Redundant CORDIC Timmermann (C)

20170204

Termination Algorithms
    Modified CORDIC
CSD (Canonic Sign Digit) Encoding
    Booth Encoding

Low Latency Time CORDIC Algorithms - Timmermann (1992)
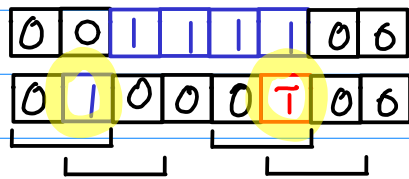Redundant and on-line CORDIC - Ercegovac & Lang (1990)

CSD (Canonic Signed Digit)

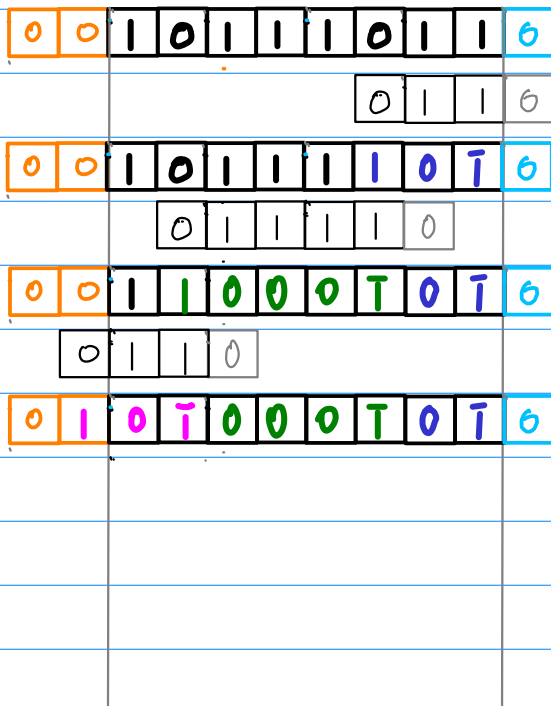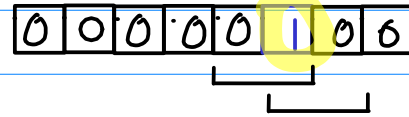like Booth encoding   (not modified Booth)

all non-zero digits are separated by zeros

$\Rightarrow$ $\boxed{\sigma_i \, \sigma_{i+1} = 0}$

·1-run

| 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | $\bar{1}$ | 0 | 0 |

$0 \cdot 1 = 0$ $0 \cdot \bar{1} = 0$
$1 \cdot 0 = 0$ $\bar{1} \cdot 0 = 0$

| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |

| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 |

| | | | | | | | 0 | 1 | 1 | 0 |

Iterative Reduction of 1-runs

| 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | $\bar{1}$ | 0 |

| | | | | 0 | 1 | 1 | 1 | 1 | 0 | |

$\boxed{\sigma_i \, \sigma_{i+1} = 0}$

| 0 | 0 | 1 | 1 | 0 | 0 | 0 | $\bar{1}$ | 0 | $\bar{1}$ | 0 |

| | 0 | 1 | 1 | 0 | | | | | | |

Unique encoding

| 0 | 1 | 0 | $\bar{1}$ | 0 | 0 | 0 | $\bar{1}$ | 0 | $\bar{1}$ | 0 |

Two successive iteration steps

$$\begin{aligned}
x_{i+1} &= x_i - m\,\sigma_i\,2^{-S(m,i)}\,y_i \\
y_{i+1} &= y_i + \sigma_i\,2^{-S(m,i)}\,x_i \\
z_{i+1} &= z_i - \sigma_i\,\alpha_{m,i}
\end{aligned}$$

(i+1)

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & -m\,\sigma_i\,2^{-i} \\ \sigma_i\,2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

(i+2)

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 & -m\,\sigma_{i+1}\,2^{-i-1} \\ \sigma_{i+1}\,2^{-i-1} & 1 \end{bmatrix} \begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -m\,\sigma_{i+1}\,2^{-i-1} \\ \sigma_{i+1}\,2^{-i-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & -m\,\sigma_i\,2^{-i} \\ \sigma_i\,2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$= \begin{bmatrix} 1 - m\,\sigma_{i+1}\,\sigma_i\,2^{-2i-1} & -m\,\sigma_i\,2^{-i} - m\,\sigma_{i+1}\,2^{-i-1} \\ \sigma_{i+1}\,2^{-i-1} + \sigma_i\,2^{-i} & -m\,\sigma_{i+1}\,\sigma_i\,2^{-2i-1} + 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$= \begin{bmatrix} 1 - m\,\sigma_i\,\sigma_{i+1}\,2^{-2i-1} & -m\left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) \\ \left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) & 1 - m\,\sigma_i\,\sigma_{i+1}\,2^{-2i-1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$
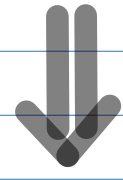
CSD

$$\begin{bmatrix} 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} & -m\left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) \\ \left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) & 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} \end{bmatrix}$$

$$\boxed{\sigma_i \neq 0} \quad \Longrightarrow \quad \boxed{\sigma_i \sigma_{i+1} = 0} \qquad \text{property of Booth encoding}$$

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & -m\,\sigma_i 2^{-i} \\ \sigma_i 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 - m\,\boxed{\sigma_i \sigma_{i+1}}\,2^{-2i-1} & -m(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}) \\ (\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}) & 1 - m\,\boxed{\sigma_i \sigma_{i+1}}\,2^{-2i-1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 & -m(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}) \\ (\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}) & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\begin{aligned} x_{i+2} &= x_i - m(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1})\, y_i \\ y_{i+2} &= (\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1})\, x_i + y_i \end{aligned}$$

