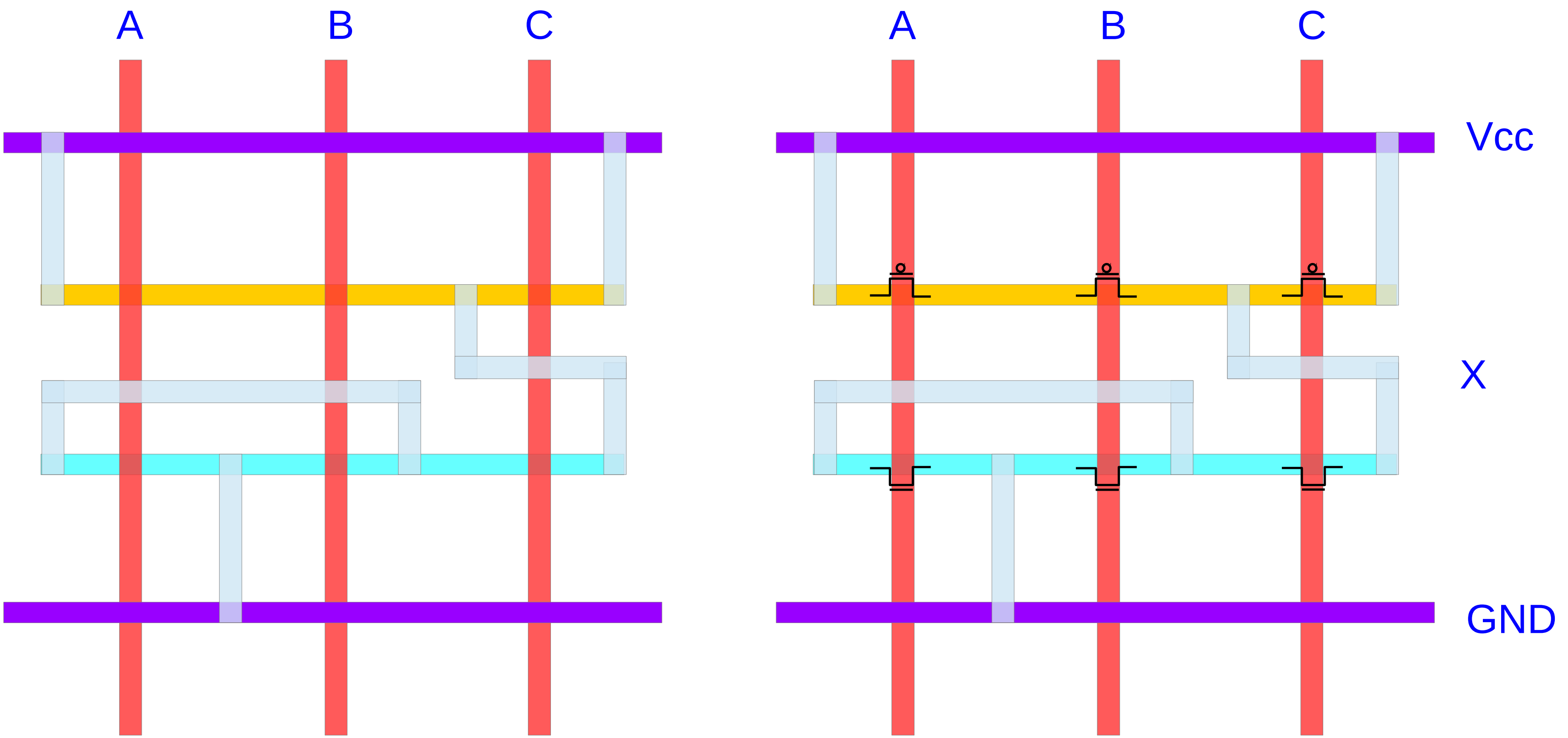
Dual Logic Graph





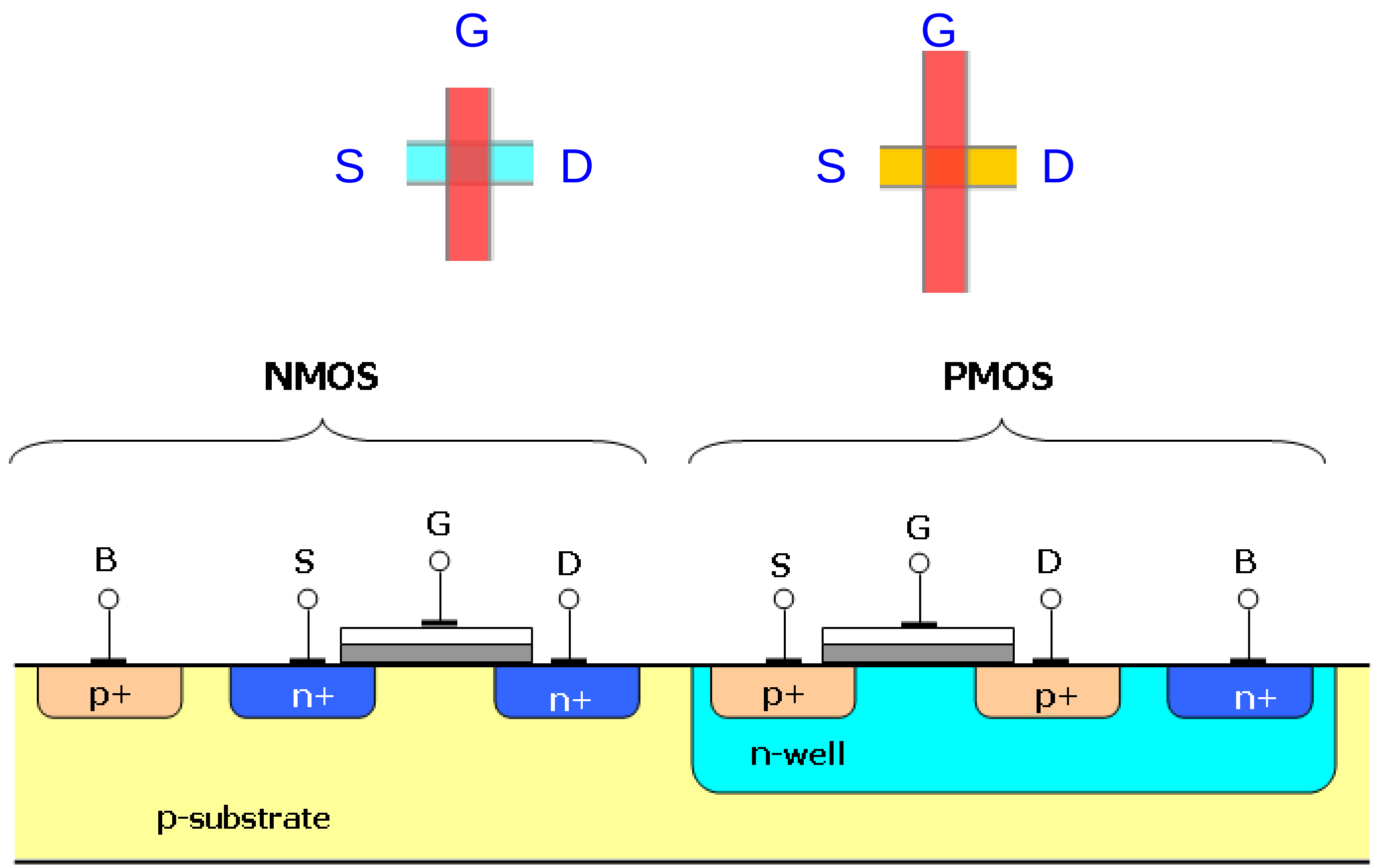
http://www.cse.psu.edu/~kxc104/class/cmpen411/11s/lec/C411L06StaticLogic.pdf

