

Random Process Background

Young W Lim

May 12, 2023

Copyright (c) 2023 - 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons
"Attribution-NonCommercial-ShareAlike 3.0 Unported"
license.



Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

- 1 Measurable Space
 - Measurable Space
 - Sigma Alebra
 - Topological Space

- 2 Stochastic Process

Outline

- 1 Measurable Space
 - Measurable Space
 - Sigma Alebra
 - Topological Space
- 2 Stochastic Process

Space (1)

- A **space** consists of selected **mathematical objects** that are treated as **points**, and selected **relationships** between these **points**.
 - the nature of the **points** can vary widely:
for example, the points can be
 - elements of a set
 - functions on another space
 - subspaces of another space
 - It is the **relationships** that define the nature of the space.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (2)

- While modern mathematics uses many types of **spaces**, such as
 - Euclidean spaces
 - linear spaces
 - topological spaces
 - Hilbert spaces
 - probability spaces
- it does not define the notion of **space** itself.

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Space (3)

- a **space** is
a **set** (or a **universe**) with some added **structure**
- It is not always clear
whether a given **mathematical object** should be considered
as a geometric **space**, or an algebraic **structure**
- A general definition of **structure** embraces
all common types of **space**

[https://en.wikipedia.org/wiki/Space_\(mathematics\)](https://en.wikipedia.org/wiki/Space_(mathematics))

Mathematical objects (1)

- A **mathematical object** is an **abstract concept** arising in mathematics.
- an **mathematical object** is anything that has been (or could be) **formally defined**, and with which one may do
 - **deductive reasoning**
 - **mathematical proofs**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (2)

- Typically, a **mathematical object**
 - can be a **value** that can be assigned to a **variable**
 - therefore can be involved in **formulas**

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (3)

- Commonly encountered **mathematical objects** include
 - numbers
 - sets
 - functions
 - expressions
 - geometric objects
 - transformations of other mathematical objects
 - spaces

https://en.wikipedia.org/wiki/Mathematical_object

Mathematical objects (4)

- **Mathematical objects** can be very *complex*;
 - for example, the followings are considered as **mathematical objects** in **proof theory**.
 - theorems
 - proofs
 - theories

https://en.wikipedia.org/wiki/Mathematical_object

Structure (1)

- a **structure** is a **set** endowed with some *additional features* on the **set**
 - e.g. an *operation*
 - *relation*
 - *metric*
 - *topology*
- Often, the *additional features* are attached or related to the set, so as to provide it with some *additional meaning* or *significance*.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Structure (2)

- A partial list of possible **structures** are
 - measures
 - algebraic structures (groups, fields, etc.)
 - topologies
 - metric structures (geometries)
 - orders
 - events
 - equivalence relations
 - differential structures
 - categories.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (1)

- A **mathematical space** is, informally, a **collection** of **mathematical objects** under consideration.
- The **universe** of **mathematical objects** within a **space** are *precisely defined entities* whose **rules** of *interaction* come baked into the **rules** of the **space**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (2)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** is also a **collection** of **objects**
 - but these **objects** are being pulled from a **space** (or **universe**) of **objects** where the **rules** and **definitions** have already been agreed upon

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (3)

- A **space** differs from a **mathematical set** in several important ways:
 - A **mathematical set** has no **internal structure**,
 - whereas a **space** usually has some **internal structure**.

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Mathematical space (4)

- having some **internal structure** could mean a variety of things, but typically it involves
 - *interactions* and *relationships* between **elements** of the **space**
 - *rules* on how to *create* and *define* **new elements** of the **space**

<https://www.localmaxradio.com/questions/what-is-a-mathematical-space>

Measurable space (1)

- A **measurable space** is any **space** with a **sigma-algebra** which can then be equipped with a **measure**
 - collection of **subsets** of the **space** following certain **rules** with a way to assign **sizes** to those sets.