CSD

$$\boxed{\begin{aligned} x_{i+2} &= x_i - m\sigma_i 2^{-i}\,\boxed{y_i} - m\sigma_{i+1} 2^{-i-1}\,\boxed{y_i} \\ y_{i+2} &= y_i + \sigma_i 2^{-i}\,\boxed{x_i} + \sigma_{i+1} 2^{-i-1}\,\boxed{x_i} \\ z_{i+2} &= z_i - \sigma_i\, \alpha_{m,i} - \sigma_{i+1}\, \alpha_{m,i+1} \end{aligned}}$$

$$\boxed{\sigma_i = 0} \implies \text{inc/dec} \quad \text{no rotation,}$$
$$\text{but compensate the scale factor.}$$

$$
\begin{aligned}
x_{i+1} &= x_i + m \cdot 2^{-2i-1} x_i \qquad &&\text{inc/dec} \\
y_{i+1} &= y_i + m \cdot 2^{-2i-1} y_i \qquad &&\text{inc/dec} \\
z_{i+1} &= z_i &&m = +1 / m = -1
\end{aligned}
$$

$$
\begin{aligned}
x_{i+1} &= \left(1 + m \cdot 2^{-2i-1}\right) x_i \\
y_{i+1} &= \left(1 + m \cdot 2^{-2i-1}\right) y_i \\
z_{i+1} &= z_i
\end{aligned}
$$

$$
\begin{aligned}
x_{i+2} &= \left(1 + m \cdot 2^{-2i-3}\right) x_{i+1} = \left(1 + m \cdot 2^{-2i-3}\right)\left(1 + m \cdot 2^{-2i-1}\right) x_i \\
y_{i+2} &= \left(1 + m \cdot 2^{-2i-3}\right) y_{i+1} = \left(1 + m \cdot 2^{-2i-3}\right)\left(1 + m \cdot 2^{-2i-1}\right) y_i \\
z_{i+1} &= z_i
\end{aligned}
$$

$$2^{-2i-3} \cdot 2^{-2i-1} = 2^{-4i-4} \ll 1$$

$$
\begin{aligned}
x_{i+2} &= \left(1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1}\right) x_i \\
y_{i+2} &= \left(1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1}\right) y_i \\
z_{i+1} &= z_i
\end{aligned}
$$

$$m=1, \quad S(m,i) = i$$

$$\boxed{CSD}$$

Cond ① $\quad 0 \leqslant i \leqslant \frac{1}{4}(n-3)$

$$x_{i+1} = x_i - \sigma_i 2^{-i} y_i$$
$$y_{i+1} = y_i + \sigma_i 2^{-i} x_i$$
$$z_{i+1} = z_i - \sigma_i \tan^{-1}(2^{-i})$$

$$x_{i+1} = x_i - \sigma_i 2^{-i} y_i$$
$$y_{i+1} = y_i + \sigma_i 2^{-i} x_i$$
$$z_{i+1} = z_i - \sigma_i \tan^{-1}(2^{-i})$$

Cond ② $\quad \frac{1}{4}(n-3) < i \leqslant \frac{1}{2}(n+1)$

$\sigma_i \neq 0$

$$x_{i+1} = x_i - \sigma_i 2^{-i} y_i$$
$$y_{i+1} = y_i + \sigma_i 2^{-i} x_i$$
$$z_{i+1} = z_i - \sigma_i \tan^{-1}(2^{-i})$$

$\sigma_i \neq 0$

$$x_{i+2} = x_i - m\sigma_i 2^{-i} \boxed{y_i} - m\widehat{\sigma_{i+1}} 2^{-i-1} \boxed{y_i}$$
$$y_{i+2} = y_i + \sigma_i 2^{-i} \boxed{x_i} + \sigma_{i+1} 2^{-i-1} \boxed{x_i}$$
$$z_{i+2} = z_i - \sigma_i \alpha_{m,i} - \sigma_{i+1} \alpha_{m,i+1}$$

$\sigma_i = 0$

$$x_{i+1} = \boxed{x_i} + m \cdot 2^{-2i-1} \boxed{x_i}$$
$$y_{i+1} = \boxed{y_i} + m \cdot 2^{-2i-1} \boxed{y_i}$$
$$z_{i+1} = z_i$$

$\sigma_i = 0$

$$x_{i+2} = \left(1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1}\right) \boxed{x_i}$$
$$y_{i+2} = \left(1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1}\right) \boxed{y_i}$$
$$z_{i+1} = z_i$$

Cond ③ $\quad \frac{1}{2}(n+1) < i$

$\sigma_i \neq 0$

$$x_{i+1} = x_i - \sigma_i 2^{-i} y_i$$
$$y_{i+1} = y_i + \sigma_i 2^{-i} x_i$$
$$z_{i+1} = z_i - \sigma_i \tan^{-1}(2^{-i})$$

$\sigma_i \neq 0$ or $\sigma_i = 0$

$$x_{i+2} = x_i - m\sigma_i 2^{-i} \boxed{y_i} - m\widehat{\sigma_{i+1}} 2^{-i-1} \boxed{y_i}$$
$$y_{i+2} = y_i + \sigma_i 2^{-i} \boxed{x_i} + \sigma_{i+1} 2^{-i-1} \boxed{x_i}$$
$$z_{i+2} = z_i - \sigma_i \alpha_{m,i} - \sigma_{i+1} \alpha_{m,i+1}$$

$\sigma_i = 0$

$$x_{i+1} = x_i$$
$$y_{i+1} = y_i$$
$$z_{i+1} = z_i$$

$$i \leftarrow i+1 \qquad\qquad\qquad i \leftarrow i+2$$

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & -m\,\sigma_i 2^{-i} \\ \sigma_i 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 - m\,\boxed{\sigma_i \sigma_{i+1}}\, 2^{-2i-1} & -m\left(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}\right) \\ \left(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}\right) & 1 - m\,\boxed{\sigma_i \sigma_{i+1}}\, 2^{-2i-1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\boxed{\sigma_i = 0} \;\Longrightarrow\; \boxed{\sigma_i \sigma_{i+1} = 0}$$

