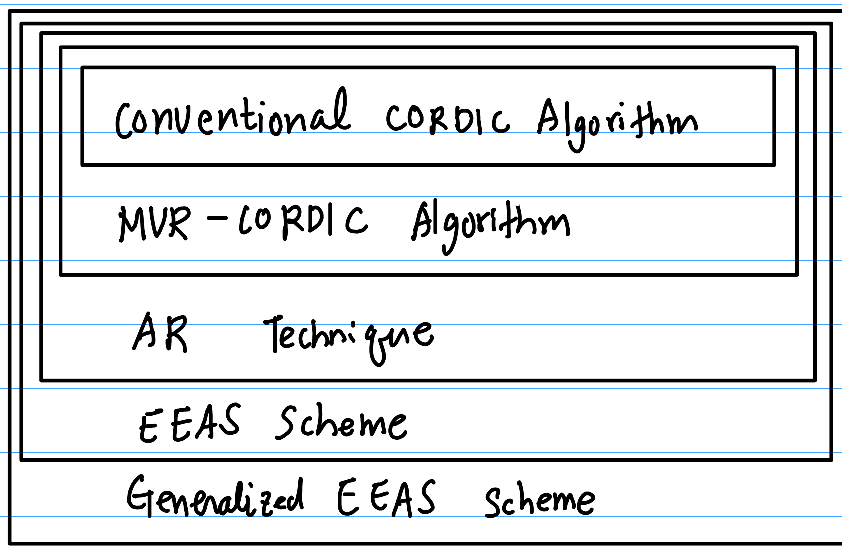
Angle Recoding CORDIC
2. Wu

20180822 Tue

Vector Rotational CORDIC



Vector Rotation Alg	Selection of Rotation Sequence	Elementary Angle Set	Micro Rotation	Angle Quantization	
				θ_i	N_A
Conventional CORDIC	$\mu = \{-1, +1\}$	EAS S	complete	$\mu(i)\, a(i)$	W Fixed
Angle Recoding	$\mu = \{-1, 0, +1\}$	EAS S_1	selective	$\tan^{-1}(\alpha(i) \cdot 2^{-s(i)})$	N' Variable
MVR-CORDIC	$\alpha = \{-1, 0, +1\}$	EAS S_1	selective	$\tan^{-1}(\alpha(i) \cdot 2^{-s(i)})$	R_m Fixed
EEAS	$\alpha_1, \alpha_2 = \{-1, 0, +1\}$	EEAS S_2	selective	$\tan^{-1}(\alpha_0(i) \cdot 2^{-s_0(i)} + \alpha_1(i) \cdot 2^{-s_1(i)})$	R_m Fixed
Generalized EEAS	$\alpha_1, \alpha_2, \cdots, \alpha_{d-1} = \{-1, 0, +1\}$	EEAS S_d $d \geq 3$	selective	$\tan^{-1}(\alpha_0(i) \cdot 2^{-s_0(i)} \ldots + \alpha_{d-1}(i) \cdot 2^{-s_{d-1}(i)})$	R_m Fixed

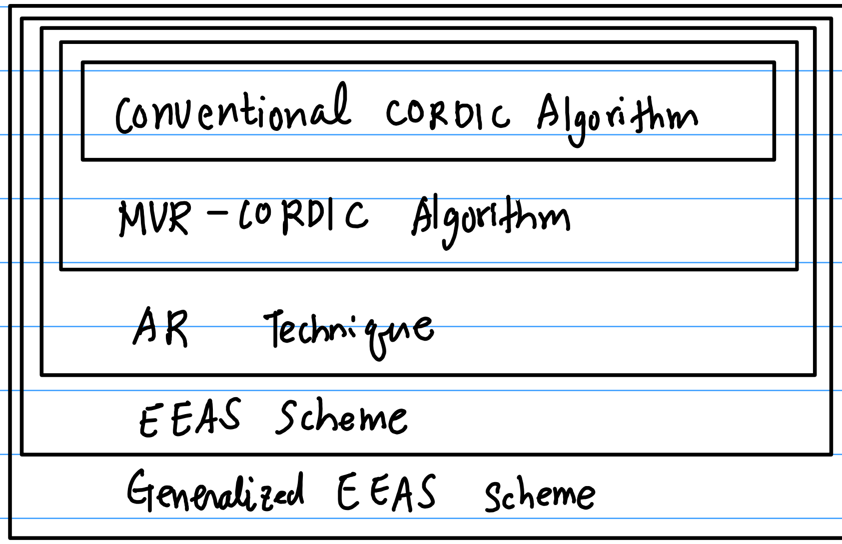
Family of Vector Rotational CORDIC

AQ process (Angle Quantization) —
- CORDIC
- AR (Angle Recoding)
- MVR - CORDIC (Modified Vector Rotation)
- EEAS (Extended Elementary Angle Set)
- Generalized EEAS

AQ process with various EAS and
and suitable combinations of subangles

to decompose the target rotational angle
into several easy-to-implement subangles

minimizing the angle quantization error ξ_m
to obtain the best precision performance



EEAS covers $\left\{ \begin{array}{l} \text{MVR-CORDIC} \\ \text{AR} \end{array} \right.$

a subset of EEAS $\hat{S}_2$ EAS S_1

MVR-CORDIC   a subset of AR

one constraint on the iteration number

Angle Quantization

Quantization process on the rotational angle θ

decompose the original rotational angle θ
into severales θ_i's

sum up those subangles to approximate
the original angle as close as possible

Minimize the angle quantization error

$$\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i$$

N_A : the number of sub-angles

$$\theta = \theta_0 + \theta_1 + \theta_2 + \cdots + \theta_{N_A-1} + \xi_m$$

design issues in the AQ process

① need to determine the sub-angles θ_i

each θ_i needs to be easy-to-implement

② how to select and combine these sub-angles ξ_m

such that the angle quantization error ξ_m

can be minimized

Angle Quantization

Quantization process on the rotational angle θ

decompose θ into several subangles θ_i's

the angle quantization error

$$\boxed{\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i}$$

N_A the number of subangles

$\theta_0, \theta_1, \cdots, \theta_{N_A-1}$

$$\theta = \theta_0 + \theta_1 + \cdots + \theta_{N_A-1} + \xi_m$$

data : W-bit word length

the iteration number : N $\quad N \leqslant W$

the restricted iteration number : R_m $\quad R_m \ll W$

Vector Rotation CORDIC Family

⓪ Conventional CORDIC

① AR

② MVR

③ EEAS

Angle Quantization

Quantization process on the rotational angle θ

decompose θ into several subangles θ_i's

the angle quantization error

$$\boxed{\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i}$$

N_A the number of subangles
$\theta_0, \theta_1, \cdots, \theta_{N_A-1}$

$$\theta = \theta_0 + \theta_1 + \cdots + \theta_{N_A-1} + \xi_m$$

data : W-bit word length		
the iteration number : N		$N \leq W$
the restricted iteration number : R_m		$R_m \ll W$