Stick Layout



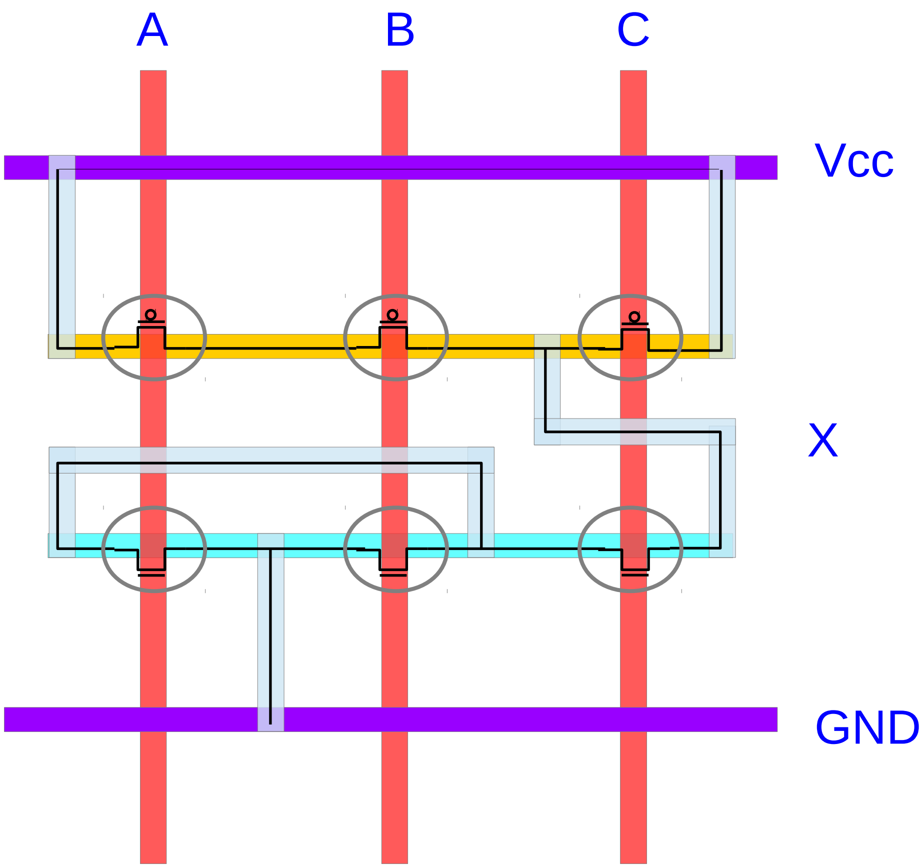
http://www.cse.psu.edu/~kxc104/class/cmpen411/11s/lec/C411L06StaticLogic.pdf