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 & -m\left(\sigma_i 2^{-i} \nearrow^{0} + \sigma_{i+1} 2^{-i-1}\right) \\ \left(\sigma_i 2^{-i} \nearrow^{0} + \sigma_{i+1} 2^{-i-1}\right) & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$x_{i+2} = x_i - m\left(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}\right) y_i$$
$$y_{i+2} = \left(\sigma_i 2^{-i} + \sigma_{i+1} 2^{-i-1}\right) x_i + y_i$$

$$x_{i+2} = x_i - m\,\sigma_i 2^{-i} y_i - m\,\sigma_{i+1} 2^{-i-1} y_i$$
$$y_{i+2} = y_i + \sigma_i 2^{-i} x_i + \sigma_{i+1} 2^{-i-1} x_i$$

$$\boxed{\sigma_i = 0} \quad \boxed{\sigma_{i+1} \neq 0}$$

$$x_{i+2} = \boxed{x_i} \qquad \boxed{\begin{aligned} &- m\,\sigma_{i+1} 2^{-i-1} y_i \\ &+ \sigma_{i+1} 2^{-i-1} x_i \end{aligned}}$$
$$y_{i+2} = \boxed{y_i}$$

$$\boxed{\sigma_i \neq 0} \quad \boxed{\sigma_{i+1} = 0}$$

$$x_{i+2} = \boxed{x_i} \qquad \boxed{\begin{aligned} &- m\,\sigma_i 2^{-i} y_i \\ &+ \sigma_i 2^{-i} x_i \end{aligned}}$$
$$y_{i+2} = \boxed{y_i}$$

$$\boxed{\sigma_i = 0} \quad \boxed{\sigma_{i+1} = 0}$$

$$x_{i+2} = \boxed{x_i}$$
$$y_{i+2} = \boxed{y_i}$$

Cond II $\frac{1}{4}(n-3) < i \leqslant \frac{1}{2}(n+1)$

$\sigma_i \neq 0$

$$x_{i+2} = x_i - m\sigma_i 2^{-i}\boxed{y_i} - m\widehat{\sigma_{i+1}} 2^{-i-1}\boxed{y_i}$$
$$y_{i+2} = y_i + \sigma_i 2^{-i}\boxed{x_i} + \sigma_{i+1} 2^{-i-1}\boxed{x_i}$$
$$z_{i+2} = z_i - \sigma_i \alpha_{m,i} - \sigma_{i+1} \alpha_{m,i+1}$$

parallel $\leftarrow$

$\sigma_i = 0$

$$x_{i+2} = (1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1})x_i$$
$$y_{i+2} = (1 + m \cdot 2^{-2i-3} + m \cdot 2^{-2i-1})y_i$$
$$z_{i+1} = z_i$$

parallel $\leftarrow$

$\sigma_i = 0$

   -- postphone this computation
     until the whole iteration process has been completed
  - use a Wallace tree to implement this in parallel

$\sigma_i \neq 0$

   - executes a rotation by either $\alpha_{m,i}$ or $\alpha_{m,i+1}$
   - multiplex the different shifts, $\boxed{\text{4-to-2 cell}}$

Cond ③ · $\frac{1}{2}(n+1) < i$

 — analogous to the termination algorithm
 — but improved design regularity
  : tree structurre are not necessary

 — the original iterations are paired
  always two subsequent iterations are
     merged into a new single iteration

 — a 4-to-2 adder cell (x_i, y_i)
   a 3-to-2 adder cell (z_i)
   a 3:1 multiplexer

σ_i	σ_{i+1}
0	0
0	1
1	0

Termination Algorithm

quit the iteration process as early as possible

termination algorithm
- T.C. Chen   IBM Journal of Research and Development
                1972
                Automatic computation of exponentials
                logarithms, ratios, and square roots

- Timmermann   Modified CORDIC algorithms

the 2nd half of the n iterations
- can be substituted by 2 multications in parallel

A fully parallel n-bit wallace tree multiplier
            - $2 \cdot \log_2 (n)$ FA time units

algorithm + prediction + termination
$(n+1)$ + $2\log_2 \left(\frac{n}{2}\right)$ + $\log_2 (n)$ = $n + 3\log_2 (n) - 1$

Modified CORDIC

Timmermann   1989   Electronics Letters

$$x_n = k_m \{ \quad x_0 \cos [\sqrt{(m)} \, \alpha] - \sqrt{(m)} \, y_0 \sin [\sqrt{(m)} \, \alpha] \}$$
$$y_n = k_m \{ 1/\sqrt{(m)} \, x_0 \sin [\sqrt{(m)} \, \alpha] + \quad y_0 \cos [\sqrt{(m)} \, \alpha] \}$$
$$z_n = z_0 + \alpha$$

k_m : the scaling factor

m : the coordinate system $(0, -1, +1)$

α : the rotation angle

$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$: the initial values depends on the iteration goal

Data dependency across iteration

$\rightarrow$ CSA no benefit

① 1st half iterations : the most significant contribution

the rotation angle $\boxed{\alpha_i = \dfrac{1}{\sqrt{(m)}} \tan^{-1} [\sqrt{(m)}\ 2^{-S(m,i)}]}$