CSD (Canonical Signed Digit) Quantization

digital filter designs

coefficients are recoded

in terms of SPT (Signed Power of Two) terms

multiplication can be easily realized

with shift-and-add operations

$h_2 = (-0.156249)_{10} \Rightarrow (0.0\bar{1}011)_2$

$W=8$, 3 non-zero digits

① CSD quantization decomposes

coefficients into several SPT terms

(sub-coefficients)

② the multiplication of a coefficient

can be reformed

through the combination of

the non-zero SPT sub-coefficients

quantize the rotation angle θ

decompose the rotation angle θ
into several sub-angles θ_i's

the rotational operation of each θ_i
should be easily realized

If each θ_i can be realized
using only shift-and-add operations

the rotation of θ can be performed
through successive applications of
sub-angle rotations
in a cost-effective way

approximation target	Coefficient h_i	Rotation angle θ
Basic Element	Non-zero digit 2^{-i}	Sub-angle $a(i) = \tan^{-1}(2^{-i})$
Basic Operation	shift-and-add operation	2 shift-and-add operations
Approximation Equation	$h_i \approx \sum_{j=0}^{N_D-1} g_j \cdot 2^{-d_j}$	$\theta \approx \sum_{j=0}^{R_m-1} \alpha(j) \cdot a(s(j))$
	$g_j \in \{-1, 0, +1\}$ $d_j \in \{0, 1, \cdots, w-1\}$ N_D = the number of non-zero digits	N_A = the number of Sub-angles

try to approach the target rotation angle θ step by step

decisions are made in each step by choosing the best combination of $\alpha(i)\, a(s(i))$

so as to minimize $|\xi_m|$

$\alpha(i)$, $a(i)$ are determined such that the error function is minimized

$$J(i) = |\theta(i) - \alpha(i)\, a(s(i))|$$

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\, a(s(m))$$

terminated if no further improvement can be found

$$J(i) \geqslant J(i-1)$$

or $\alpha(R_m - 1)$ and $s(R_m - 1)$ are determined at the end

⓪ Conventional CORDIC

elementary angle $a(i) = \tan^{-1}(2^{-i})$

the number of elementary angles N

the rotation sequence $\mu(i) = \{-1, +1\}$

$+1, -1, -1, +1, +1, \ldots\ldots$

the i-the rotation angle $a(i)$

the W-bit word length

the iteration number $N \leqslant W$

the angle quantization error

$$\boxed{\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)}$$

AQ & conventional CORDIC

EAS (Elementary Angle Set)

comprises of all $a(i)$ for $0 \le i \le N-1$

$$S = \{a(i) : 0 \le i \le N-1\}$$

the CORDIC algorithm essentially performs AQ

tries to perform the rotation

by sequential applications of

micro-rotations of all elementary angles

given a target rotation angle θ

the first rotation sequence $\mu(0)$

for the most significant elementary angle $a(0)$

the second rotation sequence $\mu(1)$

for the next most significant elementary angle $a(1)$

repeated until the last elementary angle is applied.

the sub-angle θ_i in AQ

$\theta_i = \mu(i)\, a(i)$ in CORDIC

$$\mu(i) = \{-1, +1\}$$

$$a(i) = \tan^{-1}(2^{-i})$$

the number of sub-angles

N_A in AQ

N in CORDIC

CORDIC algorithm sequentially apply all θ_i's for $i = 0, 1, \cdots, N-1$

to approximate the target angle θ

iteration number	elementary angle	value in radian
i=0	a(0)=atan(2^{-0})	
i=1	a(1)=atan(2^{-1})	
i=2	a(1)=atan(2^{-2})	
i=3	a(1)=atan(2^{-3})	
i=4	a(1)=atan(2^{-4})	
i=5	a(1)=atan(2^{-5})	
i=6	a(1)=atan(2^{-6})	
i=7	a(1)=atan(2^{-7})	

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, \alpha(i) \qquad \mu(i) = \{-1, 0, +1\}$$

$$= \theta - \sum_{j=0}^{N'} \tilde{\theta}(j)$$

$$N' \equiv \sum_{i=0}^{N-1} |\mu(i)| \qquad \{|{-1}|, |0|, |{+1}|\}$$

the effective iteration number N'

$S(j)$ the rotational sequence

determines the micro-rotation angle in the j-th iteration

$i \leftarrow S(j)$

index $0, 1, \cdots, N'$ (reduced number)

index $0, 1, \cdots, N$ (original number)

$\mu(S(j)) \leftarrow \alpha(j)$

$\Downarrow \qquad \Uparrow$

$\{-1, +1\}$

$$\mu(i) = \begin{cases} \mu(S(j)) & i = S(j) \\ 0 & i \neq S(j) \text{ --- reduced index} \end{cases}$$

er

$$
\begin{array}{rllllllll}
i = & 0, & 1, & 2, & 3, & \cdots, & N-1 & & \\
s(j) = & 0, & 1, & 2, & 3, & \cdots, & N-1 & & \text{rotational sequence} \\
\alpha(j) = & -1, & 0, & 0, & +1, & \cdots, & -1 & & \text{directional sequence} \\
j = & 0, & -, & -, & 1, & \cdots, & N'-1 & & \text{effective iteration number}
\end{array}
$$