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Measurable space (2)

- Intuitively, certain **sets** belonging to a **measurable space** can be given a **size** in a *consistent way*.

consistent way means that certain **axioms** are met:

- the **empty set** is given a **size** of zero
- if a measurable set is **contained** inside another one, then its **size** is **less than** or **equal to** the size of the **containing set**
- the size of a **disjoint union** of sets is the **sum** of the individual sets' **sizes**

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Probability space

- A **probability space** is simply a **measurable space** equipped with a **probability measure**.
- A **probability measure** has the special property of giving the entire space a size of **1**.
 - this then implies that the **size** of any disjoint union of sets (the sum of the **sizes** of the sets) in the **probability space** is less than or equal to 1

<https://www.quora.com/What-is-a-measurable-space-and-probability-space-intuitively-What-differences-do-they-have>

Outline

- 1 Measurable Space
 - Measurable Space
 - **Sigma Alebra**
 - Topological Space

- 2 Stochastic Process

Sigma algebra (1)

- We term the **structures** which allow us to use **measure** to be **sigma algebras**
- the only requirements for **sigma algebras** (on a **set** X) are:
 - the $\{\}$ and X are in the **set**.
 - if A is in the **set**, *complement*(A) is in the **set**.
 - for any **sets** E_i in the set,
 $\bigcup_i E_i$ is in the **set** (for countable i).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (2)

- The most intuitive way to think about a **sigma algebra** is that it is the kind of **structure** we can do **probability** on.
 - for example, we can assign ratios of areas and length, so the **measure** on such a **set** X tells something about the **probability** of its **subsets**.
 - we can find the **probability** of **subsets** A and B because we know their ratios with respect to a **set** X ;
 - we also know that
 - (the measure of) their **complements** are defined, and
 - their **unions** and **intersections** are defined,
 - so we know how to find the **probability** of things in this set X .

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (3)

- The **sigma algebra** which contains the **standard topology** on \mathbb{R} (that is, *all open sets* on \mathbb{R}) is called the **Borel Sigma Algebra**, and the elements of this **set** are called **Borel sets**.
- What this gives us, is the set of **sets** on which outer measure gives our list of dreams. That is, if we take a **Borel set** and we check that length follows translation, additivity, and interval length, it will always hold.

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Sigma algebra (4)

- The **set** of Lebesgue measurable sets is the **set** of **Borel sets**, along with (union) all the sets which differ from a Borel set by a **set of measure 0**.
- More intuitively, it is all the sets we can normally measure, plus a bunch of stuff that doesn't affect our ideas of area or volume (think about the **border** of the circle above).

<https://medium.com/intuition/measure-theory-for-beginners-an-intuitive-approach-f5cea0cc2e7>

Borel Sets (1-1)

- a **Borel set** is any **set** in a **topological space** that can be formed from **open sets** (or, equivalently, from **closed sets**) through the operations of
 - countable union,
 - countable intersection, and
 - relative complement.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-2)

- For a **topological space X** , the collection of all Borel sets on X forms a σ -algebra, known as the **Borel algebra** or **Borel σ -algebra**.
- The **Borel algebra on X** is the smallest **σ -algebra** containing all open sets (or, equivalently, all closed sets).

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (1-3)

- **Borel sets** are important in measure theory, since any measure defined on the open sets of a space, or on the closed sets of a space, must also be defined on all Borel sets of that space.
- Any measure defined on the Borel sets is called a **Borel measure**.
- **Borel sets** and the associated **Borel hierarchy** also play a fundamental role in descriptive set theory.

https://en.wikipedia.org/wiki/Borel_set

Borel Sets (2)

- **Borel sets** are those obtained from intervals by means of the operations allowed in a **σ -algebra**. So we may construct them in a (transfinite) "sequence" of steps:
- ... And again and again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3-1)