$S(m,i)$ the iteration shift values

α_i decreases with the increasing
iteration index i

② 2nd half iterations : can improve the accuracy
only by one bit each

Cond ①	$0 \leqslant i \leqslant \frac{1}{4}(n-3)$	$\Big\}$ 1st half
Cond ②	$\frac{1}{4}(n-3) < i \leqslant \frac{1}{2}(n+1)$	
Cond ③	$\frac{1}{2}(n+1) < i$	$\Big\}$ 2nd half

rotation $z_n \to 0$

$$x_n = k_m \{ \quad x_0 \cos [\sqrt{(m)} \propto] - \sqrt{(m)} \, y_0 \sin [\sqrt{(m)} \propto] \}$$
$$y_n = k_m \{ 1/\sqrt{(m)} \, x_0 \sin [\sqrt{(m)} \propto] + \quad y_0 \cos [\sqrt{(m)} \propto] \}$$

Vectoring $y_n \to 0$

$$x_n = k_m \sqrt{x_0^2 + m \, y_0^2}$$
$$z_n = z_0 + 1/\sqrt{(m)} \, \tan^{-1} [\sqrt{(m)} \, y_0 / x_0]$$

2nd half iterations : can improve the accuracy
only by one bit each

replace these iterations by a single rotation
after the remaining rotation angle
has been reduced using a fixed number of
pure CORDIC iterations

this truncation reduces the latency time and saves area
although the truncation requires extra hardware

the necessary minimum number of iterations

Rotation mode $(z \to 0)$

After j CORDIC rotations have been performed (x_j, y_j)
the z path contains the remaining rotation angle z_j

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \cos[\sqrt{(m)}\, z_j] & -\sqrt{(m)}\, \sin[\sqrt{(m)}\, z_j] \\ 1/\sqrt{(m)} \sin[\sqrt{(m)}\, z_j] & \cos[\sqrt{(m)}\, z_j] \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

$$x_n = k_m \left\{ x_0 \cos[\sqrt{(m)}\, \alpha] - \sqrt{(m)}\, y_0 \sin[\sqrt{(m)}\, \alpha] \right\}$$
$$y_n = k_m \left\{ 1/\sqrt{(m)}\, x_0 \sin[\sqrt{(m)}\, \alpha] + y_0 \cos[\sqrt{(m)}\, \alpha] \right\}$$
$$z_n = z_0 + \alpha$$

assume $k_m = 1$ $\leftarrow$ 2nd half iteration does not
affect scaling factors

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} \cos[\sqrt{(m)}\, z_j] & -\sqrt{(m)}\, \sin[\sqrt{(m)}\, z_j] \\ 1/\sqrt{(m)}\, \sin[\sqrt{(m)}\, z_j] & \cos[\sqrt{(m)}\, z_j] \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

Taylor Series expansions to $\sin\theta$, $\cos\theta$
take only the first terms

$$\sin\theta = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$
$$\cos\theta = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -\sqrt{(m)}\cdot\sqrt{(m)}\, z_j \\ 1/\sqrt{(m)}\cdot\sqrt{(m)}\, z_j & 1 \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -m\, z_j \\ z_j & 1 \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

for a sufficiently small z_j

the required precision of n-bit
the upper limit on the maximal remainder

$$z_j \leqslant \frac{1}{\sqrt{(m)}} \tan^{-1}[\sqrt{(m)}\, 2^{-j+1}]$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -m\, z_j \\ z_j & 1 \end{bmatrix} \begin{bmatrix} x_j \\ y_j \end{bmatrix}$$

for a sufficiently small z_j

the required precision of n-bit
the upper limit on the maximal remainder

$$\frac{1}{2} z_j^2 \leq 2^{-n} \quad \Longrightarrow \quad z_j^2 \leq 2^{-n+1} \quad \Longrightarrow \quad z_j \leq 2^{\frac{-n+1}{2}}$$

$$z_j \leq \frac{1}{\sqrt{(m)}} \tan^{-1} \left[\sqrt{(m)}\, 2^{-j+1} \right]$$

$$j > \frac{n+1}{2}$$

Cond Ⓘ	$0 \leq i \leq \frac{1}{4}(n-3)$	} 1st half
Cond Ⓘ	$\frac{1}{4}(n-3) < i \leq \frac{1}{2}(n+1)$	
Cond Ⓘ	$\frac{1}{2}(n+1) < i$	} 2nd half

Rotation mode

$$x_n = x_j - m\, z_j\, y_j \quad (j > (n+1)/2)$$
$$y_n = z_j\, x_j + y_j \quad (j > (n+1)/2)$$

Vectoring mode

$$x_n = z_j \qquad\qquad j > (n+1)/2$$
$$z_n = z_j + y_j/x_j \qquad j > (n/3) + 0.472$$

the prediction algorithm : rotation mode (OK)

vectoring mode (X)

2nd half of the n iterations in rotation mode

~ replaced by 2 multiplications in parallel

$$\begin{cases} z_j * x_j \\ z_j * y_j \end{cases}$$

A fully parallel n-bit Wallace tree multiplier : $2\log_2(n)$ FA time unit

prediction + termination.

the Truncated. CORDIC Algorithm

- reduces the number of CORDIC iterations

- Multiplication / division hardware
    Booth Technique halves the amount of partial products
    Carry Save Architecture
    .