$N' = N-2$

the j-th micro-rotation of $a(s(j))$

elementary angle

$$a(i) = \tan^{-1}(2^{-i})$$
$$a(s(j)) = \tan^{-1}(2^{-s(j)})$$

$$\alpha(j)\, a(s(j)) = \alpha(j) \tan^{-1}(2^{-s(j)}) \qquad \alpha(j) \in \{-1, +1\}$$

$$\Leftrightarrow \quad \mu(i)\, a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$= \theta - \left[\sum_{j=0}^{N'} \tilde{\theta}(j) \right]$$

$$= \theta - \left[\sum_{j=0}^{N'} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right) \right] \qquad \alpha(j) \in \{-1, +1\}$$

$$\tilde{\theta}(j) = \alpha(j) \tan^{-1}\left(2^{-s(j)} \right)$$

$$= \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)} \right)$$

$$S_1 = \left\{ \tan^{-1}\left(\boxed{\alpha} \cdot 2^{-\boxed{s}} \right) \quad \boxed{\alpha} \in \{-1, 0, +1\}, \quad \boxed{s} \in \{0, 1, 2, \cdots N-1\} \right\}$$

① AR [Hu]

skip certain micro rotations

the rotation sequence $\mu(i) = \{-1, 0, +1\}$

$\mu(i) = 0 \rightarrow$ skip

desire to minimize

$$\boxed{\sum_{i=0}^{N} |\mu(i)|}$$

so that the total number of CORDIC iterations can be minimized

Angle Recoding ← Multiplier Recoding

angle recoding method for efficient
implementation of the CORDIC algorithm
Hu & Naganathan, ISCAS 89

Greedy algorithm

the angle quantization error

$$\boxed{\xi_{m,AR} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)}$$

$\theta(0) = \theta$, $\{\mu(i) = 0,\ 0 \leqslant i \leqslant N-1\}$, $k = 0$.

repeat until $|\theta(k)| < a(N-1)$ Do

Choose i_k, $0 \leqslant i_k \leqslant N-1$

$$\big|\,|\theta(k)| - a(i_k)\big| = \min_{0 \leqslant i \leqslant N-1} \big|\,|\theta(k)| - a(i_k)\big|$$

$$\theta(k+1) = \theta(k) - \mu(i_k)\, a(i_k)$$

$$\mu(i_k) = \operatorname{Sign}(\theta(k))$$

AQ & AR

$$\xi_{m,AR} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)$$

$$= \theta - \left[\sum_{i=0}^{N'-1} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right) \right]$$

$$= \theta - \left[\sum_{i=0}^{N'-1} \tilde{\theta}(j) \right]$$

$N' \triangleq \sum_{i=0}^{N-1} |\mu(i)|$ the effective transition number

$s(j) \in \{0, 1, \cdots, N-1\}$ the rotational sequence determines the micro-rotation angle in the j-th iteration

$\alpha(j) \in \{-1, 0, +1\}$ the directional sequence controls the direction of the j-th micro-rotation of $a(s(j))$

$\tilde{\theta}(j)$ the jth micro-rotation angle

$$\tilde{\theta}(j) = \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right)$$

AR essentially tries to approximate θ with the combination of selected angle elements from a pre-defined elementary angle set (EAS).

the EAS consists of all possible values of $\tilde{\theta}(j)$'s

the EAS S_1 in AR

$$S_1 = \{ \tan^{-1}(\alpha^* \cdot 2^{-s^*}) : \alpha^* \in \{-1, 0, +1\}, \quad s^* \in \{0, 1, \cdots, N-1\} \}$$

$$\begin{aligned}
\xi_{m,AR} &\equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i) \\
&= \theta - \left[\sum_{i=0}^{N'-1} \tan^{-1}(\alpha(j) \cdot 2^{-s(j)}) \right] \\
&= \theta - \left[\sum_{i=0}^{N'-1} \tilde{\theta}(j) \right]
\end{aligned}$$

$$\xi_m \triangleq \theta - \sum_{i=0}^{N_A-1} \theta_i$$

AR performs AQ of the target angle θ

the sub-angle θ_i becomes $\hat{\theta}(i) = \tan^{-1}(\alpha(i) \cdot 2^{-s(i)})$

N_A $\quad\quad$ N'

EAS S, AR

elementary angle	value
r(1) = atan(-2^{-0})	
r(2) = atan(-2^{-1})	
r(3) = atan(-2^{-2})	
r(4) = atan(-2^{-3})	
r(5) = atan(-2^{-4})	
r(6) = atan(-2^{-5})	
r(7) = atan(-2^{-6})	
r(8) = atan(-2^{-7})	
r(9) = atan(0)	
r(10) = atan(2^{-7})	
r(11) = atan(2^{-6})	
r(12) = atan(2^{-5})	
r(13) = atan(2^{-4})	
r(14) = atan(2^{-3})	
r(15) = atan(2^{-2})	
r(16) = atan(2^{-1})	
r(17) = atan(2^{-0})	

② MVR (Modified Vector Rotational)

two modifications

① repeatition of elementary angles

each micro-rotation of elementary angle
can be performed repeatedly

- more possible combinations
- smaller ξ_m

② confinement of total micro-rotation number

confine the iteration number
in the micro-rotation phase
to R_m ($R_m \ll W$)

The role of R_m is quite similar
to the number of non-zero digit
N_D in CSD recoding scheme

the angle quantization error

$$\boxed{\xi_{m,MVR} \triangleq \theta - \sum_{i=0}^{R_m-1} \alpha(i)\, a(s(i))}$$

the rotational sequence

$s(i) \in \{0, 1, \cdots, W-1\}$

the micro-rotation angle
in the i-th iteration

the directional sequence

$\alpha(i) \in \{-1, 0, +1\}$

the direction of the i-th
micro-rotation of $a(s(i))$

$$\alpha(i)\, a(s(i)) = \tilde{\theta}(j)$$

AQ & MVR CORDIC

$$\xi_{m,MVR} \triangleq \theta - \left[\sum_{j=0}^{R_m-1} \alpha(j)\, a(s(j)) \right]$$

the rotational sequence $s(j)$

$$j = 0, 1, 2, \cdots, R_m-1$$

$$\downarrow$$

$$s(j) \in \{0, 1, \cdots, W-1\}$$ rotational sequence

determines the micro-rotation angle $a(s(j))$ in the j-th iteration

the directional sequence $\alpha(j)$

$$\alpha(j) \in \{-1, 0, +1\}$$

controls the direction of the j-th micro-rotation of $a(s(j))$

$$\alpha(j)\, a(s(j)) = \tilde{\theta}(j)$$

$$
\begin{array}{rll}
i = & 0, \boxed{1, 2,} 3, \cdots, W-1 & \\
s(j) = & 0, \boxed{1, 2,} 3, \cdots, W-1 & \text{rotational sequence} \\
\alpha(j) = & -1, \boxed{0, 0,} +1, \cdots, -1 & \text{directional sequence} \\
j = & 0, -, -, 1, \cdots, R_m-1 & \text{effective iteration number}
\end{array}
$$