1. Start with **finite unions** of **closed-open intervals**.
These sets are completely **elementary**, and they form an **algebra**.
2. **Adjoin countable unions** and **intersections** of elementary sets.
What you get already includes **open sets** and **closed sets**, **intersections** of an open set and a closed set, and so on.
Thus you obtain an **algebra**, that is still not a **σ -algebra**.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (3)

3. Again, **adjoin countable unions** and **intersections** to 2.
Observe that you get a strictly larger class, since a **countable intersection** of **countable unions** of intervals is not necessarily included in 2.
Explicit examples of sets in 3 but not in 2 include F_σ sets, like, say, the set of *rational numbers*.
4. And do the same again.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-1)

- And even after a sequence of steps we are not yet finished. Take, say, a countable union of a set constructed at step 1, a set constructed at step 2, and so on. This union may very well not have been constructed at any step yet. By axioms of σ -algebra, you should include it as well - if you want, as step ∞

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Borel Sets (4-2)

- (or, technically, the first infinite ordinal, if you know what that means).
- And then continue in the same way until you reach the first uncountable ordinal. And only then will you finally obtain the generated σ -algebra.

<https://math.stackexchange.com/questions/220248/understanding-borel-sets>

Outline

- 1 Measurable Space
 - Measurable Space
 - Sigma Alebra
 - Topological Space
- 2 Stochastic Process

Topology

- **topology**
from the Greek words
τόπος, 'place, location',
and λόγος, 'study'
is concerned with the **properties** of a **geometric object**
 - that are *preserved* under continuous deformations,
such as stretching, twisting, crumpling, and bending;
 - that is, without closing holes, opening holes,
tearing, gluing, or passing through itself.

<https://en.wikipedia.org/wiki/Topology>

Topological space (1)

- a **topological space** is, roughly speaking, a **geometrical space** in which **closeness** is defined but cannot necessarily be **measured** by a **numeric distance**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (2)

- More specifically, a **topological space** is
- a set whose elements are called points,
- along with an additional structure called a topology,
 - which can be defined as
 - a set of neighbourhoods for each point
 - that satisfy some axioms
 - formalizing the concept of closeness.

https://en.wikipedia.org/wiki/Borel_set

Topological space (3)

- There are several equivalent **definitions** of a topology, the most commonly used of which is the **definition** through **open sets**, which is easier than the others to manipulate.

https://en.wikipedia.org/wiki/Borel_set

Topological space (4)

- A **topological space** is the most general type of a **mathematical space** that allows for the definition of
 - **limits**,
 - **continuity**, and
 - **connectedness**.
- Common types of **topological spaces** include
 - **Euclidean spaces**,
 - **metric spaces** and
 - **manifolds**.

https://en.wikipedia.org/wiki/Borel_set

Topological space (5)

- Although very general, the concept of **topological spaces** is fundamental, and used in virtually every branch of modern mathematics.
- The study of **topological spaces** in their own right is called point-set topology or general topology.

https://en.wikipedia.org/wiki/Borel_set

Open set (1)

- an **open set** is a generalization of an **open interval** in the real line.
- a **metric space** is a **set** along with a **distance** defined between any two **points**
- in a **metric space**, an **open set** is a **set** that, along with every **point** P , contains all **points** that are **sufficiently near** to P
 - all **points** whose **distance** to P is less than some value depending on P

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- More generally, an **open set** is a **member** of a given **collection** of **subsets** of a given **set**, a **collection** that has the property of containing
 - every union of its **members**
 - every finite intersection of its members
 - the **empty set**
 - the **whole set** itself

https://en.wikipedia.org/wiki/Open_set

Open set (2)

- A **set** in which such a **collection** is given is called a **topological space**, and the **collection** is called a **topology**.
- These conditions are very loose, and allow enormous flexibility in the choice of **open sets**.
- For example,
 - every **subset** can be **open** (the discrete topology), or
 - no **subset** can be **open** (the indiscrete topology) except
 - the space itself and
 - the empty set .