$k_m \neq 1 \Rightarrow$ multiplication $\Rightarrow$ multiplier anyway

Modified Booth Encoding   CSD

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} - \begin{bmatrix} 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} & -m\left(\sigma_i\,2^{-i} + \widehat{\sigma_{i+1}}\,2^{-i-1}\right) \\ \left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) & 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 & -m\left(\sigma_i\,2^{-i} + \widehat{\sigma_{i+1}}\,2^{-i-1}\right) \\ \left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$x_{i+2} = x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\widehat{\sigma_{i+1}}\,2^{-i-1}\,y_i$$

$$y_{i+2} = y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i$$

$$x_{i+2} = \left(x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\sigma_{i+1}\,2^{-i-1}\,y_i\right)$$
$$\left(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,x_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,x_i\right)$$

$$y_{i+2} = \left(y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i\right)$$
$$\left(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,y_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,y_i\right)$$

※ In Timmermann's paper, the modified Booth encoding refers to CSD (Canonic Signed Digit) $\boxed{\sigma_i\,\sigma_{i+1} = 0}$ not the generally known modified Booth encoding.

$$\begin{bmatrix} x_{i+2} \\ y_{i+2} \end{bmatrix} = \begin{bmatrix} 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} & -m\left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) \\ \left(\sigma_i\,2^{-i} + \sigma_{i+1}\,2^{-i-1}\right) & 1 - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1} \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

$$x_{i+2} = \left(x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\sigma_{i+1}\,2^{-i-1}\,y_i\right) - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1}\,x_i$$

$$y_{i+2} = \left(y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i\right) - m\,\boxed{\sigma_i\,\sigma_{i+1}}\,2^{-2i-1}\,y_i$$

$$x_{i+2} = \left(x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\sigma_{i+1}\,2^{-i-1}\,y_i\right)\left(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,x_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,x_i\right)$$

$$y_{i+2} = \left(y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i\right)\left(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,y_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,y_i\right)$$

$$\lambda(\sigma_i) = 1 \quad \text{for } |\sigma_i| = 0 \qquad \{0\}$$
$$\lambda(\sigma_i) = 0 \quad \text{for } |\sigma_i| = 1 \qquad \{1, \bar{1}\}$$

$$\lambda(\sigma_{i+1}) = 1 \quad \text{for } |\sigma_{i+1}| = 0 \qquad \{0\}$$
$$\lambda(\sigma_{i+1}) = 0 \quad \text{for } |\sigma_{i+1}| = 1 \qquad \{1, \bar{1}\}$$

$$x_{i+2} = \boxed{(x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\sigma_{i+1}\,2^{-i-1}\,y_i)}\boxed{(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,x_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,x_i)}$$

$$y_{i+2} = \boxed{(y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i)}\boxed{(1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,y_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,y_i)}$$

rotation by $\alpha_{m,i}$ or $\alpha_{m,i+1}$

multiplex the diff. shifts

$\Rightarrow$ 4-to-2 cells x_{i+2}, y_{i+2}

  3-to-2 cells z_{i+2}

Timmermann's constant scaling factor
for n-bit precision.
parallelizable
late evaluation
after all iterations
Wallace Tree

σ_i's are recoded in parallel

\# of non-zero σ_i's at most half of max value w/o recoding

$$\boxed{\sigma_i \, \sigma_{i+1}} = 0$$

$$\boxed{\sigma_i \, \sigma_{i+1}}$$

case ①	0	1
case ②	1	0
case ③	0	0

$$(1 + m 2^{-2i-1})(1 + m 2^{-2i-3}) = 1 + m 2^{-2i-1} + m 2^{-2i-3}$$

$$\prod_{j=0}^{n} (1 + m 2^{-2i-2j-1}) = 1 + \sum_{j=0}^{n} m 2^{-2i-2j-1}$$

$$(1 + m 2^{-2i-0-1})(1 + m 2^{-2i-2-1})(1 + m 2^{-2i-4-1})(1 + m 2^{-2i-6-1}) \cdots$$
$$= 1 + m 2^{-2i-0-1} + m 2^{-2i-2-1} + m 2^{-2i-4-1} + m 2^{-2i-6-1}$$

Modified Booth Encoding

$$\lambda(t) = 1 \quad \text{for } |t| = 0$$
$$\lambda(t) = 0 \quad \text{for } |t| = 1 \quad \{1, \bar{1}\}$$

$$x_{i+2} = (x_i - m\,\sigma_i\,2^{-i}\,y_i - m\,\sigma_{i+1}\,2^{-i-1}\,y_i)$$
$$= (1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,x_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,x_i$$
$$y_{i+2} = (y_i + \sigma_i\,2^{-i}\,x_i + \sigma_{i+1}\,2^{-i-1}\,x_i)$$
$$= (1 + \lambda(\sigma_i)\,m\,2^{-2i-1}\,y_i + \lambda(\sigma_{i+1})\,m\,2^{-2i-3}\,y_i)$$

Modified Booth Encoding

2-bit encoding

2^1 2^0

scale factor

encoding	scale factor calc	result
all zero's	$0 \cdot 2 + 0 = +0$	$+0$
end of 1's	$0 \cdot 2 + 1 = +1$	$+X$
isolated 1	$1 \cdot 2 + \bar{1} = +1$	$+X$
end of 1's	$1 \cdot 2 + 0 = +2$	$+2X$
start of 1's	$\bar{1} \cdot 2 + 0 = -2$	$-2X$
isolated 0	$\bar{1} \cdot 2 + 1 = -1$	$-X$
start of 1's	$0 \cdot 2 + \bar{1} = -1$	$-X$
all 1's	$0 \cdot 2 + 0 = 0$	$+0$

Scale factor $\{0, \pm 1, \pm 2\}$

not the one Timmermann's paper refers to

Original Booth Encoding

n:even

Y= 0 0 | 1 0 | 1 1 | 1 0 | 1 1 | 0

| | 1 | 1 | 0 → | | 0 | T̄ | 0 : ~X
| | 1 | 0 | 1 → | | T̄ | 1 | 0 : ~X
| | 1 | 1 | 1 → | | 0 | 0 | 0 : +0
| | 1 | 0 | 1 → | | T̄ | 1 | 0 : ~X
| | 0 | 0 | 1 → | | 0 | 1 | 0 : +X

n:even

Y= 0 0 | 1 0 | 1 1 | 1 0 | 1 1 | 0

n:even

Y= 0 0 | 1 0 | 1 1 | 1 0 | 1 1 | 0

0 T̄ 0 0 1 T̄ 0 1

Y= 0 0 | 1 0 | 1 1 | 1 0 | 1 1 | 0

| | 1 | 1 | 0 → | | 0 | T̄
| | 1 | 0 | 1 → | | T̄ | 1
| | 1 | 1 | 1 → | | 0 | 0
| | 1 | 0 | 1 → | | T̄ | 1
| | 0 | 0 | 1 → | | 0 | 1