$R_m \ll W$

sub-angle $(\alpha(j)\, a(s(j)))$ ~ $\tilde{\theta}(j)$

$$\xi_{m,AR} = \theta - \left[\sum_{j=0}^{N'-1} \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right) \right]$$

$$= \theta - \left[\sum_{j=0}^{N'-1} \tilde{\theta}(j) \right], \qquad \tilde{\theta}(j) = \tan^{-1}\left(\alpha(j) \cdot 2^{-s(j)}\right)$$

$N' \triangleq \sum_{j=0}^{N-1} |\mu(j)|$ the effective iteration number

EAS formed by MVR-CORDIC
is the same as AR
also performs AQ

the EAS consists of all possible values of $\tilde{\theta}(j)$

the EAS S_1 in AR

$$S_1 = \{ \tan^{-1}(\alpha^* \cdot 2^{-s^*}) : \alpha^* \in \{-1, 0, +1\},\ s^* \in \{0, 1, \cdots, N-1\} \}$$

The major difference

1) the total number of sub-angles N_A

the total iteration number

in the micro-rotation phase

is kept fixed to a pre-defined value of R_m

$N_A = R_m$

2) the sub-angle θ_i corresponds to $\alpha(j)\, a(s(j))$

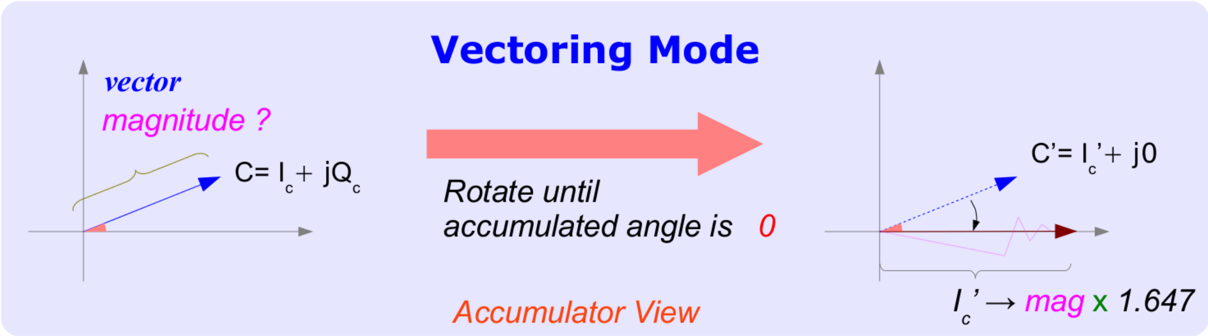
$$\theta_j = \alpha(j)\, a(s(j)) = \tilde{\theta}_j$$

MVR (Modified Vector Rotation)

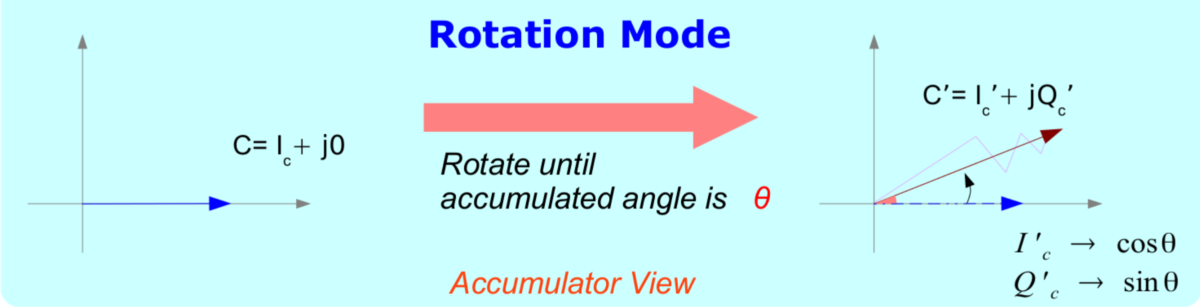
1) Repeat of Elementary Angles θ_i, θ_i

2) fixed total micro-rotation Number R_m

* Vector Rotation Mode
* and the rotation angles are known in advance



vector rotation mode



Modified Vector Rotational MVR CORDIC

- reduce the iteration number
- maintaining the SQNR performance
- modifying the basic microrotation procedure

Three Searching Algorithm

1. the selective prerotation
2. the selective scaling
3. iteration-tradeoff scheme

Optimization Problem

EAS point of view

Given θ, find the combination of R_m elementary angles from EAS S_1, such that the angle quantization error $|\xi_{m,MUR}|$ is minimized.

semi-greedy algorithm

trade offs between computational complexities and performance

key issue in the MVR-CORDIC
is to find the best sequences of
$s(i)$ and $\alpha(i)$ to minimize $|\xi_m|$
subject to the constraint that
the total iteration number is confined to R_m

1) Greedy Algorithm
2) Exhaustive Algorithm
3) Semigreedy Algorithm

1) Greedy Algorithm

try to approach the target rotation angle, θ, step by step
in each step, decisions are made on $\alpha(i)$ and $s(i)$
by choosing the best combination of $\alpha(i)\,a(s(i))$
so as to minimize $|\xi_m|$

$\alpha(i)$ and $s(i)$ are determined such that
the error function $J(i) = |\theta(i) - \alpha(i)\,a(s(i))|$ is minimited