https://en.wikipedia.org/wiki/Open_set

Open set (3)

Example:

- The *circle* represents the set of points (x, y) satisfying $x^2 + y^2 = r^2$.
- The *disk* represents the set of points (x, y) satisfying $x^2 + y^2 < r^2$.
- The *circle* set is an **open set**,
- the *disk* set is its **boundary set**, and
- the **union** of the *circle* and *disk* sets is a **closed set**.

https://en.wikipedia.org/wiki/Open_set

Open set (4)

- A **set** is a **collection** of distinct **objects**.
- Given a **set** A , we say that a is an **element** of A if a is one of the distinct **objects** in A , and we write $a \in A$ to denote this
- Given two **sets** A and B , we say that A is a **subset** of B if every element of A is also an element of B write $A \subseteq B$ to denote this.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (5) Open Balls

- We give these definitions in general, for when one is working in \mathbb{R}^n since they are really not all that different to define in \mathbb{R}^n than in \mathbb{R}^2
- An **open ball** $B_r(\mathbf{a})$ in \mathbb{R}^n centered at $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ with radius r is the set of all points $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ such that the distance between \mathbf{x} and \mathbf{a} is less than r
- In \mathbb{R}^2 an **open ball** is often called an **open disk**

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (6) Interior and boundary points

- Suppose that $S \subseteq \mathbb{R}^n$.
- A point $\mathbf{p} \in S$ is an **interior point** of S if there exists an **open ball** $B_r(\mathbf{p}) \subseteq S$.
- Intuitively, \mathbf{p} is an **interior point** of S if we can *squeeze* an entire **open ball** centered at \mathbf{p} within S
- A point $\mathbf{p} \in \mathbb{R}^n$ is a **boundary point** of S if all **open balls** centered at \mathbf{p} contain both **points** in S and **points** not in S .
- The **boundary** of S is the **set** ∂S that consists of all of the **boundary points** of S .

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (7)

- (Open and Closed Sets)
- A set $O \subseteq \mathbb{R}^n$ is **open** if every point in O is an **interior point**.
- A set $C \subseteq \mathbb{R}^n$ is **closed** if it contains all of its **boundary points**.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Open set (8)

- (Bounded and Unbounded)
- A set S
- is bounded if there is an open ball $B_M(0)$ such that $S \subseteq B$.
- Intuitively, this means that we can enclose all of the set S
- within a large enough ball centered at the origin. A set that is not bounded is called unbounded.

<https://ximera.osu.edu/mooculus/calculusE/continuityOfFunctionsOfSeveralVariables/digInOpenAnd>

Stochastic Process (1)

In probability theory and related fields, a **stochastic** (/stou'kæstɪk/) or **random** process is a mathematical object usually defined as a family of **random variables**.

The word stochastic in English was originally used as an adjective with the definition "pertaining to **conjecturing**", and stemming from a Greek word meaning "to aim at a mark, guess", and the Oxford English Dictionary gives the year 1662 as its earliest occurrence.

From Ancient Greek στοχαστικός (stokhastikós), from στοχάζομαι (stokhá-zomai, "aim at a target, guess"), from στόχος (stókhos, "an aim, a guess").

<https://en.wikipedia.org/wiki/Stochastic>
<https://en.wiktionary.org/wiki/stochastic>

Stochastic Process (2)

The definition of a **stochastic process** varies, but a **stochastic process** is *traditionally* defined as a collection of **random variables** indexed by some set.

The terms **random process** and **stochastic process** are considered synonyms and are used interchangeably, without the **index set** being precisely specified.

Both "**collection**", or "**family**" are used while instead of "**index set**", sometimes the terms "**parameter set**" or "**parameter space**" are used.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (3)

The term **random function** is also used to refer to a **stochastic** or **random process**, though sometimes it is only used when the stochastic process takes real values.