0 1 T̄ 1 0 0 T̄ 1 0 T̄
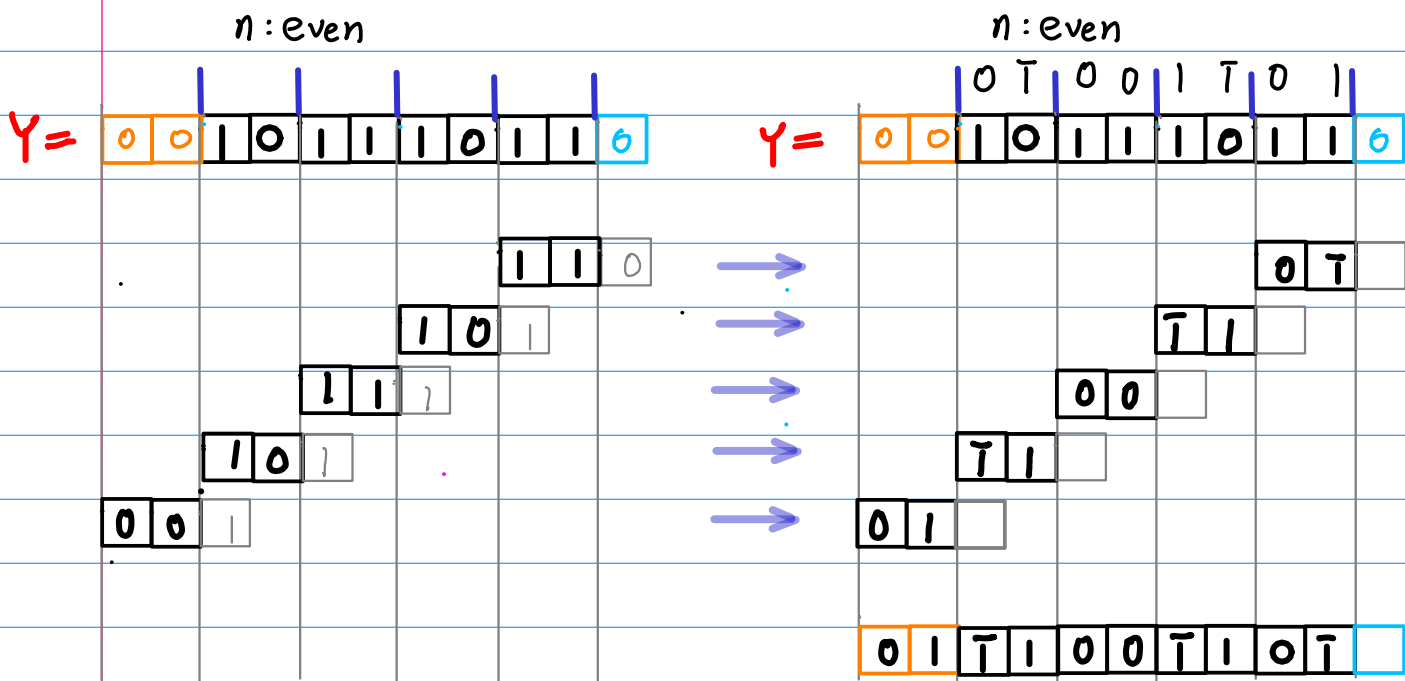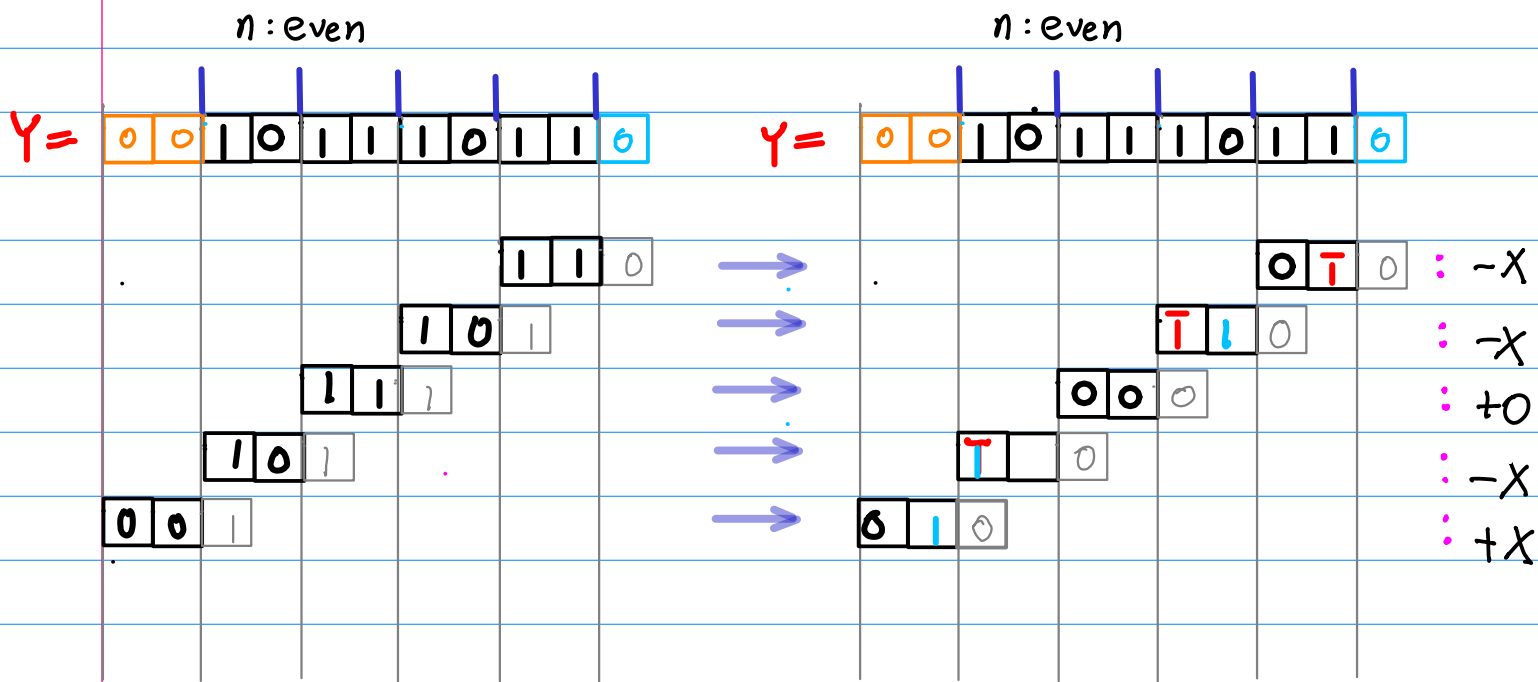
After the 1st Pass
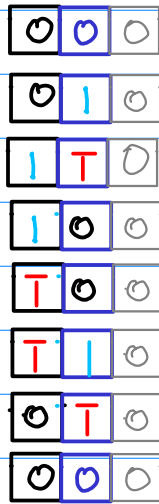
possible boundary cases
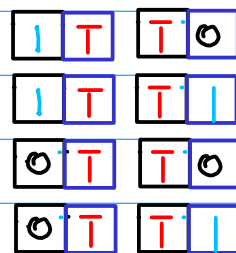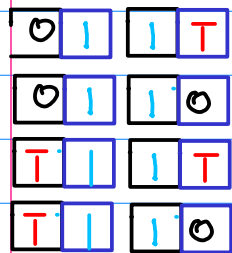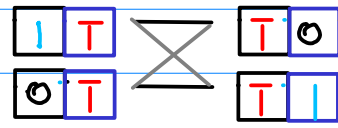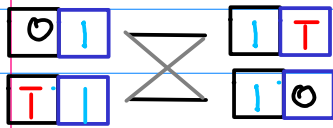
Pass 2 Operation
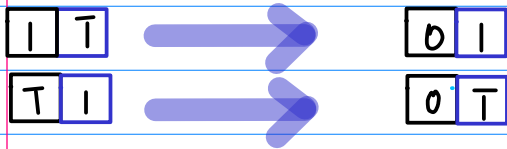