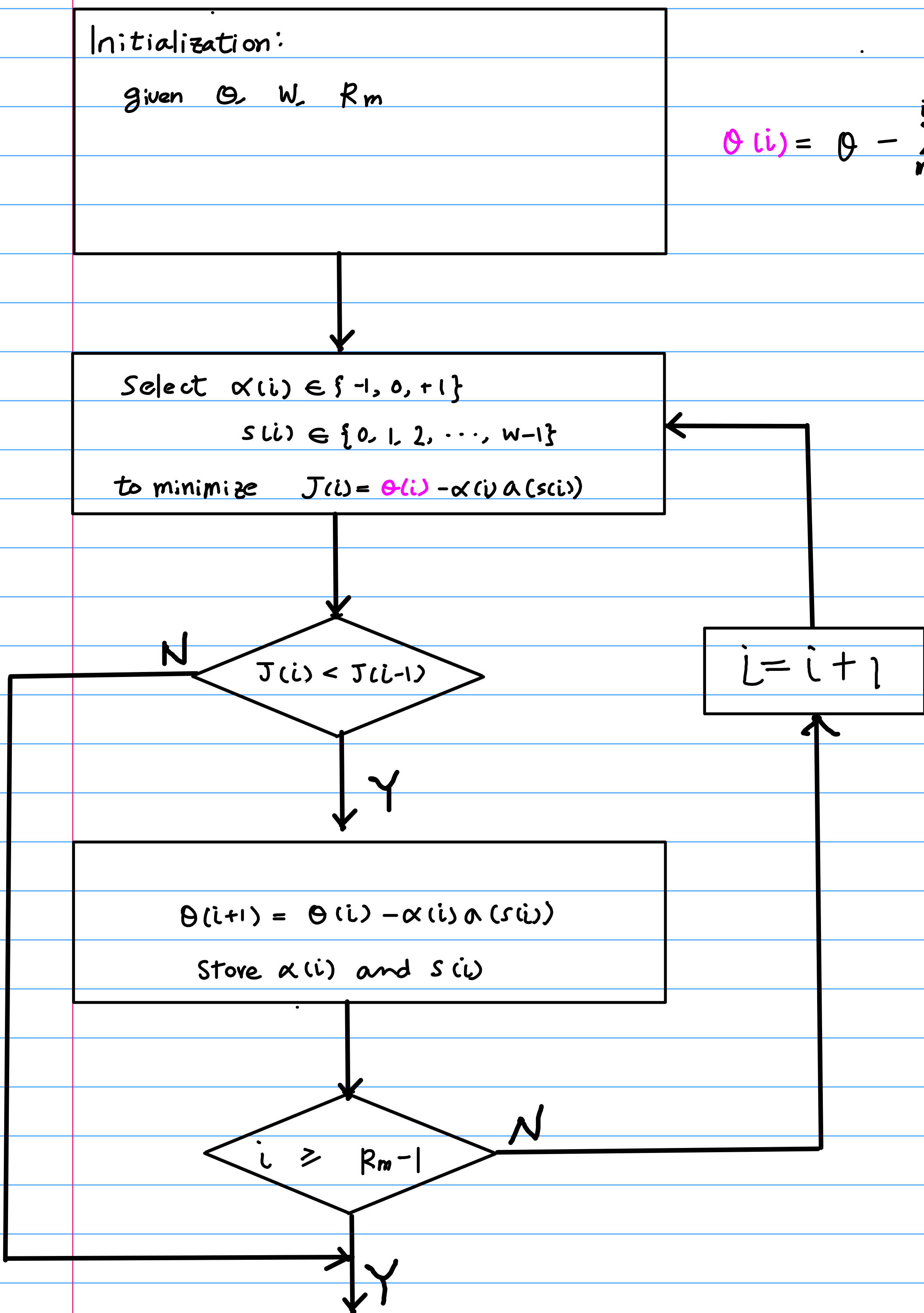
$\theta(i)$: the residue angle in the i-th step

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\,a(s(m))$$

the searching is terminated
if no further improvements can be found
$J(i) \geq J(i-1)$
$\alpha(R_m - 1)$ and $s(R_m - 1)$ are determined
at the end of the searching

the greedy algorithm terminates
only when the residue angle error
cannot be further reduced.

$$\theta(i) = \theta - \sum_{m=0}^{i-1} \alpha(m)\, a(s(m))$$



2) Exhaustive Algorithm

search for the entire solution space

all possible combinations of

$$\sum_{i=0}^{Rm-1} \alpha(i)\, a(s(i))$$

in a single step

decisions for $\alpha(i)$ and $s(i)$, $0 \leq i \leq Rm-1$

by minimizing the error function

$$J = \left| \theta - \sum_{i=0}^{Rm-1} \alpha(i)\, a(s(i)) \right|$$

global optimal solution

Initialization:

given Θ, W, R_m

let $\Theta(0) = \Theta$,

$i = 0$

$J(-1) = \infty$

Select $\alpha(i) \in \{-1, 0, +1\}$

$s(i) \in \{0, 1, 2, \cdots, W-1\}$

for $0 \leq i \leq R_m - 1$

to minimize $J(i) = \Theta - \sum_{i=0}^{R_m-1} \alpha(i)\, a(s(i))$

Store $\alpha(i)$ and $s(i)$

for $0 \leq i \leq R_m - 1$

$(3 \cdot W) \cdot (3 \cdot W) \cdots (3 \cdot W)$

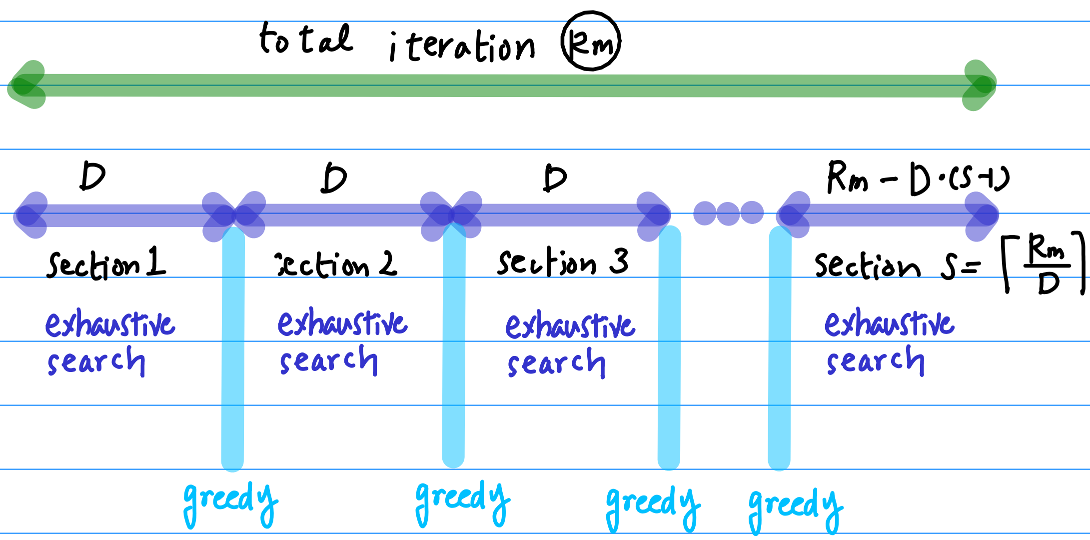
$= 3^{R_m} \cdot W^{R_m}$

3) Semi-greedy Algorithm

a combination of greedy and exhaustive algorithm

the search space of $\alpha(i)$ and $s(i)$ for $0 \le i \le R_m - 1$
are divided into several sections
with $\underbrace{D}_{\text{block length}}$ iterations as a $\underbrace{\text{segment}}_{\text{block}}$