This term is also used when the **index sets** are **mathematical spaces** other than the **real line**,

while the terms **stochastic process** and **random process** are usually used when the **index set** is interpreted as time,

and other terms are used such as **random field** when the **index set** is n -dimensional **Euclidean space** \mathbb{R}^n or a manifold

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process (4)

A **stochastic process** can be denoted, by $\{X(t)\}_{t \in T}$, $\{X_t\}_{t \in T}$, $\{X(t)\}$, $\{X_t\}$ or simply as X or $X(t)$, although $X(t)$ is regarded as an abuse of function notation.

For example, $X(t)$ or X_t are used to refer to the **random variable** with the **index** t , and not the entire **stochastic process**.

If the **index set** is $T = [0, \infty)$, then one can write, for example, $(X_t, t \geq 0)$ to denote the **stochastic process**.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (1)

A **stochastic process** is defined as a collection of **random variables** defined on a common **probability space** (Ω, \mathcal{F}, P) ,

- Ω is a **sample space**,
- \mathcal{F} is a σ -**algebra**,
- P is a **probability measure**;
- the **random variables**, indexed by some set T ,
- all take values in the same **mathematical space** S , which must be **measurable** with respect to some σ -algebra Σ

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (2)

In other words, for a given **probability space** (Ω, \mathcal{F}, P) and a **measurable space** (S, Σ) , a **stochastic process** is a **collection** of S -valued **random variables**, which can be written as:

$$\{X(t) : t \in T\}.$$

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (3)

Historically, in many problems from the natural sciences a point $t \in T$ had the meaning of time, so $X(t)$ is a **random variable** representing a value observed at time t .

A **stochastic process** can also be written as $\{X(t, \omega) : t \in T\}$ to reflect that it is actually a function of two variables, $t \in T$ and $\omega \in \Omega$.

https://en.wikipedia.org/wiki/Stochastic_process

Stochastic Process Definition (4)

There are other ways to consider a stochastic process, with the above definition being considered the traditional one.

For example, a stochastic process can be interpreted or defined as a S^T -valued **random variable**, where S^T is the space of all the possible functions from the set T into the space S .

However this alternative definition as a "**function-valued random variable**" in general requires additional regularity assumptions to be **well-defined**.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (1)

The set T is called the **index set** or **parameter set** of the **stochastic process**.

Often this set is some subset of the real line, such as the natural numbers or an interval, giving the set T the interpretation of time.

https://en.wikipedia.org/wiki/Stochastic_process

Index set (2)

In addition to these sets, the index set T can be another set with a **total order** or a more general set, such as the Cartesian plane R^2 or n -dimensional **Euclidean space**, where an element $t \in T$ can represent a point in space.

That said, many results and theorems are only possible for **stochastic processes** with a **totally ordered index set**.

https://en.wikipedia.org/wiki/Stochastic_process

State space

The **mathematical space** S of a **stochastic process** is called its **state space**.

This mathematical space can be defined using integers, real lines, n -dimensional Euclidean spaces, complex planes, or more abstract mathematical spaces.

The **state space** is defined using elements that reflect the different values that the **stochastic process** can take.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (1)

A **sample function** is a single outcome of a **stochastic process**, so it is formed by taking a single possible value of each **random variable** of the **stochastic process**.

More precisely, if $\{X(t, \omega) : t \in T\}$ is a **stochastic process**, then for any point $\omega \in \Omega$, the mapping $X(\cdot, \omega) : T \rightarrow S$, is called a **sample function**, a **realization**, or, particularly when T is interpreted as time, a **sample path** of the **stochastic process** $\{X(t, \omega) : t \in T\}$.

https://en.wikipedia.org/wiki/Stochastic_process

Sample function (2)

This means that for a fixed $\omega \in \Omega$,
there exists a **sample function**
that maps the **index set** T to the **state space** S .

Other names for a **sample function** of a **stochastic process**
include **trajectory**, **path function** or **path**

https://en.wikipedia.org/wiki/Stochastic_process