iterative application
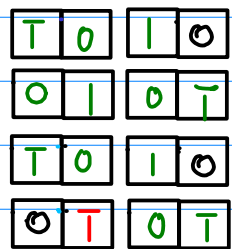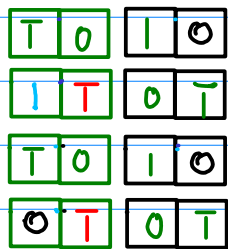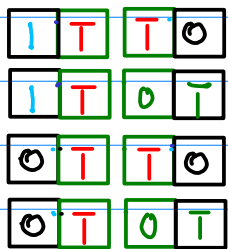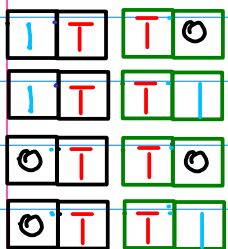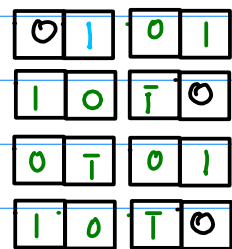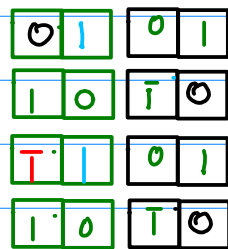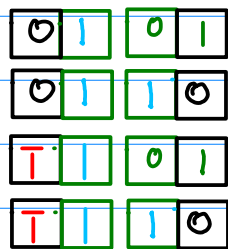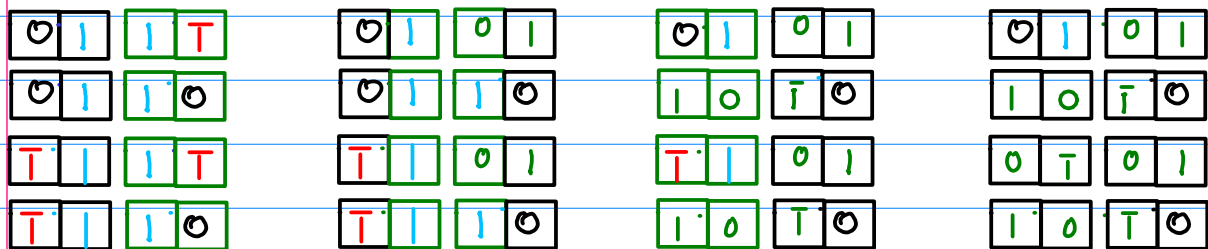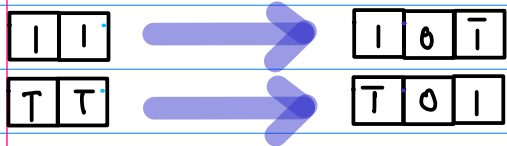
$$\sigma_i \sigma_{i+1} = 0$$