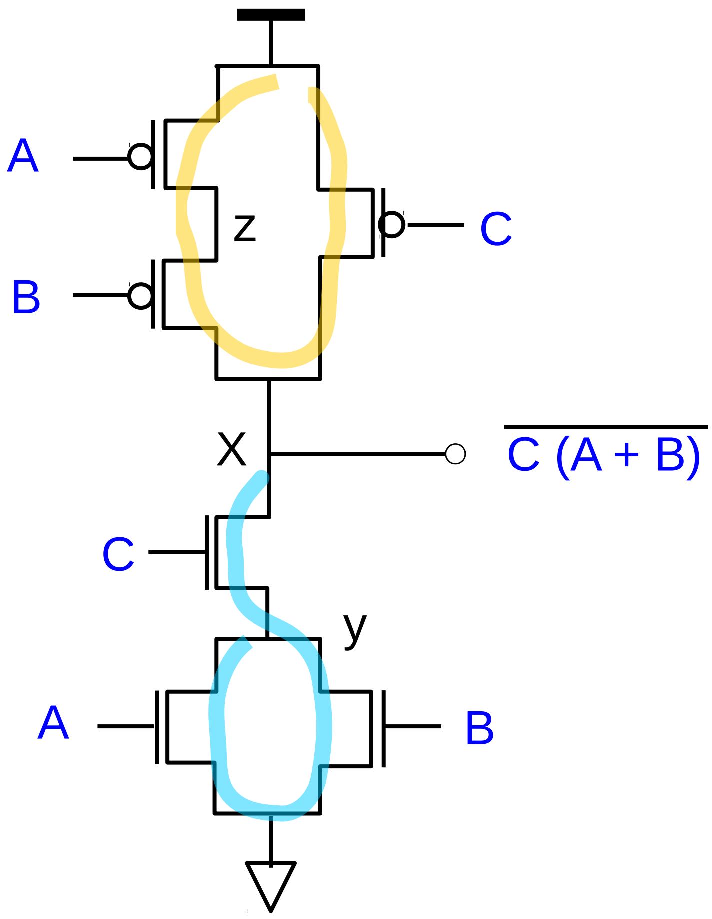
CMOS Transistors and Stick Layout



https://en.wikipedia.org/wiki/CMOS

Single-Strip Stick Graph and Logic Graph





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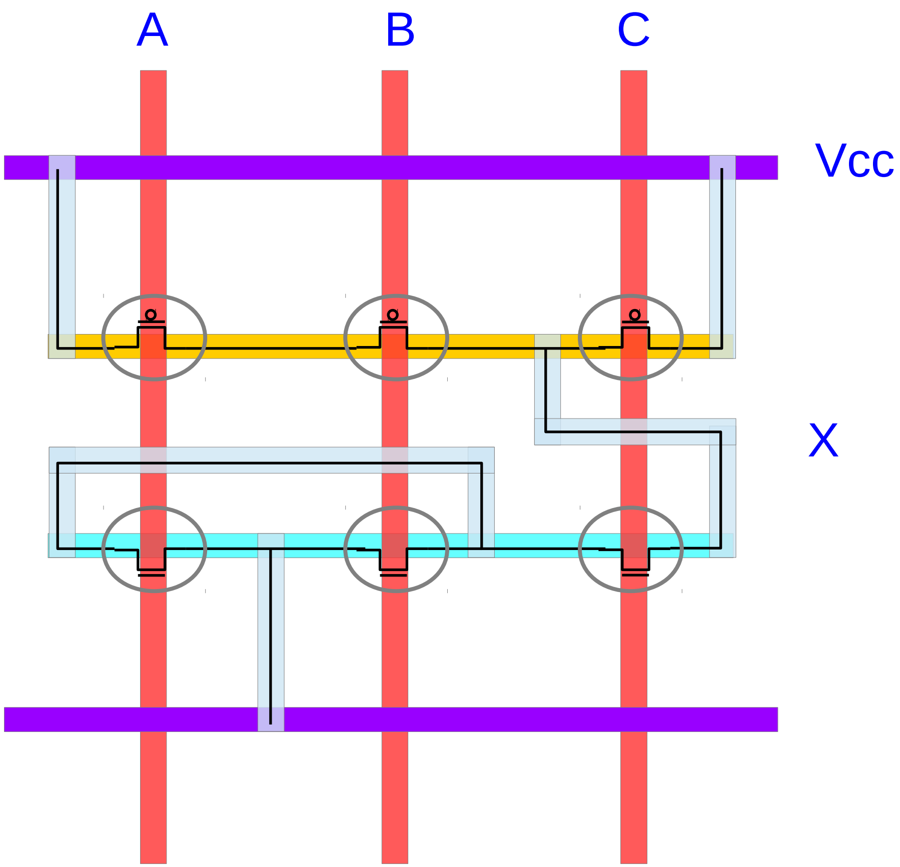
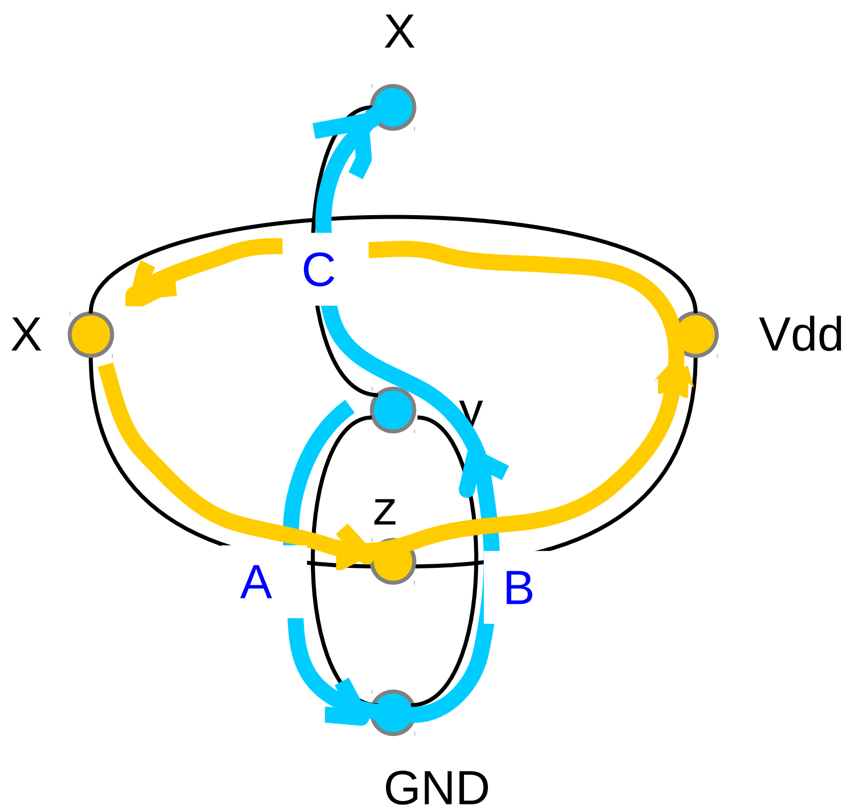
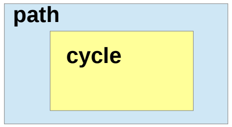
Stick Graph and Logic Diagram