the segmentation scheme



in the i-th block

decision of $\alpha(k)$ and $s(k)$ for $iD \leq k \leq (i+1)D-1$

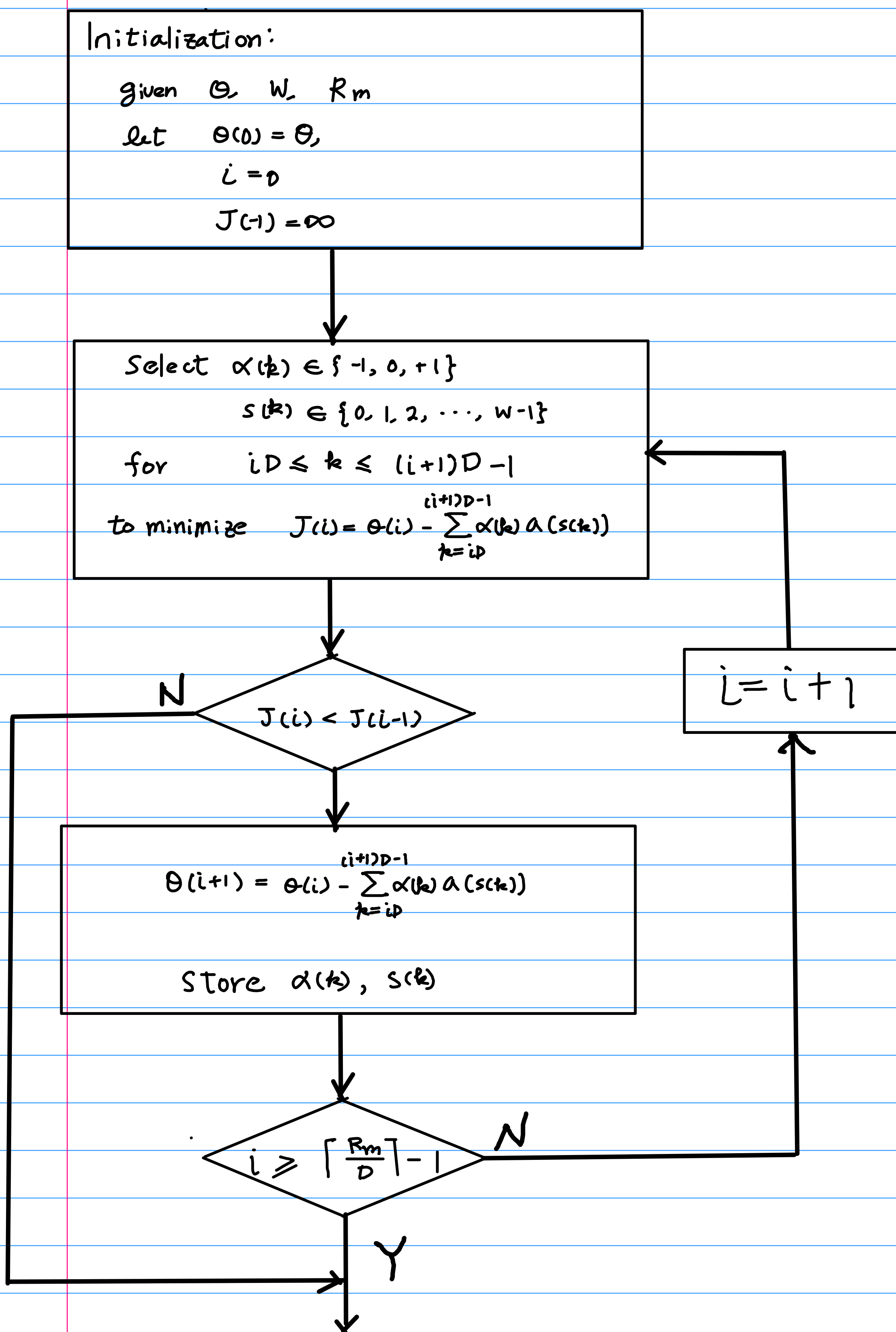
minimizes $$J = \left| \theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k)) \right|$$

where $$\theta(i) = \theta - \sum_{m=0}^{i-1} \left[\sum_{k=mD}^{(m+1)D-1} \alpha(k)\, a(s(k)) \right]$$

the residue angle in the i-th step

$S = \left\lceil \frac{R_m}{D} \right\rceil$

$$\theta(i) = \theta - \left[\sum_{k=0 \cdot D}^{1 \cdot D-1} \alpha(k)\, a(s(k)) + \sum_{k=1D}^{2 \cdot D-1} \alpha(k)\, a(s(k)) + \cdots + \sum_{k=2D}^{S \cdot D-1} \alpha(k)\, a(s(k)) \right]$$


Initialization:
given Θ, W, R_m
let $\Theta(0) = \Theta$,
$i = 0$
$J(-1) = \infty$
Select $\alpha(k) \in \{-1, 0, +1\}$
$s(k) \in \{0, 1, 2, \cdots, W-1\}$
for $iD \leq k \leq (i+1)D-1$
to minimize $J(i) = \Theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k))$
$J(i) < J(i-1)$
N
$\Theta(i+1) = \Theta(i) - \sum_{k=iD}^{(i+1)D-1} \alpha(k)\, a(s(k))$
Store $\alpha(k)$, $s(k)$
$i \geq \lceil \frac{R_m}{D} \rceil - 1$
N
Y
$i = i + 1$

③ Extended EAS-based CORDIC

$$S_2 = \{ \tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\}, \quad s_0^*, s_1^* \in \{0, 1, \cdots, W-1\} \}$$

$$\theta_i = \tan^{-1}(\alpha_0(j) \cdot 2^{-s_0(j)} + \alpha_1(j) \cdot 2^{-s_1(j)})$$

the angle quantization error

$$\boxed{|\xi_{m,EAAS}| \triangleq \left| \theta - \sum_{j=0}^{R_m - 1} \tan^{-1}(\alpha_0(j) \cdot 2^{-s_0(j)} + \alpha_1(j) \cdot 2^{-s_1(j)}) \right|}$$