the 2nd pass

$Y=$ [0][0][1][0][1][1][1][0][1][1][0] $\boxed{\sigma_i \, \sigma_{i+1}} \not= 0$

[0][1][$\bar{1}$][1][0][0][$\bar{1}$][1][0][$\bar{1}$][ ]
                               [0][$\bar{1}$]

[0][1][$\bar{1}$][1][0][0][$\bar{1}$][1][0][$\bar{1}$][ ]
                             [1][0]

[0][1][$\bar{1}$][1][0][0][$\bar{1}$][1][0][$\bar{1}$][ ]
                         [$\bar{1}$][1]

[0][1][$\bar{1}$][1][0][0][0][T][0][$\bar{1}$][ ]
                       [0][0]

[0][1][$\bar{1}$][1][0][0][0][T][0][$\bar{1}$][ ]
                     [0][0]

[0][1][$\bar{1}$][1][0][0][0][T][0][$\bar{1}$][ ]
                   [1][0]

[0][1][$\bar{1}$][1][0][0][0][T][0][$\bar{1}$][ ]
           [$\bar{1}$][1]

[0][1][0][T][0][0][0][T][0][$\bar{1}$][ ]
      [1][0]

[0][1][0][T][0][0][0][T][0][$\bar{1}$][ ] $\boxed{\sigma_i \, \sigma_{i+1}} = 0$

CSD approach

Iterative Reduction of 1-runs

$$\boxed{\sigma_i\,\sigma_{i+1}} = 0$$

Unique encoding

```
0  0 | 1 0 1 1 1 0 1 1 | 0
            0 1 1 0
0  0 | 1 0 1 1 1 1 0 1̄ | 0
          0 1 1 1 1 0
0  0 | 1 1 0 0 0 1̄ 0 1̄ | 0
        0 1 1 0
0  1 | 0 1̄ 0 0 0 1̄ 0 1̄ | 0
```

```
0 1 1 0
1 0 1̄ 0
```

```
0 1 1 1 0
1 0 0 1̄ 0
```

```
0 1 1 1 1 0
1 0 0 0 1̄ 0
```

```
0 1 1 1 1 1 0
1 0 0 0 0 1̄ 0
```

```
0 1 1 1 1 1 1 0
1 0 0 0 0 0 1̄ 0
```
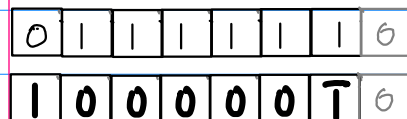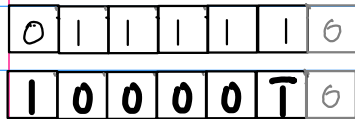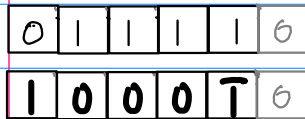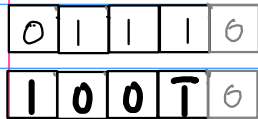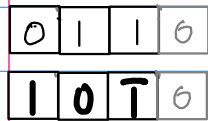
Verification

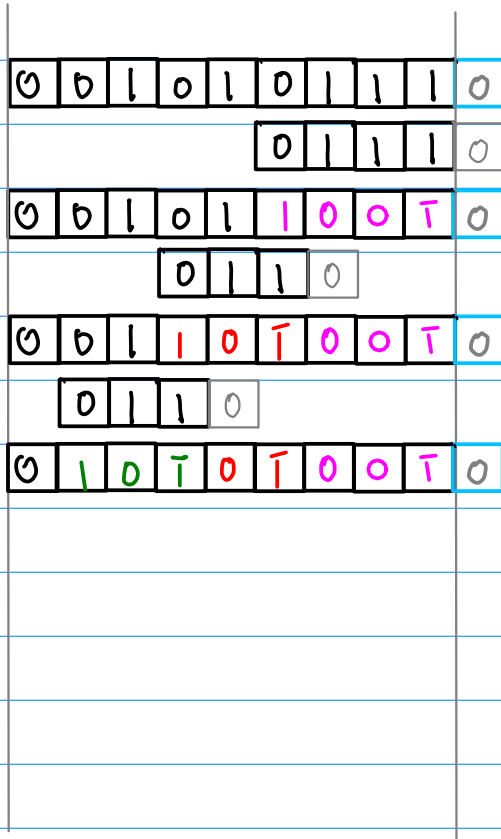|		2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0	
0	0	1	0	1	1	1	0	1	1	0
0	1	$\bar{1}$	1	0	0	$\bar{1}$	1	0	$\bar{1}$	
0	1	0	$\bar{1}$	0	0	0	$\bar{1}$	0	$\bar{1}$	
0	1	0	$\bar{1}$	0	0	0	$\bar{1}$	0	$\bar{1}$	0

$2^7 + 2^5 + 2^4 + 2^3 + 2^1 + 2^0 = 128 + 32 + 16 + 8 + 2 + 1 = 187$

$2^8 - 2^7 + 2^6 - 2^3 + 2^2 - 2^0 = 256 - 128 + 64 - 8 + 4 - 1 = 187$

$2^8 - 2^6 - 2^2 - 2^0 = 256 - 64 - 4 - 1 = 187$

$2^8 - 2^6 - 2^2 - 2^0 = 256 - 64 - 4 - 1 = 187$

0	0	1	0	1	0	1	1	1	0
					0	1	1	1	0
0	0	1	0	1	1	0	0	1	0
			0	1	1	0			
0	0	1	1	0	1	0	0	1	0
		0	1	1	0				
0	1	0	1	0	1	0	0	1	0

Canonical Signed Digit (CSD)

(1) the number of non-zero digits is minimal

(2) no two consecutive digits are both non-zero
     two non-zero digits are not adjacent