uninterrupted diffusion strip



consistent Euler **paths** (PUN & PDN)



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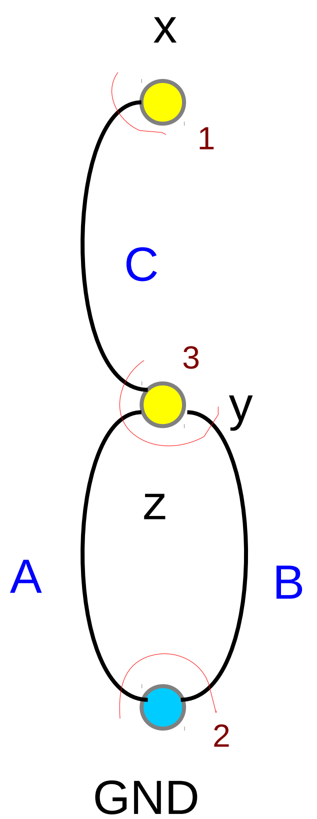
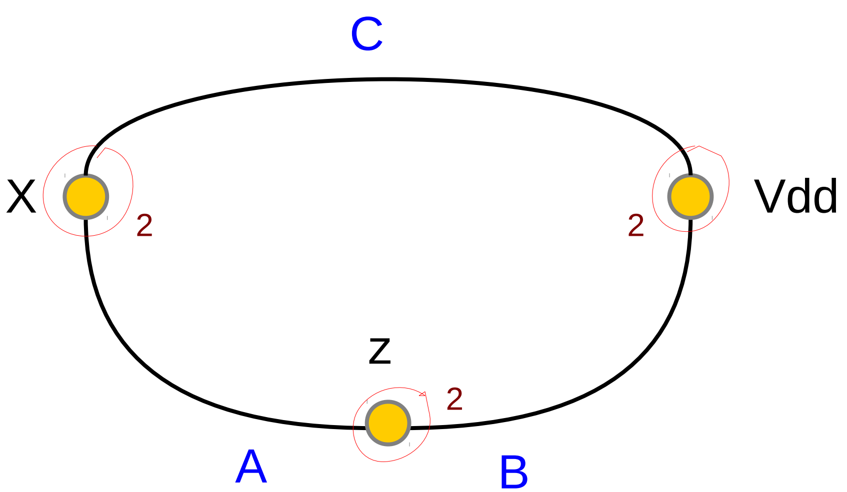
Stick Graph and Logic Diagram



Eulerian Trail



Eulerian Circuit

http://www.cse.psu.edu/~kxc104/class/cmpen411/11s/lec/C411L06StaticLogic.pdf

References

[1] http://en.wikipedia.org/
[2]