Generalized EEAS Scheme

$$S_d = \{ \tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*} \quad \alpha_{d-1}^* \cdot 2^{-s_{d-1}^*}) : \alpha_0^*, \alpha_1^*, \cdots, \alpha_{d-1}^* \in \{-1, 0, +1\},\ s_0^*, s_1^*, \cdots, s_{d-1}^* \in \{0, 1, \cdots, W-1\} \}$$

Extended EAS (EEAS) - Wu

more flexible way of decomposing the rotation angle

better
the number of iterations
the error performance

$$S_{EAS} = \{ (\sigma \cdot \tan^{-1}(2^{-r})) : \sigma \in \{+1, 0, -1\}, \; r \in \{1, 2, \cdots, n-1\} \}$$

$$S_{EEAS} = \{ (\sigma_1 \cdot \tan^{-1}(2^{-r_1}) + \sigma_2 \cdot \tan^{-1}(2^{-r_2})) : \sigma_1, \sigma_2 \in \{+1, 0, -1\}, \; r_1, r_2 \in \{1, 2, \cdots, n-1\} \}$$

The pse do-rotation
for i-th micro rotations

$$x_{i+1} = x_i - [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}]\, y_i$$
$$y_{i+1} = y_i + [\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)}]\, x_i$$

The pseudo-rotated vector $[x_{R_m}, y_{R_m}]$
after R_m (the requred number of micro-rotations)

Needs to be scaled by a factor $K = \prod K_i$

$$K_i = \left[1 + \left(\sigma_1(i) \cdot 2^{-r_1(i)} + \sigma_2(i) \cdot 2^{-r_2(i)} \right)^2 \right]^{-\frac{1}{2}}$$

$$\tilde{x}_{i+1} = \tilde{x}_i - [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}]\, \tilde{y}_i$$
$$\tilde{y}_{i+1} = \tilde{y}_i + [k_1(i) \cdot 2^{-s_1(i)} + k_2(i) \cdot 2^{-s_2(i)}]\, \tilde{x}_i$$

$\tilde{x}_0 = x_{R_m}$ $\qquad$ $k_1, k_2 \in \{-1, 0, 1\}$

$\tilde{y}_0 = y_{R_m}$ $\qquad$ $s_1, s_2 \in \{1, 2, \dots, n-1\}$

[21] C.-S. Wu, A.-Y. Wu, and C.-H. Lin, "A high-performance/low-latency vector rotational CORDIC architecture based on extended elementary angle set and trellis-based searching schemes," *IEEE Trans. Circuits Syst. II: Anal. Digital Signal Process.*, vol. 50, no. 9, pp. 589–601, Sep. 2003.

IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS—I: FUNDAMENTAL THEORY AND APPLICATIONS, VOL. 49, NO. 10, OCTOBER 2002

A Unified View for Vector Rotational CORDIC Algorithms and Architectures Based on Angle Quantization Approach

An-Yeu Wu and Cheng-Shing Wu

AR : to approximate θ
with the combination
of selected angle elements
from a pre-defined **EAS**
(Elementary Angle Set)

EAS : all possible values of $\theta(j)$

EAS $\hat{S}_1 = \{ \tan^{-1}(\alpha^* \cdot 2^{-s^*}) : \alpha^* \in \{-1, 0, +1\},$
$s^* \in \{0, 1, \cdots, N-1\} \}$

EAS $\hat{S}_1$ consists of $\tan^{-1}$(Single **S**igned **P**ower of **T**wo)
$\tan^{-1}$(Single **SPT**)
$\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

SPT-based digital filter design

to increase the coefficient resolution

→ imploy more SPT terms to represent filter coefficients

[12] H. Samueli, "An improved search algorithm for the design of multiplierless FIR filters with power-of-two coefficients," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 1044–1047, July 1989.

[13] Y. C. Lim, R. Yang, D. Li, and J. Song, "Signed power-of-two term allocation scheme for the design of digital filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 577–584, May 1999.

EAS $\mathcal{S}_1$ consists of $\tan^{-1}$(single signed power of two)

$\tan^{-1}$(single SPT)

$\tan^{-1}(\alpha^* \cdot 2^{-s^*})$

EAS $\mathcal{S}_2$ consists of $\tan^{-1}$(two signed power of two)

$\tan^{-1}$(two SPT)

$\tan^{-1}(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*})$

Two Signed-Power-of-Two terms

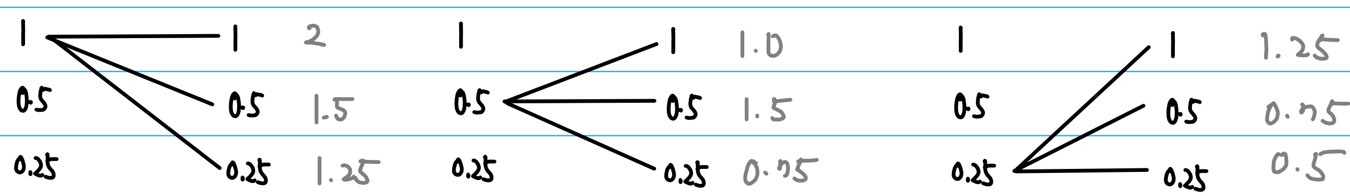
$$S_2 = \left\{ \tan^{-1}\left(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}\right) : \begin{array}{l} \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\} \\ s_0^*, s_1^* \in \{0, 1, \cdots, w-1\} \end{array} \right\}$$

S_1

1	$1 = 2^{-0}$	$\tan^{-1}(2^{-0})$
0.5	$\frac{1}{2} = 2^{-1}$	$\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\tan^{-1}(2^{-2})$

S_2

2	$1+1 = 2^{-0} + 2^{-0}$	$\pm\tan^{-1}(2^{-0} + 2^{-0})$
1.5	$1+\frac{1}{2} = 2^{-0} + 2^{-1}$	$\pm\tan^{-1}(2^{-0} + 2^{-1})$
1.25	$1+\frac{1}{4} = 2^{-0} + 2^{-2}$	$\pm\tan^{-1}(2^{-0} + 2^{-2})$
1.0	$1 = 2^{-0}$	$\pm\tan^{-1}(2^{-0})$
0.75	$\frac{1}{2}+\frac{1}{4} = 2^{-1} + 2^{-2}$	$\pm\tan^{-1}(2^{-1} + 2^{-2})$
0.5	$\frac{1}{2} = 2^{-1}$	$\pm\tan^{-1}(2^{-1})$
0.25	$\frac{1}{4} = 2^{-2}$	$\pm\tan^{-1}(2^{-2})$



$2^{-0}, \; 2^{-1}, \; 2^{-2}$ $\qquad$ $\{0, 1, 2\} = \{0, 1, w-1\}$

$w=3$

$s_0^*, s_1^* \in \{0, 1, 2\}$

$2^{s_0^*}, 2^{s_1^*} \in \{2^{-0}, 2^{-1}, 2^{-2}\}$

as the wordsize W increases,
the size of the set S_2 increases exponentially

$$\theta_i = \tan^{-1}\left(\alpha_0(j)\cdot 2^{-S_0(j)} + \alpha_1(j)\cdot 2^{-S_1(j)}\right)$$

R_m : the number of the subangle N_A

$$S_2 = \{ \theta_i \mid i = 0, 1, \cdots R_m \}$$

$$S_2 = \{ \tan^{-1}\left(\alpha_0^* \cdot 2^{-s_0^*} + \alpha_1^* \cdot 2^{-s_1^*}\right) : \alpha_0^*, \alpha_1^* \in \{-1, 0, +1\}, \; s_0^*, s_1^* \in \{0, 1, \cdots, W-1\} \}$$

the optimization problem of the EEAS-based CORDIC algorithm

given θ and R_m
find $\alpha_0(j)$, $\alpha_1(j)$, $S_0(j)$, and $S_1(j)$
the combination of elementary angles
from EEAS S_2

minimize the angle quantization error

$$|\xi_{m,\,EEAS}| \triangleq \theta - \sum_{j=0}^{R_m-1} \tan^{-1}\left(\alpha_0(j)\, 2^{-S_0(j)} + \alpha_1(j)\, 2^{-S_1(j)}\right)$$

EAS S_1

elementary angle

```
r(1) = atan(-2^{-0})
r(2) = atan(-2^{-1})
r(3) = atan(-2^{-2})
r(4) = atan(       0 )
r(5) = atan( 2^{-2})
r(6) = atan( 2^{-1})
r(7) = atan( 2^{-0})
```

EEAS S_2

```
r(1)   = atan(-2^{-0} - 2^{-0})
r(2)   = atan(-2^{-0} - 2^{-1})
r(3)   = atan(-2^{-0} - 2^{-2})
r(4)   = atan(-2^{-0}          )
r(5)   = atan(-2^{-1} - 2^{-2})
r(6)   = atan(-2^{-1}          )
r(7)   = atan(-2^{-2}          )
r(8)   = atan(       0          )
r(9)   = atan( 2^{-2}          )
r(10) = atan( 2^{-1}          )
r(11) = atan( 2^{-1}+ 2^{-2})
r(12) = atan( 2^{-0}          )
r(13) = atan( 2^{-0}+ 2^{-2})
r(15) = atan( 2^{-0}+ 2^{-1})
r(16) = atan( 2^{-0}+ 2^{-0})
```

given $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix}$

$$\begin{bmatrix} x(j+1) \\ y(j+1) \end{bmatrix} = \begin{bmatrix} 1 & \alpha_0(j)\,2^{-S_0(j)} + \alpha_1(j)\,2^{-S_1(j)} \\ \alpha_0(j)\,2^{-S_0(j)} + \alpha_1(j)\,2^{-S_1(j)} & 1 \end{bmatrix} \begin{bmatrix} x(j) \\ y(j) \end{bmatrix}$$

$$\begin{bmatrix} x_f \\ y_f \end{bmatrix} = P \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix} = \frac{1}{\prod_{j=0}^{R_0-1} \sqrt{1 + [\alpha_0(j) \cdot 2^{-S_0(j)} + \alpha_1(j) \cdot 2^{-S_1(j)}]^2}} \begin{bmatrix} x(R_m) \\ y(R_m) \end{bmatrix}$$

micro rotation procedure

the scaling operation

4 additions

increased hardware

reduced iteration steps

Rotation Angle $\theta = \frac{13\pi}{64}$

Conventional CORDIC

$\bar{\mu} = [1\ \ -1\ \ 1\ \ 1\ \ -1\ \ 1\ \ -1\ \ -1\ \ 1\ \ -1\ \ -1\ \ 1\ \ 1\ \ -1\ \ 1\ \ 1]$

Angle Recoding - Greedy

$\bar{\mu} = [1\ \ 0\ \ 0\ \ -1\ \ 0\ \ 0\ \ -1\ \ -1\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1]$

MVR - CORDIC - Greedy

$\bar{\alpha} = [1\ \ -1\ \ -1\ \ -1]$

$\bar{s} = [0\ \ 3\ \ 6\ \ 7]$

MVR - CORDIC - Semi Greedy (P=2)

$\bar{\alpha} = [1\ \ -1\ \ -1\ \ 1]$

$\bar{s} = [0\ \ 3\ \ 5\ \ 7]$

MVR - CORDIC - TBS

$\bar{\alpha} = [1\ \ 1\ \ -1\ \ -1]$

$\bar{s} = [1\ \ 2\ \ 4\ \ 7]$

EEAS - Greedy

$\bar{\alpha}_0 = [1\ \ -1]$ $\bar{\alpha}_1 = [-1\ \ -1]$

$\bar{s}_0 = [0\ \ 2]$ $\bar{s}_1 = [8\ \ 10]$

EEAS - TBS $R_m = 2$

$\bar{\alpha}_0 = [1\ \ 1]$ $\bar{\alpha}_1 = [-1\ \ -1]$

$\bar{s}_0 = [0\ \ 6]$ $\bar{s}_1 = [3\ \ 5]$

EEAS - TBS $R_m = 3$

$\bar{\alpha}_0 = [1\ \ -1\ \ 1]$ $\bar{\alpha}_1 = [-1\ \ 1\ \ -1]$

$\bar{s}_0 = [0\ \ 3\ \ 7]$ $\bar{s}_1 = [15\ \ 6\ \ 2]$